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ONLINE APPENDIX

Financial Contracting with Enforcement Externalities *

Lukasz A. Drozd and Ricardo Serrano-Padial

This document contains supplementary numerical illustrations, omitted proofs of theoretical results stated in the article, as well as a list of the data sources employed to generate some of the article's figures.

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1 Strategic Complementarities: Numerical Illustration

To highlight the impact of strategic complementarities on the provision of credit implied by the selected value of rents from lack of enforcement γ , Figure A1 compares credit provision under two scenarios. The first scenario assumes that, whenever there are multiple equilibria under common knowledge of X , the low-default equilibrium is always played ('good eq.'). The second one assumes that the highest-default equilibrium is played ('bad eq.'). As the figure shows, when $\gamma = 0.33$, the differences in credit supply range from 5 percent to 20 percent. As complementarities become stronger due to higher rents from the lack of enforcement, differences in credit go up substantially (they are in the 20-40% range for $\gamma = 0.5$). The figure also shows that the link between X and b underlying Proposition 6 is robust to the strength of complementarities, since in both the good and bad equilibrium scenarios, and for both values of γ , credit increases with X .

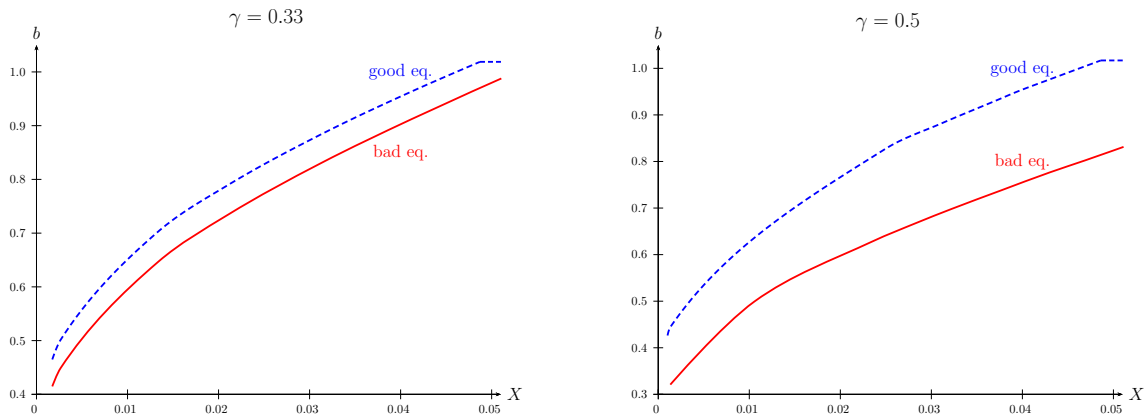


Figure A1: Credit under Common Knowledge of X .

Notes: The figure compares equilibrium credit provision assuming two alternative scenarios. Whenever there are multiple equilibria under common knowledge of X , i) the low-default equilibrium is always played ('good eq.'), and ii) the highest-default equilibrium is played ('bad eq.').

2 Omitted Proofs

Lemma 7 (belief constraint). *For any subset $W' \subseteq \mathcal{W}^*$ and any $z \in [0, 1]$,*

$$\frac{1}{\sum_{W'} f(w)} \sum_{W'} Pr(\psi(X, W') \leq z | x = k(w)) f(w) = z, \quad (\text{A1})$$

where $Pr(\cdot | x = k(w))$ is the probability assessment of $\psi(W', X)$ by an agent receiving $x = k(w)$.

Proof (based on [Sákovics and Steiner, 2012](#)). The proof of the belief constraint is given by the following steps.

First, we define “virtual signals” $\tilde{x} = x - k(w)$ for all $w \in W'$, which are a function of the random vector (x, w) , which represents the type of a player. Virtual signals exhibit a common default threshold $\tilde{k} = 0$.

Second, we show below that, given the uniform prior on X , the probability that $x = k(w)$ for a type (x, w) , conditional on the virtual threshold signal ($\tilde{x} = 0$), is given by the fraction of returns w in subset W' , i.e.,

$$Pr(k(w), w | \tilde{x} = 0) = \frac{f(w)}{\sum_{W'} f(w)}, \quad (\text{A2})$$

where $Pr(x, w | \cdot)$ denotes the conditional probability density of type (x, w) .

Third, we also show below that the default rate $\psi(X, W')$, conditional on $\tilde{x} = 0$, is uniformly distributed in $[0, 1]$. That is,

$$Pr(\psi(X, W') < z | \tilde{x} = 0) = z. \quad (\text{A3})$$

This is due to the fact the “virtual noise” associated to \tilde{x} , defined as $\tilde{\eta} = (\tilde{x} - X)/\nu$, is i.i.d.

and, hence, the aggregate action in W' satisfies the *Laplacian property* (Morris and Shin, 2003).

Finally, combining (A2) and (A3), we have that

$$\begin{aligned} z &= Pr(\psi(X, W') < z | \tilde{x} = 0) = \sum_{W'} Pr(\psi(X, W') \leq z | x = k(w)) Pr(x = k(w) | \tilde{x} = 0) \\ &= \frac{1}{\sum_{W'} f(w)} \sum_{W'} Pr(\psi(X, W') \leq z | x = k(w)) f(w). \end{aligned}$$

To prove (A2) we need to find the marginal distributions of (x, w) and \tilde{x} . First, recall that threshold signals fall in $[\nu/2, 1 - \nu/2]$, as we have shown in the proof of Lemma 6. Hence, we need to focus only on the distribution of signals $x \in [\nu/2, 1 - \nu/2]$. Next notice that, since X, ν and w are independent and $X \sim U[0, 1]$, the joint density of (x, w, X) is given by

$$Pr(x, w, X) = Pr(x|w, X)Pr(w|X)Pr(X) = h\left(\frac{x - X}{\nu}\right) \frac{1}{\nu} \frac{f(w)}{\sum_{W'} f(w)},$$

where h denotes the density of noise η . Given this, the marginal density of (x, w) is

$$Pr(x, w) = \int_{x-\nu/2}^{x+\nu/2} Pr(x, w, X) dX = \int_{x-\nu/2}^{x+\nu/2} h\left(\frac{x - X}{\nu}\right) \frac{1}{\nu} \frac{f(w)}{\sum_{W'} f(w)} dX = \frac{f(w)}{\sum_{W'} f(w)}.$$

The marginal density of the virtual signal $\tilde{x} = x - k(w)$ is given by

$$Pr(x = \tilde{x} + k(w)) = \sum_{W'} Pr(\tilde{x} + k(w), w) = 1,$$

for all \tilde{x} such that $\tilde{x} + k(w) \in [\nu/2, 1 - \nu/2]$. Since $\tilde{x} = 0$ satisfies this condition we have that

$$Pr(k(w), w | \tilde{x} = 0) = \frac{Pr(k(w), w)}{Pr(k(w))} = \frac{f(w)}{\sum_{W'} f(w)}.$$

To prove (A3) notice that virtual noise $\tilde{\eta} = (\tilde{x} - X)/\nu$ is drawn from the mixture distribution $\left\{ H\left(\tilde{\eta} + \frac{k(w)}{\nu}\right), \frac{f(w)}{\sum_{w'} f(w')} \right\}_{w \in W'}$, meaning that (i) with probability $\frac{f(w)}{\sum_{w'} f(w')}$ the virtual noise belongs to type w and, conditional on type w , (ii) its distribution is given by the noise distribution evaluated at $\eta = \tilde{\eta} + k(w)/\nu$, i.e., at the noise level associated to $\tilde{\eta}$. This mixture distribution does not depend on X and thus $\tilde{\eta}$ is i.i.d. across agents and independent of X . Denote by G the (continuous) cdf of $\tilde{\eta}$ and let $G^{-1}(z) = \inf\{\tilde{\eta} : G(\tilde{\eta}) = z\}$. The default rate $\psi(X, W')$ is given by the fraction of agents in W' whose virtual signal is lower than zero, i.e., by those with virtual noise below $-X/\nu$. Accordingly,

$$\begin{aligned} Pr(\psi(X, W') < z | \tilde{x} = 0) &= Pr(G(-X/\nu) < z | \tilde{x} = 0) = Pr(G(\tilde{\eta}) < z) \\ &= Pr(\tilde{\eta} < G^{-1}(z)) = G(G^{-1}(z)) = z. \end{aligned}$$

□

Lemma 8. *There exist a unique partition $\{W_1, \dots, W_I\}$ and a set of thresholds $k_1 > k_2 > \dots > k_I$ such that, as $\nu \rightarrow 0$, for all $w \in W_i$, $i = 1, \dots, I$, $k^\nu(w)$ uniformly converges to k_i . Moreover, thresholds $\mathbf{k} = (k_1, \dots, k_I)$ solve the system of limit indifference conditions*

$$\int_0^1 P\left(k_i, F(\bar{w}) + \sum_{\cup_{j < i} W_j} f(w') + z \sum_{W_i} f(w')\right) dA_w(z | \mathbf{k}, W_i) = \theta(w), \quad \forall w \in W_i, \forall i, \quad (\text{A4})$$

where $A_w(z | \mathbf{k}, W_i)$ represents the strategic beliefs of type- w agents in the limit and satisfies the belief constraint (22).

Proof. To prove convergence, we first partition the set of types into subsets W_i of types for sufficiently small ν as follows: (i) if we order the signal thresholds of all types, any adjacent thresholds that are within ν of each other belong to the same subset, and (ii) $j > i$ implies

that the thresholds associated with types in W_j are lower than those associated with W_i by at least ν . Also, let $Q_w^\nu(x|\mathbf{k}^\nu, z) := Pr(X \leq x | x = k^\nu(w), \psi(X, W_i) = z)$ represent the beliefs about capacity X of an agent of type $w \in W_i$ conditional on receiving her threshold signal $k^\nu(w)$ and on the event that the default rate in W_i is equal to z .

Note that a type- w agent receiving signal $x = k^\nu(w)$ knows that all agents with types in W_j are defaulting if $j < i$ and repaying if $j > i$. Also, the support of $Q_w^\nu(\cdot|\mathbf{k}^\nu, z)$ must lie within $[k^\nu(w) - \nu/2, k^\nu(w) + \nu/2]$. Given this, by the law of iterated expectations, her expected enforcement probability conditional on $x = k^\nu(w)$ can be written in terms of her strategic belief as follows:

$$E(P|\mathbf{k}^\nu; k^\nu(w)) = \int_0^1 \int_{k^\nu(w)-\nu/2}^{k^\nu(w)+\nu/2} P \left(x, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w') \right) dQ_w^\nu(x|\mathbf{k}^\nu, z) dA_w(z|\mathbf{k}^\nu, W_i). \quad (\text{A5})$$

In addition, notice that we can always express $E(P|\mathbf{k}^\nu; k^\nu(w))$ in terms of the threshold signal $k^\nu(w)$ and relative threshold differences $\Delta_{w'} = (k^\nu(w') - k^\nu(w))/\nu$. Importantly, as [Sákovics and Steiner \(2012\)](#) emphasize, strategic beliefs depend on the relative distance between thresholds $\Delta_{W_i} = \{\Delta_{w'}\}_{w' \in W_i}$ rather than on their absolute distance. That is, keeping Δ_{W_i} fixed, $A_w(z|\mathbf{k}^\nu, W_i)$ does not change with ν .¹ This implies that strategic beliefs satisfy the belief constraint when $\nu = 0$.

Fix $k^\nu(w) = k_i$ for some $w \in W_i$ and fix Δ_{W_i} , for all $i = 1, \dots, I$ and all ν sufficiently small. By fixing relative differences, the partition $\{W_i\}_{i=1}^I$ still satisfies the above definition and thus, does not change as $\nu \rightarrow 0$. We are going to show that indifference condition

¹This is straightforward to check. First, if we substitute $X = k^\nu(w) - \nu\eta$ (since agents with type w get her threshold signal) and $k(w') = \nu\Delta_{w'} + k^\nu(w)$ into (21), we find that $\psi(X, W_i)$ only depends on Δ_{W_i} and $k^\nu(w)$. But this means that $A_w(z|\mathbf{k}^\nu, W_i)$ only depends on Δ_{W_i} and $k^\nu(w)$ because h is independent of ν .

$E(P|\mathbf{k}^\nu; k^\nu(w)) = \theta(w)$ is approximated by the limit condition in the lemma for ν sufficiently small.

Note that the inner integral in (A5) is bounded below by

$$P \left(k^\nu(w) - \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w') \right)$$

and above by

$$P \left(k^\nu(w) + \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w') \right).$$

Hence,

$$\begin{aligned} \int_0^1 P \left(k_i - \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w') \right) dA_w(z|\mathbf{k}^\nu, W_i) &\leq E(P|\mathbf{k}^\nu; k^\nu(w)) \\ &\leq \int_0^1 P \left(k_i + \nu/2, F(\bar{w}) + \sum_{\cup_{j<i} W_j} f(w') + z \sum_{W_i} f(w') \right) dA_w(z|\mathbf{k}^\nu, W_i). \end{aligned} \quad (\text{A6})$$

The first term inside these integrals is Lipschitz continuous by Assumption 1. In addition, the next lemma shows that $dA_w(z|\mathbf{k}^\nu, k^\nu(w))$ is bounded for all ν .

Lemma 10. $0 \leq \frac{\partial A_w(z|\mathbf{k}^\nu, k^\nu(w))}{\partial z} \leq \frac{\sum_{W_i} f(w')}{f(w)}$ for all $w \in W_i$ and all z in the support of $A_w(\cdot|\mathbf{k}^\nu, k^\nu(w))$.

See proof below.

Hence, the LHS and the RHS of (A6) uniformly converge to each other as $\nu \rightarrow 0$, leading to limit indifference conditions (A4). Note also that $k^\nu(w) \in [-\bar{\nu}/2, 1 + \bar{\nu}/2]$ and, keeping $\{W_i\}_{i=1}^I$ fixed, $\Delta_{w'}$ are uniformly bounded for all $w' \in W_i$ since any two adjacent thresholds of types in W_i are within ν of each other. That is, the solution to the system of indifference

conditions $E(P|\mathbf{k}^\nu; k^\nu(w)) = \theta(w)$ lies in a compact set.² Accordingly, we can find $\hat{\nu}$ so that indifference conditions are within ε of the limit condition for all $\nu < \hat{\nu}$, leading to their solutions being in a neighborhood of the solution \mathbf{k} of limit indifference conditions (A4). \square

Proof of Lemma 10. Let $\psi^{-1}(z, W_i)$ be the inverse function of $\psi(X, W_i)$ w.r.t. X . The latter function is decreasing in X as long as $0 < \psi(X, W_i) < 1$, implying that ψ^{-1} is well defined and decreasing in such a range of capacities. Since the signal of an agent of type w satisfies $x = X + \nu\eta$, we can express her strategic belief as

$$A_w(z|\mathbf{k}^\nu, W_i) = \mathbb{P}(\psi^{-1}(z, W_i) \leq k^\nu(w) - \nu\eta) = H\left(\frac{k^\nu(w) - \psi^{-1}(z, \mathbf{k}^\nu, W_i)}{\nu}\right).$$

Differentiating w.r.t. z yields

$$\begin{aligned} \frac{\partial A_w(z|\mathbf{k}^\nu, W_i)}{\partial z} &= \frac{1}{\nu} h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right) \left(-\frac{\partial \psi^{-1}(z, W_i)}{\partial z}\right) \\ &= \frac{h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right)}{\frac{1}{\sum_{W_i} f(w')} \sum_{W_i} h\left(\frac{k^\nu(w') - \psi^{-1}(z, W_i)}{\nu}\right) f(w')}. \end{aligned}$$

For all $z \in (0, 1)$, we must have $h\left(\frac{k^\nu(w) - \psi^{-1}(z, W_i)}{\nu}\right) > 0$ because h is bounded away from zero in its support. Hence, the last term is positive and weakly lower than $\frac{\sum_{W_i} f(w')}{f(w)}$. \square

²If $\{W_i\}_{i=1}^I$ is not kept fixed then $E(P|\mathbf{k}^\nu; k^\nu(w))$ would be discontinuous at some ν , implying a violation of the indifference condition for some $w \in \mathcal{W}^*$.

3 Data sources

Figure 1:

- Individual and corporate bankruptcies in the U.S.: U.S. courts, <http://www.uscourts.gov/report-name/bankruptcy-filings>.
- Corporate property executions in Italy: Italian Ministry of Justice, <https://reportistica.dgstat.giustizia.it/>.
- Commercial bankruptcies in Spain: Spain's Ministry of Justice, <http://www6.poderjudicial.es>.
- Commercial bankruptcies in Spain: Spain's Ministry of Justice, <http://www6.poderjudicial.es>.
- Board of Governors of the Federal Reserve System (US), Delinquency Rate on Commercial and Industrial Loans, All Commercial Banks [DRBLACBS], retrieved from FRED, Federal Reserve Bank of St. Louis; <https://fred.stlouisfed.org/series/DRBLACBS>, October 16, 2018.

4 Numerical example with three types

Here we provide a numerical example to illustrate the impulse-response in Figure 4 using our numerical example from Section 2.2.5 with three types. We additionally assume the following parameter values: $y = 1/2$, $\mu = 2/3$, $\gamma = 3/4$, $w_1 = 1$, $w_2 = 1\frac{1}{4}$, $w_3 = 1\frac{1}{3}$.

To define the initial capacity, we derive the minimal X needed to to sustain $b = 1$ in the efficient equilibrium at the enforcement stage, i.e., the equilibrium in which only type-1

agents default while types 2 and 3 repay. In such a case, since the mass of agents of type 1 is $1/3$, the zero profit condition of lenders requires $(2/3)(\bar{b} - b) = (1/3)\mu b$, which gives $\bar{b} = 1\frac{1}{3}$ and which implies $\bar{w} = \bar{b}/(y + b) = 0.89$. Using equation (3), we calculate that $\theta_1 = 0.89$, $\theta_2 = 0.42$ and $\theta_3 = 0.33$.

The calculation of the strategy cutoff in Section 2.2.5 shows that types 2 and 3 will cluster on the same strategy profile at $k_{23}^* = 0.229$, representing the minimal X to sustain this equilibrium with only type-1 agents defaulting. This is also the only feasible equilibrium. For any X lower than this level, all agents default. If all agents default, the total liquidated value of all projects is $\mu * (\frac{1}{3}w_1 + \frac{1}{3}w_2 + \frac{1}{3}w_3)b = 0.796b$, which is unfeasible to sustain. Following Figure 4, we start from $X = 0.229$ as the initial value and assume it drops to $X = 0.2$. Repeating the above calculations, we obtain from the zero profit condition that in such a case b must decline to $b = 0.905$. This implies a drop in credit of about 10%, following a drop in enforcement resources of about 13%.

References

- Morris, Stephen and Hyun Song Shin. 2003. Global Games: Theory and Applications. In *Proceedings of the Eighth World Congress of the Econometric Society*.
- Sákovics, József and Jakub Steiner. 2012. “Who Matters in Coordination Problems?” *American Economic Review* 102(7):3439–3461.