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**WORKING PAPER NO. 02-6**

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**Abstract**

We study, theoretically and quantitatively, the equilibrium of an economy with unsecured consumer credit with the following features. Credit suppliers take deposits at a given interest rate and offer loans to households via a menu of credit levels and associated interest rates. The loan industry is competitive, with free entry and zero costs, and borrowers have a default option that resembles, in process and consequence, a bankruptcy filing under Chapter 7 of the U.S. Bankruptcy Code. The theory part of the paper demonstrates the existence of a competitive equilibrium for such an economy and characterizes the circumstances under which a household defaults on its loans. We map the theory to the data in the quantitative part of the paper. We show that the model can be specified in such a way as to account precisely for the quantitative properties of the main facts regarding bankruptcy and unsecured credit (the volume of unsecured debt, the fraction of borrowers in the market, and the percentage of defaulters). We then use the model to address the implications of two policy experiments, one of which is a policy change that eliminates the Chapter 7 bankruptcy option for households with median or above-median income (a proposal with a similar feature is currently under consideration in Congress). We find that the welfare gain from this policy experiment is substantial, being equivalent to a lump-sum transfer payment of about one-quarter of average annual U.S. earnings.

# 1 Introduction

In this paper we analyze a model economy with unsecured consumer credit that incorporates the main characteristics of U.S. consumer bankruptcy law and replicates the key empirical characteristics of unsecured consumer borrowing in the U.S. Specifically, we construct a model consistent with the following facts:

- Borrowers can default on their loans by filing for bankruptcy under the rules laid down in Chapter 7 of the U.S. Bankruptcy Code. In most cases, filing for bankruptcy results in seizure of all (non-exempt) assets and a full discharge of household debt. Importantly, filing for bankruptcy protects a household's current and future earnings from any collection actions by those to whom the debts were owed.

- Post-bankruptcy, a household's credit rating deteriorates and it has serious difficulty in getting new (unsecured) loans for a period of about 10 years.[1]

- Households that default are typically in poor financial shape.[2]

- There is free entry into the consumer loan industry and the industry behaves competitively.[3]

- There is a large amount of unsecured consumer credit.[4]

- A large number of people who take out unsecured loans default each year.[5]

Heretofore, accounting for consumer borrowing and default facts in the context of a model with rational decision-making on the part of households and firms has posed somewhat of a challenge (see, for instance, White (1998)). A key contribution of our paper is to establish

[1]This is documented in Musto (1999).

[2]This is documented in, for example, Flynn (1999).

[3]See Evans and Schmalensee (2000), Ch.10, for a compelling defense of the view that the unsecured consumer credit industry in the U.S. is competitive.

[4]The Board of Governors of the Federal Reserve System constructs a measure of revolving consumer debt that excludes debt secured by real estate, as well as automobile loans, loans for mobile homes, trailers, or vacations. This measure is probably a subset of unsecured consumer debt and it amounted to $692 billion in 2001, or almost 7 percent of the $10.2 trillion that consititutes U.S. GDP.

[5]In 2001, 1.45 million people filed for bankruptcy in the U.S., of which just over 1 million were under Chapter 7 (as reported by the American Bankruptcy Institute).

a connection between the recent facts on household debt and the bankruptcy filing rate and the theory of consumer behavior that macroeconomists routinely use to address micro and macro observations on household consumption. This connection is established by modifying the model of consumer behavior in, say, Deaton (1991)) to have default and by organizing the facts on consumer debt and bankruptcy filings in light of the model.

Turning first to the theory, we analyze an environment where households with infinitely long planning horizons choose how much to consume and how much to save or borrow. Households face uninsured idiosyncratic shocks to income and preferences and therefore have a motive to accumulate assets and to sometimes borrow in order to smooth consumption. We permit households to default on their loans. This default option resembles a Chapter 7 bankruptcy filing in that default results in a discharge of household debt, with creditors having no further recourse to the debt owed them. We abstract from the out-of-pocket expenses of declaring default (they seem to be rather minor in reality) but assume that a bankrupt household's credit rating deteriorates for some (random) length of time, which is, on average, compatible with the length of time mandated by law that a household's bankruptcy can be kept on record. Consistent with available evidence, we assume that households with a poor credit rating are shut out of the unsecured consumer credit market and experience various minor inconveniences as well (we model these minor inconveniences as a small reduction in a household's earning capability).

It should be clear from this basic setup that an indebted household will weigh the benefit of maintaining access to the unsecured credit market against the benefit of declaring default and having its debt discharged. Accordingly, credit suppliers who make unsecured loans will have to price their loans taking into account the likelihood of default. We assume a market arrangement where credit suppliers can link the price of their loans to the observable total debt position of a household and to a household's type. The first theoretical contribution of the paper is to show that when credit suppliers take the cost of funds as given in a world market, there exists a loan price schedule that is consistent with each credit supplier making zero profits, i.e., there exists a loan price schedule in which the price charged on a loan of a given size made to a household of a given type exactly compensates lenders for the objective default frequency on loans of that size made to households of that type. This demonstration is made somewhat challenging by the fact that the default option generally leads to discontinuous decision correpondences (as opposed to continuous decision rules) at the household level.

A second theoretical contribution of the paper is a characterization of default behavior and of the loan price schedules. Specifically, we demonstrate that for each level of debt and for each household type, the set of earnings that trigger default is a closed interval. While filing for bankruptcy relieves a household of its debt burden, it also induces the household to consume all of its filing-period income. Importantly, a household has no incentive to use any portion of its filing-period income for savings because any savings in the filing period is seized in the bankruptcy process. Consequently, an income-rich household is better off repaying its debt and saving, and an income-poor household is better off repaying its debt and borrowing. The "closed interval" property of the default set follows, essentially, from this point. It is worth emphasizing that besides being a nice theoretical point, the "closed interval" result is important for the computation of equilibria because it makes the task of determining equilibrium default probabilities feasible.

A third theoretical contribution is that the characterization of default behavior also allows a characterization of equilibrium loan price schedules. In particular, we demonstrate that our equilibrium loan price schedules determine, endogenously, the borrowing limit facing each household type. This is theoretically significant since borrowing constraints play a key role in a lot of empirical work regarding consumer spending. Thus, we believe it is important to provide a theory of borrowing constraints that derives from the institutional and legal features of the U.S. unsecured consumer credit market.

Turning to our quantitative work, we start by presenting some facts on consumer debt and bankruptcy that are conformable to the model set out in the theoretical section. For the most part, these facts serve as the targets for the calibration of the model. In our model, the optimal response of individuals to uninsurable idiosyncratic risk generates an endogenous wealth distribution and heterogeneous bankruptcy filings. We organize the data from the 1998 Survey of Consumer Finances to be consistent with the reasons, cited by PSID survey participants who declared bankruptcy between 1984 and 1995, that can be captured by our model. Our baseline model successfully matches these statistics and is qualitatively consistent with a further set of "overidentifying statistics."

We then examine the properties of equilibrium borrowing rates and credit limits, as well as the wealth distribution in our baseline model. We show that for small values of debt (up to 40 percent of the value of average yearly earnings), households pay low interest rates. As the size of debt increases past this point, households begin to default (either voluntarily or involuntarily). Recognizing the household decision rule, firms raise equilibrium interest rates substantially past that point and eventually refuse to make loans above an endogenously determined limit. This limit varies across household types, being more stringent for those households most likely to default. Our model allows for demographic turnover and generates a pattern of the wealth distribution that is typical of overlapping generations models and broadly consistent with U.S. data.

Finally, we use our calibrated model to answer two regulatory questions. First we ask what is the quantitative significance of a 50 percent reduction in the length of time that credit bureaus can legally store the information that agents have filed for bankruptcy. Second, we seek the quantitative significance of a proposal currently under consideration in Congress to prevent "above-median-income" households from filing under Chapter 7. We find the first experiment has minimal quantitative effect but that the second has a substantial impact. Under such a policy there is almost a three-fold increase in the level of debt extended, without a significant increase in the total amount defaulted. We also find a significant welfare effect; households are willing (on average) to pay a once-and-for-all transfer up to a quarter of their earnings to implement such a policy (the flow value of this amount is 0.125 percent of earnings).

Our paper is related to several recent strands of literature on unsecured debt. One strand studies optimal contractual arrangements in the presence of commitment problems. For instance, Kocherlakota (1996) designs state (earnings) contingent bilateral contracts where the threat of punishment to autarky is sufficient to ensure that a given household does not default. Similarly, Kehoe and Levine (2001) embed this idea in a general equilibrium framework. These papers have the implication that it is in the state where earnings are high that households want to default, but the binding individual rationality constraint prevents equilibrium default. To model equilibrium default, we depart from this literature in an important way. In our case a contract between the lending institution and a household is incomplete. While contract terms can depend on such things as the household's current total debt, credit rating, and demographic characteristics that provide partial information on a household's earnings prospects (such as its zip code), it cannot depend on earnings. This assumption is motivated by the typical credit card arrangement.[6] In this regard, our paper is closer to the literature on default with incomplete markets as in Dubey, Geanokoplos, and Shubik (2000) and Zame (1994). As in these papers, we simply take the incompleteness as given and explore the consequences. Zame's work is particularly relevant because he shows that with incomplete markets, it may be efficient to allow a bankruptcy option to debtors.

There are several papers that study issues of default in a quantitative framework. In innovative work, Athreya (1999) analyzes a model that includes a default option with stochastic punishment spells. In his economy, competitive credit suppliers precommit to long-term credit contracts. The credit limit is effectively exogenous. He also imposes that ex-ante expected profits have to be zero as an equilibrium condition. However, in his model economy reducing the credit limit increases profits. Also, Lehnert and Maki (2000) have a model with competitive credit suppliers and borrowers that can both precommit to long-term credit contracts. In their model, ex–ante profits on contracts are zero, and there are numerous periods where firms are committed to making negative profits. Livshits, MacGee, and Tertilt (2001) follow our approach where the zero profit condition is applied to loans of varying size. However, they assume that creditors can garnish wages of a bankrupt person in the period in which that person files for bankruptcy and that a person has unrestricted access to unsecured credit in the period immediately following default. [7]

The paper is organized as follows. We start, in Section 2, by describing bankruptcy in the U.S. in terms of the part of the Bankruptcy Code we are interested in and in terms of the facts that surround default. In Section 3, we turn to a description of the model economy and a characterization of the problem of the household and of the structure of the unsecured credit industry. We prove existence of equilibrium in Section 4. We describe and discuss our calibration targets in Section 5. We then describe the properties of the baseline model economy in Section 6. In Section 7, we pose and answer two quantitative questions that relate to the implications of changing certain features of the law surrounding bankruptcy in the U.S. Section 8 concludes. The proofs and a description of the computational procedures are in the Appendix.

[6] For more detail on the form of the standard credit card "contract" see Section III of Gross and Souleles (2002).

[7] Zha (2001) models bankruptcy as a state where a borrower is unable, in a legal sense, to make contractually agreed payments. He also permits bankrupt individuals to borrow in periods immediately following default.

## 2 Bankruptcy in the U.S: Process and Consequences

In the United States, the right to petition for relief from burden of debt has existed since colonial times. Article I, section 8, of the U.S. constitution authorizes Congress to "enact uniform Laws on the subject of Bankruptcies." Under this authority various bankuptcy laws have been enacted over the years, and, at present, the Bankruptcy Act of 1978 provides federal guidelines for debt relief. These guidelines are described in the various chapters of the U.S. Bankruptcy Code of which Chapters 7 and 13 are the ones most relevant to individuals. Most individuals who seek relief from debt do so under Chapter 7.

There has been a large and growing number of individuals who filed for bankruptcy each year since 1978. In 1998 alone, more than 1 million individuals, or about one percent of U.S. houesholds, filed for bankruptcy under Chapter 7, and an additional 379,000 filed under other chapters. Since the late 1960s, Chapter 7 filings have annually averaged around 70 percent of all individual filings.

The procedure for completing a filing under Chapter 7 is as follows. An individual debtor seeking relief fills out a set of standardized forms that collect information on his or her existing debts, income, property, and monthly living expenses. The individual then files for bankruptcy in a special bankruptcy court, and the court informs the creditors listed by the individual in the filing of that fact. Once the creditors learn of the filing they are required by law to cease all actions to collect their debts. In about a month's time, the creditors meet with the debtor to determine whether there are any non-exempt assets that can be liquidated to pay off unsecured debts.[8] Although the meeting also affords an occasion for creditors to verify the income and expense information on the debtor's filing forms, they rarely do so in practice.

While some unsecured debts (such as student loans) are not dischargeable, unsecured consumer loans such as credit card debt are dischargeable. A discharge releases the debtor from personal liability for discharged debts and prevents the creditors owed those debts from taking any future action against the debtor or his property to collect the debts. From start to finish, the process takes, on average, about four months and costs filers about $200,

[8] Some assets are exempt from liquidation (allowable exemptions vary by state: in Texas and Florida an individual's home equity is exempt while in Iowa the total value of exemptions permitted is just $500).

not including attorney's fees. After filing, the individual loses the right to file for another bankruptcy under Chapter 7 for six years.

In contrast to Chapter 7, filing under Chapter 13 leads to a rescheduling of debt rather than immediate discharge. The rescheduling generally results in a situation where the debtor promises to make additional payments on existing debts over a period of three to five years, followed by a discharge of any remaining debt. By and large, a Chapter 13 filing is not as beneficial to an individual as a Chapter 7 filing and hence is not the preferred Chapter for most individuals seeking debt relief.[9] In any case, for many Chapter 13 filings the resulting rescheduling does not succeed and leads ultimately to a filing under Chapter 7.

This brief description of the bankruptcy procedure should make clear that defaulting on unsecured consumer loans does not take much time or money. But while easy to do, filing for bankruptcy does have some adverse consequences. These stem from the fact that a bankruptcy filing remains on record on an individual's credit history for a period of 10 years from the date of filing. After the 10-year period is over, federal law mandates that the record of the filing be deleted from the household's credit history. During those 10 years, the individual's access to unsecured consumer loans is demonstrably impaired. As documented carefully in Musto (1999), individuals who filed for bankruptcy enjoy better access to unsecured consumer credit when the record of their filing disappears from the view of potential creditors after 10 years. This is apparent in the improvement in their overall credit score, in the number and borrowing capacity of their credit cards, and in their credit relationships more generally. Provided their credit history is not poor for other reasons, individuals with a record of bankruptcy typically see their total credit card balance rise from well under $1000 in the first six post-filing years to well over $2000 in the 11th post-filing year. In addition, credit-card credit granted during the early post-filing years may be secured against a deposit.

The bankruptcy flag in an individual credit history has other consequences as well. These arise because credit histories are also accessed by entities other than credit-granting agencies. For instance, a landlord may request a credit report on a prospective renter, and a potential employer may wish to see a job applicant's credit history. In such cases, an adverse credit

[9] In some cases an individual may be denied a petition to file under Chapter 7 if the bankruptcy court feels that use of Chapter 7 constitutes an abuse of the law. In such cases, the individual has the option to file under Chapter 13.

history is likely to impose costs.

To summarize, filing for a discharge of unsecured consumer debt under Chapter 7 is not very costly in terms of time or money. Doing so discharges existing debts but makes it difficult for the individual to get new unsecured loans for the next 10 years. In addition, the individual suffers costs that result from having an impaired credit history. These are the key institutional features we incoporate in the model presented in the next section.

# 3 The Model Economy

We begin by describing the default option and the market arrangement in our model economy. This is followed by a recursive formulation of the household's decision problem and a description of profit-maximing behavior of firms serving the unsecured credit industry.

## 3.1 The Default Option and Market Arrangement

We model the default option to resemble, in procedure and consequences, a Chapter 7 bankruptcy filing. Let $h \in \{0, 1\}$ denote the "bankruptcy flag" for a household, where $h = 1$ indicates a record of a bankruptcy filing in the household's credit history and $h = 0$ denotes the absence of any such record. In what follows, we will refer to $h$ as simply the household's credit rating, with the rating being either good ($h = 0$ ) or bad ($h = 1$).

Consider a household that starts the current period with a good credit rating and some unsecured debt. If the household files for bankruptcy (and we permit a household to do so irrespective of its current income or past consumption level), the following things happen:

1. The household's beginnning of period liabilities are set to zero (i.e., its debts are discharged), and the household is not permitted to save in the current period. The latter assumption is a simple way to recognize that a household's attempt to accumulate assets during the filing period will result in those assets being seized by creditors.

2. The household begins next period with a bad credit rating (i.e., with a bankruptcy flag in its credit history).

3. A household whose beginning-of-period credit rating is bad (i.e., $h = 1$) cannot get any new loans, an assumption that is broadly consistent with the experience of bankrupt

individuals in Musto (1999). Also, a household with a bad credit rating experiences a loss equal to a fraction $0 < \gamma < 1$ of income, a loss intended to capture the pecuniary costs of a bad credit rating.[10]

4. There is an exogenous positive probability $\lambda$ that a household with a bad credit rating will have a good credit rating in the following period. This is a simple, albeit idealized, way of modelling the fact that a bankruptcy flag remains on an individual's credit history for only a finite number of years.

The addition of the default option necessitates a departure from the conventional modeling of borrowing and lending opportunities. In particular, we need to posit a market arrangement where unsecured loans of different sizes for different types of agents are treated as distinct financial assets. This expansion of the "asset space" is required in order to correctly handle the competitive pricing of default risk, a risk that will vary with the size of the loan and type of the household. Thus, in our model, a household of type $\eta \in S$ can borrow or save by purchasing a single one-period pure discount bond with a face value in a finite set $L \subset \mathbb{R}$. The set $L$ contains 0 and positive and negative elements. We will denote the largest and smallest elements of $L$ by $\ell_{\max} > 0$ and $\ell_{\min} < 0$, respectively. A purchase of a discount bond with a nonnegative face value $\ell$ means that the household has entered into a contract where it will receive $\ell \geq 0$ units of the consumption good next period. A purchase of a discount bond with a negative face value $\ell$ means that the household has entered into a contract where it promises to deliver, conditional on not declaring bankruptcy, $-\ell > 0$ units of the consumption good next period; if it declares bankruptcy, the household delivers nothing. Associated with each element of $L$ is a nonnegative contract price denoted $q_{\ell,\eta} \geq 0$ available to households of type $\eta$. Thus, a purchase of a discount bond with a negative face value $\ell$ "costs" a type $\eta$ household $q_{\ell,\eta} \cdot \ell$ in period-$t$ consumption goods (i.e., the household receives $q_{\ell,\eta} \cdot (-\ell)$ units of the period-$t$ consumption good). The total number of financial assets traded is $N_L \times N_S$, where $N_L$ and $N_S$ are the cardinality of the sets $N$ and $S$, respectively. Define the entire set of prices $q \in \mathbb{R}^{N_L \times N_S}$ as the price vector $\{q_{\ell,\eta}, (\ell, \eta) \in L \times S\}$. The price vector is not indexed by $t$ because the equilibrium price vector will, in fact, turn out to be stationary.

[10] For instance, there are substantial annual fees associated with secured credit cards.

### 3.2 Households

We next describe household preferences, earnings capabilities, and demographics. Demographic turnover will generate a relatively large number of households with wealth close to zero, which makes them prone to indebtebtedness. We also include preference shocks (a strong desire for current consumption). These features will allow us to account for the data, since they both increase the default rate and the amount of indebtedness.[11] Although we don't exploit it in our computational work, we also permit shocks to the stochastic process for household income. Such shocks allow for persistent changes in household earnings.

Specifically, let the household's type $\eta \in S$ denote the realization of a finite-state Markov chain. Realizations of this process affect the household's current period utility function and the probability distribution of the household's next period earnings shock. We will refer to realizations of $\eta$ as a household's type-shock and assume that this shock is publicly observable. There is also the possibility that a household may die at the end of a period with probability $1 - \delta$.

The preferences of a household are given by the expected value of a discounted sum of momentary utility functions:

$$E_0 \left\{ \sum_{t=0}^{\infty} (\beta\delta)^t \, u(c_t, \eta_t) \right\}, \tag{1}$$

where $0 < \beta < 1$ is the discount factor, $0 < \delta < 1$ is the survival probability, $c_t$ is consumption in period $t$, $\eta_t$ is the realization of the type-shock in period $t$, $u : I\!R_+ \times S \to I\!R$ is a utility function that is continuous, strictly increasing, and strictly concave for each $\eta \in S$. Let $\underline{\eta}$ and $\overline{\eta}$ be elements of $S$ such that $u(c, \underline{\eta}) \leq u(c, \eta) \leq u(c, \overline{\eta})$.

A surviving household draws its period $t$ earnings from an atomless probability space $(E, \mathcal{B}(E), \mu_{\eta_{t-1}})$, where $e \in E = [\underline{e}, \overline{e}] \subset I\!R_{++}$ and draws its realization of the current-period type-shock, $\eta_t$, according to the transition law $\Gamma_{\eta,\eta'}$.[12] In addition to the surviving households, a measure $\rho_\eta \geq 0$ of type $\eta$ households are born every period, where $\sum_\eta \rho_\eta =$

[11]In a pure infinite-horizon model without preference shocks, it's hard to simultaneously obtain a sizable debt and a large number of bankruptcies; households avoid default by saving a large amount of wealth.

[12]Specifically, $\mathcal{B}(E)$ is the Borel $\sigma$–algebra generated by $E$ and the assumption that the probability measure $\mu_\eta$ is atomless is simply that $\mu_\eta(e) = 0$ for all $e \in E$.

$(1-\delta)$. Newborns of type $\eta$ are identical to survivors of type $\eta$ with good credit ratings ($h=0$) and zero assets ($\ell = 0$ ). We assume that both losses and gains resulting from death are absorbed by firms (or perfect annuity markets). That is, a household that purchases a negative face value bond honors its obligation only if it survives and does not declare bankruptcy, and, symmetrically, a firm that sells a positive face value discount bond is released from its obligation if the household to which the contract was sold is not around to collect.

We now turn to a recursive formulation of the household decision problem. The nature of a household's current period budget correspondence depends on the household's exogenous type, which is the pair $\{\eta, e\}$, its beginning of period asset position $\ell$, its credit rating $h$, and whether or not the household exercises its default option $d$. Denote a household's current period budget correspondence by $B_{\ell,h,\eta,d}(e,q)$, where $d=1$ indicates that the household is exercising its default option and $d=0$ indicates that it's not. In what follows, dependence on discrete variables is indicated by placing those variables as subscripts while dependence on continuous variables is indicated by putting those variables within parentheses. This notation, while slightly nonstandard, is very natural from a computational point of view. Then $B_{\ell,h,\eta,d}(e,q)$ has the following form:

1. If household type $\eta$ has a good credit rating ($h=0$), has no debt or has debt but chooses not to default ($d=0$), then

$$B_{\ell,0,\eta,0}(e,q) = \{c \in I\!R_{+}, \ell' \in L : c + q_{\ell',\eta}\ \ell' \leq e + \ell\}, \tag{2}$$

where $c$ is current consumption and $\ell'$ is the household's end-of-period (beginning of next period) asset position. This is the standard case where the household chooses how much to consume and how much to save, given that its resources are its inherited assets and its current earnings. A nonstandard aspect is that we permit $B_{\ell,0,\eta,0}(e,q)$ to be empty; in particular it could be empty if the household is deep in debt, earnings are low, and new loans are expensive. Allowing $B_{\ell,0,\eta,0}(e,q)$ to be empty permits us to analyze both "voluntary" and "involuntary" default (for the meaning of this distinction, see below).

2. If household type $\eta$ has a good credit rating ($h=0$), has debt and chooses to default

$(d = 1)$, then

$$B_{\ell,0,\eta,1}(e, q) = \{c \in I\!R_+, \ell' = 0 : c \leq e\}. \tag{3}$$

In this case, inherited debts disappear from the budget constraint and no savings is possible during the default period.

3. If household type $\eta$ has a bad credit rating $(h = 1)$, then

$$B_{\ell,1,\eta,0}(e, q) = \{c \in I\!R_+, \ell' \in L^+ : c + q_{\ell',\eta}\, \ell' \leq e(1 - \gamma) + \ell\}, \tag{4}$$

where $L^+ = L \cap I\!R_+$. With a bad credit rating, the household cannot borrow and is subject to pecuniary costs of a bad credit rating.

To set up the household's decision problem, define $\mathcal{L}$ to be the set $L^{--} \times \{0\} \times S \cup L^+ \times \{0, 1\} \times S$ where $L^{--} = L \cap I\!R_{--}$. The set $\mathcal{L}$ lists all $(\ell, h, \eta)$ pairs possible, given that only households with good credit rating can have debt. Let $N_{\mathcal{L}}$ be the cardinality of $\mathcal{L}$. We will now restrict the vector $q$ to lie in a compact set $[0, \overline{q}]^{N_L \times N_S}$ where $1 > \overline{q} > 0$. Later, we will interpret $\overline{q}$ as the reciprocal of the (gross) risk-free savings rate available to households so that $Q$ restricts the implicit interest rates on loans to be at least as large as the risk-free savings rate.

Let $w_{\ell,h,\eta}(q) : \mathcal{L} \times Q \to I\!R$ be a function that assigns a value, for each $q$, to the triple $(\ell, h, \eta)$. For a given $q$, think of this value as a candidate for the expected life-time utility of a household that starts the current period with $(\ell, h)$ but does not yet know its current-period earnings draw or its current-period type shock. It depends on the exogenous type $\eta$ because the realization of its current type is Markov. Typically we will use $\beta\delta\, w_{\ell',h',\eta}(q)$ to refer to the expected value today of tomorrow's lifetime utility if tomorrow's assets are $\ell'$, tomorrow's credit rating is $h'$ and today's type is $\eta$. Let $w(q)$ be the vector-valued function $\{w_{\ell,h,\eta}(q) : \{\ell, h, \eta\} \in \mathcal{L}\}$. In what follows, $w(q)$ will be the unknown in a functional equation.

To develop this functional equation we specify the space in which $w(q)$'s may lie. Let $\mathcal{W}$

be the set of all continuous (vector-valued) functions $w : Q \rightarrow I\!R^{N_{\mathcal{L}}}$ such that:

$$w_{\ell,h,\eta}(q) \in \left[\frac{u[\underline{e}(1-\gamma),\underline{\eta}]}{(1-\beta\delta)}, \frac{u(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta})}{(1-\beta\delta)}\right], \qquad \forall(\ell,h,\eta) \in \mathcal{L},\ \forall q. \tag{5}$$

$$\ell^0 \geq \ell^1 \Rightarrow w_{\ell^0,h,\eta}(q) \geq w_{\ell^1,h,\eta}(q), \qquad \forall(\ell^0,h,\eta) \text{ and } (\ell^1,h,\eta) \in \mathcal{L},\ \forall q. \tag{6}$$

$$w_{\ell,0,\eta}(q) \geq w_{\ell,1,\eta}(q), \qquad \forall(\ell,0,\eta) \text{ and } (\ell,1,\eta) \in \mathcal{L},\ \forall q. \tag{7}$$

$$u(\underline{e}(1-\gamma),\eta) + \beta\delta\ w_{0,1,\eta}(q) > u(0,\eta) + \beta\delta\ w_{\ell_{\max},0,\eta}(q), \qquad \forall q. \tag{8}$$

These restrictions on the expected value function are, for the most part, quite intuitive. The first restriction is essentially a boundedness condition. The particular bounds anticipate the fact that household consumption in our model will never fall below $\underline{e}(1-\gamma)$ or exceed $\overline{e}+\ell_{\max}-\ell_{\min}$ in any period. The next two restrictions impose monotonicity requirements on $w(q)$. The first of these requirements is that, holding fixed a household's credit rating, life-time expected utility should be increasing in $\ell$. The requirement recognizes the fact that a household's budget set increases with $\ell$. The second monotonicity requirement is that, holding fixed the household's asset position, life-time expected utility should be decreasing in $h$. This requirement recognizes the fact that a household with a good credit rating does not suffer any income loss and has the option to borrow. The final restriction is an implicit restriction on the utility function that guarantees that if faced with a level of debt so high that not defaulting on the debt will lead to zero current period consumption, the household will prefer to default. In effect, this is a restriction on how large $u(0,\eta)$ can be. Later we'll supply an explicit upper bound on $u(0,\eta)$ that will justify this restriction.

Next we verify that any constant vector-valued function that satisfies the first bound restriction will satisfy all the others and so,

**Lemma 1.** $\mathcal{W}$ is non-empty.

We now define an operator that yields the maximum life-time utility achievable when the household's current earnings draw is $e$, its current type-draw is $\eta$, and its future life-time utility is assessed according to a given function $w(q)$. Specifically, for $w \in \mathcal{W}$ and $q \in Q$, define $T_1(w)(\ell, h, \eta, e, q)$ as follows:

1. For $\ell < 0$, $h = 0$, and $B_{\ell,0,\eta,0}(e, q) = \emptyset$:

$$T_1(w)\,(\ell, 0, \eta, e, q) \;=\; \max_{c, \in B_{\ell,0,\eta,1}(e,q)} \; u(c, \eta) \;+\; \beta\delta w_{\ell',1,\eta}(q) = \; u(e, \eta) \;+\; \beta\delta w_{0,1,\eta}(q). \quad (9)$$

2. For $\ell < 0$, $h = 0$, and $B_{\ell,0,\eta,0}(e, q) \neq \emptyset$:

$$T_1(w)\,(\ell, 0, \eta, e, q) = \max\left\{ \max_{c,\ell' \in B_{\ell,0,\eta,0}(e,q)} \; u(c, \eta) + \beta\delta w_{\ell',0,\eta}(q), \;\; u(e, \eta) + \beta\delta w_{0,1,\eta}(q) \right\}. \quad (10)$$

3. For $\ell \geq 0$, $h = 0$:

$$T_1(w)\,(\ell, 0, \eta, e, q) = \max_{c,\ell' \in B_{\ell,0,\eta,0}(e,q)} \; u(c, \eta) \;+\; \beta\delta w_{\ell',0,\eta}(q). \quad (11)$$

4. For $\ell \geq 0$, $h = 1$:

$$T_1(w)\,(\ell, 1, \eta, e, q) = \max_{c,\ell' \in B_{\ell,1,\eta,0}(e,q)} \; u(c, \eta) \;+\; \beta\delta \left[\lambda \; w_{\ell',1,\eta}(q) \;+ (1 - \lambda) \; w_{\ell',0,\eta}(q)\right]. \quad (12)$$

The first part of this definition says that if the household has debt and the budget set conditional on not defaulting is empty, the household must default. In this case, the expected life-time utility of the household is simply the sum of the utility from consuming the current endowment and the discounted expected utility of starting next period with no assets and a bad credit rating. The second part says that if the household has debt and the budget set conditional on not defaulting is not empty, the household chooses whichever default option yields higher life-time utility. In the case where both options yield the same utility the household may choose either. The difference between default under part 1 and default under part 2 is the distinction between "involuntary" and "voluntary" default. In the first case, default is the *only* option while in the second case it's the *best* option. The final two parts apply when the household has no debt (so default is not an option) but distinguish between a good and bad credit rating. Recall that when the household's credit rating is bad, there is some probability that it will continue in that state in the following period.

Denote the image of $w$ under $T_1$ by $v_{\ell,h,\eta}(e,q;w) = T_1(w)(\ell,h,\eta,e,q)$ and by $v(e,q;w)$ the vector-valued function whose component functions are $\{v_{\ell,h,\eta}(e,q;w), (\ell,h,\eta) \in \mathcal{L}\}$. Some important properties of these functions are noted in the following Lemma.[13]

**Lemma 2.** For any $w \in \mathcal{W}$ and $(\ell,h,\eta) \in \mathcal{L}$, $v_{\ell,h,\eta}(e,q;w)$ is (i) continuous in $e$ and $q$, (ii) increasing in $e$ and $\ell$, and (iii) is integrable with respect to probability measures $\mu_\eta$, $\eta \in S$.

Next, let $V$ be the set of all vector-valued functions $v : E \times Q \to I\!R^{N_{\mathcal{L}}}$ such that each coordinate function is continuous in $e$ and $q$ and integrable with respect to the probability measure that corresponds to the $\eta$ appearing in that coordinate. For $v \in V$ and $q \in Q$, define the operator $T_2(v)(\ell',h',\eta,q)$ as:

$$T_2(v)(\ell',h',\eta,q) = \sum_{\eta'} \left[ \int_E \; v_{\ell',h',\eta'}(e',q)\; d\mu_\eta(e') \right] \Gamma_{\eta,\eta'}, \tag{13}$$

For the recursive formulation of the household's decision problem to be well-posed, there must be a unique $w(q)$ in $\mathcal{W}$ such that

$$T_2(T_1(w))(\ell',h',\eta,q) = w_{\ell',h',\eta}(q) \qquad\qquad \forall \ell',h',\eta \in \mathcal{L}. \tag{14}$$

If we define $T(w)(q) = \{T_2(T_1(w))(\ell',h',\eta,q),\ \forall \ell',h',\eta \in \mathcal{L}\}$, all $q$, then the requirement in equation (14) can be compactly expressed as $T(w)(q) = w$. The following assumption provides a sufficient condition under which this is true (it is used to guarantee that if faced with a level of debt so high that not defaulting will lead to zero consumption, the household chooses to default).

**Assumption 1.** For every $\eta \in S$

$$u[\underline{e}(1-\gamma),\eta] - u(0,\eta) > \left(\frac{\beta\delta}{1-\beta\delta}\right) \left[u\left(\overline{e} + \ell_{\max} - \ell_{\min}, \overline{\eta}\right) - u\left(\underline{e}\left(1-\gamma\right), \underline{\eta}\right)\right].$$

[13]Since $L$ contains only a finite number of elements, there always exists a solution to these maximization problems.

**Lemma 3.** Let $\|w\| = \max_{\ell,h,\eta}\{\sup_{q\in Q} \mid w_{\ell,h,\eta}(q) \mid\}$ be the norm on $\mathcal{W}$. Then (i) $(\mathcal{W}, \|\cdot\|)$ is a complete metric space, (ii) given Assumption 1, $T(\mathcal{W}) \subset \mathcal{W}$, and (iii) $T$ is a contraction mapping with modulus $\beta\delta$.

Summarizing, we then have,

**Theorem 1.** (The Recursive Formulation Is Well-Defined) There exists a unique $w^* \in \mathcal{W}$ such that $w^* = T(w^*)$.

### 3.2.1 Characterization of the Default Decision

Since the option to default is the novel feature of this paper, it's useful to establish some results on the manner in which the decision to default varies with a household's level of earnings and with its level of debt. Here $w$ is always the unique fixed point $w^*$ and objects with asterisks indicate that they are conditioned on $w^*$. For $\ell < 0$, let $\overline{D}^*_{\ell,\eta}(q) = \{e : v^*_{\ell,h,\eta}(e,q) = u(e,\eta) + \beta\delta w^*_{0,1,\eta}(q)\}$. This is the set of $e$'s for which either $B_{\ell,0,\eta,0}(e,q)$ is empty or $B_{\ell,0,\eta,0}(e,q)$ is not empty but the value from not defaulting does not exceed the value from defaulting. $\overline{D}^*_{\ell,\eta}(q) \subseteq E$ can be interpreted as household type $\eta$'s maximal default set for $\ell$ if we assume that a household who is indifferent between defaulting and not defaulting always defaults.

**Theorem 2.** (The Maximal Default Set is a Closed Interval) If $\overline{D}^*_{\ell,\eta}(q)$ is non-empty, then it's a closed interval.

The intuition for this result can be seen in the following way. Suppose that there are two earnings levels, say $e_1$ and $e_2$ with $e_1 < e_2$, for which it is optimal for the household to default on its debt. Now consider an earnings level $\widehat{e}$ that's intermediate between $e_1$ and $e_2$. Suppose that the household prefers to maintain access to the credit market at $\widehat{e}$, even though it defaults at a higher earnings level $e_2$. It seems intuitive that the reason for not defaulting at the lower earnings level $\widehat{e}$ must be that the household finds it optimal to consume more than its earnings and incur even more debt. On the other hand, the fact that the household defaults at the earnings level $e_1$ but maintains access to the credit market at the higher earnings level $\widehat{e}$ suggests that the reason for not defaulting at $\widehat{e}$ must be that the household finds it optimal to consume less than its earnings and reduce its level of indebtedness. Since the household cannot simultaneously be consuming more and less than $\widehat{e}$, this implies that if the household defaults at $e_1$ and $e_2$ it must default at all intermediate earning levels as well.

**Theorem 3.** (The Maximal Default Set Increases with Indebtedness) If $\ell^0 > \ell^1$, then $\overline{D}^*_{\ell^0,\eta}(q) \subseteq \overline{D}^*_{\ell^1,\eta}(q)$.

The result follows from the property that $v^*_{\ell,0,\eta}(e,q)$ is increasing in $\ell$ and that utility from default is independent of the level of debt. Figure 1 helps to visualize this.

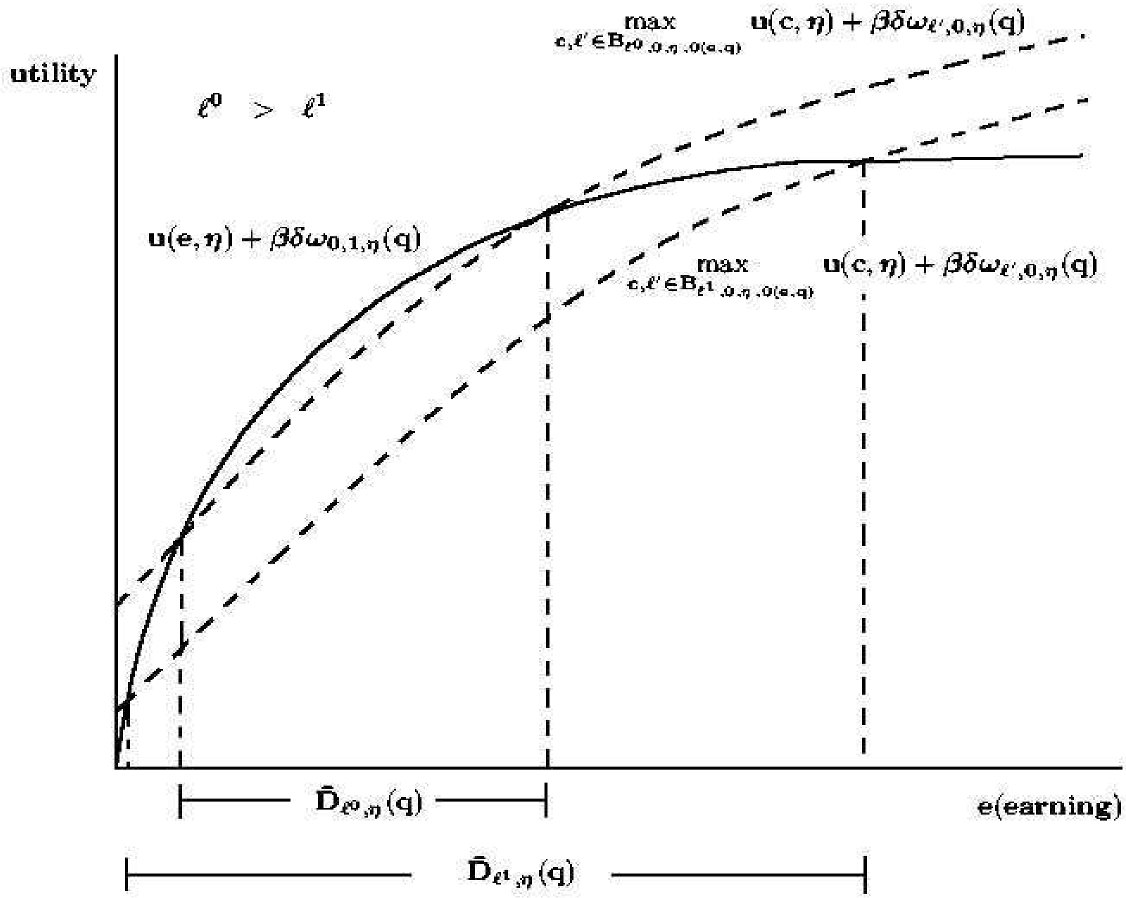


Figure 1: **Typical Default Sets Conditional on Household Type**

### 3.3 Unsecured Credit Industry

Firms serving the consumer credit industry have access to an international credit market in which they can borrow or lend as much as needed at a constant risk-free rate $\widehat{r} > 0$. For $\ell' \in L_{--}$, households of type $\eta$ with a good credit rating can buy contracts in which a firm provides $q_{\ell',\eta} \cdot (-\ell')$ units of the consumption good today in exchange for the household's promise to deliver, conditional on not defaulting, $(-\ell')$ units of the consumption good next period.[14] For $\ell' \in L_{+}$, all households can buy (regardless of credit rating) contracts in which a firm promises to provide, for sure, $\ell'$ units of the consumption good next period in exchange for $q_{\ell',\eta} \cdot \ell'$ units of the consumption good today.

The profit on a contract with $\ell' < 0$ is the present discounted value of inflows less the current value of outflows and the profit on a contract with $\ell' > 0$ is the current value of inflows less the present discounted value of outflows. Therefore, if $a_{\ell',\eta} \geq 0$ is the measure of size $\ell'$ contracts sold to households of type $\eta$, then the (percapita) expected profits of the firm on $(\ell', \eta)$ contracts is

$$\pi[a_{\ell',\eta}; q] = \begin{cases} a_{\ell',\eta}\,(1+\widehat{r})^{-1}\delta[1-p_{\ell',\eta}](-\ell') - a_{\ell',\eta}\,q_{\ell',\eta}\,(-\ell') & \text{if } \ell' < 0 \\ a_{\ell',\eta}\,q_{\ell',\eta}\ell' - a_{\ell',\eta}\,(1+\widehat{r})^{-1}\delta\ell' & \text{if } \ell' \geq 0 \end{cases}, \tag{15}$$

where $p_{\ell',\eta}$ is the fraction of households expected to default on a contract of type $(\ell', \eta)$. Note that the implicit interest charged on loans takes into account that some borrowers will not survive to repay their loans and the implicit interest rate on deposits is also higher to take account of the fact that some depositors may not survive to collect on their deposits. The full expected profits of a firm is then simply $\sum_{(\ell',\eta)\in L\times S} \pi[a_{\ell',\eta}; q]$. The decision problem of each firm is to maximize $\sum_{(\ell',\eta)\in L\times S} \pi[a_{\ell',\eta}; q]$ subject to $a_{\ell',\eta} \geq 0$ for all $(\ell', \eta) \in L \times S$.[15]

[14] We interpret the assumption that firms do not lend to households with a record of a bankruptcy filing in their credit history as a legal restraint on firm behavior. The central banking authority restricts the type of assets that can be held by credit firms. In addition, note that this restriction has the full support of the incumbent firms in the unsecured credit industry. Lifting this restriction will reduce the costs of filing for bankruptcy for consumers, resulting in more defaults than expected and losses for the incumbent credit firms.

[15] Here, and in the household's decision problem, we assumed that a household enters into a single contract with some firm. This simplifies the situation in that a household's end-of-period asset holding is the same as $\ell'$, the size of the single contract entered into by the household. However, this is without loss of generality in the following sense. Let households write any collection of contracts $\{\ell'^k \in L\}$ as long as $\ell' = \sum_k \ell'^k \in L$. Consistent with the procedures of a Chapter 7 bankruptcy filing, assume that a household has the option

## 4 Equilibrium

In this section we define and establish the existence of an equilibrium price schedule and characterize some of its properties. Our strategy is to develop a correspondence whose fixed points satisfy all conditions of an equilibrium price schedule and then show, via the Kakutani Fixed Point Theorem, that a fixed point of the correspondence exists.

There are two key requirements for the proof of existence to work. The first requirement is that the earnings distribution have no mass points. Otherwise, small changes in the price vector $q$ may induce large changes in default behavior, and hence, in default probabilities, and the "equilibrium correspondence" developed below may fail to be continuous. The second requirement is that there not be any tie-breaking rule for households that are indifferent between defaulting and not defaulting. If the set of indifferent households is of positive measure (something that could happen), a tie-breaking rule could lead to a situation where small changes in $q$ induce large changes in default behavior. The first requirement is taken care of by assumption; recall that the measure $\mu_\eta$ is assumed to be atomless. The second requirement is met by letting the mapping between a household's state and its default decision be a *correspondence* rather than a function and developing the fixed point argument in terms of correspondences rather than functions.

For $\ell < 0$, let $d^*_{\ell,\eta}(e, q)$ be an indicator function that takes on the value 1 if either the budget set $B_{\ell',0,\eta,0}(e, q)$ is empty or if the utility from defaulting strictly exceeds the utility from not defaulting, it takes on the value 1 *or* 0 if the utility from defaulting is exactly the same as the utility from not defaulting, and it takes on the value 0 if utility from defaulting is strictly less than the utility from not defaulting. We may interpret $d^*_{\ell,\eta}(e, q)$ as a default decision function. If there are earnings levels at which the household is indifferent between defaulting and not defaulting, there will be more than one default decision function with

to either (i) default on all negative face value subcontracts (i.e., loans) or (ii) not default on any of them. In case of default, assume that creditor-firms can liquidate any positive face value subcontracts held by the household and use the proceeds to recover their loans in proportion to the size of each loan. With these bankruptcy rules in place, the price charged on any subcontract in the collection $\{\ell'^k \in L\}$ must be the price that applies to the single contract of size $\ell'$. Consequently, as long as credit suppliers can condition their loan price on total end-of-period debt position of a household, there is a market arrangement in which the household is indifferent between writing a single contract or a collection of subcontracts with the same total value. Parlour and Rajan (2001) analyze equilibrium in a two-period model of unsecured consumer debt when such conditioning is not possible.

the functions differing in value on at least one such point of indifference. Let $M^*_{\ell,\tilde{\eta}}(q)$ be a set of nonnegative numbers such that if $m \in M^*_{\ell,\tilde{\eta}}(q)$ then there is a $d^*_{\ell,\eta}(e,q)$ such that $\sum_\eta \Gamma_{\tilde{\eta},\eta} \int_E d^*_{\ell,\eta}(e,q)\, d\mu_{\tilde{\eta}}(e) = m$. Then, $M^*_{\ell',\eta}(q)$ is the set of default probabilities consistent with optimizing behavior for debt level $\ell'$ and type $\eta$, given the price schedule $q$.

It is clear from $\pi[a_{\ell',\eta}; q]$ that profits on a contract of type $(\ell',\eta)$ depend linearly on $a_{\ell',\eta}$. We are interested in analyzing a situation where the set of contracts actually sold by firms is entirely demand determined. This requires that the profit from selling any contract in the set $L \times S$ be exactly zero. In other words, it requires that

$$q_{\ell',\eta} = \begin{cases} (1+\widehat{r})^{-1}\delta(1-p_{\ell',\eta}) & \text{if } \ell' < 0 \\ (1+\widehat{r})^{-1}\delta & \text{if } \ell' \geq 0 \end{cases}. \tag{16}$$

Since $p_{\ell',\eta} \geq 0$, the zero-profit requirement implies that the $\overline{q} \in Q = [0,\overline{q}]^{N_L \times N_S}$ is $\delta(1+\widehat{r})^{-1}$.

We can now put the restrictions imposed by zero profits and household optimization. Let $\varphi_{\ell,\eta}(q)$ be a correspondence that takes points in $Q$ to subsets of $[0,\overline{q}]$ given by

$$\varphi_{\ell',\eta}(q) = \begin{cases} \{y : y = \overline{q}(1-m) \text{ for some } m \in M^*_{\ell',\eta}(q)\} & \text{if } \ell' < 0 \\ \overline{q} & \text{if } \ell' \geq 0. \end{cases}$$

Then, $\varphi_{\ell',\eta}(q)$ is the set of prices for a loan of type $(\ell',\eta)$ that are consistent with zero profits given the price vector $q$. Then, we have:

**Definition 1.** A price vector $q^*$ is an equilibrium if, for all $(\ell',\eta) \in L \times S$, $q^*_{\ell',\eta} \in \varphi_{\ell',\eta}(q^*)$.

The equilibrium price vector has the property that positive face value discount bonds bear the risk-free rate and negative face value discount bonds bear a rate that reflects the risk-free rate and a premium for the objective default probability on the bond. There is no mention of quantities in this definition for two reasons. First, the market arrangement precludes any cross-subsidization across loans of different sizes; in particular, it's not possible for firms to charge more than the cost of funds on (small-sized) low-risk loans in order to offset losses on (large-sized) higher-risk loans. For if there were positive profits in some contracts that were offsetting the losses in others, firms could enter the market for those particularly profitable sized loans. In contrast, in the environment of Athreya (1999) and

Lehnert and Maki (2000) such cross-subsidization occurs and the calculation of firm profits requires knowledge of the distribution of customers across various loan sizes. In our model, the equilibrium price schedule is consistent with any distribution of households over the $(\ell, h, \eta)$ space. Second, the assumption that firms can borrow or lend as much as needed in the international credit market means we can ignore any resource balance condition. If this "open-economy" assumption is dropped and the risk-free rate is determined by the marginal product of capital (as in Aiyagari (1994)) then the distribution of households over $\ell$ will become relevant for equilibrium.

**Theorem 4.** A competitive equilibrium exists.

It is possible for $q^*_{\ell',\eta}$ to be zero for some $\ell < 0$. This will happen if the default set corresponding to $\ell'$ is $E$ for all $\eta$. Then, $M^*_{\ell',\eta}(q) = \{1\}$ (i.e., a household of type $\eta$ with a loan of size $\ell'$ will default for any realization of the earnings draw) and $q^*_{\ell',\eta} = 0$. Even in this case, firms are indifferent as to how many loans of type $(\ell', \eta)$ they "sell"; "selling" these loans doesn't cost the firms anything (since the price is zero) and they (rationally) expect the loans to generate no payoff in the following period. From the perspective of a household, taking out one of these free loans gets nothing in the current period but saddles the household with a liability. Since a household of type $\eta$ can do better by choosing $\ell' = 0$ in the current period, there is no demand for such loans either.

We now deal with the limits of the set $L$, for a given $\eta$. Models of precautionary savings have the property that when $\beta\delta < \overline{q}$ there is an upper bound on the amount of assets a household will accumulate. This upper bound arises because as wealth gets larger and larger, the coefficient of variation of income goes to zero, and hence the role of consumption smoothing vanishes.[16] Since ours is also a model of precautionary saving, the same argument applies and $\ell_{\max}$ exists. With respect to the debt limit, $\ell_{\min}$, it can be set to any value less than or equal to $-\overline{e}/(1-\overline{q})$. Note that $-\overline{e}/(1-\overline{q})$ is the largest debt level that could be paid back by the luckiest household facing the lowest possible interest rate and is the polar opposite of the one in Huggett (1993), Aiyagari (1994) and Athreya (1999). As we show in the next theorem, a loan of this size or larger would have a price of zero in any equilibrium. Hence, as long as the lower limit is at least as low as $-\overline{e}/(1-\overline{q})$, it will not have any effect on the equilibrium price schedule.

[16]See Huggett (1993) and Aiyagari (1994) for a detailed argument.

Finally, note that we have not said anything about the existence of a stationary distribution of agents over the state space. Again, we don't really need to because such statements are not necessary for the proof of existence of the equilibrium price schedule. In our computational experiments we have no difficulty in converging to a stationary distribution.[17]

We now turn to characterizing the equilibrium price schedule.

**Theorem 5.** (Characterization of Equilibrium Prices) In any competitive equilibrium: (i) $q^*_{\ell',\eta} = \overline{q}$ for $\ell' \geq 0$; (ii) if the grid for $L$ is sufficiently fine, there exists $\ell^0 < 0$ such that $q^*_{\ell^0,\eta} = \overline{q}$; (iii) $q_{\ell^1,\eta} \geq q_{\ell^2,\eta}$ for $0 > \ell^1 > \ell^2$ ; and (iv) when $\ell_{\min} \leq -\overline{e}/(1-\overline{q})$, $q^*_{\ell_{\min},\eta} = 0$.

The first property simply says that firms charge the risk-free rate on deposits. The second property says that if the grid is taken to be fine enough, there is always a level of debt for which it is never optimal for households to default. As a result, competition leads firms to charge the risk free rate on these loans as well. The third property says that the price on loans falls with the size of loans, i.e., the implied interest rate on loans rises with the size of the loan. The final property says that the prices on loans eventually become zero; in particular, the price on a loan of size $-\overline{e}/(1-\overline{q})$ or larger is always zero in every equilibrium. In other words, the equilibrium delivers an endogenous credit limit for each household type.

## 5 Calibrating the Model to U.S. Data

We now turn to mapping the model to data. This means that we use the structure that we have laid out to reproduce some properties of the U.S. data, and once the model is specified, we use it to answer some quantitative questions. Calibrating the model amounts to choosing parameters so that the statistics in the model economy are the same as those that we have selected as most representative of the features of the U.S. that we are interested in. We start by listing the parameters of our baseline model that need to be calibrated in Section 5.1. We then set the calibration targets in Section 5.2. This requires some discussion of the properties of the wealth (and debt) distribution in the U.S., of bankruptcy, and of how we think they relate to our theory.

[17] We conjecture that the existence of a long-run stationary distribution in our model basically follow from the arguments in Hopenhayn and Prescott (1992).

## 5.1 Parameters to Set in the Calibration

The baseline model has 10 parameters that have to be set. They include parameters on demographics, preferences, technology, the legal system, and the processes for earnings and type shocks. These parameters are (where (#) denotes the number of parameters under each category):

- (1) Demographic parameter. The parameter $\delta$ yields both the the mass of new entrants (young people) and the probability of death.
- (2) Preference parameters. We assume standard time–separable constant relative risk aversion preferences that are characterized by two parameters, the discount rate, $\beta$ and the risk aversion coefficient, $\sigma$.
- (2) Technology. There are two parameters that we place under this heading: the rate of return of the storage technology (or world interest rate) $\widehat{r}$, and the fraction of lost earnings while a household is bankrupt, $\gamma$.
- (1) The legal system is characterized by the average length of the exclusion from access to credit, $(\lambda)$.
- (2) The process for earnings. Despite the general setup of our model economy, here we assume the type shock does not affect the earnings distribution, *i.e.*, earnings are an *i.i.d* process over time. We choose a rather flexible form for its distribution function. Given the range of earnings $[\underline{e}, \overline{e}]$ this functional form is characterized by one parameter. Of the two parameters that define the bounds, only one is relevant as the other can be used to define the units. The distribution function of earnings is given by

$$F(e^*) = P[e \leq e^*] = \left[ \frac{e^* - \underline{e}}{\overline{e} - \underline{e}} \right]^{\epsilon}. \tag{17}$$

- (2) The process for the type shock. We assume that the type shock affects only utility. That is, we posit a multiplicative preference shock to the utility function that essentially increases today's marginal utility of consumption inducing agents to consume a lot. We assume that this increase cannot occur two periods in a row, which implies two parameters: $\theta$, the size of the shock, and $p_\theta$, the probability of it ocurring.

## 5.2 Calibration Targets

Some of the calibration targets are quite uncontroversial and they uniquely determine one parameter value. These include the population turnover rate, that we set at 2.5 percent, which implies an average length of adult life of 40 years, and is a good compromise for an economy without population growth. We also choose the degree of risk aversion and we set it to a value of 1.6. As is standard in the macro literature, it is a little larger than that of log preferences. We assume, consistent with the U.S. Bankruptcy Code, that filing for bankruptcy implies a 10-year exclusion (on average). Finally, we set the rate of return of storage at 0.5 percent. The reason for this low value is that we are interested in the after-tax real rate of return of assets of people who are not very rich, and we think that the rate of return of a checking account is a good guide for this purpose. The remaining calibration targets concern statistics that involve the distribution of households over assets and bankruptcy status and require some discussion.

In our model, the optimal response of individuals to uninsurable idiosyncratic risk generates an endogenous wealth distribution. Therefore, a natural set of statistics to use in the calibration are those that pertain to the distribution of earnings and wealth. However, as is well known,[18] accounting simultaneously for the earnings and wealth distribution in the U.S. economy is no easy task, given the extreme wealth concentration observed in the data. The task requires modelling households with the motive and opportunity to save vast amounts of resources. While this can be done following the method described in Castañeda, Díaz-Giménez, and Ríos-Rull (2000), the associated computations are quite demanding. Given the objective of our paper, we avoid this computational burden so as to effectively meet the (computational) demands of modelling consumer loans and default.[19] Furthermore, we are more interested in the distribution of assets among households that face the possibility of filing for bankruptcy; these are most households except the relatively old,[20] and the richest households.

[18] See for example Quadrini and Ríos-Rull (1997), Krusell and Smith (1998), and Castañeda, Díaz-Giménez, and Ríos-Rull (2000).

[19] See Appendix B for a description of the computational procedures that we have used.

[20] According to Sullivan and Westbrook (2000) the average age of filers is 38 years while the average age of U.S. males over 20 is 45. For Canada, Ramsay (1999) reports an average age of filers of 38.5 and a median age of filers of 37.

In addition, our theory concerns *voluntary loans* that households may or may not pay back later. As we will see, the amounts lent to individual households on this basis are not large, with the maximum debt associated with a positive mass of households being slightly over one year's average income. In the data, the negative asset position of many households arises for reasons such as lawsuits, fines, hospital bills,and business losses[21] and are not the result of a loan made voluntarily by a credit-granting institution.

For these reasons, we match our model to a subset of the U.S. population: to those households whose head is younger than 65 years of age; those who are not in the top quintile of the wealth distribution; and those whose debt does not exceed 120 percent of average earnings of households included in this subset. This reduces our sample of households by about one-third.

Some of the main statistics of the distribution of the described subsample are in Table 1. They describe the wealth to earnings ratio, the size of negative assets, the Gini indices, and the mean-to-median ratio for both earnings and wealth.

Table 1: **Main Characteristics of Household Assets and Earnings in our Sub-Sample of the 1998 Survey of Consumer Finances**

| Statistic | Value |
|---|---|
| Average Earnings | 100.0 |
| Total assets | 153.0 |
| Assets held by households with negative wealth | 2.8 |
| Percentage of households with negative assets | 11.4 |
| Earnings Gini | 0.44 |
| Mean to Median Earnings | 1.19 |
| Wealth Gini | 0.63 |
| Mean to Median Wealth | 1.86 |

Turning next to the distribution of households over bankruptcy status, we note that the percentage of households who file for bankruptcy has risen over the years. In 1990 there were 506,940 filers under Chapter 7 (and 208,666 more under Chapter 13) and in 2000 there

[21]By business loans we mean negative asset positions resulting from contracts between business partners.

were 838,576 filers under Chapter 7 (and 378,366 more under Chapter 13). In those years, the number of people above 20 years of age was 178,059,000 and 196,879,000, respectively. Focusing on the recent statistics, the ratio of filers to total population is 0.426 percent. Since our sub-sample of the SCF consists of two-thirds of the U.S. population, the ratio of filers to total population relevant for us is about 0.64 percent (strictly speaking, this number needs to be adjusted down a bit because in the data there are a few filers over 65 years of age).

A key aspect of the calibration is to determine what fraction of the 0.64 percent figure applies to our model. Recall that our model is about default associated with *voluntary loans*. In particular, it does not relate to defaults that result from household liabilities incurred for other reasons. To address this issue, we use self-reporting to sort filers into categories that should be accounted for by our theory. Specifically, Chakravarty and Rhee (1999) report the reasons stated by PSID survey participants that filed for bankruptcy between 1984 and 1995. We reproduce their partition in Table 2. Of the reasons given, we identify the first three – loss of job, marital distress, and credit mismanagement – as motives for defaulting that should be accounted for by our model. Another 16 percent of filers claim health care reasons. This item results from both high hospital bills and reduced earning capability. Our theory covers this item in so far as it can capture health-related shocks to earnings but it is not designed to cover high hospital bills. Households that point to lawsuits and harassment are probably filing because of liabilities that weren't approved by any credit-granting institution.

Table 2: **Reasons Cited for Filing Bankruptcy**

| Reason | Percentage |
|---|---|
| Loss of job | 12.2 |
| Marital Distress | 14.3 |
| Credit Mismanagement | 41.3 |
| Health Care | 16.4 |
| Lawsuits and Harassment | 15.9 |

These considerations suggest setting a target for the fraction of filers in the model economy that's about 80 percent of 0.64, i.e., a target of 0.5 percent of defaulters. To be consistent, we also adjust down by small amounts the targets for the fraction of population with negative assets and the target for the average size of debt of households with negative assets. The calibration targets for the baseline model economy are, therefore, those reported in Table 3.

In addition to the calibration targets, we also report three other statistics, namely, the Gini index for wealth, the mean-to-median ratio of wealth, and the ratio between the lowest possible earning and average earnings. These will serve as "over-identifying statistics" for our model. Note also that we partition the target statistics into two groups. The first group is easy to calibrate to since each of these statistics is determined by one, or at most two, parameters ($\widehat{r}$, $\sigma$, $\lambda$, $\delta$, $\epsilon$, and $\overline{e}/\underline{e}$). The second group is used to set values for the remaining parameters but this process is more complicated since it involves solving a nonlinear system of four equations and four unknowns.

Table 3: **Calibration Targets for the Baseline Model Economy**

| Statistic | Value |
|---|---|
| Population Turnover in % per year | 2.5 |
| Earnings Gini | 0.44 |
| Mean to Median Earnings | 1.19 |
| Degree of Risk Aversion | 1.6 |
| Length of the Punishment period in years | 10. |
| Rate of return of the storage technology (in %) | 0.5 |
| Wealth to earnings ratio in % | 153.0 |
| Assets held by households with negative wealth (in % of earnings) | 2.5 |
| Percentage of households with negative assets | 10.0 |
| Percentage of defaulters | 0.50 |

## 6 The Baseline Model Economy

The calibration process was successful and we found parameter values that generate the target statistics for the model economy.[22] Table 4 reports the values of the target statistics in the data and in the model economy as well as the list of the parameters and the values

[22]Note that nonlinear equation systems are not guaranteed to have solutions.

that they take. As noted above, we place them in order so that those in the top panel can be chosen by simple calculations (one or two equations) without having to compute the whole equilibrium of the model. In fact, the average length of punishment, $\lambda$, the risk-free rate of return facing firms, $\widehat{r}$, the population turnover rate, $\delta$, and the coefficient of risk aversion, $\sigma$ are readily determined. The parameters $\epsilon$ and the ratio $\overline{e}/\underline{e}$ are those that solve the equations that obtain from setting two statistics generated by the distribution function in (17) (the earnings Gini and the mean-to-median earnings ratio) to the desired targets. The parameters in the middle panel are chosen by solving a system of equations that involves computing the equilibrium of the economy.

Table 4: **Statistics in the Baseline Model Economy and its Parameter Values**

| Statistic | Target | Model | Parameter | Value |
|---|---|---|---|---|
| Targets determined by one or two parameters | | | | |
| Average length of punishment | 10 years | 10 years | $\lambda$ | 0.1 |
| Risk Free Rate of Return | 0.5% | 0.5% | $\widehat{r}$ | 0.005 |
| Population Turnover Rate | 2.5% | 2.5% | $\delta$ | 0.025 |
| Coefficient of Risk Aversion | 1.6 | 1.6 | $\sigma$ | 1.6 |
| Earnings Gini | 0.44 | 0.44 | $\epsilon$ | 0.60422 |
| Mean to Median Earnings | 1.19 | 1.19 | $\overline{e}/\underline{e}$ | 71.6 |
| Targets determined jointly by various parameters | | | | |
| Assets to Earnings Ratio | 153. | 153. | $\gamma$ | 0.004 |
| Negative assets | 2.50 | 2.53 | $\theta$ | 20.154 |
| Percentage of Defaulters | 0.50 | 0.54 | $p_\theta$ | .07 |
| Percentage of the Population with Debt | 10.0 | 10.0 | $\beta$ | 0.8192 |
| Non-Targetted Statistics | | | | |
| Wealth Gini | 0.63 | 0.48 | | |
| Mean to Median Wealth | 1.86 | 1.11 | | |
| Lowest to Mean Earnings (in %) | – | 9.01 | | |

The statistics in the bottom panel of Table 4 have not been targeted. As we can see, the model falls short of reproducing the concentration of wealth found in the data for our sub-sample. Still, while these statistics are quantitatively different they do have properties that are the same as those in the data: *i)* wealth is more concentrated than earnings, *ii)* the wealth distribution is skewed to the right, (the mean-to-median ratio is bigger than 1) and *iii)* the implied ratio between the lowest and mean earnings imply lowest possible annual earnings of about $3,500 which seems reasonable. Overall, the model economy is capable of generating the amount of default observed in the data. We turn next to exploring some of the properties of the baseline model.

## 6.1 Some Properties of the Baseline Model Economy

While our main interest is of course to understand default, it helps to begin by discussing the endogenous distribution of wealth for our model economy. The histogram of the wealth distribution is shown in Figure 2. The asset holdings of households with good and bad credit histories are plotted separately. For households with good credit histories, the model generates a pattern of the wealth distribution that is typical of overlapping generations models. There is a relatively large mass of agents at zero wealth, mostly reflecting the newborn. Because most households accumulate some savings, there is another peak of the histogram at about mean wealth. The distribution has a long right tail reflecting households that have had high earnings realizations and haven't experienced the preference shock for some time. Importantly, there is also a relatively large number of households with a small amount of debt and there is another mass of households with debt in the neighborhood of average annual earnings. Households with a bad credit history consist mostly of households with very few assets. No one in this group has debt because these households are precluded from borrowing. The right tail of this distribution is relatively long, indicating that some households remain with a bad credit history for many periods, have high earnings realizations, and have not experienced the preference shock for some time.

Figure 3 plots the price schedules faced by households with high or low marginal utility of consumption in the current period (that is, $q^*_{\ell',\eta}$ for both values of $\eta$). As predicted, the function is increasing in the value of assets (decreasing in the value of debt). For small values of debt (up to 40 percent of the value of average yearly earnings), households pay low interest rates, since they will not default as our theory suggested. As the size of debt increases so does the interest rate required for the loans, and hence the present value of higher liabilities, what we denote $q$, is lower. Eventually there is a value of debt that is so

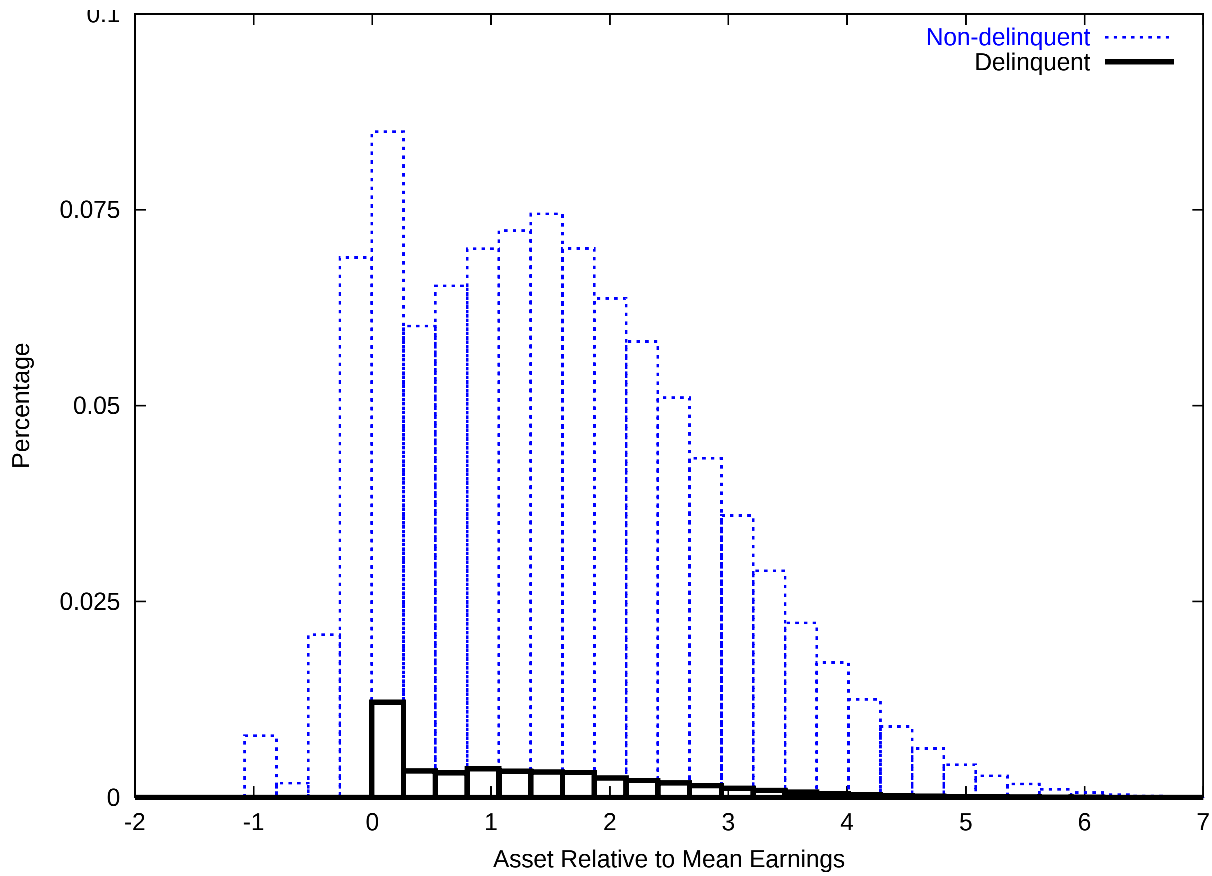


Figure 2: **Distribution of Wealth among Households**

high that households always refuse to pay the debt and the present value of such a liability is zero. This occurs when debt is almost 2.5 times average yearly earnings. This value of the endogenous debt limit is a lot lower in absolute value than the absolute value of the exogenous debt limit given by $\overline{e}(1-\overline{q})$ (which takes the value of 524 times average earnings) or even the debt limit that would exist in a world where there is no bankruptcy and all debt has to be repaid with probability one that is given by $\underline{e}(1-\overline{q})$ (which takes the value of 72 in this calibration). We also note that households that have a high marginal utility of consumption in the current period face slightly larger prices (i.e., their borrowing rates are somewhat lower) because according to our assumptions about the stochastic process $\Gamma$ they will not receive the shock next period and hence are less likely to default.

Figure 4 shows the probabilities of default on loans taken out in the current period conditional on the value of the preference shock next period. For low levels of debt there is never any default. However, starting at about the same level of debt (around 37 percent of average earnings) both types of households (those that experience the preference shocks and those who don't) begin to display positive default probabilities. Households that experience the preference shock (i.e., those with $\eta'$ high) default for a much larger set of earnings and this set expands rapidly with a rise in debt. By around a debt level of 40 percent of average earnings, a household experiencing the preference shock defaults for all realizations of earnings. In contrast, for households that do not experience the preference shock, default probabilities are considerably lower (i.e., default sets are much smaller) and these probabilities do not rise as rapidly with an increase in debt. The relatively high default probabilities of (indebted) households that'll end up experiencing the preference shock next period account for a higher loan price schedule (lower interest rates) for households that experience the preference shock in the current period. Recall, again, that according to our assumptions about the stochastic process $\Gamma$, a household that has $\eta$ high in the current period has $\eta'$ low in the following period for certain. Although default probabilities (for different debt levels) conditional on experiencing the preference shock are quite high, the difference in the loan price schedules is not very big because the probability of experiencing the preference shock next period, conditional on not having experienced it in the current period, is only 7 percent.

Figure 4 would seem to suggest that most default occurs as a result of households' experiencing the preference shock. In a sense this is true, but the connection is a bit subtle. From Figure 2 of the wealth distribution, we see that a little over 16 percent of households

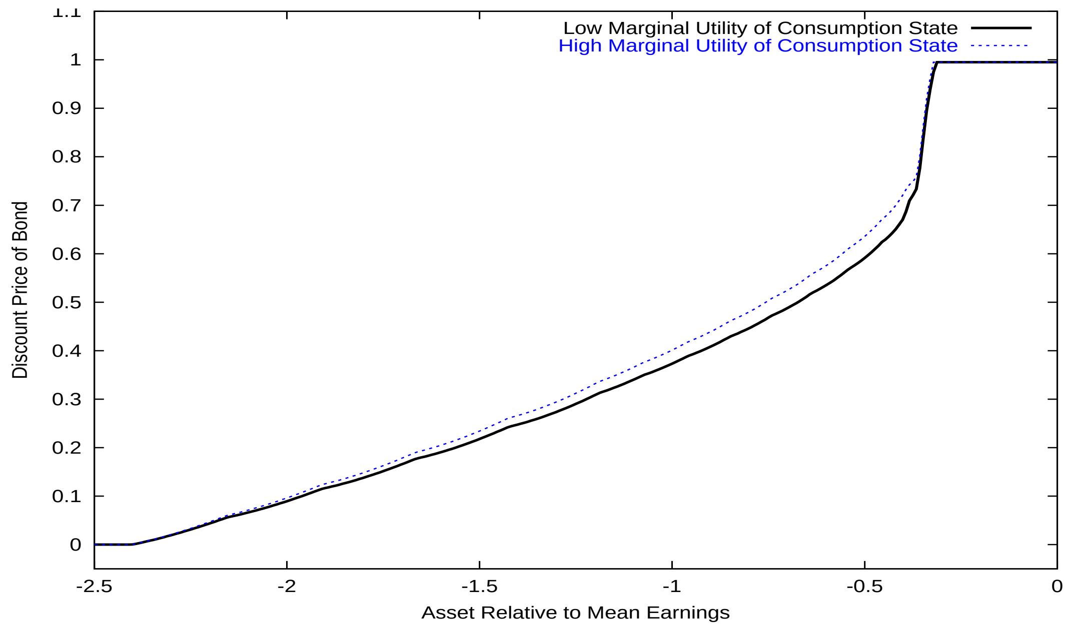


Figure 3: **Dependence of Loan Price on Loan Size and the Value of the Current Preference Shock.**

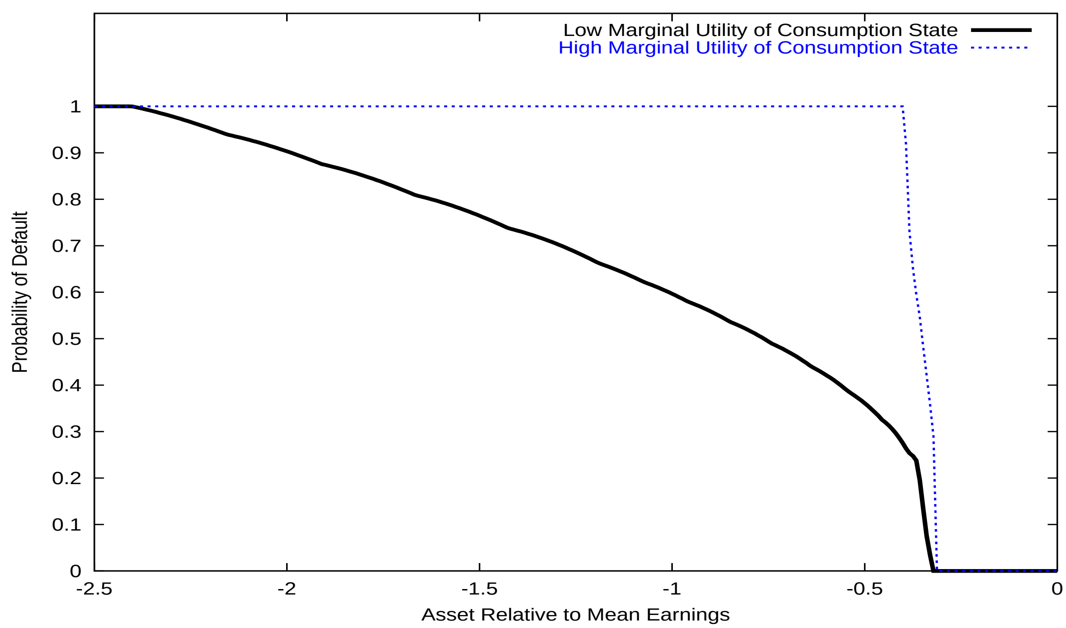


Figure 4: **Dependence of the Probability of Default on Loan Size and the Value of Tomorrow's Preference Shock**

have small amounts of savings or debt (i.e., are situated around zero wealth). A household in this region of asset holdings, when hit by a preference shock in the current period, is induced to borrow significantly both because of the high marginal utility of current consumption and because it faces a slightly lower interest rate (higher q schedule in Figure 3). This shows up in Figure 2 as the small spike in the measure of households (a little under 1.0 percent) at asset holdings of -1. In the following period, these highly indebted households default with a probability of around 60 percent (this can be read off the smooth curve in Figure 4), leading to a steady-state default frequency in the vicinity of 0.5 percent.

As this description indicates, the channel through which the preference shock contributes to default is by inducing households to load up on debt even though the interest rate on unsecured loans is quite high. Once households have loaded up on debt, the event that generally triggers default is a low earnings realization. Indeed, we find that in the steady-state, 75 percent of all defaulters experienced the preference shock in the previous period (despite being only 6 percent of the entire population). If we associate credit mismanagement and some part of marital distress with the preference shock, the three-quarters figure we find is fairly close to the fraction of those defaulting for those reasons cited by Chakravarty and Rhee (1999) relative to the total amount of default that is consistent with our model.[23] To summarize, in our model most households that default are not experiencing a consumption binge currently but did so in the recent past.

# 7 Changing the Bankruptcy Laws

Given that our model matches the relevant U.S. statistics on consumer debt and default, it is possible for us to now examine the consequences of changes in regulation that affect unsecured consumer credit. In the context of our formulation of the U.S. Bankruptcy Code, we perform two policy experiments. These are: (1) a reduction from 10 to 5 years of the (average) length of time that credit bureaus can retain record of a bankruptcy filing, and (2) the imposition of an upper limit on the earnings of those who can file for bankruptcy.

[23] In particular, if credit mismanagement and half of those associated with marital distress are associated with the preference shock, then we would find 72 percent.

### 7.1 A Reduction of Record Keeping to 5 Years

Table 5 reports the long-run implications of reducing the length of the punishment period from 10 to 5 years.

Table 5: **Changing the Law: Shorter Punishment Period**

| Statistic | Baseline 10 years | Shorter Punishment 5 years |
|---|---|---|
| Prob($h' = 0 \mid h = 1$) in % | 10. | 20. |
| Earnings | 100.00 | 100.00 |
| Total assets | 153.204 | 153.830 |
| Negative assets | -2.528 | -2.453 |
| Total Defaulted amount | 0.522 | 0.615 |
| Percentage of Defaulters | 0.541 | 0.655 |
| Percentage with Bad Credit Rating | 4.428 | 2.985 |

As we can see from Table 5 the implications of the change are qualitatively those that were expected, given that a lower punishment period implies a lower degree of commitment.

1. There is less borrowing.

2. There are higher interest rates for all levels of debt (see Figure 5).

3. There is more default.

4. There are fewer households with bad credit ratings.

However, while these effects are consistent with what we would expect, they are quantitatively quite small. This is probably due to the fact that our estimate of the discount factor of households is low (0.82 versus the value of 0.95 that macroeconomists conventionally associate with this parameter) and that the policy change results in differences in punishment that are relatively far into the future.

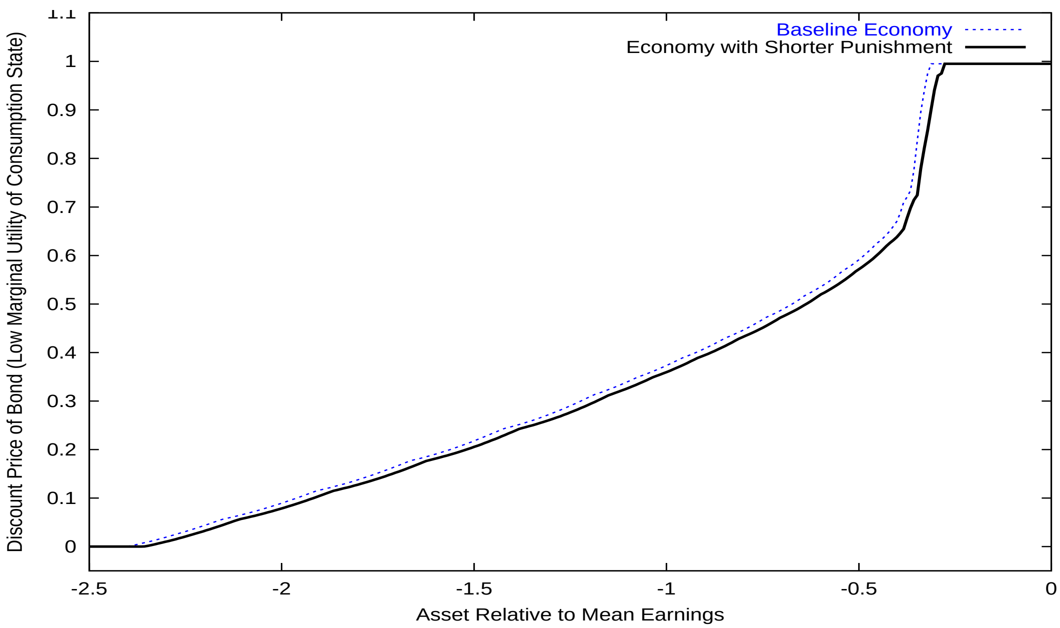


Figure 5: **Price of Loans for the Economy with Shorter Punishment.**

### 7.2 Limit on the Earnings of Bankruptcy Filers

The second policy change we study is aimed at evaluating a proposal currently under consideration in Congress to prevent "above-median-income" households from filing under Chapter 7.[24] This policy change has a lot of bite as indicated by Figure 6. We see that equilibrium interest rates fall quite a lot (the value of $q$ is much higher), indicating that this policy change imposes a strong form of commitment on households. The reason for this dramatic change is that the policy is binding for many agents. In fact, of those that default in the baseline model economy about 15 percent would not be able to do so under the new rules. As a result of this enforced inability to declare bankruptcy, the statistics of the model economy also change dramatically, as indicated by Table 6. This table also shows the results for a weaker policy change, one that limits the right to file for bankruptcy to those with less than 1.68 times

[24]The actual proposal discussed in Congress is that a person cannot file under Chapter 7 (and effectively would have to pursue Chapter 13) if all of the following three conditions are met: (1) Filer's income is at least 100 percent of the national median income for families of the same size up to four members; larger families use median income for a family of four plus an extra \$583 for each additional member over four. (2) The minimum percentage of unsecured debt that could be repaid over 5 years is 25 percent or \$5000, whichever is less. (3) The minimum dollar amount of unsecured debt that could be repaid over 5 years is \$5000 or 25 percent, whichever is less. We summarize these criteria by restricting filing to those with lower than median earnings.

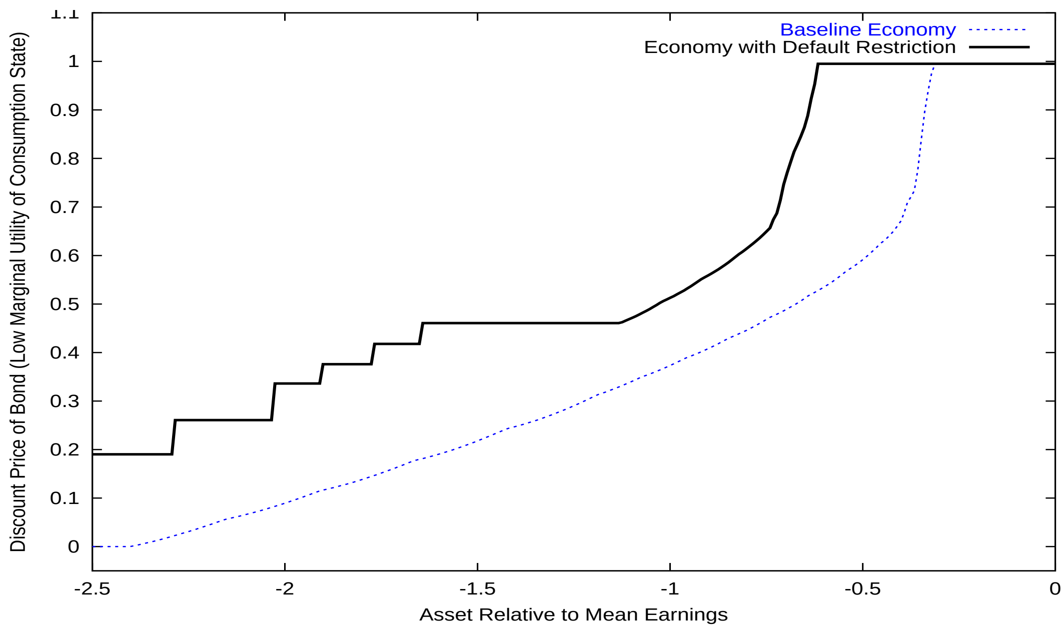


Figure 6: **Price of Loans for the Baseline Economy and the Economy with Restriction on the Ability to Default**

median earnings (or 1.5 times average earnings).[25] The first thing to note is the dramatic decrease of assets held, about 20 percent, under the policy proposal, indicating how important loans are for agents to smooth consumption. Because of the added commitment and the consequent decline in interest rates, the ability to borrow is greatly enhanced, inducing an almost threefold increase in the level of debt. All of this is achieved without a significant increase in the total amount defaulted. The reason is that the percentage of defaulters may actually go down (we can see from the last column that the equilibrium relationship between the percentage of defaulters and the earnings limit is not monotonic).

### 7.3 Welfare Analyses of the Policy Changes

Steady-state comparisons do not say anything about the benefits of a policy change because such comparisons do not take differences in initial conditions into account. To be able to assess whether a policy change is desirable, transition paths need to be computed as well. Moreover, in environments with multiple agents there generally will be no agreement between

[25] This less extreme rule would have prevented about 2.5 percent of the defaults in the baseline model economy.

Table 6: **Changing the Law: Earnings Limits for Defaulters**

| Statistic | Baseline | Tight Earnings Limit | Loose Earnings Limit |
|---|---|---|---|
| Maximum Earnings for Filing (in % of Median Earnings) | No Limit | 100. | 150. |
| Average Earnings | 100.00 | 100.00 | 100.00 |
| Total assets | 153.204 | 124.603 | 138.778 |
| Negative assets | -2.528 | -6.907 | -4.765 |
| Total Defaulted amount | 0.522 | 0.842 | 0.997 |
| Percentage of Defaulters | 0.541 | 0.534 | 0.574 |
| Percentage with Bad Credit Rating | 4.428 | 4.356 | 4.585 |

the agents as to the desirability of policy change and some form of aggregation is necessary. In Table 7 we report the desirability of the two policy changes for two different aggregation criteria. The first criterion asks how many people will be better off from the policy change and the second criterion reports the average amount that households will be willing to pay for the policy change. This amount is measured as a once-and-for-all transfer (not a flow transfer). With different households facing different interest rates, the once-and-for-all measure is more appropriate.

Table 7: **Welfare Comparisons of Alternative Policies Relative to Status Quo**

| | Reduction of Punishment Length to 5 Years | Limit Defaulter's Income to Median Earnings |
|---|---|---|
| Percentage of popular support | 5.40 | 99.99 |
| Average transfer required to be indifferent as % of Mean Earnings | -0.99 | 24.83 |

The policy that reduces the punishment to five years has little support, with only one in 20 households in favor of it. Essentially, those who support the policy change are households with a bad credit history and those who would default under the new policy but not the old one. Next, we see that on average agents have to be compensated by about 1 percent of average earnings to accept this policy change. In welfare terms, this is a sizable number even if it's not a flow measure.

The situation is quite different for the proposal to limit default to those with below-median earnings. Only one in 10,000 households oppose this change and the group that does is a small set of people with a large debt, above-median earnings, the high marginal utility-of-consumption shock, and who would choose to default if they could. Even though the measure of such households is small, they have a large influence on the equilibrium lending patterns as we saw in Figure 6. The much lower equilibrium interest rates translate into large welfare gains. Households are willing (on average) to pay up to a quarter of their earnings to have access to the borrowing technology. Even though the flow value of this amount is 0.125 percent of earnings, in welfare terms this is a huge amount.[26]

## 8 Conclusion

In this paper we have constructed an equilibrium model of credit and default that is consistent with U.S. bankruptcy law and matches U.S. data on the volume of unsecured debt, the fraction of households holding it, the percentage of households that default, and other relevant statistics. In our model, suppliers of unsecured debt price loans of different sizes in a competitive market, taking into account the objective probability of default consistent with optimal household behavior. This leads naturally to an endogenous debt limit below which no firm would offer loans. Thus our paper provides a rationale for borrowing constraints that derives from the institutional and legal features of the U.S. unsecured consumer credit market. Given that our model matches the relevant U.S. statistics on consumer debt and default, we examine the consequences of changes in regulation of bankruptcy and find that certain legal changes can have important effects on the level of unsecured debt and welfare.

There are many other interesting questions that can be answered by extensions of this model. For example, one can measure the relative contributions of changes in information technology and the elimination of usury laws in the late 1970s for the subsequent steep increase in unsecured consumer credit and bankruptcies. We can also study, quantitatively,

[26]Note, however, that our calculation abstracts from any negative effects of a tougher bankrupcty law on work incentives.

phenomena such as “financial fragility” where waves of bankruptcies propagate through the economy. Finally, models of this type are particularly well-suited to assess the implications of interest-rate setting by monetary authorities.

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## Appendix

# A Proofs of Lemmas 1–3 and Theorems 1–5

This appendix provides all the proofs for Lemmas 1 – 3 and Theorems 1–5.

## A.1 The Household Problem

**Lemma 1.** $\mathcal{W}$ is non-empty.

*Proof.* Pick a constant vector-valued function whose value in $\left[\frac{u(\underline{e}(1-\gamma),\underline{\eta})}{(1-\beta\delta)}, \frac{u(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta})}{(1-\beta\delta)}\right]^{N_{\mathcal{L}}}$. Such a function is continuous and obviously satisfies (5), (6), and (7). Since the function is a constant function, (8) is equivalent to $u(\underline{e}(1-\gamma),\eta) - u(0,\eta) > 0$, which is satisfied by virtue of $\underline{e}(1-\gamma) > 0$ and the strict monotonicity of $u(\cdot,\eta)$. □

In what follows let $\{c_{\ell,h,\eta}(e,q;w), \ell'_{\ell,h,\eta}(e,q;w)\}$ be a pair of optimal decision rules, given $w \in \mathcal{W}$. And, when $h = 0$, let $h'_{\ell,0,\eta}(e,q;w)$ be the household's optimally chosen credit rating at the start of next period (when $\ell < 0$, this credit rating is optimally chosen). Before proving Lemma 2 we need the following two results.

**Lemma App. 1.** *For any $w \in \mathcal{W}$, $c_{\ell,h,\eta}(e,q;w) > 0$ for all $(\ell,h,\eta) \in \mathcal{L}$ and for all $(e,q) \in E \times Q$.*

*Proof.* (i) Suppose that $h = 0$ and $\ell < 0$. Since $u(e,\eta) + \beta\delta w_{0,1,\eta}(q) > u[e(1-\gamma),\eta] + \beta\delta\, w_{0,1,\eta}(q) > u(0,\eta) + \beta\delta\, w_{\ell_{\max},0,\eta}(q)$, it follows that $c = 0$ cannot be an optimal choice for any associated choice of $\ell'$. (ii) Suppose, $h = 0$ and $\ell \geq 0$. Then, $c = e$ and $\ell' = 0$ is a feasible choice. Since $u(e,\eta) + \beta\delta\, w_{0,0,\eta}(q) > u[e(1-\gamma),\eta] + \beta\delta\, w_{0,1,\eta}(q)$, it follows from $u[e(1-\gamma),\eta] + \beta\delta\, w_{0,1,\eta}(q) > u(0,\eta) + \beta\delta\, w_{\ell_{\max},0,\eta}(q)$ that $c = 0$ cannot be an optimal choice for any associated choice of $\ell'$. (iii) Suppose $h = 1$ and $\ell \geq 0$. Again, $c = e$ and $\ell' = 0$ is a feasible choice. Since $u[e(1-\gamma),\eta] + \beta\delta[\lambda\, w_{0,1,\eta}(q) + (1-\lambda)\, w_{0,0,\eta}(q)] \geq u[e(1-\gamma),\eta] + \beta\delta\, w_{0,1,\eta}(q)$ and $u(0,\eta) + \beta\delta\, w_{\ell_{\max},0,\eta}(q) \geq u(0,\eta) + \beta\delta[\lambda w_{\ell_{\max},1,\eta}(q) + (1-\lambda) w_{\ell_{\max},0,\eta}(q)]$, it follows $u[e(1-\gamma),\eta] + \beta\delta[\lambda w_{0,1,\eta}(q) + (1-\lambda)\, w_{0,0,\eta}(q)]$ strictly exceeds $u(0,\eta) + \beta\delta[\lambda w_{\ell_{\max},1,\eta}(q) + (1-\lambda)\, w_{\ell_{\max},0,\eta}(q)]$. Hence, $c = 0$ cannot be an optimal choice for any associated choice of $\ell'$. Combining (i)–(iii) we have $c_{\ell,h,\eta}(e,q;w) > 0$ for all $(\ell,h,\eta) \in \mathcal{L}$ and for all $(e,q) \in E \times Q$. □

**Lemma App. 2.** *Let $\{(e^n, q^n) \subset E \times Q$ be a sequence converging to $(e,q)$ in $E \times Q$. Then, $\ell'_{\ell,h,\eta}(e,q;w)$ is also a feasible choice for endowment $e^n$ and price vector $q^n$, provided $n \geq N$ for some $N$.*

*Proof.* First consider the case where $h = 0$. Then, by Lemma App. 1,

$$c_{\ell,0,\eta}(e,q;w) = e + \ell \cdot \left[1 - h'_{\ell,0,\eta}(e,q;w)\right] - q_{\ell'_{\ell,0,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,0,\eta}(e,q;w) > 0.$$

Define

$$c^n = e^n + \ell \cdot \left[1 - h'_{\ell,0,\eta}(e,q;w)\right] - q^n_{\ell'_{\ell,0,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,0,\eta}(e,q;w).$$

Since $\lim_{n\to\infty} c^n = c_{\ell,0,\eta}(e,q;w)$, it follows that there must exist $N_0$ such that $c^n > 0$ for $n \geq N_0$. Hence, $\ell'_{\ell,0,\eta}(e,q;w)$ is a feasible choice when endowment is $e^n$ and the price vector is $q^n$ for $n \geq N_0$.

Now consider the case where $h = 1$. Again, by Lemma App. 1

$$c_{\ell,1,\eta}(e,q;w) = e(1-\gamma) + \ell - q_{\ell'_{\ell,1,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,1,\eta}(e,q;w) > 0.$$

Define

$$c^n = e^n(1-\gamma) + \ell - q^n_{\ell'_{\ell,1,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,1,\eta}(e,q;w).$$

Since $\lim_{n\to\infty} c^n = c_{\ell,1,\eta}(e,q;w)$, there exists $N_1$ such that $c^n > 0$ for all $n \geq N_1$. Hence, $\ell'_{\ell,1,\eta}(e,q;w)$ is a feasible choice when the endowment is $e^n$ and the price vector is $q^n$ for $n \geq N_1$. Combination these two results means $\ell'_{\ell,h,\eta}(e,q;w)$ is a feasible choice when the endowment is $e^n$ and the price vector is $q^n$ for all $n \geq N$ for some $N$. □

We now establish Lemma 2. We do it in parts.

**Lemma 2 (i).** For any $w \in \mathcal{W}$ and $(\ell,h,\eta) \in \mathcal{L}$, $v_{\ell,h,\eta}(e,q;w)$ is continuous in $e$ and $q$.

*Proof.* Step 1: Let $\{(e^n,q^n) \subset E \times Q$ be a sequence converging to $(e,q) \in E \times Q$. We need to show that $\lim_{n\to\infty} v_{\ell,h,\eta}(e^n,q^n;w) = v_{\ell,h,\eta}(e,q;w)$. Suppose, to get a contradiction, that there is an $\varepsilon_0 > 0$ such that for any $\overline{N}$ there is a $N(\varepsilon_0,\overline{N}) > \overline{N}$ for which $\mid v_{\ell,h,\eta}(e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})};w) - v_{\ell,h,\eta}(e,q;w) \mid \geq \varepsilon_0$ ($N$ also depends on $e$, $q$, and $\eta$ but we supress this dependence).

Step 2: Consider first the case where $h = 0$. By Lemma App. 2, there exists an $N_1^0$ such that for all $n \geq N_1^0$

$$v_{\ell,0,\eta}(e^n,q^n;w) \geq u\left[e^n + \ell - q^n_{\ell'_{\ell,0,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,0,\eta}(e,q;w), \eta\right] + \beta\delta\, w_{\ell'_{\ell,0,\eta}(e,q;w),h'_{\ell,0,\eta}(e,q;w),\eta}(q^n). \quad (18)$$

Furthermore, since $\lim_{n\to\infty}\{e^n + \ell - q^n_{\ell'_{\ell,0,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,0,\eta}(e,q;w)\} = c_{\ell,0,\eta}(e,q;w)$ and $u(\cdot,\eta)$ and $w(.)$ are continuous, it follows that for any $\varepsilon_1 > 0$ there is an $N_2^0$ such that

$$\left| u\left[e^n + \ell - q^n_{\ell'_{\ell,0,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,0,\eta}(e,q;w), \eta\right] + \beta\ \delta\ w_{\ell'_{\ell,0,\eta}(e,q;w),h'_{\ell,0,\eta}(e,q;w),\eta}(q^n) - v_{\ell,0,\eta}(e,q;w) \right| < \varepsilon_1$$

for all $n \geq N_2^0$. Then, for all $n \geq N^0 = \max(N_1^0, N_2^0)$,

$$v_{\ell,0,\eta}(e^n, q^n; w) \geq v_{\ell,0,\eta}(e, q; w) - \varepsilon_1.$$

Step 3: Consider now the case where $h = 1$. Again, by Lemma App. 2, there exists an $N_1^1$ such that for all $n \geq N_1^1$

$$v_{\ell,1,\eta}(e^n, q^n; w) \geq u\left[(1-\gamma)e^n + \ell - q^n_{\ell'_{\ell,1,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,1,\eta}(e, q; w), \eta\right] + \beta\delta\left[\lambda\, w_{\ell'_{\ell,1,\eta}(e,q;w),1,\eta}(q^n) + (1-\lambda)\, w_{\ell'_{\ell,1,\eta}(e,q;w),0,\eta}(q^n)\right]$$

Again, since $\lim_{n\to\infty}\left\{(1-\gamma)e^n + \ell - q^n_{\ell'_{\ell,1,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,1,\eta}(e, q; w)\right\} = c_{\ell,1,\eta}(e, q; w)$ and $u(\cdot, \eta)$ and $w(.)$ are continuous, it follows that for any $\varepsilon_1 > 0$ there is an $N_2^1$ such that for all $n \geq N_2^1$

$$\left|u\left[(1-\gamma)e^n + \ell - q^n_{\ell'_{\ell,1,\eta}(e,q;w),\eta} \cdot \ell'_{\ell,1,\eta}(e, q; w), \eta\right] + \beta\delta\left[\lambda\, w_{\ell'_{\ell,1,\eta}(e,q;w),1,\eta}(q^n) + (1-\lambda)\, w_{\ell'_{\ell,1,\eta}(e,q;w),0,\eta}(q^n)\right] - v_{\ell,1,\eta}(e, q; w)\right| < \varepsilon_1.$$

Then, for all $n \geq N^1 = \max(N_1^1, N_2^1)$,

$$v_{\ell,1,\eta}(e^n, q^n; w) \geq v_{\ell,1,\eta}(e, q; w) - \varepsilon_1.$$

Step 4: Combining Steps 2 and 3, we have that for all $(\ell, h, \eta) \in \mathcal{L}$ and all $n \geq \max(N^0, N^1)$,

$$v_{\ell,h,\eta}(e^n, q^n; w) \geq v_{\ell,h,\eta}(e, q; w) - \varepsilon_1.$$

Step 5: From Step 1, we have either

$$v_{\ell,h,\eta}(e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w) \geq v_{\ell,h,\eta}(e, q; w) + \varepsilon_0,$$

or

$$v_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w\right] \leq v_{\ell,h,\eta}(e, q; w) - \varepsilon_0.$$

Since the choice of $\overline{N}$ is arbitrary, take $\overline{N}$ to be greater than $\max(N^1, N^2)$. Now suppose $\varepsilon_1 < \varepsilon_0$. Since $v_{\ell,h,\eta}(e^n, q^n; w) \geq v_{\ell,h,\eta}(e, q; w) - \varepsilon_1$ for all $n \geq \max(N^1, N^2)$ (by Step 4), it follows that $v_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w\right] \leq v_{\ell,h,\eta}(e, q; w) - \varepsilon_0$ is not possible. So, the only possibility is that

$$v_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w\right] \geq v_{\ell,h,\eta}(e, q; w) + \varepsilon_0.$$

Now define:

$$c_{\ell,h,\eta}(N(\varepsilon_0, \overline{N})) = e - \gamma \cdot e \cdot h + \ell - q_{\ell'_{\ell,h,\eta}(e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w),\eta} \cdot \ell'_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}, q^{N(\varepsilon_0,\overline{N})}; w\right].$$

By choosing $\overline{N}$ sufficiently large, $c_{\ell,h,\eta}(N(\varepsilon_0,\overline{N}))$ can be made arbitrarily close to $c_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}\right.$, $\left.q^{N(\varepsilon_0,\overline{N})};w\right]$ because $\{e^{N(\varepsilon_0,\overline{N})},q^{N(\varepsilon_0,\overline{N})}\}$ becomes arbitrarily close to $(e,q)$. Since $c_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})}\right.$, $\left.q^{N(\varepsilon_0,\overline{N})};w\right]$ is strictly positive (Lemma App. 1), this implies that given $\ell,h,e,\eta$, and $q$ the pair $\left\{c_{\ell,h,\eta}\left[N(\varepsilon_0,\overline{N})\right],\ell'_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})},q^{N(\varepsilon_0,\overline{N})};w\right]\right\}$ is feasible and, because $u(\cdot,\eta)$ and $w(\cdot)$ are continuous, delivers lifetime discounted utility arbitrarily close to $v_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})},q^{N(\varepsilon_0,\overline{N})};w\right]$. But $v_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})},q^{N(\varepsilon_0,\overline{N})};w\right]$ exceeds $v_{\ell,h,\eta}(e,q;w)$ by at least $\varepsilon_0>0$, so, for $\overline{N}$ sufficiently large, the pair $\left\{c_{\ell,h,\eta}\left[N(\varepsilon_0,\overline{N})\right],\ell'_{\ell,h,\eta}\left[e^{N(\varepsilon_0,\overline{N})},q^{N(\varepsilon_0,\overline{N})};w\right]\right\}$ delivers lifetime discounted utility strictly greater than $v_{\ell,h,\eta}(e,q;w)$. This contradicts the definition of $v_{\ell,h,\eta}(e,q;w)$. Hence, we have that $\lim_{n\to\infty} v_{\ell,h,\eta}(e^n,q^n;w)=v_{\ell,h,\eta}(e,q;w)$. Thus $v_{\ell,h,\eta}(e,q;w)$ is continuous in $e$ and $q$. □

**Lemma 2 (ii).** For any $w\in\mathcal{W}$, $v_{\ell,h,\eta}(e,q;w)$ is increasing in $e$ and $\ell$.

*Proof.* Given $\eta$, let $e^0\geq e^1$, both elements of $E$. It's easy to verify that for $\ell<0$ and $h=0$, $B_{\ell,0,\eta,d}(e^1,q)\subseteq B_{\ell,0,\eta,d}(e^0,q)$; for $h=0$ and $\ell\geq 0$, $B_{\ell,0,\eta,0}(e^1,q)\subseteq B_{\ell,0,\eta,0}(e^0,q)$; and for $h=1$ and $\ell\geq 0$, $B_{\ell,1,\eta,0}(e^1,q)\subseteq B_{\ell,1,\eta,0}(e^0,q)$. Hence, $v_{\ell,h,\eta}(e^0,q;w)\geq v_{\ell,h,\eta}(e^1,q;w)$.

Let $\ell^0\geq\ell^1$, both elements of $L$. If $0>\ell^0\geq\ell^1$, then $B_{\ell^1,0,\eta,d}(e,q)\subseteq B_{\ell^0,0,\eta,d}(e,q)$; if $\ell^0\geq 0>\ell^1$ then $B_{\ell^1,0,\eta,d}(e,q)\subseteq B_{\ell^0,0,\eta,0}(e,q)$; and if $\ell^0\geq\ell^1\geq 0$ then $B_{\ell^1,h,\eta,0}(e,q)\subseteq B_{\ell^0,h,\eta,0}(e,q)$. It follows from the definition of the $T_1(w)(\ell,h,\eta,e,q)$ operator that $T_1(w)$ $(\ell,h,\eta,e,q)$ is increasing in $\ell$. Hence, $v_{\ell,h,\eta}(e,q;w)$ is increasing in $\ell$. □

**Lemma 2 (iii).** For any $w\in\mathcal{W}$, $v_{\ell,h,\eta}(e,q;w)$ is integrable with respect probability measures $\mu_\eta$, $\eta\in S$.

*Proof.* For any $w\in\mathcal{W}$, $(\ell,h,\eta)\in\mathcal{L}$, and $q\in Q$, $v_{\ell,h,\eta}(e,q;w)$ is a continuous and increasing function of $e$ defined over the compact set $E$. Hence, $v_{\ell,h,\eta}(e,q;w)$ as a function of $e$ is measurable with respect to $B(E)$ and bounded. Therefore, it's integrable with respect to any probability measure defined over $B(E)$. Hence, $v_{\ell,h,\eta}(e,q;w)$ is $\mu_\eta$ -integrable for all $\eta\in S$. □

We now establish Lemma 3 in three parts where $\|w\|=\max_{\ell,h,\eta}\{\sup_{q\in Q}\mid w_{\ell,h,\eta}(q)\mid\}$ be the norm on $\mathcal{W}$.

**Lemma 3 (i).** $(\mathcal{W},\|.\|)$ is a complete metric space.

*Proof.* Let $C(Q)$ be the set of all bounded and continuous (vector-valued) functions from $Q\to R^{N_{\mathcal{L}}}$. Then, $(C(Q),\|.\|)$ is a complete metric space. Since any closed subset of a complete metric space is also a complete metric space, it is sufficient to show that $\mathcal{W}\subset C(Q)$ is closed in the norm $\|.\|$. Let $w^n(q)$ be a sequence of functions in $\mathcal{W}$ converging to $w^*$ in norm, i.e., $\lim_{n\to\infty}\|w^n(q)-w^*\|=0$. We need to show that $w^*\in\mathcal{W}$. If $w^*$ violates any of the range and monotonicity properties of $\mathcal{W}$, there must be some $w^n(q)$, for $n$ sufficiently large, that violates those properties. But that would contradict the assertion that $w^n(q)$ belong to $\mathcal{W}$ for all $n$. Hence, $w^*$ must satisfy all the range and monotonicity properties (5)-(7).

To prove that $w^*(q)$ is continuous we proceed as follows. Let $\varepsilon > 0$ and let $O_r(x)$ denote an open ball of radius $r$ around $x$. Choose $k$ such that $\|w^k(q) - w^*(q)\| < \varepsilon/3$. Fix $q$ at $\tilde{q}$. Choose $r > 0$ such that for all $q \in O_r(\tilde{q})$, $w^k(q) \in O_{\varepsilon/3}(w^k(\tilde{q}))$. This is possible because $w^k(q)$ is continuous. Now consider the Euclidean distance between $w^*(q)$ and $w^*(\tilde{q})$, denoted $\|w^*(q) - w^*(\tilde{q})\|_E$ , for $q \in O_r(\in \tilde{q})$. We have:

$$\|w^*(q) - w^*(\tilde{q})\|_E \leq \|w^*(q) - w^k(q)\|_E + \|w^k(q) - w^k(\tilde{q})\|_E + \|w^k(\tilde{q}) - w^*(\tilde{q})\|_E.$$

Since $\|w^k(q) - w^*(q)\| < \varepsilon/3$, it follows that $\|w^*(q) - w^k(q)\|_E < \varepsilon/3$ for all $q$. Hence, the first and last terms on the r.h.s. of the above inequality are both less than $\varepsilon/3$. The middle term is less than $\varepsilon/3$ for all $q \in O_r(\tilde{q})$ by the choice of $\delta$. Therefore, $\|w^*(q) - w^*(\tilde{q})\|_E < \varepsilon$ for all $q \in O_r(\tilde{q})$. Hence $w^*(q)$ is continuous. □

**Lemma 3 (ii).** $T(\mathcal{W}) \subset \mathcal{W}$.

*Proof.* Recall that $T(w)(q) = \{T_2\,[T_1(w)]\,(\ell', h', \eta, q)$ for all $\{\ell', h', \eta\} \in \mathcal{L}\}$. By Lemma 2, $T_1(w) = v_{\ell,h,\eta}(e, q; w)$ is a continuous function of $q$. Since $Q$ is compact, $v_{\ell,h,\eta}(e, q; w)$ is also bounded. Let $\{q^n\}$ be a sequence in $Q$ converging to $q^* \in Q$. Then

$$\lim_{n\to\infty} \sum_{\eta'} \left[\int v_{\ell',h',\eta'}(e', q^n; w) d\mu_\eta(e')\right] \Gamma_{\eta,\eta'} = \sum_{\eta'} \left[\lim_{n\to\infty} \int v_{\ell',h'\eta'}(e', q^n; w) d\mu_\eta(e')\right] \Gamma_{\eta,\eta'}.$$

By the Lebesgue Dominated Convergence Theorem and the continuity of $v_{\ell,h,\eta}(e, q; w)$,

$$\begin{aligned} \lim_{n\to\infty} \int v_{\ell',h',\eta'}(e', q^n; w)\, d\mu_\eta(e') &= \int \lim_{n\to\infty} v_{\ell',h',\eta'}(e', q^n; w)\, d\mu_\eta(e') \\ &= \int v_{\ell',h',\eta'}(e', q^*; w)\, d\mu_\eta(e'). \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \sum_{\eta} \left[\int v_{\ell',h',\eta'}(e', q^n; w)\, d\mu_\eta(e')\right] \Gamma_{\eta,\eta'} = \sum_{\eta'} \left[\int v_{\ell',h',\eta'}(e', q^*; w)\, d\mu_\eta(e')\right] \Gamma_{\eta,\eta'}$$

In other words,

$$\lim_{n\to\infty} T_2[T_1(w)(\ell', h', \eta, q^n)] = T_2[T_1(w)(\ell', h', \eta, q^*)]$$

Hence, $T(w)(q)$ is also a continuous function of $q$.

We now need to verify that $T(w)(q)$ satisfies the range and monotonicity properties of $\mathcal{W}$. We do this in steps. Pick $w \in \mathcal{W}$.

Step 1: For any $(\ell, h, \eta) \in \mathcal{L}$, and $(e, q) \in E \times Q$,

$$\begin{aligned} v_{\ell,h,\eta}(e,q;w) &\leq u(\overline{e}+\ell_{\max}-\ell_{\min},\eta)+\beta\,\delta\;w_{\ell_{\max},0,\eta} \\ &\leq u(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta})+\beta\,\delta\;\left\{\frac{1}{(1-\beta\delta)}\;u(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta})\right\} \\ &= \frac{1}{(1-\beta\delta)}\;u(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta}). \end{aligned}$$

Hence,

$$\sum_{\eta'}\left[\int v_{\ell',h',\eta'}(e',q;w)\;\mu_\eta(e')\right]\Gamma_{\eta,\eta'} \leq \frac{1}{(1-\beta\delta)}\;u\left[\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta}\right].$$

Since $c_{\ell,h,\eta}(e,q;w)=(1-\gamma)\underline{e}$ is a feasible choice for all $\ell$, $h$, $e$, $\eta$, and $q$, it follows that $v_{\ell,h,\eta}(e,q;w) \geq \frac{1}{(1-\beta\delta)}\;u[(1-\gamma)\underline{e},\underline{\eta}]$. Therefore,

$$\sum_{\eta'}\left[\int v_{\ell',h',\eta'}(e',q;w)\;d\mu_\eta(e')\right]\;\Gamma_{\eta,\eta'} \geq \frac{1}{(1-\beta\delta)}\;u[(1-\gamma)\underline{e},\underline{\eta}].$$

Hence

$$T(w)(q) \in \left[\frac{1}{(1-\beta\delta)}\;\;u[\underline{e}(1-\gamma),\underline{\eta}]\;,\;\frac{1}{(1-\beta\delta)}\;\;u[\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta}]\right]^{N_{\mathcal{L}}}.$$

Step 2: By Assumption 1,

$$u[\underline{e}(1-\gamma),\eta]-u(0,\eta) > \beta\,\delta\;\left[\frac{1}{1-\beta\delta}\;\;u\left(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta}\right)-\frac{1}{1-\beta\delta}\;u\left(\underline{e}\,(1-\gamma)\,,\underline{\eta}\right)\right].$$

From step 1 we have that $T_2(T_1(w))(\ell',h',\eta,q) \leq \frac{1}{1-\beta\delta}u\left(\overline{e}+\ell_{\max}-\ell_{\min},\overline{\eta}\right)$ and we also have that $T_2(T_1(w))(\ell',h',\eta,q) > \frac{1}{1-\beta\delta}u\left(\underline{e}\,(1-\gamma)\,,\underline{\eta}\right)$. Hence,

$$u[\underline{e}(1-\gamma),\eta]-u(0,\eta) > \beta\,\delta\;\left[T_2(T_1(w))(\ell_{\max},0,\eta,q)-T_2(T_1(w))(0,1,\eta,q)\right].$$

Re-arranging gives:

$$u[\underline{e}(1-\gamma),\eta]+\beta\,\delta\;T_2(T_1(w))(0,1,\eta,q) > u(0,\eta)+\beta\,\delta\;T_2(T_1(w))(\ell_{\max},0,\eta,q)$$

Step 3: By Lemma 2, $v_{\ell,h,\eta}(q;w)$ is increasing in $\ell$. Therefore,

$$\sum_{\eta'}\left[\int v_{\ell',h',\eta'}(e',q;w)\;d\mu_\eta(e')\right]\Gamma_{\eta,\eta'}$$

is increasing in $\ell$. Hence, $T_2(T_1(w))(\ell',h',\eta,q)$ is increasing in $\ell$.

Step 4: Given that $w \in \mathcal{W}$, we can verify from the definition of the $T_1(w)(\ell,h,\eta,e,q)$ operator that for $\ell \geq 0$,

$$T_1(w)(\ell, 0, \eta, e, q) \geq T_1(w)(\ell, 1, \eta, e, q).$$

In other words,

$$v_{\ell,0,\eta}(e, q; w) \geq v_{\ell,1,\eta}(e, q; w).$$

Therefore,

$$\sum_{\eta'} \left[ \int v_{\ell',0,\eta'}(e', q; w) \;\; d\mu_\eta(e') \right] \Gamma_{\eta,\eta'} \geq \sum_{\eta'} \left[ \int v_{\ell',1,\eta'}(e', q; w) \;\; d\mu_\eta(e') \right] \Gamma_{\eta,\eta'}.$$

Hence

$$T_2(T_1(w))(\ell', 0, \eta, q) \geq T_2(T_1(w))(\ell', 1, \eta, q).$$

□

**Lemma 3 (iii).** *$T : \mathcal{W} \to \mathcal{W}$ is a contraction mapping with modulus $\beta\delta$.*

*Proof.* The first step is to establish the analogue of the Blackwell monotonicity and discounting properties. Monotonicity: Let $w$, $w' \in \mathcal{W}$ and $w(q) \leq w'(q)$ for all $q$. From the definitions of the $T_1$ and $T_2$ operators it's clear that $T(w) \leq T(w')$. Discounting: It is also clear that for any $c \in R_+^{N_\mathcal{L}}$, $T(w + c)(q) = T(w)(q) + \beta\, \delta\, c$. We can now show that $T$ is a contraction mapping. From the definition of $\| \cdot \|$, it follows that for any $w, w' \in \mathcal{W}$, $w(q) \leq w'(q) + \overline{\|w - w'\|}$, where $\overline{\|w - w'\|}$ is a $N_\mathcal{L}$-element vector with $\|w - w'\|$ as each component element. Hence, $T(w) \leq T\left(w' + \overline{\|w - w'\|}\right) = T(w') + \beta\delta\overline{\|w - w'\|}$. Reversing the roles of $w$ and $w'$ gives $T(w') \leq T(w) + \beta\delta\overline{\|w - w'\|}$. Combining these two inequalities shows that $T(w) - T(w') \leq \beta\delta\overline{\|w - w'\|}$ for all $q$ and $T(w') - T(w) \leq \beta\delta\overline{\|w - w'\|}$ for all $q$ . Hence, $\sup_q \mid T(w)(q) - T(w')(q) \mid \leq \beta\, \delta\, \overline{\|w - w'\|}$. Therefore, $\max_{\ell,h,\eta}\{\sup_q \mid T(w)(q) - T(w')(q) \mid\} \leq \beta\, \delta\, \|w - w'\|$. Hence, $\|T(w) - T(w')\| \leq \beta\, \delta\, \|w - w'\|$. This establishes that $T$ is a contraction mapping with modulus $\beta\delta$. □

**Theorem 1.** (The Recursive Formulation Is Well-Defined) There exists a unique $w^* \in \mathcal{W}$ such that $w^* = T(w^*)$.

*Proof.* Follows directly from Lemma 1 and Lemma 3. □

We turn now to the characterization of the default sets. Recall that $\overline{D}^*_{\ell,\eta}(q) = \{e : v^*_{\ell,h,\eta}(e, q) = u(e, \eta) + \beta\, \delta\, w^*_{0,1,\eta}(q)\}$. We first prove the following two Lemmas.

**Lemma App. 3.** *Let $\widehat{e} \in (\overline{D}^*_\ell(\eta, q))^c$ and $e > \widehat{e}$. If $e \in \overline{D}^*_{\ell,\eta}(q)$ then $c^*_{\ell,0,\eta}(\widehat{e}, q) > \widehat{e}$.*

*Proof.* Since $\widehat{e} \in \left[\overline{D}^*_{\ell,\eta}(q)\right]^c$, we have

$$u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] + \beta\,\delta\; w^*_{\ell'^*_{\ell,0,\eta}(\widehat{e},q),0,\eta}(q) > u(\widehat{e},\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q). \tag{19}$$

Let $\Delta = e - \widehat{e} > 0$. Clearly, the pair $\{\underline{c} = c^*_{\ell,0,\eta}(\widehat{e},q) + \Delta, \underline{\ell'} = \ell'^*_{\ell,0,\eta}(\widehat{e},q)$ belongs in $B_{\ell,0,\eta,0}(e,q)$. By optimality, utility obtained by not defaulting when endowment is $e$ must satisfy the inequality

$$u\left[\tilde{c}_{\ell,0,\eta,0}(e,q)\right] + \beta\,\delta\; w^*_{\tilde{\ell}'_{\ell,0,\eta,0}(e,q),0,\eta}(q) \geq u(\underline{c},\eta) + \beta\,\delta\; w^*_{\underline{\ell'},0,\eta}(q), \tag{20}$$

where $\tilde{c}_{\ell,0,\eta,0}(e,q)$ and $\tilde{\ell}'_{\ell,0,\eta,0}(e,q)$ are the optimal choices of $c$ and $\ell'$ conditional on not defaulting.

Since $e \in \overline{D}^*_{\ell,\eta}(q)$, then,

$$u\left[\tilde{c}_{\ell,0,\eta,0}(e,q),\eta\right] + \beta\,\delta\; w_{\tilde{\ell}'_{\ell,0,\eta,0}(e,q),0,\eta}(q) \leq u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q). \tag{21}$$

By (20) and the fact that $\widehat{e} + \Delta = e$, (21) can be re-written

$$u(\underline{c},\eta) + \beta\delta w^*_{\underline{\ell'},0,\eta}(q) \leq u(\widehat{e} + \Delta,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q). \tag{22}$$

Then (22) minus (19) implies

$$\begin{aligned} u(\underline{c},\eta) + \beta\,\delta\; w^*_{\underline{\ell'},0,\eta}(q) - u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] - \beta\,\delta\; w^*_{\ell'^*_{\ell,0,\eta}(\widehat{e},q),0,\eta}(q) < \\ u(\widehat{e} + \Delta,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q) - u(\widehat{e},\eta) - \beta\,\delta\; w^*_{0,1,\eta}(q). \end{aligned} \tag{23}$$

Or, by definition of $(\underline{c}, \underline{\ell'})$,

$$u\left[c^*_{\ell,0,\eta}(\widehat{e},q) + \Delta,\eta\right] - u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] < u(\widehat{e} + \Delta,\eta) - u(\widehat{e},\eta).$$

Since $u(\cdot,\eta)$ is strictly concave, the last inequality implies $c^*_{\ell,0,\eta}(\widehat{e},q) > \widehat{e}$. □

**Lemma App. 4.** *Let $\widehat{e} \in (\overline{D}^*_{\ell,\eta}(q))^c$ and $e < \widehat{e}$ . If $e \in \overline{D}^*_{\ell,\eta}(q)$ then $c^*_{\ell,h,\eta}(\widehat{e},q) < \widehat{e}$.*

*Proof.* Since $\widehat{e} \in \left[\overline{D}^*_{\ell,\eta}(q)\right]^c$, we have

$$u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] + \beta\,\delta\; w^*_{\ell'^*_{\ell,0,\eta}(\widehat{e},q),0,\eta}(q) > u(\widehat{e},\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q). \tag{24}$$

Let $\Delta = \widehat{e} - e > 0$. Consider the quantity $c^*_{\ell,0,\eta}(\widehat{e},q) - \Delta$. If $c^*_{\ell,0,\eta}(\widehat{e},q) - \Delta \leq 0$ then it must be the case that $c^*_{\ell,0,\eta}(\widehat{e},q) < \widehat{e}$ because $\widehat{e} - \Delta = e > 0$. So, we only need to consider the case where $c^*_{\ell,0,\eta}(\widehat{e},q) - \Delta > 0$. Clearly the pair $\{\underline{c} = c^*_{\ell,0,\eta}(\widehat{e},q) - \Delta,\ \underline{\ell'} = \ell'^*_{\ell,0,\eta}(\widehat{e},q)\}$ belongs in $B_{\ell,0,\eta,0}(e,q)$. By optimality, utility obtained by not defaulting when endowment is $e$ must satisfy the inequality

$$u\left[\tilde{c}_{\ell,0,\eta,0}(e,q),\eta\right] + \beta\,\delta\; w^*_{\tilde{\ell}'_{\ell,0,\eta,0},0,\eta}(q) \geq u(\underline{c},\eta) + \beta\,\delta\; w^*_{\underline{\ell'},0,\eta}(q), \tag{25}$$

where, once again, $\tilde{c}_{\ell,0,\eta,0}(e,q)$ and $\tilde{\ell}'_{\ell,0,\eta,0}(e,q)$ are the optimal choices of $c$ and $\ell'$ conditional on not defaulting.

Since $e \in \overline{D}^*_{\ell,\eta}(q)$, then

$$u\left[\tilde{c}_{\ell,0,\eta,0}(e,q),\eta\right] + \beta\,\delta\, w^*_{\tilde{\ell}'_{\ell,0,\eta,0}(e,q),0,\eta}(q) \leq u(e,\eta) + \beta\delta w^*_{0,1,\eta}(q). \tag{26}$$

By (25) and the fact that $\widehat{e} + \Delta = e$ , (26) can be re-written

$$u(\underline{c},\eta) + \beta\delta w^*_{\underline{\ell}',0,\eta}(q) \leq u(\widehat{e}-\Delta,\eta) + \beta\delta w^*_{0,1,\eta}(q). \tag{27}$$

Then (27) minus (24) implies

$$u(\underline{c},\eta) + \beta\,\delta\, w^*_{\underline{\ell}',0,\eta}(q) - u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] - \beta\,\delta\, w^*_{\ell'^*_{\ell,0,\eta}(\widehat{e},q),0,\eta}(q) < \\ u(\widehat{e}-\Delta,\eta) + \beta\delta w^*_{0,1,\eta}(q) - u(\widehat{e},\eta) - \beta\,\delta\, w^*_{0,1,\eta}(q). \tag{28}$$

Or, by definition of $(\underline{c},\, \underline{\ell}')$,

$$u\left[c^*_{\ell,0,\eta}(\widehat{e},q),\eta\right] - u\left[c^*_{\ell,0,\eta}(\widehat{e},q)-\Delta,\eta\right] > u(\widehat{e},\eta) - u(\widehat{e}-\Delta,\eta). \tag{29}$$

Since $u(\cdot,\eta)$ is strictly concave, the last inequality implies $c^*_{\ell,0,\eta}(\widehat{e},q) - \Delta < \widehat{e} - \Delta$, or, $c^*_{\ell,0,\eta}(\widehat{e},q) < \widehat{e}$. □

**Theorem 2.** (A Non-empty Maximal Default Set is a Closed Interval) *If $\overline{D}^*_{\ell,\eta}(q)$ is non-empty, then it's a closed interval.*

*Proof.* Let $e_L = \inf \overline{D}^*_{\ell,\eta}(q))$ and $e_U = \sup \overline{D}^*_{\ell,\eta}(q)$. Since $\overline{D}^*_{\ell,\eta}(q)$ is nonempty and bounded (i.e. $\overline{D}^*_{\ell,\eta}(q) \subset E$, which is bounded), the Completeness Property of Real Numbers assures us that both $e_L$ and $e_U$ exist. If $e_L = e_U$, the default set contains only one element $e = e_L = e_U$ and the result is trivially true. Suppose, then, that $e_L < e_U$. Let $\widehat{e} \in (e_L, e_U)$ and assume that $\widehat{e} \notin \overline{D}^*_{\ell,\eta}(q)$. Then there is a $e \in \overline{D}^*_{\ell,\eta}(q)$ such that $e > \widehat{e}$ (if not, then $e_U = \widehat{e}$ which contradicts the assertion that $\widehat{e} \in (e_L, e_U)$). Then, by Lemma App. 3, $c^*_{\ell,0,\eta}(\widehat{e},q) > \widehat{e}$. Similarly, there is an $e \in \overline{D}^*_{\ell,\eta}(q)$ such that $e < \widehat{e}$. Then, by Lemma App. 4, $c^*_{\ell,0,\eta}(\widehat{e},q) < \widehat{e}$. But $c^*_{\ell,0,\eta}(\widehat{e},q)$ cannot be both greater and less than $\widehat{e}$. Hence, the assertion $\widehat{e} \notin \overline{D}^*_{\ell,\eta}(q)$ must be false and $(e_L, e_U) \subset \overline{D}^*_{\ell,\eta}(q)$.

To show that $e_U \in \overline{D}^*_{\ell,\eta}(q)$, pick a sequence $\{e^n\} \subset (e_L, e_U)$ converging to $e_U$. Then, $v^*_{\ell,0,\eta}(e^n,q) - u(e^n,\eta) = \beta\,\delta\, w^*_{0,1,\eta}(q)$ for all $n$. Since $e_U$ is clearly in $E$, by the continuity of $v^*_{\ell,0,\eta}(e,q)$ and $u$, it follows that $\lim_{n\to\infty}\{v^*_{\ell,0,\eta}(e^n,q) - u(e^n,\eta)\} = v^*_{\ell,0,\eta}(e_U,q) - u(e_U,\eta)$. Since every element of the sequence $\{v^*_{\ell,0,\eta}(e^n,q) - u(e^n,\eta)\}$ is equal to $\beta\,\delta\, w^*_{0,1,\eta}(q)$, it must be the case that $v^*_{\ell,0,\eta}(e_U,q) - u(e_U,\eta) = \beta\,\delta\, w^*_{0,1,\eta}(q)$. Hence, $e_U \in \overline{D}^*_{\ell,\eta}(q)$. By analogous reasoning, $e_L \in \overline{D}^*_{\ell,\eta}(q)$. Hence, $[e_L, e_U] \subseteq \overline{D}^*_{\ell,\eta}(q)$. But by the definition of $e_L$ and $e_U$, $\overline{D}^*_{\ell,\eta}(q) \subset [e_L, e_U]$. Hence $[e_L, e_U] = \overline{D}^*_{\ell,\eta}(q)$. □

**Theorem 3.** (The Maximal Default Set Increases with Indebtedness) *If* $\ell^0 > \ell^1$, *then* $\overline{D}^*_{\ell^0,\eta}(q) \subseteq \overline{D}^*_{\ell^1,\eta}(q)$.

*Proof.* Since $v^*_{\ell,0,\eta}(e,q)$ is increasing in $\ell$ and that utility from default is independent of $\ell$, it follows that if $v^*_{\ell^0,0,\eta}(e,q) = u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$, then it must be the case that $v^*_{\ell^1,0,\eta}(e,q) = u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$. Hence any $e$ that belongs in $\overline{D}^*_{\ell^0,\eta}(q)$ must also belong in $\overline{D}^*_{\ell^1,\eta}(q)$.

□

## A.2 Equilibrium

We demonstrate the existence of at least one competitive equilibrium via Kakutani's Fixed Point Theorem (FPT). This is done in three parts. In the first part, we develop the correspondence whose fixed points are competitive equilibria. In the second part, we establish that this correspondence is convex-valued and closed. In the third part, we invoke the Kakutani FPT to establish the existence of a competiive equilibrium and establish some properties of the equilibrium price vector.

### A.2.1 The Equilibrium Correspondence

We start by defining the default correspondence, $\Delta^*_{\ell,\eta}(e,q)$, for $\ell < 0$, as the following mapping from $E \times S \times Q$ to $(\mathcal{P}(\{0,1\}))$, the power set of $\{0,1\}$:

$$\Delta^*_{\ell,\eta}(e,q) = \begin{cases} \{1\} & \text{if } B_{\ell,0,\eta,0}(e,q) = \emptyset, \\ \{1\} & \text{if } \tilde{v}_{\ell,0,\eta,0}(e,q) < u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q) \\ \{0,1\} & \text{if } \tilde{v}_{\ell,0,\eta,0}(e,q) = u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q) \\ \{0\} & \text{if } \tilde{v}_{\ell,0,\eta,0}(e,q) > u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q) \end{cases} \tag{30}$$

where we have defined

$$\tilde{v}_{\ell,0,\eta,0}(e,q) = \max_{(c,\ell')\in B_{\ell,0,\eta,0}(e,q)\neq\emptyset} u(c,\eta) + \beta\,\delta\; w^*_{\ell',0,\eta}(q).$$

Let $I_{\ell,\tilde{\eta},\eta}(e,q)$ be the set of $\mu_{\tilde{\eta}}$–integrable functions $d^*_{\ell,\eta} : E \times Q \to I\!R$ which have the property that $d^*_{\ell,\eta}(e,q) \in \Delta^*_{\ell,\eta}(e,q)$. Define the integral of $I_{\ell,\tilde{\eta},\eta}(e,q)$ as the set $M^*_{\ell,\tilde{\eta},\eta}(q)$ where $m_\eta \in M^*_{\ell,\tilde{\eta},\eta}(q)$ implies there is an $d^*_{\ell,\eta}(e,q) \in I_{\ell,\tilde{\eta},\eta}(e,q)$ such that $\int_E d^*_{\ell,\eta}(e,q)\; d\mu_{\tilde{\eta}}(e) = m_\eta$. Since $\mu_{\tilde{\eta}}$ is atomless, it follows from Hildenbrand (1974) (Theorem 3, p. 62) that $M^*_{\ell,\tilde{\eta},\eta}(q)$ is a convex set. In particular, define $\overline{d}^*_{\ell,\eta}(e,q) \in \Delta^*_{\ell,\eta}(e,q)$ as a function with the property that whenever $\tilde{v}^*_{\ell,0,\eta,0}(e,q) = u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$, then $\overline{d}^*_{\ell,\eta}(e,q) = 1$ and define $\underline{d}^*_{\ell,\eta}(e,q) \in \Delta^*_{\ell,\eta}(e,q)$ as a function with the property that whenever $\tilde{v}^*_{\ell,0,\eta,0}(e,q) = u(e,\eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$, then $\underline{d}^*_{\ell,\eta}(e,q) = 0$. Both functions are $\mu_{\tilde{\eta}}$-integrable and so belong to $I_{\ell,\tilde{\eta},\eta}(e,q)$. Let the integral of the first function be $\overline{m}^*_{\ell,\tilde{\eta},\eta}(q)$ and that of the second be $\underline{m}^*_{\ell,\tilde{\eta},\eta}(q)$. Since every element of $I_{\ell,\tilde{\eta},\eta}(e,q)$ is bounded above and below by $\overline{d}^*_{\ell,\eta}(e,q)$ and $\underline{d}^*_{\ell,\eta}(e,q)$, respectively, it follows that $M^*_{\ell,\tilde{\eta},\eta}(q) = [\underline{m}^*_{\ell,\tilde{\eta},\eta}(q), \overline{m}^*_{\ell,\tilde{\eta},\eta}(q)]$.

Now we integrate over the last component of $M^*$, $\underline{m}^*$ and $\overline{m}^*$ and we have (with a slight abuse of notation)

$$\begin{aligned} M^*_{\ell,\tilde\eta}(q) &= \{m : m = \sum_\eta m_\eta \, \Gamma_{\tilde\eta,\eta} \qquad \text{for some } m_\eta \in M^*_{\ell,\tilde\eta,\eta}(q)\}, \\ \underline{m}^*_{\ell,\tilde\eta}(q) &= \sum_\eta \left[\underline{m}^*_{\ell,\tilde\eta,\eta}(q)\right] \, \Gamma_{\tilde\eta,\eta}, \\ \overline{m}^*_{\ell,\tilde\eta}(q) &= \sum_\eta \left[\overline{m}^*_{\ell,\tilde\eta,\eta}(q)\right] \, \Gamma_{\tilde\eta,\eta}. \end{aligned}$$

Since each of the sets $M^*_{\ell,\tilde\eta,\eta}(q)$ is convex, it follows that $M^*_{\ell,\tilde\eta}(q)$ is also convex (see, for instance, Rockafellar (1970) (p.17)), and, $M^*_{\ell,\tilde\eta}(q) = [\underline{m}^*_{\ell,\tilde\eta}(q), \overline{m}^*_{\ell,\tilde\eta}(q)]$. The set $M^*_{\ell,\tilde\eta}(q)$ is the set of default probabilities on a loan of size $\ell$ taken out by a household of type $\tilde\eta$ that is consistent with optimization, given the price vector $q$.

Define the equilibrium correspondence as

$$\varphi_{\ell',\eta}(q) = \begin{cases} \{y : y = \bar q \, (1-m) \quad \text{for some } m \in M^*_{\ell',\eta}(q)\} & \text{if } \ell' < 0, \\ \bar q & \text{if } \ell' \geq 0. \end{cases}$$

Then, $q^*$ is an equilibrium price vector if $q^*_{\ell',\eta} \in \varphi_{\ell',\eta}(q^*)$ for all $\ell', \eta \in L \times S$.

### A.2.2 Properties of the Equilibrium Correspondence

**Lemma App. 5.** *For each $q$, $\varphi_{\ell',\eta}(q)$ is a closed interval in $I\!R$.*

*Proof.* For $\ell' \geq 0$, $\varphi_{\ell',\eta}(q) = \bar q$ and for $\ell' < 0$, $\varphi_{\ell',\eta}(q) = \left[\bar q \left(1 - \overline{m}^*_{\ell',\eta}(q)\right), \bar q \left(1 - \underline{m}^*_{\ell',\eta}(q)\right)\right]$. In either case, $\varphi_{\ell',\eta}(q)$ is a closed interval in $I\!R$. □

The second property we wish to establish is that the correspondence $\varphi_{\ell,\eta}(q)$ has a closed graph. In order to do that, we need to following two prelimnary results.

**Lemma App. 6.** *Let $\{q^n\} \subset Q$ converging to $q \in Q$. For a given $\tilde\eta$ and $\eta$, let $\{d^*_{\ell,\eta}(e,q^n)\}$ be a sequence such that $d^*_{\ell,\eta}(e,q^n) \in I_{\ell,\tilde\eta,\eta}(e,q^n)$ for all $n$ and let $D\left[d^*_{\ell,\eta}(e,q^n)\right] = \{e : d^*_{\ell,\eta}(e,q^n) = 1\}$. Define*

$$E_n = \cup_{k \geq n} \left[\overline{D}^*_{\ell,\eta}(q)\right]^c \cap D\left[d^*_{\ell,\eta}(e,q^k)\right].$$

*Then $E_{n+1} \subseteq E_n$ for all $n \geq 1$ and $\cap_{n=1}^\infty E_n = \emptyset$.*

*Proof.* Given $\eta$, $E_n$ is the set of all $e$ which belong to the complement of the maximal default set for $q$ but for which there is default for some $q^k$, $k \geq n$. Clearly, if $e \in E_{n+1}$, then $e \in E_n$. Hence $E_{n+1} \subseteq E_n$ for all $n \geq 1$. Let $\widehat e \in [\overline{D}^*_{\ell,\eta}(q)]^c$. By definition of $\overline{D}^*_{\ell,\eta}(q)$, it follows that $v^*_{\ell,0,\eta}(\widehat e, q) > u(\widehat e, \eta) + \beta\,\delta\, w^*_{0,1,\eta}(q)$. By Lemma 2 and the continuity of $u(\cdot,\eta)$ and $w^*$, there must exist $N(\widehat{e})$ such that for all $n \geq N(\widehat{e})$, $v^*_{\ell,0,\eta}(\widehat{e}, q^n) > u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{0,1,\eta}(q^n)$. Hence $\widehat{e} \notin E_n$ for $n \geq N(\widehat{e})$. It follows that $\cap_{n=1}^{\infty} E_n = \emptyset$. □

**Lemma App. 7.** *Let $\{q^n\} \subset Q$ converging to $q \in Q$. For a given $\tilde{\eta}$ and $\eta$, let $\{d^*_{\ell,\eta}(e, q^n)\}$ be a sequence such that $d^*_{\ell,\eta}(e, q^n) \in I_{\ell,\tilde{\eta},\eta}(e, q^n)$ for all $n$ and let $D\left[d^*_{\ell,\eta}(e, q^n)\right] = \{e : d^*_{\ell,\eta}(e, q^n) = 1\}$. Let $\underline{D}^*_{\ell,\eta}(q) = D\left[\underline{d}^*_{\ell,\eta}(e, q)\right]$, and define*

$$H_n = \cup_{k \geq n} \;\; \underline{D}^*_{\ell,\eta}(q) \;\; \cap [D(d^*_{\ell,\eta}(e, q^k))]^c.$$

*Then $H_{n+1} \subseteq H_n$ for all $n \geq 1$ and $\cap_{n=1}^{\infty} H_n = \emptyset$.*

*Proof.* Given $\eta$, $H_n$ is the set of all $e$ which belong to the minimal default set for $q$, but for which there is no default for some $q^k$, $k \geq n$. Clearly, if $e \in H_{n+1}$,then $e \in H_n$. Hence $H_{n+1} \subseteq H_n$ for all $n \geq 1$. Let $\widehat{e} \in \underline{D}^*_{\ell,\eta}(q)$,. Then either $B_{\ell,0,\eta,0}(\widehat{e}, q)$ is empty or $\tilde{v}^*_{\ell,0,\eta,0}(\widehat{e}, q) < u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$. If $B_{\ell,0,\eta,0}(\widehat{e}, q)$ is empty then there must be $N'(\widehat{e})$ such that for all $n \geq N'(\widehat{e})$, $B_{\ell,0,\eta,0}(\widehat{e}, q^n)$ is also empty. If $\tilde{v}_{\ell,0,\eta,0}(\widehat{e}, q) < u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{\ell,0,\eta}(q)$, then we claim that there exists $N'(\widehat{e})$ such that for all $n \geq N'(\widehat{e})$ either $B_{\ell,0,\eta,0}(\widehat{e}, q^n)$ is empty or $\tilde{v}_{\ell,0,\eta,0}(\widehat{e}, q^n) < u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{\ell,0,\eta}(q^n)$. Suppose not, then there exists a sequence $q^{n_k} \to q$, such that $\tilde{v}_{\ell,0,\eta,0}(\widehat{e}, q^{n_k}) \geq u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{\ell,0,\eta}(q^{n_k})$. Because having zero consumption is strictly worse than default, it must be the case that $\tilde{c}_{\ell,0,\eta,0}(\widehat{e}, q^{n_k}) > \kappa > 0$ for all $n_k$. Then for $q^{n_k}$ sufficiently close to $q$, $\tilde{\ell}'_{\ell,0,\eta,0}(\widehat{e}, q^{n_k})$ is feasible for $(\widehat{e}, q)$ and the pair $\{\widehat{e} + \ell - q_{\tilde{\ell}'_{\ell,0,\eta,0}(\widehat{e}, q^{n_k}),\eta} \cdot \tilde{\ell}'_{\ell,0,\eta,0}(\widehat{e}, q^{n_k}), \tilde{\ell}'_{\ell,0,\eta,0}(\widehat{e}, q^{n_k})\}$ delivers lifetime utility arbitrarily close to $u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$. But this contradicts the fact that $\tilde{v}_{\ell,0,\eta,0}(\widehat{e}, q) < u(\widehat{e}, \eta) + \beta\,\delta\; w^*_{0,1,\eta}(q)$. In either case, there is a $N(\widehat{e})$ such that $\widehat{e} \notin H_n$ for $n \geq N(\widehat{e})$. It follows that $\cap_{n=1}^{\infty} H_n = \emptyset$. □

We can now establish

**Lemma App. 8.** *The correspondence $\varphi_{\ell',\eta}(q)$ has a closed graph.*

*Proof.* For $\ell' \geq 0$, the result is obvious. We need only consider the case that $\ell' < 0$. Let $\{(x^n, q^n)\}$ be any sequence in $Q \times Q$ such that $x^n \in \varphi_{\ell',\eta}(q^n)$ and $\{(x^n, q^n)\}$ converges to $(x, q)$, $q \in Q$. We need to establish that $x \in \varphi_{\ell',\eta}(q)$. Since $x^n \in \varphi_{\ell',\eta}(q^n)$, there exists a $y^n \in M^*_{\ell',\eta}(q^n)$ such that $x^n = \bar{q}(1 - y^n)$. Since $\{x^n\}$ converges to $x$, it is clear that $y^n$ converges to $y$ satisfying $x = \bar{q}(1 - y)$. Hence, there is a sequence $\{(y^n, q^n)\}$ such that $y^n \in M^*_{\ell',\eta}(q^n)$ and $\{(y^n, q^n)\}$ converges to $(y, q)$, $q \in Q$. Now, it is sufficient to show that $y \in M^*_{\ell',\eta}(q)$ for then we can conclude that $x = \bar{q}(1 - y)$ belongs in $\varphi_{\ell',\eta}(q)$.

Let $d^*_{\ell',\eta'}(e', q^n)$ be any sequence of functions in $I_{\ell',\eta,\eta'}(e', q)$ such that $\int d^*_{\ell',\eta'}(e', q^n)\; d\mu_\eta(e') = y^n$.

Step 1: $\quad y \leq \overline{m}^*_{\ell',\eta}(q)$.

Since $D\left[d^*_{\ell',\eta'}(e',q^n)\right] = D\left[d^*_{\ell',\eta'}(e',q^n)\right] \cap \left\{\overline{D}^*_{\ell',\eta'}(q) \cup \left[\overline{D}^*_{\ell',\eta'}(q)\right]^c\right\}$, we have that

$$\mu_\eta\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right)\right] = \mu_\eta\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right) \cap \overline{D}^*_{\ell',\eta'}(q)\right] + \mu_\eta\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right) \cap [\overline{D}^*_{\ell'\eta'}(q)]^c\right].$$

Since $\mu_\eta\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right)\right] = y^n$, and since $\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right) \cap \overline{D}^*_{\ell',\eta'}(q)\right] \subseteq \overline{D}^*_{\ell',\eta'}(q)$, and also $\left[D\left(d^*_{\ell',\eta'}(e',q^n)\right) \cap \left[\overline{D}^*_{\ell'\eta'}(q)\right]^c\right] \subseteq E_n$ it follows that

$$y^n \le \mu_\eta[\overline{D}^*_{\ell',\eta'}(q)] + \mu_\eta[E_n].$$

Since $E_n$ is a decreasing sequence with $\mu[E_n] < +\infty$, $\lim_{n\to\infty}\mu[E_n] = \mu[\cap_{i=1}^\infty E_n]$ (see, for instance, Royden (1988) (Proposition 14, pp. 62). But by Lemma App. 6, $\cap_{i=1}^\infty E_n = \emptyset$ and, so, $\lim_{n\to\infty}\mu[E_n] = 0$. Hence

$$\lim_n y^n \le \overline{m}^*_{\ell',\eta,\eta'}(q), \qquad \text{or} \qquad y \le \overline{m}^*_{\ell',\eta,\eta'}(q).$$

Hence

$$y \le \sum_{\eta'}\left[\overline{m}^*_{\ell',\eta,\eta'}(q)\right]\Gamma_{\eta,\eta'} = \overline{m}^*_{\ell',\eta}(q).$$

<u>Step 2:</u> $\qquad y \ge \underline{m}^*_\ell(q)$.

Since $\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c = \left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c \cap \left\{\underline{D}^*_{\ell',\eta'}(q) \cup \left[\underline{D}^*_{\ell',\eta'}(q)\right]^c\right\}$, we have that

$$\mu_\eta\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c = \mu_\eta\left[\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c \cap \underline{D}^*_{\ell',\eta'}(q)\right] + \mu_\eta\left[\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c \cap [\underline{D}^*_{\ell'\eta'}(q)]^c\right].$$

Since $\mu_\eta\left[\left\{D\left(d^*_{\ell',\eta'}(e',q^n)\right)\right\}^c\right] = 1 - y^n$, and since $\left[\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c \cap \underline{D}^*_{\ell',\eta'}(q)\right] \subseteq H_n$, and also $\left[\left\{D\left[d^*_{\ell',\eta'}(e',q^n)\right]\right\}^c \cap \left[\underline{D}^*_{\ell'\eta'}(q)\right]^c\right] \subseteq \left[\underline{D}^*_{\ell',\eta'}(q)\right]^c$ it follows that

$$1 - y^n \le \mu_\eta[H_n] + \left[1 - \underline{m}^*_{\ell',\eta,\eta'}(q)\right].$$

Since $H_n$ is a decreasing sequence with $\mu[H_n] < +\infty$, $\lim_{n\to\infty}\mu[H_n] = \mu[\cap_{i=1}^\infty H_n]$. But by Lemma App. 7, $\mu[\cap_{i=1}^\infty H_n] = 0$. Hence

$$-\lim_n y^n \le -\underline{m}^*_{\ell',\eta,\eta'}(q), \qquad \text{or} \qquad y \ge \underline{m}^*_{\ell',\eta,\eta'}(q).$$

Hence,

$$y \ge \sum_{\eta'}\left[\underline{m}^*_{\ell',\eta,\eta'}(q)\right]\Gamma_{\eta,\eta'} = \underline{m}^*_{\ell',\eta}(q).$$

<u>Step 3:</u> Therefore $y \in M^*_{\ell',\eta}(q)$ and, hence, $x \in \varphi_{\ell',\eta}(q)$.

□

### A.2.3 Existence of Equilibrium and Properties of the Equilibrium Price Vector

**Theorem 4.** *A competitive equilibrium exists.*

*Proof.* For any $q \in Q$, let $\varphi(q) \subset Q$ be the product correspondence $\prod_{\ell',\eta \in L\times S} \varphi_{\ell',\eta}(q)$. Since $\varphi_{\ell',\eta}(q)$ is convex-valued for each $\ell', \eta$, (by Lemma App. 5), $\varphi(q)$ is convex-valued as well. Furthermore, since $\varphi_{\ell',\eta}(q)$ has a closed graph for each $\ell', \eta$, the product correspondence $\varphi(q)$ has a closed graph as well (see, for instance, Border (1985), (Proposition 11.25 (c), pp.60). Thus, $\varphi(q)$ is a closed, convex-valued correspondence that takes elements of the compact, convex set $Q$ and returns sets in $Q$. By Kakutani's FPT there is $q^* \in Q$ such that $q^* \in \varphi(q^*)$. In other words, there exists $q^*$ such that $q^*_{\ell',\eta} \in \varphi_{\ell',\eta}(q^*)$ for all $(\ell',\eta) \in L \times S$. Hence a competitive equilibrium exists. □

**Theorem 5.** *In any competitive equilibrium: (i) $q^*_{\ell',\eta} = \bar{q}$ for $\ell' \geq 0$; (ii) if the grid for $L$ is sufficiently fine, there exists $\ell^0 < 0$ such that $q^*_{\ell^0,\eta} = \bar{q}$; (iii) $q_{\ell^1,\eta} \geq q_{\ell^2,\eta}$ for $0 > \ell^1 > \ell^2$ ; and (iv) $q^*_{\ell_{\min},\eta} = 0$.*

*Proof.* (i) Follows from the zero profit condition of firms and the definition of $\varphi_{\ell',\eta}(q)$.

(ii) Let the grid be fine enough so that there is at least one $\ell^0 < 0$ for which $\underline{e} + \ell^0 > 0$. For a household with endowment $e$ and type shock $\eta$, the utility from defaulting on a loan of size $\ell^0$ can be expressed as

$$u(e,\eta) + \beta\,\delta \sum_{\eta'} \Gamma_{\eta,\eta'} \cdot$$
$$\int \left\{ u\left[c^*_{0,1,\eta'}\left(e',q^*\right),\eta'\right] + \beta\,\delta\,\left[\lambda\, w_{\ell'^*_{0,1,\eta'}(e',q^*),1,\eta'}(q^*) + (1-\lambda)\, w^*_{\ell'_{0,1,\eta}(e,q^*),0,\eta'}(q^*)\right]\right\} d\mu_\eta(e').$$

Since $\underline{e} + \ell^0 > 0$, an alternative to not defaulting is to pay off the loan, consume the remaining endowment, and in the following period set consumption equal to $c^*_{0,1,\eta'}(e',q^*) + \gamma e'$. The utility from this choice

$$u(e+\ell^0,\eta) + \beta\,\delta \sum_{\eta'} \Gamma_{\eta\eta'} \int \left\{ u\left[c^*_{0,1,\eta'}(e',q^*) + \gamma e',\eta'\right] + \beta\,\delta\, w^*_{\ell'^*_{0,1,\eta'}(e',q^*),0,\eta'}(q^*)\right\} d\mu_\eta(e').$$

In view of (7), the utility-gain from not defaulting must be at least as large as

$$u(e+\ell^0,\eta) - u(e,\eta) + \beta\,\delta \sum_{\eta'} \Gamma_{\eta,\eta'} \int \left\{ u\left[c^*_{0,1,\eta'}\left(e',q^*\right) + \gamma e',\eta'\right] - u[c^*_{0,1,\eta'}\left(e',q^*\right),\eta']\right\} d\mu_\eta(e').$$

Since consumption is bounded above by $\overline{e} + \ell_{\max} - \ell_{\min} = \overline{c}$ and the $u(\cdot,\eta)$ is strictly concave, for each $\eta$ the integral in the above expression is bounded below by $\int [u(\overline{c} + \gamma e',\eta') - u(\overline{c},\eta')] d\mu_\eta(e')$.

Hence, the utility-gain from not defaulting must be also be at least as large as

$$u(e+\ell^0,\eta)-u(e,\eta)+\beta\,\delta\,\sum_{\eta'}\Gamma_{\eta,\eta'}\left[\int[u(\bar{c}+\gamma e',\eta')-u(\bar{c},\eta')]\;d\mu_\eta(e')\right]. \tag{31}$$

Notice that the integral in the above expression is strictly positive and independent of the fineness of the grid for $L$. Since $u(\cdot,\eta)$ is continuous, we can make expression (31) strictly positive. Hence, for a sufficiently fine grid there exists an $\ell^0<0$ for which defaulting is never optimal. Therefore, $q^*_{\ell^0,\eta}=\bar{q}$.

(iii) Let $d^*_{\ell,\eta}(e,q^*)$ be the equilibrium default functions. It is sufficient to establish that if $d^*_{\ell^1,\eta}(e,q^*)=1$ then $d^*_{\ell^2,\eta}(e,q^*)=1$. If $B_{\ell^2,0,\eta,0}(e,q^*)$ is empty, then by the definition of a default function, $d^*_{\ell^2,\eta}(e,q^*)$ must equal 1. Suppose then that $B_{\ell^2,0,\eta,0}(e,q^*)\neq\emptyset$. Then, clearly $B_{\ell^1,0,\eta,0}(e,q^*)\neq\emptyset$ and hence $\tilde{v}^*_{\ell^1,0,\eta,0}(e,q^*)\leq u(e,\eta)+\beta\,\delta\;w^*_{0,1,\eta}(q^*)$. Since $e+\ell^1-q^*_{\ell',\eta}\cdot\ell'<e+\ell^2-q^*_{\ell',\eta}\cdot\ell'$ for all $\ell'\in L$, it follows (recalling that $u$ is strictly increasing) that $\tilde{v}^*_{\ell^2,0,\eta,0}(e,q^*)<\tilde{v}^*_{\ell^1,0,\eta,0}(e,q^*)$. Hence, $\tilde{v}^*_{\ell^2,0,\eta,0}(e,q^*)<u(e,\eta)+\beta\,\delta\;w^*_{0,1,\eta}(q)$. This implies that $d^*_{\ell^2,\eta}(e,q^*)=1$.

(iv) Take $\ell_{\min}=-\bar{e}/(1-\bar{q})$. If a household enters any period with endowment $e$ and $\ell_{\min}$, its consumption conditional on not defaulting is bounded above by $e+\ell_{\min}+\max_{\ell'\in L}\{-q_{\ell',\eta}\cdot\ell'\}$. Since $e$ is bounded above by $\bar{e}$ and $\max_{\ell'\in L}\{-q_{\ell',\eta}\cdot\ell'\}\leq-\bar{q}\cdot\ell_{\min}$, it follows that consumption conditional on not defaulting is bounded above by $\bar{e}+\ell_{\min}-\bar{q}\cdot\ell_{\min}=0$. This means either that the set $B_{\ell_{\min},0,\eta,0}(e,q)$ is empty or that the only feasible consumption is zero consumption. In the first case default is the only option and in the second case it's the best option by (8). Therefore in any competitive equilibrium $q^*_{\ell_{\min},\eta}$ must be zero. By (iii) above,it follows that $q^*_{\ell',\eta}$ must be zero for any $\ell'$ less than $-\bar{e}/(1-\bar{q})$.

□

## B Computational Procedure

This appendix outlines the procedure to compute the steady state equilibrium of the model economy.

First we set grids on the spaces of loan holdings and earnings. We denote the grid on the space of loan holdings by $L=\{\ell_{\min},...,0,...,\ell_{\max}\}$ and in the space of earnings by $E=\{\underline{e},...,\bar{e}\}$. We have to ensure that the earnings grid is sufficiently fine so that the results from the computation do not change if we add additional grid points. Even though, as we discussed in Section 4, there is a lower bound on assets given by $-\bar{e}/(1-\bar{q})$ (because it is guaranteed that $q_{\ell_{\min},\eta}=0$ for all $\eta$), this turns out to be too big in absolute value. We can pick an $\ell_{\min}$ which is much bigger than $-\bar{e}/(1-\bar{q})$, and still have the equilibrium price of $\ell_{\min}$ be zero for all $\eta$. The upper bound, $\ell_{\max}$, is chosen so that the saving decision of households is not binding. As we discussed in the section on equilibrium, concavity of the utility function guarantees the existence of such a nonbinding upperbound.

Since all state variables are discrete and we use a grid on earnings, we can store the earnings process function $F_\eta(e)$, the price schedule of the discounted loans $q_{l,\eta}$, the expected value function $w_{l,h,\eta}$, and the measure of households $x_{\ell,h,\eta}$, as finite-dimensional arrays.

Then, we use the following procedure to compute an equilibrium. Functions defined on the earning space are stored by the values on the grid points on the earning space and are interpolated when needed.

1. Guess an initial price of discounted loans $q^0_{\ell,\eta} \in Q$.
2. Given a price for loans, $q^0_{\ell,\eta}$, we solve the household's optimization problem. This procedure includes finding the value function as well as the default interval $\overline{D}_{\ell,\eta}(q^0)$ for every $(\ell, \eta) \in L \times S$. In particular, we follow the steps below:
   (a) Guess an expected value function $w^0_{\ell,h,\eta}(q^0)$.
   (b) Taking the price of the loans $q^0_{\ell,\eta}$ and the value functions $w^0_{\ell,h,\eta}(q^0)$ as given, solve the household's optimization problem and compute the value associated with each individual state $v_{\ell,h,\eta}(e, q^0; w^0)$. We use the grid search method in finding the optimal asset possition. For the problem of a household with debt and a good credit rating, the optimal default choice $\overline{d}_{\ell,\eta}(e, q^0; w^0)$ requires comparison between the implications of defaulting and not defaulting. This comparison also enables us to calculate the corresponding default interval $\overline{D}_{\ell,\eta}(q^0)$ as a part of the solution. Note also that we only need to store at most two numbers, $e_L$ and $e_U$ for each $(\ell, \eta)$ with $\ell < 0$ in storing the default interval. This is due to the fact that a non-empty maximal default interval is a closed interval.
   (c) Numerically integrate the value associated with each individual state $v_{\ell,h,\eta'}(e', q^0; w^0)$ over the probability measure of the earning process $F_\eta(e)$ and the transition matrix associated with type $\eta$, and derive a new expected value function $w^1_{\ell,h,\eta}(q^0)$. Notice that the fact that the default set is an interval makes the integration relatively easy.
   (d) If $w^1_{\ell,h,\eta}(q^0)$ is sufficiently close to $w^0_{\ell,h,\eta}(q^0)$, stop iterating on $w$ and go to step 3. Otherwise, update the guess for the value function $w_{\ell,h,\eta}(q^0)$ and go back to the step 2b. In the early stages of the iteration on $q_{\ell,\eta}$, the guess for $q_{\ell,\eta}$ is not typicically close to the equilibrium one which suggests that there is no need for a lot of accuracy on the convergence criteria of $w_{\ell,h,\eta}(q^0)$. Therefore, we use a dynamic accuracy adjustment, i.e. the convergence criteria of the iteration of $w_{\ell,h,\eta}(q^0)$ starts out being large but shrinks as the price of loans $q_{\ell,\eta}$ gets close to an equilibrium.
3. Using the default interval $\overline{D}_{\ell,\eta}(, q^0)$ derived in step 2 and the zero profit condition for every $(\ell, \eta) \in L \times S$, we compute the new price of discounted loans $q^1_{\ell,\eta}$. If $q^1_{\ell,\eta}$ is sufficiently close to $q^0_{\ell,\eta}$, stop iterating on $q$ and go on to the step 4, else go back to step 2. Note that, since the price of discounted loans does not depend on the measure of households for each $(\ell, \eta) \in L \times S$, we do not need to find a stationary distribution of households to update the price.
4. Verify that the household's optimization problem is not constrained by the upper bound of loan holdings $\ell_{\max}$. If it is, increase $\ell_{\max}$ and start from the step 1 again. Also, we have to confirm that our choice of $\ell_{\min}$ guarantees that $q_{\ell_{\min},\eta} = 0$ for all $\eta$.

5. Using the optimal decision rules for loan holdings and default, and the exogenous transition probabilities, we compute the associated stationary distribution $x_{\ell,h,\eta}$. As we mentioned earlier, we can compute the stationary measure after deriving the optimal decision rules and prices. We also compute the relevant aggregate statistics as moments with respect to the invariant measure.

In addition, we interpret the above steps as a function from a set of parameter values to a measure of a loss. The loss is defined as a weighted sum of the differences between some aggregate statistics and their target values. We use a numerical minimization routine to find a set of parameter values which minimizes the loss. In particular, we use four parameters $(\gamma, \theta, p_\theta, \beta)$ to match four statistics (assets–to–earnings ratio, total assets of those households that have negative assets, percentage of defaulters, and percentage of the population with debt). As we have shown in the calibration section, the minimization works well for our case.