WORKING PAPERS
RESEARCH DEPARTMENT

WORKING PAPER NO. 13-37
DO SUPPLY RESTRICTIONS RAISE THE VALUE OF URBAN LAND?
THE (NEGLECTED) ROLE OF PRODUCTION EXTERNALITIES

Satyajit Chatterjee
Federal Reserve Bank of Philadelphia

Burcu Eyigungor
Federal Reserve Bank of Philadelphia

September 24, 2013

Supersedes Working Paper 12-25

RESEARCH DEPARTMENT, FEDERAL RESERVE BANK OF PHILADELPHIA
Ten Independence Mall, Philadelphia, PA 19106-1574 • www.philadelphiafed.org/research-and-data/

Do Supply Restrictions Raise the Value of Urban Land? The (Neglected) Role of Production Externalities[1]

Satyajit Chatterjee and Burcu Eyigungor

Federal Reserve Bank of Philadelphia

September 24, 2013

[1]Corresponding Author: Satyajit Chatterjee, Research Department, Federal Reserve Bank of Philadelphia, Ten Independence Mall, Philadelphia PA, 19106; (215) 574-3861, satyajit.chatterjee@phil.frb.org. The authors thank Roc Armenter, Gilles Duranton, and participants at the Federal Reserve Bank of Philadelphia Lunch Time Seminar series and the Wharton Macro Lunch series for thoughtful comments. The views expressed here are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System. This paper is free of charge at www.philadelphiafed.org/research-and-data/publications/working-papers/.

Abstract

Restriction on the supply of new urban land is commonly thought to raise the value of existing urban land. Our paper questions this view. We develop a tractable production-externality-based circular city model in which firms and workers choose locations and intensity of land use. Consistent with evidence, the model implies exponentially decaying density and price gradients. For plausible parameter values, an increase in the demand for urban land can lead to a smaller increase in urban rents in cities that cannot expand physically because they are less able to exploit the positive external effect of greater employment density.

Keywords: Land use, density gradients, agglomeration economies, commuting costs
JEL Codes: E10, R30, R52

1 Introduction

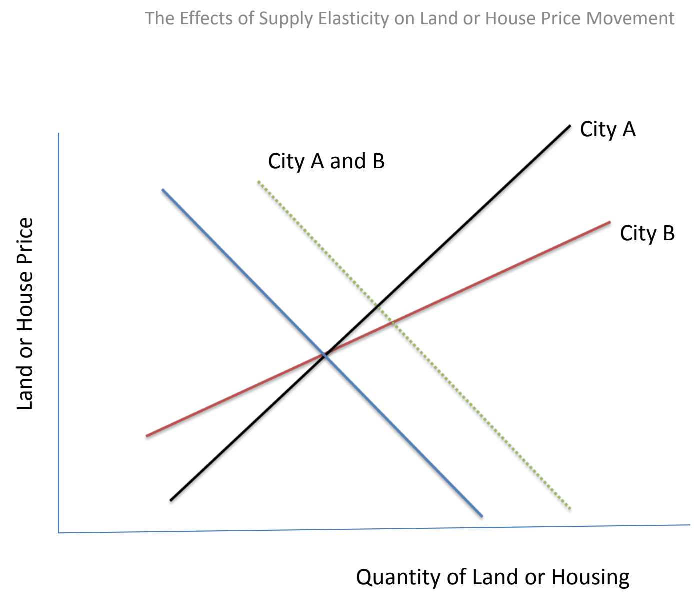
Recent studies have drawn attention to the fact that variations in the price elasticity of housing supply across metropolitan areas are related to differences in land use regulation (Green, Malpezzi, and Mayo (2005)) or to differences in the constraints imposed on urban expansion by geography (Saiz (2010)): More stringent regulations or more difficult geography reduce the price elasticity of housing supply.

Within a standard demand-supply framework, an implication of this finding is that all else the same, a similar increase in demand for urban land will lead to a bigger increase in land prices (and, therefore, house prices) in a city with more restrictions on urban expansion. Following Glaeser, Gyourko, and Saks (2006), Figure 1 plots the supply curve for urban land for two cities with different restrictions on urban expansion. The city with more stringent restrictions corresponds to the more steeply sloped (less elastic) supply function, labeled city A. If there is a shock that increases demand for urban land equally in the two cities—in the figure this is the movement from the solid (blue) downward-sloping line to the dotted (green) downward- sloping line—city A will see a smaller increase in supply and a higher increase in prices compared to city B.

Indeed, recent studies have attempted to link the size of house price movements over time to restrictions on urban development. Glaeser, Gyourko, and Saks (2006), Glaeser, Gyourko, and Saiz (2008) and Huang and Tang (2013) find that cities that have more stringent restrictions or more difficult topography have generally experienced bigger movements in house prices. But there are some puzzles. Davidoff (2010) notes that cities in the so-called "sand states" have experienced some of the most dramatic boom and bust in house prices even though the supply of housing in these cities is relatively elastic (Glaeser, Gyourko, and Saiz (2008) make a similar observation). In the same vein, Kuminoff and Pope (2013) observe that, within cities, the areas that appear to have the most elastic housing supply display the greatest volatility in house and land prices.

The goal of our paper is to caution against expecting a tight link between the size of

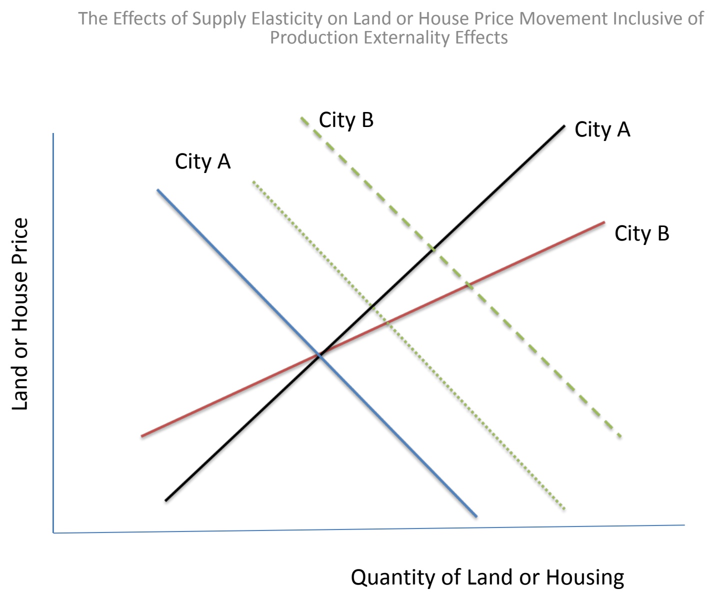
Figure 1



land (and house) price movements and restrictions on urban development. What Figure 1 misses is the reason why cities exist in the first place, namely, the presence of production externalities that lead to urban agglomerations.[1] Since these production externalities are tied to total employment in the city, constraints on the physical expansion of a city (induced by policy or topography) mean that city A cannot absorb as many new workers as city B and, therefore, cannot benefit as much from the production externality. This may hurt worker productivity in city A and, therefore, wages. Lower wages in city A relative to city B would, of course, be associated with relatively less demand for urban land in city A relative to city B. This effect would show up as *unequal* shifts in the demand curve for urban land, i.e., the *same* fundamental shock can lead to a larger shift in the demand curve in city B—shown as the dashed (green) downward-sloping line in Figure 2—than in city A. If this external effect

[1]There is an an extensive empirical literature documenting the existence of urban agglomeration economies: Larger cities confer a production advantage to businesses that locate there. Combes, Duranton, Gobillon, Puga, and Roux (2012) is one recent paper that estimates the size of the external effect. Rosenthal and Strange (2004) and Melo, Graham, and Noland (2009) provide a survey of the previous studies.

Figure 2



is strong enough, land and house prices might rise *more* in city B than in city A.

In this paper we analyze, both theoretically and quantitatively, the effect of restrictions on the growth of urban land on land rents when cities owe their existence to a production externality.[2] To accomplish our goal, we develop a tractable model of a production-externality-based city in which both firms and workers choose location as well as the intensity of land use. Thus the two key margins along which we may expect businesses and households to adapt to restrictions on the supply of new urban land are part of the model. The analytical simplicity of the model allows us to easily discern the main factors that determine whether supply restriction can actually be a force in favor of lower rather than higher urban land prices. We show that there are good reasons to expect the relationship between the elasticity of urban land supply and the response of urban land rents to demand shocks to be ambiguous, once production externalities are taken into account. This ambiguity is consistent with some of the puzzling contradictory evidence reported in the papers cited earlier.

[2]Although we focus on restrictions on the growth of urban land caused by topography or policy (such as an urban growth boundary), our results should also apply to restrictions of other types such building height restrictions that put an upper limit on urban density.

There is an extensive empirical literature on the impact of regulations aimed at constraining the growth of urban land (see, for instance, Fischel (1990) and the many references cited therein).[3] Furthermore, as noted in Glaeser, Gyourko, and Saks (2005a) and Glaeser, Gyourko, and Saks (2005b) there has been an important shift toward more stringent regulation of new housing construction over the last three decades, as well as a gradual strengthening of the power of neighborhood groups to obstruct new construction in expensive cities. While these facts and trends are well documented, theoretical analyses of the impact of such regulations have been relatively sparse. Brueckner (1990), Ding, Knaap, and Hopkins (1999), and Brueckner (2007) study the impact of urban growth boundaries in the context of the standard monocentric city model with a (negative) congestion externality, while Bertaud and Brueckner (2005) examine the impact of building-height restrictions, again in the standard monocentric city model. However, none of these studies allow for production externalities and therefore miss the production-side effects of such controls, which are the focus of this paper.

Our analytical framework is a variant of the Lucas and Rossi-Hansberg (2002) circular city model, which in turn builds on the seminal contribution of Fujita and Ogawa (1982). Lucas & Rossi-Hansberg worked out the nature of the spatial equilibrium when the productivity of a firm is positively affected by proximity to other firms. The only difference between our model and theirs is how proximity between firms is measured. In our model, the proximity between two firms located at different points in the plane is measured as the sum of the lengths of the rays connecting the two points to the city center; in Lucas and Rossi-Hansberg, proximity is measured as Euclidean distance or distance "as the crow flies." In our model, the internal structure of the city—which is endogenous as in Lucas & Rossi-Hansberg —can take exactly one of two forms. Either the city has a decentralized form, where firms are in every location in the city and workers reside at the same location as the firm they work for, or it has a monocentric form with a circular central business district (CBD) of positive radius and a

[3]Starting in the early 1970s, more than a dozen US states have enacted legislation to control/channel urban growth with the express purpose of increasing urban density and preventing environmentally sensitive areas from being lost to urbanization (see, for instance, Paulsen (2013) and references cited therein).

surrounding residential ring.

The analytical tractability of our model stems from the fact that, regardless of which city form prevails, the intra-city variation in all endogenous variables—business and residential rents, employment and residential densities, and wages—are predicted to be negative exponentials (over their relevant domains): $x(r) = x(0)e^{-\phi_x r}$, where r is distance from the city center (which is indexed by 0) and ϕ_x depends only on preference and technology parameters and city form. For each city form, the equilibrium of the model *uniquely* determines the heights of these exponential functions, or, equivalently, the value of endogenous variables at the city center (the values $x(0)$). Thus a potentially very complicated equilibrium problem involving *functions* is reduced to one involving a small number of scalars. Given the ubiquity of the exponential specification in empirical work (see, for instance, the survey by Anas, Arnott, and Small (1998)), we view our derivation of this specification from fundamentals as an independent contribution that strengthens the links between urban theory and empirics.[4]

Our findings are as follows. First, we show that a city that has a binding urban growth boundary (the restricted city) will experience a *smaller* increase in employment density at the city center than will a city that has no urban growth boundary (unrestricted city). This result is true regardless of the internal structure of the cities (decentralized or mononcentric) and is a general property of this class of models. Since productivity of a firm at the city center depends positively on a (weighted) average of employment densities at all locations in the city, a corollary to this result is that *business* rents at the city center will rise *less* in the restricted city.

The impact on *residential* rents is more complex. In the case where the cities are decentralized, residential land rents inherit the same ordering across the restricted and unrestricted cities as business rents since firms and workers compete for land in every location of the city.

[4] For instance, Glaeser and Kahn (2001) use the negative exponential specification to document that employment density gradients in US metropolitan areas have flattened over the last 50 years or so. By providing a structural interpretation to the exponents of e in a more general setting, our model can potentially shed light on forces that shape the observed historical trends in population, employment, and land price and rent gradients.

In other words, in decentralized cities, residential land rents will also rise *less* in the restricted city. If the city is monocentric, the behavior of residential rents depends on how the urban growth boundary affects wages at the city center. There are two countervailing forces at work. Since the restricted city has lower employment density at the city center relative to the unrestricted city, firms there use relatively more land per worker. This land-intensity effect is a force in favor of higher wages in the restricted city. On the other hand, since employment density is lower in the restricted city, the positive impact of the production externality is lower there as well. This externality effect is a force in favor of lower wages in the restricted city. If the production externality is at least as important in production as land, the externality effect will dominate and the restricted city will have lower wages and lower residential land rents. In contrast, if land is more important in production than the production externality, the restricted city will tend to have higher wages and, therefore, higher residential land rents.

Given these results, it is natural to ask, what is known empirically about the size of the two countervailing effects? We defer the answer to this question until later in the paper, after the model has been presented. It will turn out that there are good empirical reasons to think that the two effects are quite close in magnitude so that the effects of urban growth restrictions on land rents can be expected to go either way and differently for different cities. Furthermore, we show that the dominance of the externality effect need not contradict the empirical finding that elasticity of housing supply is lower for cities with more stringent land supply restrictions.

There is, however, one general objection to the dominance of the externality effect that we wish to address at the outset. If the externality effect dominates, the wage at the city center is predicted to be increasing in city population. This raises the possibility that, in this case, the only stable equilibrium configuration of cities is one in which everyone ends up living in one giant city. Indeed, previous studies (e.g., Henderson (1974)) have generally assumed that the land is more important in production than the production externality so as to rule out such equilibria. In our model, such an assumption is certainly sufficient to rule out the “one giant city” outcome, *but it is not necessary.* Provided land is sufficiently important in workers’ *utility*, welfare of workers will be (eventually) decreasing in population—the so-called “no-black-hole” condition—even if the production externality is more important to production than land itself. Therefore, as a matter of theory, the requirement of stability does not rule out the dominance of the externality effect over the land intensity effect. And, as an empirical matter, the no-black-hole condition is easily satisfied for plausible parameter values (including those that imply the dominance of the externality effect).[5]

The paper is organized as follows. Section 2 describes the environment. Sections 3 and 4 develop the equilibrium implications of this environment for the location decisions and intensity of land use by firms and workers. Section 5 analyzes how an urban growth boundary affects business and residential rents and other variables of interest when there is an increase in the demand for urban land, and discusses the empirical evidence on the magnitude of the externality and land-intensity effects. Section 6 concludes.

2 Model

Space is modeled as a flat, featureless plain extending infinitely in all directions, with an arbitrary point marked off as the center. In polar coordinates, the center is the point $(0, 0)$ and all other points have coordinates (r, θ), where r is the length of a straight line connecting the point to the center and θ is the angle this line makes at $(0, 0)$. Given that each point in space is physically indistinguishable from any other, it is natural to focus on allocations that are symmetric relative to the center. A location is then described fully by the radius, r, of the circle centered at $(0, 0)$ on which it resides, and there is a continuum of locations for each r (all the points on the circle of radius r).

Utility function of a worker depends on the consumption of the single *numeraire* good avaliable in this economy and on the service flow from land. A worker who resides in location

[5]Lucas (2001) and Lucas and Rossi-Hansberg (2002) also ruled out the dominance of the externality effect, but for a different reason noted later in this paper.

r has utility

$$U = c^{\beta}(r) l(r)^{1-\beta}, \;\; \beta \in (0,1), \tag{1}$$

where $l(r)$ is the consumption of land in location r and $c(r)$ is consumption at location r.

A firm has a technology to produce the single consumption good. The production function of a firm that uses one unit of land at location s is

$$Y(s) = Az(s)^{\gamma} n^{\alpha}(s), \;\; \alpha \in (0,1), \gamma > 0, \tag{2}$$

where $n(s)$ is the number of workers per unit of land at location s, A is a TFP term that is common to all firms in the city, and $z(s)^{\gamma}$ is a term—to be defined more precisely below—that captures the efficiency gain that comes from proximity to workers employed by firms in other locations.

A key assumption is that the proximity between any two firms is measured by the sum of the distance of the two firms from the city center. In other words, if one firm is located on a circle of radius r and the other firm is located on a circle of radius s, the distance of the firms to each other is simply $(r + s)$. The assumption that distance between two firms is measured by the sum of the lengths to the city center is reasonable if communication between workers in different firms requires travel to a central meeting place and the road system is radial. A second justification of this assumption is given below.

If we let $N(s)$ denote the number of workers employed by a firm at location s, the level of the production externality enjoyed by a firm at location r is

$$z(r) = \int_0^{\infty} 2\pi s \exp\left(-\delta\left(r + s\right)\right) N(s) ds.$$

Since $z(0) = \int_0^{\infty} 2\pi s \exp\left(-\delta s\right) N(s) ds$, the above definition implies

$$z(r) = z(0) \exp\left(-\delta r\right). \tag{3}$$

Thus, irrespective of the distribution of employment across the city, the level of the production externality decays at the rate δ with distance from the city center. The spatial distribution of employment affects the level of the production externality at any location only through the $z(0)$ term.

As will become evident, (3) is the reason why our model predicts that all density and price gradients follow exponential functions (and it is also the reason why our model is tractable). Given the importance of (3), we might ask, what other distance measures generate (3)? If we denote the general distance function as $\nu(r,s)$ and require that $\nu(r,s)=\nu(s,r)$ (symmetry), then it is straightforward to show that any symmetric distance function that generates (3) must be a linear transform of $r+s$.[6] Thus a second justification for our distance measure is that it is the only (symmetric) measure that is consistent with (3) and, therefore, with exponential density and price gradients.

There is a technology for commuting. This technology allows workers to commute to any firm that is located on the straight line that connects the worker's residential location to the city center. We follow Anas, Arnott, and Small (2000) and Lucas and Rossi-Hansberg (2002) and assume that a worker who resides in location s and commutes to a firm at location r has $\exp(-\kappa|s-r|)$ unit of time to devote to production, where $\kappa>0$.[7]

There is also a technology for converting land from its natural state into land that can be used by workers and firms. The cost of converting a unit of natural land into developed land is d units of the consumption good.

Finally, following convention, it is assumed that all land in the economy is owned by

[6] For the general distance function $z(r)=\int_0^S \exp(-\nu(r,s))N(s)ds$. We require that $z(r)=z(0)\exp(-\delta r)$, where δ is some positive constant. Then $z(0)=\int_0^S \exp(-\nu(r,s)+\delta r)N(s)ds$. Since this relationship must hold for any r, it follows that $\nu(r,s)$ must be of the form $a+\delta r+f(s)$. From symmetry $a+\delta r+f(s)=a+f(r)+\delta s$, which in turn implies $f(s)-f(r)=\delta\cdot(s-r)$. Hence $\nu(r,s)=A+\delta\cdot(r+s)$.

[7] As noted in Anas, Arnott, and Small (2000), this assumption is key to obtaining an exponentially declining land rent and population density function without making counterfactual assumptions on the structure of preferences for land. Coupled with our assumption regarding how proximity between firms is calculated, we can extend the negative exponential form to commercial rents as well as employment density. Note also that, to a first-order approximation, the (net) income of a commuter is $w(r)[1-\kappa|s-r|]$, which corresponds to the common assumption that the commuting cost is proportional to the hourly wage and linear in the distance traveled.

entities outside of the model. These entities decide whether to convert any given unit of natural land into developed land and then rent the developed land to workers and firms.

3 On the Internal Structure of the City

In this section, we show that the spatial organization of the city is consistent with only one of two forms, depending on technology and preference parameters. Either the city is *decentralized* with workers choosing to live next to their firms, or it is *monocentric* with a CBD of positive radius and a surrounding residential ring. This result can be rigorously established by applying the method described in Lucas and Rossi-Hansberg (2002) (section 3, pp. 1453-1462) to the case where $z(r)$ is of the form in (3), namely, $z(0)\exp(-\delta r)$. In the interest of brevity, what we do here is simply derive conditions on parameter values under which the two forms can prevail and show that these conditions are mutually exclusive and exhaust the parameter space.

We will assume that the set of developed locations comprises all points on and inside of a circle of radius S (this circle defines the city boundary). The question we want to answer is, how is this developed land allocated between commercial and/or residential use? It is customary in urban economic theory to approach land use in terms of bid rent functions (Alonso (1964) and Fujita (1989)). Let $w(r)$ be the market wage at location r. Turning first to firms, we let $q_F(r)$ be the maximum rent a firm would be willing to pay for a unit of land at location r. This quantity is simply $Az(r)^\gamma n^*(r)^\alpha - w(r)n^*(r)$, where $n^*(r)$ is the optimal choice of n conditional on locating at r, and is given by

$$n^*(r) = \left[A\alpha z(r)^\gamma / w(r)\right]^{1/(1-\alpha)}. \tag{4}$$

Then,

$$q_F(r) = \left[(1-\alpha)/\alpha\right]\left[\alpha A z(r)^\gamma w(r)^{-\alpha}\right]^{1/(1-\alpha)}. \tag{5}$$

As is intuitive, the maximum rent a firm is willing to pay depends positively on the location's productivity and negatively on the location's wage.

Turning to households, we let $q_H(r,s)$ be the maximum rent a worker would be willing to pay for a unit of land at location r, given that he will work at location s. Conditional on paying $q_H(r,s)$ in rent, a worker's optimal choices of c and l at location r are

$$c^*(r,s) = \beta w(s) \exp(\kappa|s-r|) \text{ and } l^*(r,s) = (1-\beta) w(s) \exp(\kappa|s-r|)/q_H(r,s), \tag{6}$$

and his optimal utility is $\beta^\beta (1-\beta)^{1-\beta} w(s) \exp(\kappa|s-r|) q_H(r,s)^{-(1-\beta)}$. If U is the maximum utility a worker can obtain from locating somewhere else, then

$$q_H(r,s) = (1-\beta)\, \beta^{\beta/(1-\beta)} (w(s) \exp(\kappa|s-r|)/U)^{1/(1-\beta)}. \tag{7}$$

As is intuitive, the maximum rent a worker is willing to pay for land at r depends positively on the wage he earns and negatively on the utility he can get elsewhere.

Consider first the decentralized city where firms and their workers co-locate. In this case, the bid rent functions $q_F(r)$ and $q_H(r,r)$ must coincide for all $r \in [0,S]$. Setting s equal to r in the bid rent function for households, setting the resulting bid rent function equal to the bid rent function for firms, and using the expression in (3) for $z(r)$ implies

$$w(r) = w(0) \exp\left(-\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} r\right) \text{ for } r \in [0,S]. \tag{8}$$

Thus wages decline exponentially from the city center, reflecting the fact that the production externality is felt most strongly at the center. However, for this wage profile to be an equilibrium, it must be the case that workers do not have an incentive to commute to a job closer to the city center to take advantage of higher wages. This requires that the rise in wages as a worker commutes toward the center not exceed the loss in working time due to commuting, namely,

$$\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} \leq \kappa. \tag{9}$$

If commuting costs are high (κ is large), if the production externality is weak (γ is small), and if communication between workers in different locations is not too difficult (δ is low), then decentralized urban form can be sustained in equilibrium. When (8) holds, (7) implies that

$$q(r) = q(0) \exp\left(-\frac{\delta\gamma}{1-\alpha\beta} r\right) \text{ for } r \in [0, S]. \tag{10}$$

We now turn to the case in which the city has a monocentric structure. In this case, there is an endogenously determined boundary $S_F < S$ such that all $r \in [0, S_F)$ are devoted to production and all $s \in (S_F, S]$ are devoted to residential use. The boundary S_F can be devoted to either use. If there is a CBD, workers must be indifferent between working at different locations within this district. This implies that in the business district the wages must satisfy the condition

$$w(r) = w(0) \exp(-\kappa r) \text{ for } r \in [0, S_F]. \tag{11}$$

Substituting this into the expression for $n^*(r)$ and using the expression for $z(r)$ in (3) yields

$$q_F(r) = q_F(0) \exp\left(-\frac{\delta\gamma - \kappa\alpha}{1-\alpha} r\right) \text{ for } r \in [0, S_F), \tag{12}$$

which is declining in r provided $\delta\gamma - \kappa\alpha > 0$.

Given that workers earn the same regardless of where they work, we do not need to know their place of work in order to determine their bid rent for a particular location in the city. It is convenient, however, to imagine that the place of work is in the city center. Then, the maximum rent a worker is willing to pay for land at location $r \in [0, S]$ and still get a utility of U is

$$q_H(r) = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}} \left(\frac{w(0)\exp(-\kappa r)}{U}\right)^{\frac{1}{1-\beta}} = q_H(0)\exp\left(-\frac{\kappa}{1-\beta}\right) \text{ for } r \in [0, S]. \quad (13)$$

For the monocentric structure to be an equilibrium outcome, the two bid rents must be the same at the boundary of the CBD, and the slope of the firm's bid rent function must be steeper than the slope of the worker's bid rent function. These requirements impose a constraint on the admissible value of κ. Observe that the slope of the worker's bid rent function at S_F is $[-\kappa/(1-\beta)]q_H(S_F)$ and the slope of the firm's bid rent function at S_F is $[(\kappa\alpha - \delta\gamma)/(1-\alpha)]q_F(S_F)$. Since at the boundary of the CBD $q_H(S_F) = q_F(S_F)$, the necessary slope condition boils down to $-(\kappa\alpha - \delta\gamma)/(1-\alpha) > \kappa/(1-\beta)$. This implies that

$$\kappa < \frac{(1-\beta)\gamma\delta}{(1-\beta\alpha)}, \quad (14)$$

which is the exact complement of the condition (9). Since both α and β are less than unity, (14) implies that $\gamma\delta > \alpha\kappa$. Therefore, when (14) holds, the firm's bid rent function is downward sloping, as assumed. This shows that the internal structure of the city can be only one of these two types. For completeness, we note the CBD analog of equation (10):

$$q(r) = \begin{cases} q_F(0)\exp\left(-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}r\right) & \text{for } r \in [0, S_F) \\ q_H(0)\exp\left(-\frac{\kappa}{1-\beta}r\right) & \text{for } r \in [S_F, S]. \end{cases} \quad (15)$$

4 Equilibrium

The goal of this section is to show how the equilibrium of the model is determined. We will approach this discussion in terms of a *closed city*, wherein the city's population, P, is taken as given and the equilibrium determines the city's geographic size, S, and the utility it can deliver to its residents, U. We will develop the equilibrium conditions for the decentralized and the monocentric cities in parallel since the arguments are (for the most part) very similar.

The determination of equilibrium can be broken down into two parts. In the first part, P and S are taken as given and the equilibrium employment and residential density functions along with the equilibrium wage and rent functions are determined as functions of P and S. In the second part, S as well as U are determined as functions of P.

The task of determining the various equilibrium functions is made very simple by the fact that all these functions are negative exponentials, where the only unknowns are the values of these functions at $r = 0$ (the city center). Furthermore, these unknown values are all determined once $n(0)$ and $z(0)$ are determined. To see this, note that, in either type of city, $w(0)$ is simply the marginal product of labor at the city center. Therefore

$$w(0) = \alpha A z(0)^{\gamma} n(0)^{\alpha-1}. \tag{16}$$

And, in any type of city, there are businesses operating at the city center and therefore $q(0)$ must be output at 0 minus the wage bill at 0 (since all "surplus" must go to the owners of land). Therefore $q(0) = Az(0)^{\gamma} n(0)^{\alpha} - w(0)n(0)$. This implies

$$q(0) = [(1-\alpha)/\alpha]w(0)n(0) = (1-\alpha)Az(0)^{\gamma}n(0)^{\alpha}. \tag{17}$$

For the decentralized city, $q(0)$ is the only unknown for the rent function since the bid rent functions for businesses and workers coincide. For the monocentric city, (17) determines $q_F(0)$. To determine $q_H(0)$, we use the fact that the bid rents for businesses and workers are the same at S_F, which implies $q_F(0)\exp(-[\delta\gamma-\kappa\alpha]/[1-\alpha]S_F) = q_H(0)\exp(-[\kappa/(1-\beta)]S_F)$. Therefore

$$q_H(0) = q_F(0)\exp\left(\frac{\kappa - \delta\gamma + \beta\delta\gamma - \beta\kappa\gamma}{(1-\alpha)(1-\beta)} S_F\right). \tag{18}$$

While this equation depends on S_F, we will show below that S_F is, in fact, pinned down by S alone (recall that we are taking both P and S as parametrically given in this part). Therefore, the first part of the equilibrium problem boils down to simply determining $n(0)$

and $z(0)$.

To proceed, we observe that the expression for $n^*(r)$, along with the expressions for $w(r)$ in (8) and $z(r)$ in (3), gives the following employment density equation for the decentralized city:

$$n(r) = n(0) \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) \text{ for } r \in [0, S] \tag{19}$$

and, using (11), gives the following employment density equation for the monocentric city:

$$n(r) = n(0) \exp\left(-\frac{\delta\gamma - \kappa}{1-\alpha} r\right) \text{ for } r \in [0, S_F]. \tag{20}$$

For either type of city, the values of $n(0)$ and $z(0)$ are determined by invoking two market-clearing conditions. First, there is the labor-market-clearing condition. For the decentralized city, since a firm and its workers co-locate, each location is a labor market in which demand and supply for labor have to match. Letting $\theta(r)$ denote the fraction of land devoted to production in location r, we can express this location-by-location labor market balance requirement as

$$n^*(r)\theta(r) = [1 - \theta(r)]/l^*(r). \tag{21}$$

Since $n^*(r) = [\alpha q(r)]/[(1-\alpha)w(r)]$ and $l^*(r) = (1-\beta)w(r)/q(r)$, we find that $\theta(r) = [1-\beta]/[1-\alpha\beta] = \theta$.[8] Thus, the proportion of land devoted to production is constant across all locations in the city, and therefore the *level* of employment in location r, $N(r)$, is simply $\theta n(r)$.

For the monocentric city, we already know that $\theta(s) = 1$ for $s \in [0, S_F]$ and $\theta(s) = 0$ for $s \in (S_F, S]$. What the labor-market-clearing condition determines for this case is the

[8]The decentralized city case has also been analyzed in Wheaton (2004) for an exogenously given productivity gradient and exogenously given land use intensities for firms and workers. Wheaton does not impose the local labor-market-clearing condition (21). Instead, the fraction of land in use by firms (or workers) at any location is determined by the relative magnitude of the rent levels for each use.

location of the commercial district boundary, namely, S_F. To develop this condition, we note that the total supply of labor time available at the border of the CBD, taking into account the time lost in commuting, is $\int_{S_F}^{S}[2\pi r/l(r)]e^{-\kappa(r-S_F)}dr$. If the employment density at a CBD location r is $n(r)$, the labor time needed at the border of the commercial district to fulfill this demand is $e^{\kappa(S_F-r)}n(r)$. Therefore, the total time needed at the border of the CBD to satisfy total labor demand inside the commercial district is $\int_0^{S_F} 2\pi rn(r)e^{\kappa(S_F-r)}dr$. Equality of labor demand and supply then requires

$$\int_0^{S_F} 2\pi rn(r)\exp\left(\kappa(S_F-r)\right)dr = \int_{S_F}^{S}\frac{2\pi r}{l(r)}\exp\left(-\kappa(r-S_F)\right)dr,$$

which, using the fact that $l(r) = (1-\beta)w(0)e^{-\kappa r}/q_H(r)$ and the expressions for $n(r)$ and $q_H(r)$ derived earlier, simplifies to

$$n(0)w(0)(1-\beta)\int_0^{S_F} r\exp\left(-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}r\right)dr = q_H(0)\int_{S_F}^{S} r\exp\left(-\frac{\kappa}{(1-\beta)}r\right)dr.$$

Using (17) and (18) we can further simplify this equation to

$$\left[\int_{S_F}^{S} r\exp\left(-\frac{\kappa}{1-\beta}r\right)dr\right] =$$
$$\frac{(1-\beta)}{(1-\alpha)}\alpha\left[\int_0^{S_F} r\exp\left(-\frac{\gamma\delta-\alpha\kappa}{1-\alpha}r\right)dr\right]\exp\left(\frac{-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F\right). \qquad (22)$$

Observe that this is an equation that implicitly defines S_F as a function of S. The following Lemma establishes that there is a unique S_F corresponding to each S that is strictly increasing in S and converging to a finite limit as S increases unboundedly.

Lemma 1 *For each $S > 0$, (22) uniquely determines $S_F(S) \in (0, S)$. Furthermore, $S_F(S)$ is strictly increasing in S and* $\lim_{S\to\infty} S_F(S) = \bar{S}_F > 0$.

Proof. See Appendix.

The second market-clearing condition requires that the total number of residents in the city must equal the total population of the city, P. For the decentralized city, this requires

$$P = \int_0^S 2\pi r[1-\theta]/l^*(r)dr.$$

Using (19) and (21), we see that the above implies

$$n(0) = \frac{P}{2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{23}$$

Knowing $n(0)$ and the fact that $\theta(s) = \theta$ also allows us to pin down $z(0)$:

$$z(0) = 2\pi \int_0^S r \exp(-\delta r) N(r) ds = 2\pi\theta n(0) \int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr, \tag{24}$$

or

$$z(0) = P \frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{25}$$

For the monocentric city, the analogous requirement is $P = \int_{S_F}^S [2\pi r/l(r)]dr$. Since $l(r) = (1-\beta)w(0)\exp(-\kappa r)/q_H(0)\exp\left(-\frac{\kappa}{1-\beta} r\right)$, this implies

$$P = \frac{q_H(0)}{(1-\beta)w(0)} \int_{S_F}^S 2\pi r \exp\left(-\frac{\beta\kappa}{(1-\beta)} r\right) dr.$$

Using (17), (18), and (22), we obtain

$$n(0) = \frac{P}{2\pi} \frac{1}{\left[\int\limits_0^{S_F(S)} r \exp\left(-\frac{\gamma\delta-\alpha\kappa}{1-\alpha} r\right) dr\right]} \frac{\left[\int\limits_{S_F(S)}^{S} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{S_F(S)}^{S} r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{26}$$

Again, knowing $n(0)$ allows us to pin down the level of the external effect at the city center.

Since $z(0) = n(0) \int_0^{S_F} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha} + \delta\right]\right) r dr$, we have

$$z(0) = \frac{P}{2\pi} \frac{\left[\int\limits_0^{S_F} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha} + \delta\right] r\right) dr\right]}{\left[\int\limits_0^{S_F} r \exp\left(-\frac{\gamma\delta-\alpha\kappa}{1-\alpha} r\right) dr\right]} \frac{\left[\int\limits_{S_F}^{S} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{S_F}^{S} r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{27}$$

This completes the first part of the equilibrium determination problem.

Before proceeding to the second part, it is useful to report how the equilibrium is affected by changes in the demand and supply of urban land, separately considered.[9] Consider, first, the effects of a change in the general demand for urban land. In the model, this could come about through a change in A, which changes the productivity of firms located in the city, or a change in P, which changes the numbers of city residents. If P and S are held constant, a change in A will leave both $n(0)$ and $z(0)$ unchanged, since A does not appear in (23)-(27). Given this, it follows that a change in A will simply shift the wage and bid rent functions proportionally, and there will be no change in any relative price or in the intensity of land use in any location. If A and S are held constant, a change in P will change $n(0)$ and $z(0)$ proportionally since both quantities depend proportionally on P for both types of cities. From this fact, we can infer, using (16), (17) and the fact that $U = \beta^\beta(1-\beta)^{1-\beta} w(r) q(r)^{-(1-\beta)}$, the following:

Proposition 1 *(The Effects of a Change in Population): If A and S are held constant, (i) employment density and the level of the production externality change proportionately with P, (ii) the elasticity of rents in any location with respect to P is $\alpha + \gamma$, (iii) the elasticity of wage in any location with respect to P is $\alpha + \gamma - 1$, and (iv) elasticity of U with respect to P is $\beta(\alpha + \gamma) - 1$.*

We turn now to the effects of change in the supply of urban land. Consider, first, a change in S for the decentralized city, holding A and P constant. From (23) we see immediately that

[9] Of course, in full equilibrium, changes in the demand for urban land will induce changes in its supply. This interaction is the focus of the next section of this paper.

$n(0)$ is decreasing in S: Employment density at the city center is lower in a more spread-out city. The effect on $z(0)$ is not so clear because an increase in S increases the geographic reach of the externality—the numerator term in (25). Notice, however, that both integrals calculate a "mean distance" with weights that decline exponentially with distance and the weights decline *faster* for the numerator term (since $\delta > 0$). Intuitively, we would expect an increase in S to increase the numerator proportionately less than the denominator, and that, indeed, is true. Since this sort of logic will be used repeatedly to sign expressions, we state it as a Lemma here:

Lemma 2 *Let* $0 \leq s_L < s_U$. *Let* $\Lambda(s_L, s_U) = [\int_{s_L}^{s_U} se^{k_2 s} ds]/[\int_{s_L}^{s_U} se^{k_1 s} ds]$. *Then,* $\Lambda(s_L, s_U)$ *is increasing (decreasing) in both* S_U *and* S_L *if* $k_1 < (>) k_2$.

Proof. See Appendix.

Therefore, by virtue of Lemma 2, it is the case that $z(0)$ is declining in S as well.

Turning to the monocentric city, recall that $S_F(S)$ is increasing in S (Lemma 1). Therefore, by Lemma 2 again, the ratio of the integrals in the expression for $n(0)$ in (26) is decreasing in S. Since the remaining fractional term is clearly decreasing in S, employment density at the city center is decreasing in S for the monocentric city as well. The effect on $z(0)$ is potentially ambiguous for the same reason as in the decentralized city: While employment density is decreasing in S, the geographic reach of the external effect is increasing in S_F and therefore in S. However, if $\delta > \kappa$ (communication is harder than commuting) then, by Lemma 2 again, the first of the two ratios of integrals in (27) is decreasing in S. And, since $\beta < 1$, the second ratio of integrals is also decreasing in S. Henceforth, we will always operate under the assumption that $\delta > \kappa$. Then, it is easy to verify that the following is true:

Proposition 2 *(The Effect of Change in City Size* S*): If* A *and* P *are held constant, (i) employment density (and employment), the level of the production externality, and rents at the city center are decreasing in* S*, (ii) if* $\alpha+\gamma \leq 1$*, wages at the city center are increasing in*

S, otherwise the effect is ambiguous and (iii) if $\beta(\alpha+\gamma) \leq 1$, U is increasing in S, otherwise the effect is ambiguous.

We now turn to the second part of equilibrium determination, namely, the determination of S and U, given A and P. Since it costs d units of the consumption good to convert one unit of undeveloped land into urban land, developers (the entities that own all land in this economy) will continue to develop urban land until the rent at the city boundary S is equal to the cost of development. Therefore, S is determined by

$$q(S; A, P) = d, \tag{28}$$

where $q(S; A, P)$ is the rent at the city boundary when TFP is A and population is P. The following Lemma establishes how $q(S; A, P)$ varies with S.

Lemma 3 *$q(S; A, P)$ is strictly decreasing in S and strictly increasing in A and P. Furthermore,* $\lim_{S\to 0} q(S; A, P) = \infty$ *and* $\lim_{S\to\infty} q(S; A, P) = 0$.

Proof. See Appendix.

All else the same, rents fall with S because workers who live at the boundary earn the least. Complementing this effect is the fact that, recorded in Proposition 2, rents at the city *center* are also declining with S. The latter effect pushes down rents in all locations in the city, including the boundary. The "Inada-type" conditions of $q(S; A, P)$ are also intuitive: Rents in locations very far from the city center must be very low to compensate for very low wages in those locations (for the decentralized city) or for the very large amount of time lost in commuting to a job (for the monocentric city). If the boundary is very close to the city center, employment density at the center must be very high, which would require very high rents there and, by extension, at the city boundary. Given Lemma 3, it follows that, for any A, P, and d, there is a unique S, denoted $S_d(A, P)$ that solves (28).

The following proposition describes how S is affected by changes in TFP, population, and costs of development. These properties follow directly from Lemma 3.

Proposition 3 *$S_d(A, P)$ is strictly increasing in A and P and strictly decreasing in d. Furthermore,* $\lim_{P \to 0} S_d(A, P) = 0$ *and* $\lim_{P \to \infty} S_d(A, P) = \infty$.

Finally, we come to the relationship between U, the utility deliverable by a city, and A and P when the city boundary adjusts so that rent at the boundary is d. We will denote this relationship by the function $U_d(A, P)$. We are primarily interested in understanding how this function behaves with respect to variations in P, since migration in or out of the city is the key adjustment mechanism for cities. It is a convenient feature of the model that this function can be expressed as a composition of two functions: An "outer" function, denoted $V_d(A, S)$, which gives the utility deliverable by a city given A and S and rent at the boundary of d, and an "inner" function, which is just $S_d(A, P)$. Thus, $U_d(A, P) = V_d(A, S_d(A, P))$. The benefit of this decomposition is that the $V_d(A, S)$ function has a closed-form expression that allows easy assessment of its shape with respect to variations in S. And, since $S_d(A, P)$ is strictly increasing in P (Proposition 3), the shape of $U_d(A, P)$ with respect to P is simply a shape-preserving rescaling of $V_d(A, S)$.

To develop the $V_d(A, S)$ function, we use two conditions. The first condition is that rent at the city boundary must be d. For the decentralized city, this condition implies $d = q(0) \exp(-\delta\gamma/(1-\alpha\beta)S)$, and for the monocentric city it implies $d = q_H(0) \exp(-\kappa/(1-\beta))S$. This condition implies that S and d pin down rents in the city center. We have already seen, however, that rents at the city center are determined by A, $n(0)$, and $z(0)$. Since $z(0)$ is itself pinned down by $n(0)$, it follows that the first condition fully determines $n(0)$ as a function of A, S, and d.

The second condition equates the utility obtained by a worker who resides at the city boundary when the city size is S and rent at the boundary is d to the utility delivered by the city to any worker, which is U. For the decentralized city this condition is $U = \beta^\beta(1 - \beta)^{1-\beta} d^{-(1-\beta)} w(0) \exp(-[\delta\gamma(1 - \beta]/[1 - \alpha\beta]S)$, and for the monocentric city the condition is $U = \beta^\beta(1 - \beta)^{1-\beta} d^{-(1-\beta)} w(0) \exp(-\kappa/(1 - \beta)S)$. These conditions imply that U, S, and d completely determine wages at the city center. Since wages at the city center are also fully determined by A, $n(0)$, $z(0)$, the second condition fully determines $n(0)$ as a function of A, S, d, and U.

Equating the two expressions for $n(0)$ and rearranging terms yields $V_d(A,S)$. To determine the shape of this function with respect to S, it is convenient to examine $\ln(V_d(A,S))$. Collecting terms that do not depend on S into a "constant" D, for the decentralized city we have

$$\begin{aligned}&\ln(V_d(A,S)) = \\ &D + \frac{\gamma}{\gamma+\alpha}\left\{\ln\left[\int_0^S r\exp\left(-\frac{\delta\,(1-\beta\alpha+\gamma\beta)}{1-\beta\alpha}r\right)dr\right] - \frac{\delta\,(1-\beta(\alpha+\gamma))}{(1-\beta\alpha)}S\right\}.\end{aligned} \tag{29}$$

Thus, on the logarithmic scale, $V_d(A,S)$ has a component that starts at 0 and declines linearly with S provided $1-\beta(\alpha+\gamma)>0$, and a component that starts at $-\infty$ and rises at most logarithmically with S. Since the rate of change of $\ln(x)$ is infinite at $x=0$, $\ln(V_d(A,S))$ must be increasing at $S=0$. Furthermore, since the derivative of the ln term declines monotonically to 0 with S, there is some $\hat{S}>0$ at which $\ln(V_d(A,S))$ peaks and then declines monotonically, asymptoting to $-\infty$. It follows that $V_d(A,S)$ is hump-shaped, with $\lim_{S\to 0} V_d(A,S) = \lim_{S\to\infty} V_d(A,S) = 0$.

For the monocentric city, the corresponding expression is

$$\begin{aligned}&\ln(V_d(A,S)) = \\ &D + \frac{\gamma}{\alpha+\gamma}\ln\left[\int_0^{S_F(S)} r\exp\left(-\frac{\delta\,(\gamma+1-\alpha)-\kappa}{1-\alpha}r\right)dr\right] + \\ &-\kappa\frac{1-\beta\,(\alpha+\gamma)}{(1-\beta)\,(\gamma+\alpha)}S + \left(\frac{\gamma+\alpha-1}{\gamma+\alpha}\right)\frac{-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma}{(1-\alpha)\,(1-\beta)}S_F(S).\end{aligned} \tag{30}$$

The presence of $S_F(S)$ introduces a new element that is not present in the decentralized city. From Lemma 1, however, we know that $\lim_{S\to 0} S_F(S) = 0$ and $\lim_{S\to\infty} S_F(S) = \bar{S}_F$. Therefore, it is still true that $\lim_{S\to 0}\ln(V_d(A,S)) = \lim_{S\to\infty}\ln(V_d(A,S)) = -\infty$. Whether the function generally has a single peak is not easy to establish, but, for the region of the parameter space that matters empirically, it is likely to be monotonically declining beyond some value of S. Empirically, $\alpha+\gamma \approx 1$ and $\delta >> \kappa$. Assume for the moment that $\alpha+\gamma = 1$. Then the r.h.s. of (30) simplifies to $D + \ln[\int_0^{S(F)} r \exp[\kappa/(1-\alpha)r]dr - [\kappa(1-\beta(\alpha+\gamma))/(1-\beta)(1-\alpha)]S$. If $\delta >> \kappa$, then (22) implies that S_F changes very little in response to any change in S. In this case, the behavior of the r.h.s. of (30) is effectively dominated by the term involving S. Therefore, beyond some initial (potentially non-monotone) segment, the function will decline with S. To summarize:

Lemma 4 *Assume* $1 - \beta(\alpha + \gamma) > 0$. *Then,* $\lim_{S\to 0} V_d(A, S) = \lim_{S\to\infty} V_d(A, S) = 0$. *In addition, for the decentralized city,* $V_d(A, S)$ *is single-peaked. For the monocentric city,* $V_d(A, S)$ *is eventually monotonically declining in* S*, provided* $\alpha + \gamma \approx 1$ *and* $\delta >> \kappa$.

As mentioned earlier, because $S_d(A, P)$ is strictly increasing in P, $U_d(A, P)$ inherits all the properties of $V_d(A, S)$. Therefore, we have the following proposition:

Proposition 4 *Assume* $1 - \beta(\alpha + \gamma) > 0$. *Then,* $\lim_{P\to 0} U_d(A, P) = \lim_{P\to\infty} U_d(A, P) = 0$. *In addition, for the decentralized city,* $U_d(A, P)$ *is single-peaked. For the monocentric city,* $U_d(A, P)$ *is eventually monotonically declining in* P*, provided* $\alpha + \gamma \approx 1$ *and* $\delta >> \kappa$.

It is worth noting that Proposition 4 is consistent with the results in Propositions 1 and 2. Proposition 1 states that if $1-\beta(\alpha+\gamma) > 0$, utility in the city is decreasing in P (when S is fixed), while Proposition 2 states that, under the same condition, utility is increasing in S (when P is fixed). Thus, as S increases and the city fills up with people so that the rent at the boundary is d, there are two offsetting forces working on utility obtained by residents of the city. When the city is physically small, the utility-enhancing effect of S is stronger than the utility-decreasing effect of higher population. Eventually, though, the utility-depressing effect of higher population dominates and utility declines with P.

To understand why utility is increasing when P is low but declining when P is high, it is helpful to think of the case in which the city cannot expand at all. In this case, utility deliverable by the city declines as population increases. With population growth, even if the wages increase (which happens when $\alpha + \gamma > 1$), utility declines because the increase in wages, and the implied increase in c, is not large enough to compensate for the lower consumption of residential space. This comes from the condition $\beta(\alpha + \gamma) < 1$. In the case in which the city can expand at the cost of d, the city does expand with higher P and allows workers to increase their consumption of land, but only when the city is small. As we see from equation (23), employment density at the center (which is inversely proportional to the consumption of land per worker at the center) becomes increasingly insensitive to an increase in S as S gets higher. In the limit, equilibrium allocations at the center become similar to the case in which S does not change. Although some people move to the outskirts when S goes up, they form an increasingly small portion of the general population, so this reshuffling has little effect on employment and residential densities in the center.

The condition $1 - \beta(\alpha + \gamma) > 0$ is our analog of what Fujita, Krugman, and Venables (1999) call the "no-black-hole condition." If this condition is violated, then, as is evident from the expression of $\ln(V_d(A, S))$, utility deliverable by the city would be increasing in S. Since $S_d(A, P)$ is strictly increasing in P, utility deliverable by the city would be strictly increasing in P. The model would then imply that the entire population of an economy would tend to gravitate to one giant city—the "black hole," so to speak. To rule this out, the strength of increasing returns must be bounded above.[10]

5 Demand Shocks, Growth Controls, and Land Rents

In this section, we use the model to explore the effects of urban growth controls on land rents when there is an increase in the demand for urban land. The goal is to assess the

[10] Lucas and Rossi-Hansberg (2002) (and also Lucas (2001)) assume a condition that is stronger, namely, $\alpha + \gamma < 1$. Although this condition is also labeled a "no-black-hole condition," it is needed to rule out a different kind of black hole, one in which all firms pile up at 0 (the city center) with each firm using a vanishingly small amount of land but enjoying unboundedly high external effect, i.e., it is needed to rule out the case where $z(0)$ diverges to ∞. However, this case is not a concern for us because $z(r)$ is known to have the negative exponential form and, hence, productivity at the city center is naturally bounded above by city size and total population, as seen in (25).

conventional wisdom that, all else the same, locations that have difficulty expanding (due to topography or policy) will experience a larger increase in the value of land (i.e., rents) as demand for urban land increases. We show that this intuition needs to be modified when production externalities are present: A city that can expand easily benefits more from the production externality, and this effect may cause a bigger increase in rents in such a city.

We will consider two cities that, prior to the shock, are identical. However, following the shock, one city is free to convert undeveloped land into developed land at the cost d (i.e., expand the boundary of the city), but the other city is prohibited from doing so (i.e., it cannot push out the boundary of the city). We will consider different ways in which the demand for urban land can increase. In one case there is an increase in the population of the economy due to population growth or immigration. The other case is an increase in the demand for urban land in the two (identical) cities resulting from a positive TFP shock.

The effects of the two types of shocks on the population of these two cities—henceforth called the unrestricted and restricted cities, respectively—are illustrated in Figures 2 and 3. Figure 2 shows the impact of an increase in economy-wide population. The solid hump-shaped line plots $\ln(U_d(A, P))$ against P for the unrestricted city. The solid horizontal line is the utility available to a worker in any other city in the economy prior to the increase in population. We assume that the city is at the point labeled A (which corresponds to a stable equilibrium in the usual sense). The increase in population then results in a drop in the utility deliverable by cities in general, so the horizontal line shifts down to the dotted one. People move into the city until the city reaches the point labeled B. The dashed line in the figure is the utility curve for the restricted city. Since this city cannot expand its physical boundary, the decline in utility in response to increased in-migration is larger (at any level of P) relative to the unrestricted city. People move into the city until the restricted city reaches the point labeled C.

Figure 3 shows what happens if the two cities receive a positive TFP shock. The solid lines have the same interpretation as in Figure 1, and the initial position of both cities is A. The shock leaves the general level of utility (the solid horizontal line) unchanged, but shifts

Figure 3

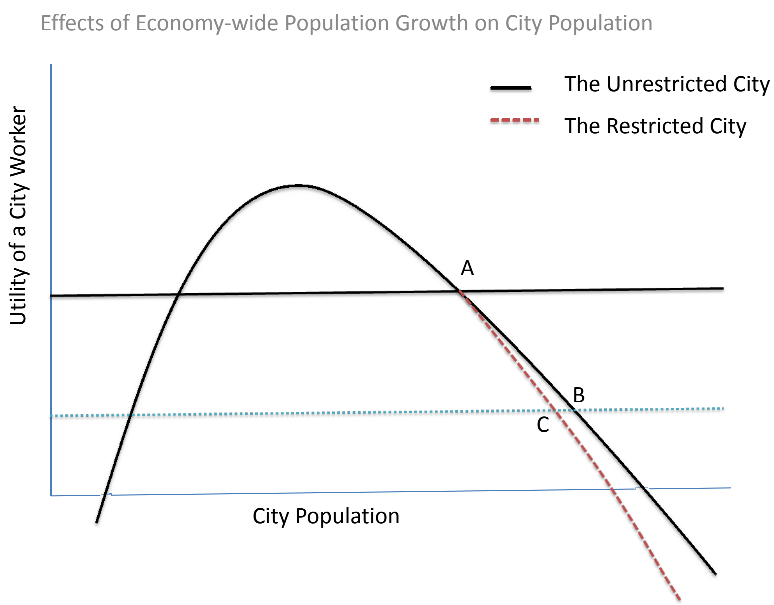
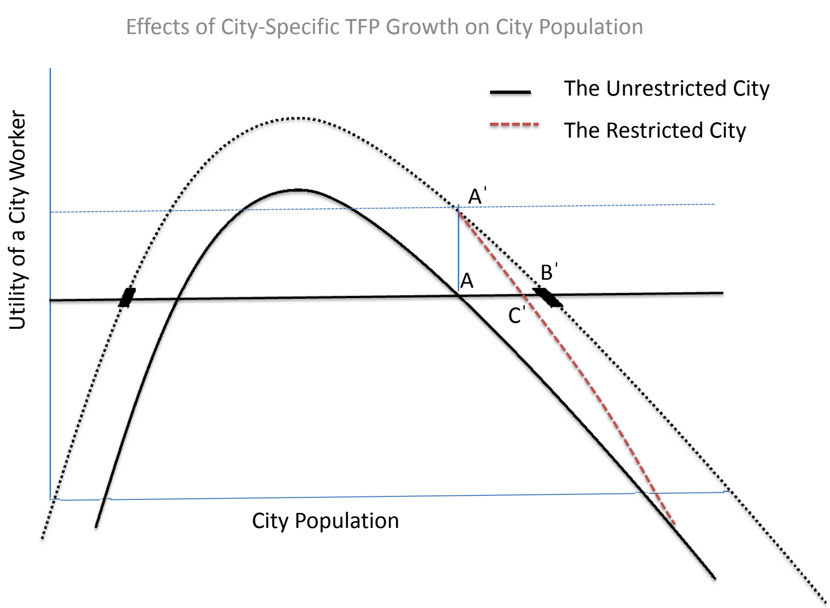


Figure 4



the $\ln(U_d(A, P))$ for the two cities upward to the dotted line. The city draws in population from elsewhere in the economy until it reaches point B. The dashed line is the utility curve for the restricted city. Once again, it lies below the utility curve for the unrestricted city. The restricted city also draws in people from the rest of the economy until it reaches point C. We can, however, view this adjustment as happening from A' to B and C, which is exactly like a drop in the general level of utility from what is available at A' to the solid horizontal line. Thus the effects of the two types of demand shocks are fundamentally similar.

What we take from Figures 2 and 3 is that, following the increase in total population or TFP, both cities will experience increases in population. Proposition 3 implies that the unrestricted city will be physically larger than the restricted city following the shock. From the figures, it is also clear that population will increase more in the unrestricted city than in the restricted one. Of course, in the new equilibrium, both cities will deliver the same utility to workers residing there. In what follows, we analyze the impact of demand shocks on employment density, wages, and land rents in the two cities. We analyze land rents last because it is easier to understand why land rents behave as they do once we understand how employment density and wages are affected by the demand shocks.

5.1 Demand Shocks and Employment Density

We will focus on employment density at the city center since that will determine what employment density will be in any other location. Consider first the case where both cities have decentralized employment. Examining the expression for $n(0)$ in (23), we see that it is not immediately possible to tell how $n(0)$ compares across restricted and unrestricted cities: The unrestricted city has higher P and larger S relative to the restricted city. However, if we use the fact that, in equilibrium, both cities must deliver the same utility to workers, then it becomes possible to compare employment densities.

Observe that, in both the restricted city and the unrestricted city, the firm's bid rent and the worker's bid rent for the city center coincide. This implies $(1-\alpha)\,Az(0)^{\gamma}n(0)^{\alpha} =$

$(1-\beta)\,\beta^{\frac{\beta}{1-\beta}}\left(\frac{w(0)}{U}\right)^{\frac{1}{1-\beta}}$, where U is the common utility delivered by the two cities. Using the fact that both $w(0)$ and $z(0)$ can be expressed in terms of only $n(0)$, S, and other parameters, it is possible to express $n(0)$ in terms of U, S, and other parameters:

$$n(0) = KA^{\frac{\beta}{1-\beta(\alpha+\gamma)}}U^{\frac{-(1-\beta)}{1-\beta(\alpha+\gamma)}}\left[\int_0^S r\exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right)dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}}, \tag{31}$$

where K is, again, some positive constant. Since the unrestricted city is physically larger in the new equilibrium, and since A and U are common across the two cities, it follows that employment density at the center of the unrestricted city must exceed the employment density at the center of the restricted city. For CBDs, the corresponding expression for $n(0)$ is

$$\begin{aligned} n(0) &= KA^{\frac{\beta}{1-\beta(\alpha+\gamma)}}U^{\frac{-(1-\beta)}{1-\beta(\alpha+\gamma)}}\times \\ &\exp\left(\frac{[\delta\gamma-\kappa]-\beta[\delta\gamma-\alpha\kappa]}{(1-\alpha)(1-\beta(\alpha+\gamma))}S_F\right)\left[\int_0^{S_F} 2\pi r\exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right]r\right)dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}}. \end{aligned} \tag{32}$$

By virtue of the "no-black-hole condition" $1-\beta(\alpha+\gamma)>0$ and the upper bound on κ in (14) it follows that $n(0)$ is increasing in S_F. Since S_F is strictly increasing in S (Lemma 1), it follows again that, in the new equilibrium, employment density in the center of the unrestricted city must exceed that in the center of the restricted city. We summarize this discussion in the following proposition:

Proposition 5 *If two cities have the same fundamentals and one of the cities has an urban growth boundary that is just binding, an increase in the demand for urban land will cause employment density in the unrestricted city to rise more than in the restricted city.*

5.2 Demand Shocks and (Product) Wages

As in the case of employment density, it is sufficient to consider what happens to wages offered by firms locating at the city center, namely, $w(0)$. In any city, $w(0)=\alpha Az(0)^{\gamma}n(0)^{\alpha-1}$. Using the relationship between $z(0)$ and $n(0)$ for decentralized cities, this implies

$$w(0) = \alpha A \left[2\pi\theta \int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta \right] r \right) \right]^\gamma n(0)^{\gamma+\alpha-1}, \tag{33}$$

and, for monocentric cities, it implies

$$w(0) = \alpha A \left[2\pi \int_0^{S_F} r \exp\left(-\left[\frac{\delta\gamma - \kappa}{1-\alpha} + \delta \right] r \right) \right]^\gamma n(0)^{\gamma+\alpha-1}. \tag{34}$$

We already know that, in the new equilibrium, the unrestricted city will be larger in size and it will have a higher employment density. From the above expressions it follows that if $\alpha + \gamma \geq 1$ then wages at the city center will be higher in the unrestricted city relative to the restricted one. By continuity this ordering will also prevail when $\alpha + \gamma$ is slightly less than 1, but it may or may not prevail when $\alpha + \gamma$ is substantially less than 1. Since $(1 - \alpha)$ is simply the exponent to land in the production function and γ is the exponent to the level of agglomeration in the city, wages at the center of the unrestricted city will exceed those in the center of the restricted city, provided agglomeration is more important in production than land. Summarizing, we have the following proposition:

Proposition 6 *If two cities have the same fundamentals and one of the cities has an urban growth boundary that is just binding, and if $\gamma \geq 1 - \alpha$, an increase in the demand for urban land will cause the wage offered at the center of the unrestricted city to exceed the wage offered at the center of the restricted city.*

5.3 Demand Shocks and Urban Land Rents

It is helpful to break up the discussion in terms of how demand shocks affect business rents and how they affect residential rents. Once again, it is sufficient to focus on the rents at the city center.

For any type of city, there are always firms that locate at the center of the city. The bid rent for a firm at the city center, $q_F(0)$, is $(1-\alpha)Az(0)^{\gamma}n(0)^{\alpha}$. Using the relationship between $z(0)$ and $n(0)$ for decentralized cities, this implies

$$q_F(0) = (1-\alpha)A\left[2\pi\theta\int_0^S r\exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right)\right]^{\gamma} n(0)^{\gamma+\alpha}, \tag{35}$$

and, for monocentric cities, it implies

$$q_F(0) = (1-\alpha)A\left[2\pi\int_0^{S_F} r\exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right]r\right)\right]^{\gamma} n(0)^{\gamma+\alpha}. \tag{36}$$

We already know that, in the new equilibrium, the unrestricted city will be larger in size and it will have a higher employment density. Therefore, business rents at the center of the restricted city will be higher than business rents at the center of the restricted city. Summarizing, we have the following proposition:

Proposition 7 *If two cities have the same fundamentals and one of the cities has an urban growth boundary that is just binding, an increase in the demand for urban land will cause rents paid by businesses at the center of the unrestricted city to exceed the rents paid by businesses at the center of the restricted city.*

Turning to residential rents, we can proceed by considering again what happens to the bid rent for residential space at the centers of the two cities. For the decentralized city we already know the answer: Since workers' and firms' bid rents coincide, residential rents equal business rents. By the above proposition, it follows that residential rents at the center of the unrestricted city will be higher than those in the center of the restricted city. For monocentric cities, we have

$$q_H(0) = \beta^{\beta/(1-\beta)}(1-\beta)w(0)^{1/(1-\beta)}U^{-1/(1-\beta)}. \tag{37}$$

Since U is the same for both cities, the ordering of workers' bid rent for space at the center of the city depends on the ordering of wages at the center of the city. Therefore, the conditions that govern the ranking of $w(0)$ also govern the ranking of $q_H(0)$: Specifically, if $\gamma \geq 1-\alpha$, then wages at the center will be higher in the unrestricted city in the new equilibrium and therefore workers will be willing to bid more for land at the center of the unrestricted city than in the restricted city. Summarizing, we have the following:

Proposition 8 *If two decentralized cities have the same fundamentals and one of the cities has an urban growth boundary that is just binding, an increase in the demand for urban land will cause the bid rent for residential space at the center of the unrestricted city to exceed the bid rent for residential space at the center of the restricted city. For the same result to hold for monocentric cities, it is sufficient that* $\gamma \geq 1-\alpha$.

Since rent in any location is simply $\max\{q_F(r), q_H(r)\}$, the immediate implication of Propositions 7 and 8 is that, in the new equilibrium, land rents may be higher in every comparable location in the unrestricted city relative to the restricted city. Specifically, denoting the physical size of the restricted city by $\bar{S}$ and land rents in the restricted and unrestricted cities by $q_R(r)$ and $q_{UR}(r)$, respectively, we have the following corollary:

Corollary 1 *If* $\gamma \geq (1-\alpha)$ *then, in the new equilibrium,* $q_{UR}(r) > q_R(r)$ *for all* $r \in [0, \bar{S}]$.

We can now address the empirical plausibility (or otherwise) of the externality effect dominating the land intensity effect. As noted above, this boils down to a simple comparison of γ and $(1-\alpha)$. Turning first to γ, Melo, Graham, and Noland (2009) (Table 2, p. 355) report that estimates of this parameter (across various types of datasets, methodology and measures of agglomeration) range between -0.366 and 0.319 for the US, with the mean (and median) estimate at 0.036. The median estimate for other developed countries ranges between 0.028 (Canada) and 0.083 (UK). Across industrial groupings, the median estimate of γ for manufacturing is 0.036 and for services it is 0.142. Studies that use some measure of market potential to measure agglomeration, the median estimate for γ is 0.076. Measures based on average density or size imply median estimates of 0.039 and 0.030, respectively.

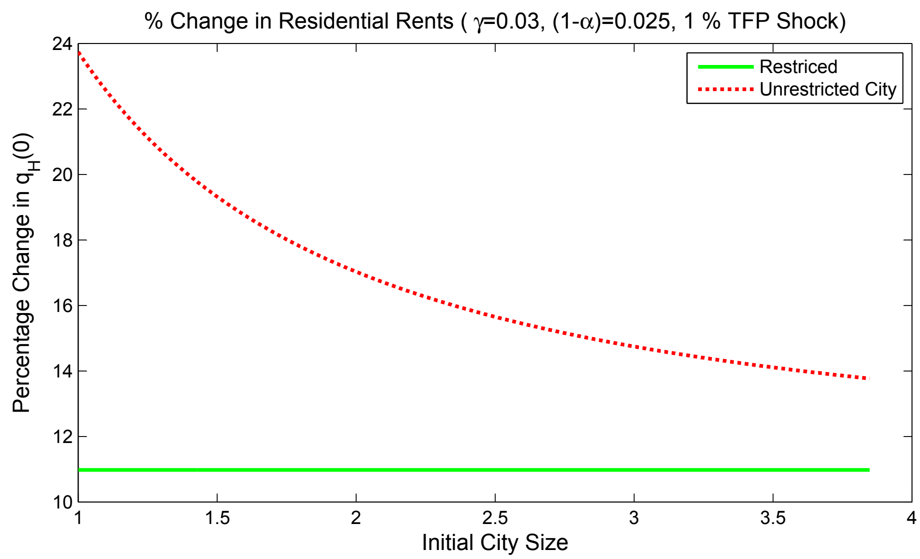
For estimates of $(1-\alpha)$, there does not appear to be any source that compares estimates for different countries and industrial groupings. Indeed, direct information on $(1-\alpha)$ seems sparse. An estimate can be constructed using information reported in Kiyotaki, Michaelides, and Nikolov (2011). They estimate the share of productive tangible assets in the production of nonhousing final output from NIPA data for the period 1952-2005 to be 0.258. In addition, for the share of land in the production of tangible assets, they use 0.10 following Haughwout and Inman (2001) and Davis and Heathcote (2005). These estimates suggest that a plausible value of $(1-\alpha)$ is around $0.025 \approx 0.258 \times 0.10$.

For $(1-\beta)$, we follow Davis and Heathcote (2007) who estimate that land accounts for 36 percent of the value of aggregate housing stock. Given that households spend about 25 percent of their budget on housing (which includes the services from structures and land), a plausible value of $(1-\beta)$ is around $0.10 \approx 0.25 \times 0.36$.

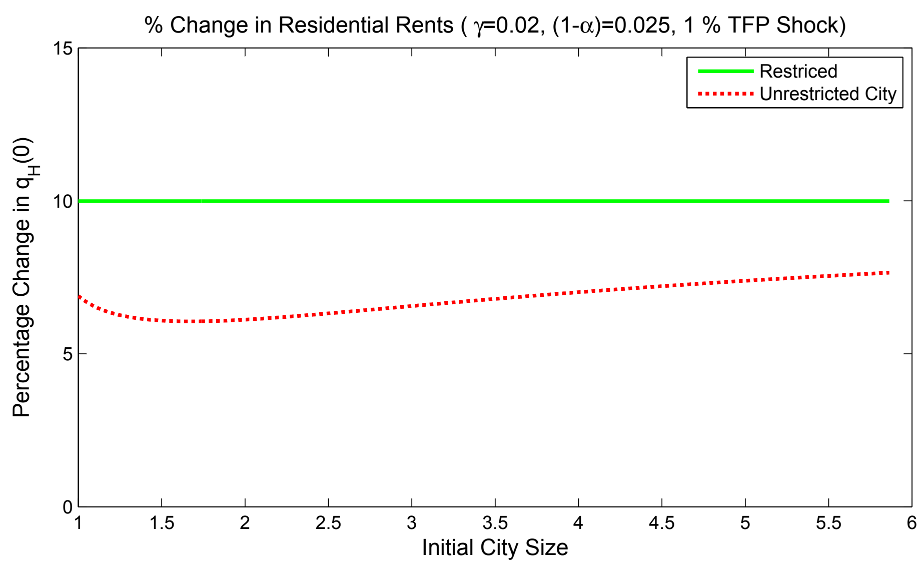
What we take from this brief discussion of the empirical magnitudes of γ and $(1-\alpha)$ is that these values are roughly similar. Thus the empirical evidence does not rule out the possibility that, for some cities, the externality effect may actually dominate the land-intensity effect. Service-oriented cities are more likely to display a dominant externality effect than manufacturing-oriented cities. Furthermore, with $\alpha + \gamma \approx 1$ and $\beta \approx 0.90$, the no-black-hole condition is generally satisfied.

This comparision, however, leaves open the magnitude of the difference in the growth of land rents across the two cities when hit with a common shock. To investigate this issue, we numerically simulate our model. We fix $(1-\alpha) = 0.025$ and $(1-\beta) = 0.10$. For other parameters, we set $\delta = 6$ and $\kappa = 0.005$. These values are not calibrated to any given facts but are consistent with (i) cities being monocentric, and (ii) the gradient in commercial rents being much, much steeper than the gradient in residential rents. They are also the values used in Lucas and Rossi-Hansberg (2002). For γ, we experiment with both 0.02 and 0.03. In the former case, land is more important in production than the externality and in the latter case the opposite is true. We set $d = 1$ and initial $A = 1$ and compute the value of S (and therefore P) for which the utility of the worker is highest. The initial city size is always larger than this “optimal” value.

Figure 5



% Change in Residential Rents (γ=0.03, (1-α)=0.025, 1 % TFP Shock)
Restriced
Unrestricted City
Percentage Change in $q_H(0)$
Initial City Size
24
22
20
18
16
14
12
10
1
1.5
2
2.5
3
3.5
4



% Change in Residential Rents (γ=0.02, (1-α)=0.025, 1 % TFP Shock)
Restriced
Unrestricted City
Percentage Change in $q_H(0)$
Initial City Size
15
10
5
0
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

The top panel of Figure 5 displays percentage change in $q_H(0)$ for different initial city sizes for the case where $\gamma = 0.03$.[11] The horizontal axis records the initial city size relative to the size that gives the highest utility. The vertical axis records the percentage change in residential bid rent at the city center. As is evident from the figure, the percentage change is higher for the unrestricted city than for the restricted one for all values of the initial city size. It is also evident that the percentage change for the restricted city is constant and independent of initial city size, while that for the unrestricted city is declining in initial city size. However, when the city is very large the expansion of the city does not affect employment density at the city center much because so few people move into the newly developed land. Consequently, the price elasticity in the two cities behave similarly. In contrast, when the city is small, the expansion of the city does change the employment density at the city center significantly and that leads to larger percentage increase in residential rents. The bottom panel of Figure 5 plots the case where the land intensity effect dominates the externality effect. As we would expect, the ordering of the percentage change in rents is reversed and, once again, the gap is larger the closer the intial city size is to the optimal one. The important quantitative point made by Figure 5 is that effects of differences in the elasticity of urban land supply in response to a common demand shock will be more dicernable when the cities are close to their optimal size.

Finally, as another check on the plausibility of the externality effect dominating the land-intensity effect, we investigate whether it has counterfactual implications for the relationship between the stringency of regulations on new development and the price elasticity of housing supply. As noted in the introduction, the empirical evidence indicates that urban growth restrictions reduce the price elasticity of housing. This is implied for sure by our model in the case where restrictions on urban growth raise workers' wages (i.e., when the land-intensity effect dominates): In this case, rents rise more in the restricted city and population rises less. In the case where urban growth restrictions lower workers' wages (i.e., when the production

[11] Since a worker's bid rent for any location is some fixed fraction (independent of A) of his bid rent for space at the city center, this is as good a measure of the percentage change in residential rent as any.

externality effect dominates), both population and land rents rise less in the restricted city so it is not clear whether elasticity of housing supply is lower or higher in the restricted city compared to the unrestricted city.

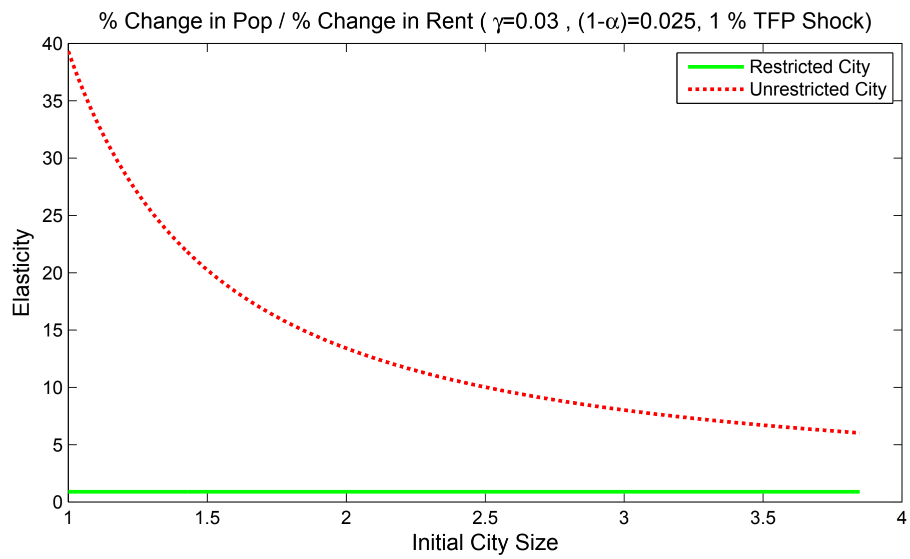
To investigate this issue we use the same model parameters as those used in Figure 5. The demand shock hitting the two cities is, again, a 1 percent increase in A. The elasticity of housing supply is measured as the ratio of the percentage change in the total population of the city to the percentage change in the worker's bid rent for space at the city center, namely, $q_H(0)$. The top panel of Figure 6 displays this elasticity for different initial city sizes for the case where $\gamma = 0.03$. Once again, the elasticity for the restricted city is independent of its initial size. And, as is evident from the figure, elasticity is *higher* for the unrestricted city than for the restricted one for all values of the initial city size. This is true despite the fact that the increase in residential rent is *larger* for the unrestricted city than the restricted city. The bottom panel of Figure 6 displays the case where $\alpha = 0.02$. Not surprisingly, the same picture emerges: The price elasticity of housing is higher for the unrestricted city with the difference becoming small for large initial city size. We conclude from these results that the dominance of the externality effect need not conflict with the empirical evidence that restrictions on new urban developments tend to lower the price elasticity of housing supply.

6 Reprise

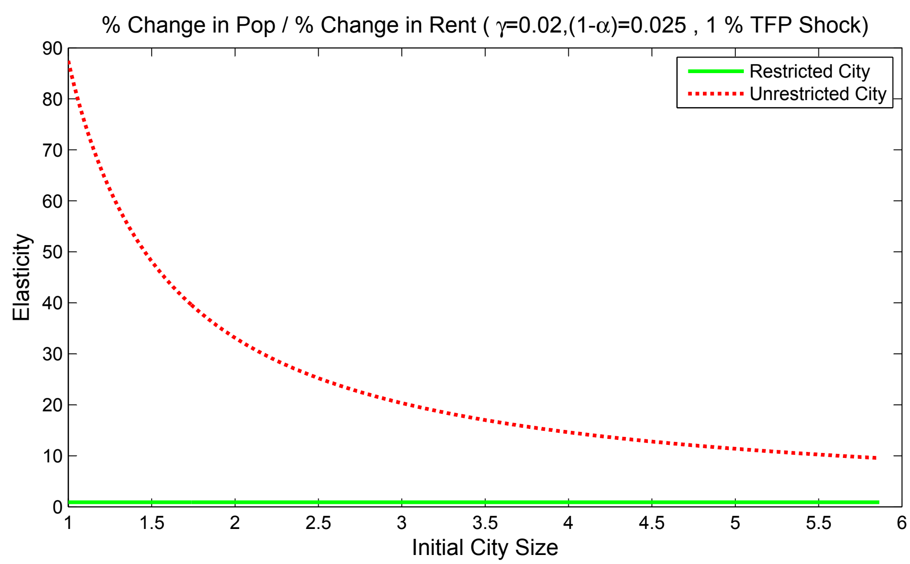
We close the paper by reproducing the model analog of Figure 2. The parameter configuration is the one underlying the top panel of Figure 5, where S has been chosen so that the city is close to its optimal size.

Figure 7 plots the bid rents for workers in the new equilibrium. The solid (black) vertical line is the location of the urban growth boundary for the restricted city. The dashed (blue) line is the initial (common) bid rents of workers. The dotted (green) line is the bid rent of workers in the restricted city and the dashed-dotted (magenta) line is the new bid rent of workers in the unrestricted city. The solid (red) horizontal line is the cost of expansion for the

Figure 6

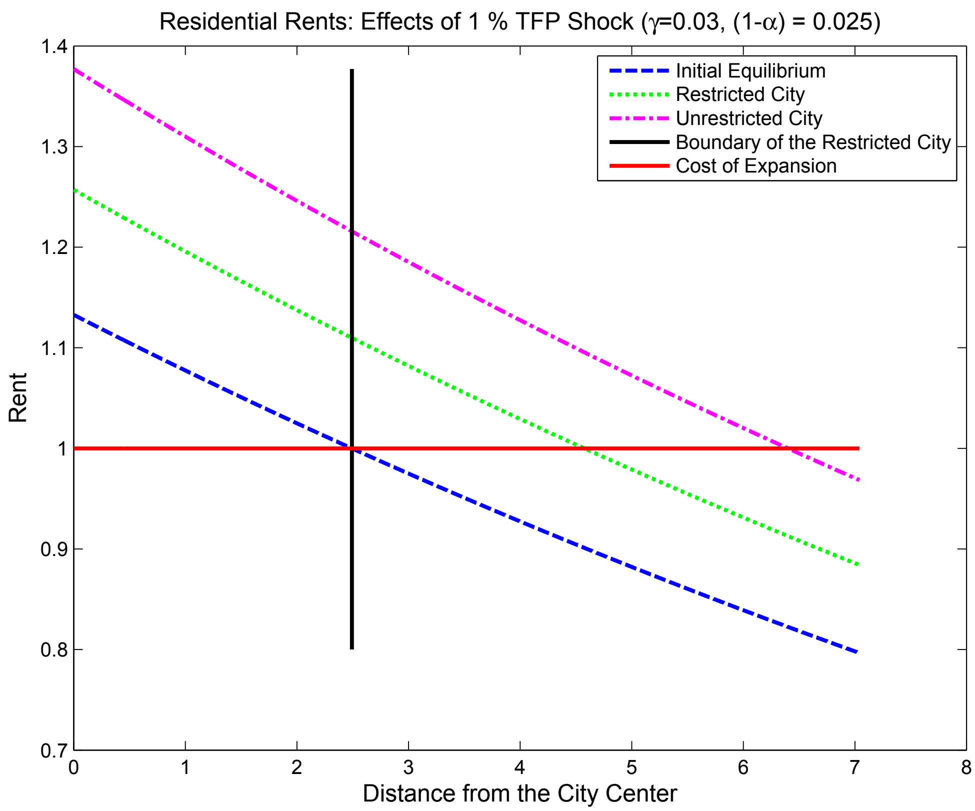


% Change in Pop / % Change in Rent (γ=0.03 , (1-α)=0.025, 1 % TFP Shock)
Restricted City
Unrestricted City
Elasticity
Initial City Size
0
5
10
15
20
25
30
35
40
1
1.5
2
2.5
3
3.5
4



% Change in Pop / % Change in Rent (γ=0.02,(1-α)=0.025 , 1 % TFP Shock)
Restricted City
Unrestricted City
Elasticity
Initial City Size
0
10
20
30
40
50
60
70
80
90
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

Figure 7


Residential Rents: Effects of 1 % TFP Shock (γ=0.03, (1-α) = 0.025)
Initial Equilibrium
Restricted City
Unrestricted City
Boundary of the Restricted City
Cost of Expansion
Rent
Distance from the City Center
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
1
2
3
4
5
6
7
8

unrestricted city. In the new equilibrium, the unrestricted city expands to the point where the magenta line intersects the red line. This implies that despite the underlying shock being the same in both cities, the bid rent functions for workers shift up by different amounts. In particular, the bid rent function of workers in the unrestricted city shifts up more than in the restricted city, reflecting the fact that wages are higher in the unrestricted city relative to the restricted city. The implication is that, for all locations to the left of the solid vertical line, the bid rent of workers is higher in the unrestricted city than in the restricted city. Thus, allowing the city to expand actually raises the value of *existing* residential land.

References

Alonso, W. (1964): *Location and Land Use: Toward a General Theory of Land Rent.* Harvard University Press, Cambridge, Massachusetts.

Anas, A., R. J. Arnott, and K. A. Small (1998): "Urban Spatial Structure," *Journal of Economic Literature*, 36, 1426–1464.

——— (2000): "The Panexponential Monocentric City Model," *Journal of Urban Economics*, 47, 165–179.

Bertaud, A., and J. K. Brueckner (2005): "Analyzing Building-Height Restrictions: Predicted Impacts and Welfare Costs," *Regional Science and Urban Economics*, 35, 109–125.

Brueckner, J. K. (1990): "Growth Controls and Land Values in an Open City," *Land Economics*, 66(3), 237–248.

——— (2007): "Urban Growth Boundaries: An Effective Second-Best Remedy for Unpriced Traffic Congestion?," *Journal of Housing Economics*, 16(3-4), 263–273.

Combes, P.-P., G. Duranton, L. Gobillon, D. Puga, and S. Roux (2012): "The Productivity Advantages of Large Cities: Distinguishing Agglomeration from Firm Selection," *Econometrica*, 80(6), 2543–2594.

Davidoff, T. (2010): "Supply Elasticity and the Housing Cycle of the 2000s," Unpublished, Sauder School of Business, University of British Columbia.

Davis, M. A., and J. Heathcote (2005): "Housing and the Business Cycle," *International Economic Review*, 46(3), 751–784.

——— (2007): "The Price and Quantity fo Residential Land in the United States," *Journal of Monetary Economics*, 54, 2595–2620.

Ding, C., G. K. Knaap, and L. D. Hopkins (1999): "Managing Urban Growth with Urban Growth Boundaries: A Theoretical Analysis," *Journal of Urban Economics*, 46(1), 53–68.

Fischel, W. A. (1990): "Do Growth Controls Matter? A Review of Empirical Evidence on the Effectiveness and Efficiency of Local Government Land Use Regulation," Working Paper, Lincoln Institute of Land Policy.

Fujita, M. (1989): *Urban Economic Theory*. Cambridge University Press, Cambridge, UK.

Fujita, M., P. Krugman, and A. J. Venables (1999): *The Spatial Economy*. The MIT Press, Cambridge, Massachusetts.

Fujita, M., and H. Ogawa (1982): "Multiple Equilibria and Structural Transition of Non-Monocentric Urban Configurations," *Regional Science and Urban Economics*, 12, 161–196.

Glaeser, E. L., J. Gyourko, and A. Saiz (2008): "Housing Supply and Housing Bubbles," *Journal of Urban Economics*, 64, 198–217.

Glaeser, E. L., J. Gyourko, and R. E. Saks (2005a): "Why Have House Prices Gone Up?," *American Economic Review*, 95(2), 329–333.

——— (2005b): "Why Is Manhattan So Expensive? Regulation and the Rise in Housing Prices," *Journal of Law and Economics*, 48(2), 331–369.

——— (2006): "Urban Growth and Housing Supply," *Journal of Economic Geography*, 6(1), 71–89.

Glaeser, E. L., and M. E. Kahn (2001): "Decentralized Employment and the Transformation of the American City," in *Brookings-Wharton Papers on Urban Affairs*, pp. 1–63. The Brookings Institution, Washington, D.C.

Green, R. K., S. Malpezzi, and S. K. Mayo (2005): "Metropolitan-Specific Estimates of the Price Elasticity of Supply of Housing, and Their Sources," *American Economic Review*, 95(2), 334–339.

Haughwout, A., and R. P. Inman (2001): "Fiscal Policies in Open Cities with Firms and Households," *Regional Science and Urban Economics*, 31, 147–180.

Henderson, J. V. (1974): "The Sizes and Types of Cities," *American Economic Review*, 64(4), 640–656.

Huang, H., and Y. Tang (2013): "Residential Land Use Regulation and the US Housing Price Cycle Between 2000 and 2009," *Journal of Urban Economics*, 71(1), 93–99.

Kiyotaki, N., A. Michaelides, and K. Nikolov (2011): "Winners and Losers in Housing Markets," *Journal of Money, Credit, and Banking*, 43(2-3), 255–296.

Kuminoff, N. V., and J. C. Pope (2013): "The Value of Residential Land and Structures During the Great Housing Boom and Bust," *Land Economics*, 89(1), 1–29.

Lucas, R. E. (2001): "Externalities and Cities," *Review of Economic Dynamics*, 4, 245–274.

Lucas, R. E., and E. Rossi-Hansberg (2002): "On the Internal Structure of Cities," *Econometrica*, 70(4), 1445–1476.

Melo, P., D. Graham, and R. Noland (2009): “A Meta-Analysis of Estimates of Urban Agglomeration Economies,” *Regional Science and Urban Economics*, 39(3), 332–342.

Paulsen, K. (2013): “The Effects of Growth Management on Spatial Extent of Urban Development, Revisited,” *Land Economics*, 89(2), 193–210.

Rosenthal, S. S., and W. Strange (2004): “Evidence on the Nature and Sources of Agglomeration Economies,” in *Handbook of Regional and Urban Economics, Vol. 4*, ed. by V. Henderson, and J.-F. Thisse, pp. 2119–2171. North Holland: Amsterdam.

Saiz, A. (2010): “Geographic Determinants of Housing Supply,” *Quaterly Journal of Economics*, 125(3), 1253–1296.

Wheaton, W. C. (2004): “Commuting, Congestion, and Employment Dispersal in Cities with Mixed Land Use,” *Journal of Urban Economics*, 55, 417–438.

APPENDIX

Proof of Lemma 1

Given any $S > 0$, (14) (the upper bound on κ) implies that the r.h.s. of (22) is increasing in S_F. The l.h.s. of (22) is clearly decreasing in S_F. Furthermore, the r.h.s. is 0 for $S_F = 0$ while the l.h.s. is strictly positive, and the r.h.s. is strictly positive for $S_F = S$ while the l.h.s. is 0. Therefore, for each $S > 0$ there is a unique $S_F \in (0, S)$ that ensures (22) is satisfied. Observe also that as S goes up and S_F does not change, the integral on the l.h.s. goes up. Since the r.h.s. is increasing in S_F, the equilibrium S_F must be strictly higher. Thus $S_F(S)$ is strictly increasing in S.

To prove the second part, we observe that since $S_F(S) < S$ for all S, it must be the case that $\lim_{S\to 0} S_F(S) = 0$. To prove the other limiting result, we will first establish that $\lim_{S\to\infty} S_F(S)$ is bounded above. Let S_n be an increasing sequence diverging to ∞. Let $S_F(S_n)$ be a corresponding sequence of S_F that satisfies (22). Then $S_F(S_n)$ is also a strictly increasing sequence. Next, observe that

$$\int\limits_{S_F(S_n)}^{S_n} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds = -\left[\frac{(1-\beta)}{\kappa}\right]^2 \left[e^{-\frac{\kappa}{(1-\beta)}s}(ks+1)\right]_{S_F(S_n)}^{S_n}.$$

If $S_F(S_n)$ diverges to infinity along with S_n, the above integral will converge to 0. This will imply that the l.h.s. of (22) will converge to 0 while the r.h.s. will diverge to ∞, which is impossible. Hence, $S_F(S_n)$ must be bounded above. Since $S_F(S)$ is strictly increasing, it follows that $\lim S_F(S)$ must converge to some number $\bar{S}_F > 0$. ■

Proof of Lemma 2

We will first establish the following two sets of inequalities. If $k_1 < k_2$, then

$$e^{(k_2-k_1)s_L} < \frac{\int\limits_{s_L}^{s_U} se^{k_2 s}ds}{\int\limits_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_U}, \tag{38}$$

and if $k_2 < k_1$, then

$$e^{(k_2-k_1)s_U} < \frac{\int\limits_{s_L}^{s_U} se^{k_2 s} ds}{\int\limits_{s_L}^{s_U} se^{k_1 s} ds} < e^{(k_2-k_1)s_L}. \tag{39}$$

Turning first to the l.h.s. inequality in 38, we observe that $se^{k_2 s} = se^{s_L k_2 + (s-s_L)k_2}$and $se^{k_1 s} = se^{s_L k_1 + (s-s_L)k_1}$. Multiplying both sides of the latter equation by $e^{(k_2-k_1)s_L}$ yields $e^{(k_2-k_1)s_L} se^{k_1 s} = se^{s_L k_2 + (s-s_L)k_1} \leq se^{s_L k_2 + (s-s_L)k_2} = se^{k_2 s}$, where the inequality follows because $k_2 > k_1$ and $s - s_L \geq 0$. Furthermore, the inequality is strict for all $s \in (s_L, s_U]$. Therefore, integrating the first and last expressions in the chain with respect to s, we have

$$e^{(k_2-k_1)s_L} \int\limits_{s_L}^{s_U} se^{k_1 s} ds < \int\limits_{s_L}^{s_U} se^{k_2 s} ds.$$

Turning to the r.h.s. of the inequality, we observe that $se^{k_2 s} = se^{s_U k_2 + (s-s_U)k_2}$and $se^{k_1 s} = se^{s_U k_1 + (s-s_U)k_1}$. Multiplying both sides of the latter equation by $e^{(k_2-k_1)s_U}$ yields

$$e^{(k_2-k_1)s_U} se^{k_1 s} = se^{k_2 s_U + (s-s_U)k_1} \geq se^{s_U k_2 + (s-s_U)k_2} = se^{k_2 s},$$

where the inequality follows since $k_2 > k_1$ and $s - s_U \leq 0$. Furthermore, the inequality is strict for all $s \in [s_L, s_U)$. Therefore, integrating the first and last terms in the chain with respect to s, we have

$$e^{(k_2-k_1)s_U} \int\limits_{s_L}^{s_U} se^{k_1 s} ds > \int\limits_{s_L}^{s_U} se^{k_2 s} ds. \quad \blacksquare$$

The proof of 39 is entirely analogous.

We now turn to the proof of the Lemma. We begin with the case in which $k_1 < k_2$.

Observe that

$$\frac{\partial \ln(\Lambda(s_L, s_U))}{\partial s_U} = \frac{s_U \exp\left(k_2 s_U\right)}{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds} - \frac{s_U \exp\left(k_1 s_U\right)}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds}.$$

Suppose, to get a contradiction, that $\partial\Lambda(s_L, s_U)/\partial s_U \leq 0$. Then, we must have

$$\frac{s_U \exp\left(k_2 s_U\right)}{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds} \leq \frac{s_U \exp\left(k_1 s_U\right)}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds}.$$

Or, given that all elements are positive, we have

$$\exp\left(\left[k_2 - k_1\right] s_U\right) = \frac{s_U \exp\left(k_2 s_U\right)}{s_U \exp\left(k_1 s_U\right)} \leq \frac{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds}.$$

But this contradicts the r.h.s. inequality in Lemma 1. Therefore, $\partial\Lambda(s_L, s_U)/\partial s_U > 0$. Analogous proof can be given for the case in which $k_2 < k_1$. ■

Remark: Let $I(s_U, s_L, k) = \int\limits_{s_L}^{s_U} s \exp\left(-ks\right) ds$. Then (i) $\lim_{s_U, s_L \to \infty} I(s_U, s_L, k) = 0$ and (ii) $\lim_{s_U \to \infty, s_L \to \underline{s}} I(s_U, s_L, k) = \bar{I} > 0$.

Observe that

$$\int\limits_{s_L}^{s_U} s e^{-ks} ds = \frac{s_U e^{-ks_U} - s_L e^{-ks_L}}{-k} - \frac{e^{-ks_U} - e^{-ks_L}}{k^2}.$$

To prove (i), we notice that, as s_U and s_L go to infinity, the second term goes to 0, and the first term (on an application of L'Hospital's Rule to s/e^{ks}) also goes to 0. To prove (ii), we observe that if s_U goes to infinity and s_L converges to $\underline{s}$, then $I(s_U, s_L, k)$ converges to

$$\frac{-\underline{s} e^{-k\underline{s}}}{-k} + \frac{e^{-k\underline{s}}}{k^2} > 0. \quad ■$$

Proof of Lemma 3

Decentralized City: From (10) we have that

$$q(S; A, P) = q(0) \exp\left(-\frac{\gamma\delta}{1-\beta\alpha} S\right).$$

Holding fixed A and P, we see that $q(0)$ is decreasing in S by Proposition 2. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha} S\right)$ is strictly decreasing in S, it follows that $q(S; A, P)$ is strictly decreasing in S. And, holding fixed S and P, we see that $q(0)$ is proportional to A and therefore $q(S; A, P)$ is increasing in A. And, if S and A are held fixed, $q(0)$ is increasing in P by Proposition 1. Therefore $q(S; A, P)$ is increasing in P.

To establish the limit properties, we use (17), (23), and (25) to express $q(0)$ in terms of P and S:

$$q(0) =$$
$$(1-\alpha)AP^{(\alpha+\gamma)} \left[\frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}\right]^{\gamma} \times \left[2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr\right]^{-\alpha}.$$

As S approaches 0, the term involving the ratio of integrals approaches 1 (this follows from an application of L'Hospital's Rule) and the remaining integral term approaches infinity. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha} S\right)$ approaches 1, it follows that $\lim_{S\to 0} q(S; A, P) = \infty$. Going the other way, as S approaches ∞, by Lemma 2 the ratio of integrals term approaches 0 and the integral term approaches a positive constant. Hence, $\lim_{S\to\infty} q(S; A, P) = 0$.

Monocentric City: To prove the first part, we note that $q_H(S; A, P) = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S}$. Since $e^{-\frac{\kappa}{(1-\beta)}S}$ is decreasing in S, it is sufficient to show that, if we hold A and P constant, $q_H(0)$ is decreasing in S. To begin, note that $q_F(0)e^{-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}S_F} = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S_F}$, which implies that $q_F(0)/q_H(0) = e^{-\frac{-\kappa+\delta\gamma+\alpha\beta\kappa-\beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F}$. By (14), the r.h.s. of the latter equation is increasing in S_F. Since $S_F(S)$ is increasing in S, it follows that $q_F(0)/q_H(0)$ is increasing in S. From Proposition 2 we know, holding A and P constant, that $q_F(0)$ is decreasing in S. Therefore

$q_H(0)$ must be decreasing in S. And, if we hold fixed S and P, $q_H(0)$ is proportional to A and therefore $q(S;A,P)$ is increasing in A. And, holding fixed S and A, we see that $q_H(0)$ is increasing in P by Proposition 1. Therefore $q(S;A,P)$ is increasing in P.

We now turn to limiting behavior of $q_H(S;A,P)$.

Part (i): $\lim_{S\to\infty} q_H(S;A,P) = 0$. Consider

$$q_H(S;A,P) = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}} \left(\frac{w(0)\exp(-\kappa S)}{U}\right)^{\frac{1}{1-\beta}}.$$

Using (16), (18), (26), and (27), we can express the ratio of $w(0)$ to U as

$$\begin{aligned}\frac{w(0)}{U} &= KP^{(1-\beta)(\gamma+\alpha)}A^{-1}\left(\int\limits_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds\right)^{-(1-\beta)(\gamma+\alpha)} \times \\ &\left(\int_0^{S_F} s\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)ds\right)^{\gamma(1-\beta)} \times \\ &\exp\left(\frac{(-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma)(\gamma+\alpha-1)}{(1-\alpha)}S_F\right),\end{aligned}$$

where K is a positive constant. Given that $\lim_{S\to\infty} S_F(S) = \bar{S}_F$, the last two terms approach finite numbers. And, by Lemma 2, $\int\limits_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds$ appoaches a strictly positive finite number. Thus, we can conclude that, as $S\to\infty$, the ratio $w(0)/U$ approaches a finite number as well. Therefore, the limiting behavior of $q_H(S;A,P)$ is governed by the limiting behavior of $\exp(-\kappa S)$. Hence, $\lim_{S\to\infty} q_H(S;A,P) = 0$.

Part (ii): $\lim_{S\to 0} q_H(S;A,P) = \infty$

Since $S > S_F(S), S\to 0$ implies $S_F(S)\to 0$. Then, it is easiest to show that $q_F(0) = (1-\alpha)\,z(0)^\gamma n(0)^\alpha$ goes to infinity, which would imply that $q_H(S;A,P)$ goes to infinity also.

Turning first to $n(0)$, we observe that

$$n(0) = \left[\frac{\int_{S_F}^{S} s \exp\left(-\frac{\kappa}{1-\beta} s\right) ds}{\int_{S_F}^{S} s \left(\exp \frac{-\kappa\beta}{1-\beta} s\right) ds} \right] \frac{P}{2\pi \left[\int_{0}^{S_F} s \exp\left(\frac{\alpha\kappa - \gamma\delta}{1-\alpha} s\right) ds \right]}.$$

We know from Lemma 2 that

$$\exp\left(\kappa S_F\right) < \frac{\left[\int_{S_F}^{S} s \left(\exp \frac{-\kappa\beta}{1-\beta} s\right) ds \right]}{\left[\int_{S_F}^{S} s \exp\left(-\frac{\kappa}{1-\beta} s\right) ds \right]} < \exp\left(\kappa S\right).$$

This implies that as S and S_F converge to 0 (and so both $\exp\left(\kappa S_F\right)$ and $\exp\left(\kappa S\right)$ converge to 1) the term in square brackets converges to 1. We also know that $\left[\int_{0}^{S_F} s \exp\left(\frac{\alpha\kappa - \gamma\delta}{1-\alpha} s\right) ds \right]$ goes to zero as S_F goes to zero, so $n(0)$ goes to infinity as S goes to zero.