WORKING PAPERS
RESEARCH DEPARTMENT

WORKING PAPER NO. 15-37
A TRACTABLE CITY MODEL FOR AGGREGATIVE ANALYSIS

Satyajit Chatterjee
Federal Reserve Bank of Philadelphia

Burcu Eyigungor
Federal Reserve Bank of Philadelphia

October 2015

RESEARCH DEPARTMENT, FEDERAL RESERVE BANK OF PHILADELPHIA
Ten Independence Mall, Philadelphia, PA 19106-1574 • www.philadelphiafed.org/research-and-data/

A Tractable City Model for Aggregative Analysis

Satyajit Chatterjee* and Burcu Eyigungor

Federal Reserve Bank of Philadelphia

October 2015

*Corresponding author: Satyajit Chatterjee, Research Department, Federal Reserve Bank of Philadelphia, Ten Independence Mall, Philadelphia PA, 19106; (215) 574-3861, satyajit.chatterjee@phil.frb.org. This paper combines material from two earlier papers circulated as "A Tractable Circular City Model with Endogenous Internal Strucure" and "Agglomeration Economies, Geography and the Value of Urban Land." The authors thank Roc Armenter, Jeffrey Brinkman, Daniele Coen-Pirani, Russel Cooper, Gilles Duranton, Jeffrey Lin, Esteban Rossi-Hansberg, Maisy Wong, and seminar participants at the Federal Reserve Bank of Philadelphia, the Wharton School of the University of Pennsylvania, Iowa State University, Pennsylvania State University, and participants at the SED 2012, RSAI 2014, HULM Fall 2014, and CURE 2014 conferences for their thoughtful comments.

Abstract

An analytically tractable city model with external increasing returns is presented. The equilibrium city structure is either monocentric or decentralized. Regardless of which structure prevails, intracity variation in endogenous variables displays exponential decay from the city center, where the decay rates depend only on parameters. Given population, the equilibrium of the model is generically unique. Tractability permits explicit expressions for when a central business district (CBD) will emerge in equilibrium, how external increasing returns affect the steepness of downtown rent gradients, and how wages and welfare vary with population. An application to urban growth boundary is presented.

Keywords: agglomeration economies, central business district, rent gradient, urban growth boundary

JEL Codes: Z10, R30

It seems to me that the force we need to postulate for the central role of cities in economic life is of exactly the same character as the external human capital I have postulated as a force to account for certain features of aggregative development. If so, land rents should provide an indirect measure of this force ... What can people be paying Manhattan or downtown Chicago rents for, if not for being near other people?

—Robert E. Lucas Jr. (1988)

"On the Mechanics of Economic Development," p. 39

1 Introduction

Increasing returns are a key element in several explanations of economic growth (Arrow (1962), Romer (1986), Lucas (1988)). Often times, external increasing returns accompany and sustain the spatial concentration of industries, as famously noted by Alfred Marshall (1890) long ago.[1] Indeed, when placing external human capital at the center of the process of economic development, Lucas (1988) linked cities and growth. His intent (expressed in the quote above) was to bring facts regarding the internal structure of cities — in particular, the concentration of businesses and the generally high value of land in the center of cities — to both validate the existence of increasing returns and learn about their empirical importance.

Our paper is motivated by this link between cities and growth. It presents a tractable model of a city in which industry scale improves the efficiency of firms depending on how close physically firms are to each other. Our model has the same setup as the Lucas and Rossi-Hansberg (2002) circular city model but alters the way physical proximity between firms is defined. Specifically, the proximity between two firms located at different points in the plane is measured as the sum of the lengths of the rays connecting each point to the city center.[2] With this change, we can show that the city either has a "monocentric" structure

[1] *Principles of Economics*, Book IV, Chapter 10: "The Concentration of Specialized Industries in Particular Localities."

[2] In most studies, including Lucas and Rossi-Hansberg, proximity is taken to mean Euclidean distance

with businesses concentrating in the city center, or it has a "decentralized" structure in which firms locate next to their workers. Regardless of which form prevails, the intracity variation in all endogenous variables — residential and commercial rents, employment and residential densities, and wages — displays (over their relevant domains) exponential decay from the city center, in which the rates of decay depend only on preference and technology parameters. Furthermore, for a given population size, the equilibrium outcome is generically unique.

Analytic tractability has several useful consequences. First, we are able to give explicit conditions on preference and technology parameters under which a business core, or central business district (CBD), will emerge in equilibrium. Second, the explicit expressions for the CBD rent gradients reveal how the strength of increasing returns, in conjunction with other parameters, affects the steepness of downtown rent gradients. Third, we are able to characterize the equilibrium relationship between downtown rents and wages and population.

Analytic tractability also allows us to characterize the equilibrium relationship between welfare and population. The shape of this relationship is important for explaining why modern economies are organized around a finite number of large cities. If city welfare is monotonically increasing or decreasing in population, we should expect all economic activity to concentrate in one giant city or be dispersed across an infinity of miniscule locations. The fact that it is neither suggests that welfare is initially increasing but eventually declining in population. Visually, the relationship must resemble an inverted-U. Our characterization of the relationship gives explicit conditions under which the inverted-U shape will emerge for both the mononcentric and decentralized cities.[3]

Finally, tractability permits informative comparative static analyses. We study the im-

between two points. It is unclear, however, what measure of proximity is relevant for urban agglomerations. For instance, if distance between firms matters because of commuting costs, one might need to consider that most cities have a radial highway network. Our definition is consistent with this and has the benefit of tractability.

[3]Studies on the system of cities derive this relationship in a reduced form fashion: It is *assumed* that firms locate at the city center to benefit from local increasing returns, and it is (typically) *assumed* that the city area grows with population (see the survey by Abdel-Rahman and Anas (2004)).

pact of an urban growth boundary on equilibrium outcomes in the context of rising demand for urban land.[4] Intuition might suggest that urban growth controls reduce housing affordability by increasing rents. We show that this intuition ignores the local increasing returns that are central to urban agglomerations. When productivity is increasing in population, cities that can expand will experience *larger* increases in rents relative to cities whose expansion is constrained by a growth boundary. On the other hand, if factors other than land, such as structures, are also fixed — as they might be in the short run — then productivity is declining in population and rents are predicted to rise more in the constrained city. Thus, the model predicts that the effects of urban growth boundaries on housing affordability will vary with the time horizon under consideration. Empirical evidence on this point is discussed later in the paper (Section 7).

There are several precursors to the theory presented in this paper. The paper by Fujita and Ogawa (1982) is an early precursor that examined for a linear city (and mostly numerically) the conditions under which one or more business districts could emerge in equilibrium.[5] Lucas (2001) studied the connection between the magnitude of increasing returns and the steepness of business rent gradients in a model in which the residential location choice of workers was suppressed. Thus, the conditions under which a business core would emerge were not addressed. This question was addressed in Lucas and Rossi-Hansberg (2002) for a circular city model. They gave an innovative proof of existence of a competitive equilibrium and numerically explored equilibrium city structures for different parameter values.[6] What sets our paper apart from these studies is that we are able to give an analytic characterization of equilibrium configurations for the case in which both firms and workers choose locations,

[4] Constraints on urban development are growing trends in the U.S. and a well-established policy in most of Western Europe (Pendall, Martin, and Fulton (2002)). In emerging economies, where the rate of urbanization is expected to increase dramatically in the coming years, urban growth containment policies are of prime interest.

[5] There is extensive literature in urban economics that theoretically examines residential land use in a city (see, for instance, the surveys by Brueckner (1987) and Duranton and Puga (2014)). This literature assumes that businesses locate in the city center and, therefore, makes no predictions regarding the emergence, size, or rent gradients of business districts.

[6] Recently, Dong and Ross (forthcoming) have succeeded in giving a characterization of employment and rent gradients around the city center for the Lucas and Rossi-Hansberg model.

and production and utility functions take forms that are standard in aggregative analyses.

The paper is organized as follows. Section 2 describes the environment. Section 3 derives the condition under which the city features a business core. Section 4 constructs the equilibrium of such a city for a given population (the case of the city without a business core is given in Appendix C). Section 5 discusses some of the key implications of the model. Section 6 characterizes the relationship between welfare and population. Section 7 contains the analysis of urban growth boundaries, and Section 8 concludes. Several appendixes contain supplementary results. To aid the reader, a glossary of terms is provided after the concluding section.

2 Environment

Space is modeled as a flat plane extending infinitely in all directions, with a point marked off as the city center. Each point other than the center is physically indistinguishable from any other, so we focus on allocations that are symmetric relative to the center. A location is then described fully by its distance r from the center.

2.1 Workers

A population of P workers lives in the city. Workers decide where to live and where to commute for work within the city. Workers can commute to any firm located on the ray that connects the worker's residential location to the city center (since the equilibrium is symmetric around the city center, this is without loss of generality). We follow Anas, Arnott, and Small (2000) and Lucas and Rossi-Hansberg (2002) and assume that a worker who resides in location s and commutes to a firm at location r has $\exp(-\kappa|s-r|)$ unit of time to devote to production, where $\kappa > 0$.[7]

[7] This specification is a convenient abstraction (but see Anas, Arnott, and Small (2000) for a substantive defense of it). Note that when κ is small, the (net) income of a commuter is approximately $w(r)[1-\kappa|s-r|]$, which is the commonly used specification.

Workers take the land rent and wage schedules, $q(r)$ and $w(r)$, respectively, along any ray from the city center as parametrically given. Then, a worker who resides at location s solves the following decision problem:

$$
\begin{aligned}
u(s) &= \max_{c \geq 0,\, l \geq 0} c^{\beta} l^{1-\beta}, \quad \beta \in (0,1) \\
&\text{s.t. } c + q(s)l = W(s), \text{ where} \\
W(s) &= \max_{r \geq 0} (w(r) \exp(-\kappa\,|s - r|).
\end{aligned}
\tag{1}
$$

Here, c denotes consumption of the single numeraire good, l is the consumption of land, and $W(s)$ is the maximum income a worker can earn when he lives at location s and optimally chooses where to commute for work. We will refer to $W(s)$ as the earnings schedule. The solution gives

$$
c(s) = \beta W(s), l(s) = (1-\beta)\frac{W(s)}{q(s)} \text{ and } u(s) = \beta^{\beta}(1-\beta)^{(1-\beta)}\frac{W(s)}{q(s)^{(1-\beta)}}. \tag{2}
$$

2.2 Firms

There is free entry of firms at each location, and firms have constant returns to scale technology to produce the single numeraire good. The production function of a firm that uses one unit of land at location s is

$$
Az(s)^{\gamma} n(s)^{\alpha}, \quad A > 0, \alpha \in (0,1), \gamma \in (0,1].
$$

where n is the number of units of worker time per unit of land, A is a total factor productivity (TFP) term that is common to all firms, and $z(s)$ is a variable — defined more precisely below — that captures how many other workers are in close proximity to the firm located at s. The parameter γ controls how important proximity to other businesses is in production.

Firms also take the rent and wage schedules as parametrically given. A firm that chooses to set up production in location s solves the following decision problem:

$$\pi(s) = \max_{n \geq 0} Az(s)^{\gamma} n^{\alpha} - nw(s) - q(s). \tag{3}$$

The solution gives

$$n(s) = \left(A\alpha z(s)^{\gamma}/w(s)\right)^{1/(1-\alpha)} \text{ and } \pi(s) = \left(\alpha A z(s)^{\gamma} w(s)^{-\alpha}\right)^{\frac{1}{1-\alpha}} [1/\alpha - 1] - q(s). \tag{4}$$

Turning to the determination of $z(s)$, we assume that distance between any two firms is measured by the sum of the distance of the two firms from the city center. In other words, if one firm is located on a circle of radius r and the other firm is located on a circle of radius s, the distance of the firms to each other is simply $(r + s)$. This assumption is reasonable if communication between workers at different firms requires travel to a central meeting place and the road system is radial. Let $N(r)$ denote the total units of worker time employed at location r. Then, proximity to other workers enjoyed by a firm at location s is

$$z(s) = \int_0^{\infty} 2\pi r \exp\left(-\delta\left(s + r\right)\right) N(r) dr, \delta > 0.$$

Since $z(0) = \int_0^{\infty} 2\pi r \exp\left(-\delta r\right) N(r) dr$, we have

$$z(s) = z(0) \exp\left(-\delta s\right). \tag{5}$$

For any location s, the measure of proximity to other firms is the measure of proximity at the city center discounted by its distance from the center. Thus, the distribution of employment within the city affects $z(s)$ only through its affect on $z(0)$. This property is key for the tractability of the model.[8]

[8]Since workers commute along rays from the center, it might seem more reasonable to require that proximity to workers *on the same ray as the firm* is simply Euclidean distance. Given that the measure of workers on the same ray as the firm is zero, this interpretation is consistent with our definition of $z(s)$.

2.3 Landowners

Following convention, we assume that all land in the economy is owned by absentee landlords. These landlords supply land inelastically to the rental market, with a reservation rental price of $d > 0$ units of consumption good.

2.4 Competitive Equilibrium

Now, we are ready to give the definition of a competitive equilibrium. To express these conditions, we let $\theta_F(r)$ and $\theta_H(r)$ denote the fraction of land at location r devoted to business and residential use, respectively. We use $m(s, r)$ to denote the fraction of workers living in location s who commute to location r for work.

Definition 1 *Given population $P > 0$ and value of land in nonurban use $d > 0$, a symmetric city equilibrium is (i) a wage and earning schedule $w^*(\cdot) \geq 0$ and $W^*(\cdot) \geq 0$; (ii) a rent schedule $q^*(\cdot) \geq d$; (iii) an employment density schedule $n^*(\cdot) \geq 0$; (iv) an intensity of residential land use schedule $l^*(\cdot) \geq 0$ and consumption $c^*(\cdot) \geq 0$; (v) a schedule of the fraction of land devoted to business use $1 \geq \theta_F^*(s) \geq 0$; (vi) a schedule of the fraction of land devoted to residential use $1 \geq \theta_H^*(s) \geq 0$; (vii) for each s for which $\theta_H^*(s) > 0$, a schedule of the fraction of residents commuting to r, $1 \geq m^*(s, r) \geq 0$; and (viii) a utility level U^*, such that*

1. *$n^*(s)$ and $\pi^*(s)$ solve (3) for $w(s) = w^*(s)$ and $q(s) = q^*(s)$.*

2. *$\pi^*(s) \leq 0$, and $\pi^*(s)\theta_F^*(s) = 0$.*

3. *$l^*(s)$, $c^*(s)$, and $u^*(s)$ solve (1) for $W(s) = W^*(s)$ and $q(s) = q^*(s)$.*

4. *$u^*(s) \leq U^*$ and $[u^*(s) - U^*]\theta_H^*(s) = 0$.*

5. *$\theta_F^*(s) + \theta_H^*(s) \leq 1$ and $[1 - \theta_F^*(s) - \theta_H^*(s)](q^*(s) - d) = 0$.*

6. *For* $\theta^*_H(s) > 0$, $\int_0^\infty m^*(s,r)dr = 1$, *where* $m^*(s,r) = 0$ *if* $w^*(r)\exp(-\kappa|s-r|) < W^*(s)$.

7. *For each* $r \geq 0$,

$$\int_0^r \theta^*_F(k)n^*(k)2\pi k\,dk = \int_0^\infty \int_0^r \mathbb{1}_{\{\theta^*_H(s)>0\}} m^*(s,k)\frac{\theta^*_H(s)}{l^*(s)}2\pi s\exp\left(-\kappa|k-s|\right)\,dk\,ds.$$

8.

$$\int_0^\infty \theta^*_H(s)\frac{1}{l^*(s)}2\pi s ds = P.$$

9. *For each* $r \geq 0$,

$$z^*(r) = \exp(-\delta r)\int_0^\infty 2\pi s\theta^*_F(s)n^*(s)\exp(-\delta s)ds.$$

Conditions (1)–(4) spell out the requirements of optimization. Because of free entry of firms, profits must be nonnegative in any location and must be exactly zero where some portion of the land is devoted to business use (condition (2)). Similarly, because of free mobility of workers within the city, utility available in any location cannot exceed U^* and must be exactly U^* where some portion of the land is devoted to residential use (condition (4)). Condition (5) requires that, if some portion of the land in a location is devoted to nonurban use, land rent there must be d.

The specifically spatial aspects of the equilibrium show up in conditions (6)–(9).

Condition (6) requires that no worker commutes to any location that gives earnings less than $W^*(s)$.

Condition (7) imposes labor market balance. The l.h.s. gives the total demand for worker time on the disk (of radius) r. On the r.h.s., the inner integral $\int_0^r m^*(s,k)[\theta^*_H(s)/l^*(s)]\exp(-\kappa|k-s|)2\pi s\,dk$ gives the total worker time supplied to the disk r by workers residing on the circle s. The outer integral gives the total supply of worker time to the disk r.

Condition (8) imposes that the total number of city residents must equal the population. On the l.h.s., $1/l^*(s)$ is the residential population density at location s and $\theta_H^*(s)/l^*(s)2\pi s$ are all the people living on circle s. The integral over s must equal P.

Finally, condition (9) gives the proximity to businesses enjoyed by a firm at location r as a function of the distribution of worker time across all locations in the city. Observe that the external effect depends on distribution of worker *time* rather than workers and, thus, the time spent commuting does not contribute to the productivity of workers.

3 Internal Structure

In this section, we characterize the internal structure of a city (existence and uniqueness of equilibrium, given P, is demonstrated in Section 4). The main result is that, generically, the structure of the city can take only one of two forms: It has either a CBD with a surrounding residential ring or firms and workers colocating throughout the city. In preparation, we state some lemmas and introduce some definitions that come in handy. Proofs of all lemmas appear in Appendix A; in the text, we give intuition for why each result holds.

Lemma 1 *In any equilibrium* $z^*(0) \leq P$*.*

The index of proximity at the city center is bounded above by the total amount of time devoted to production in the city and, since each individual worker can devote at most one unit of time to production, this quantity is bounded above by the measure of workers P. This bound applies to all locations in the city, not just the center, and it will apply for any distance measure, not just the one we are working with.

The next lemma asserts that a city that delivers positive equilibrium utility to its workers must have a boundary. The lemma follows from the fact that firms farther away from the center are less productive ($z(r)$ declines exponentially from the city center), commuting to firms is costly, and land has strictly positive value in nonurban use.

Lemma 2 *In any equilibrium with $U^* > 0$, there is $S(U^*) > 0$ such that for all $s > S(U^*)$, $\theta_F^*(s) = \theta_H^*(s) = 0$.*

Next, we introduce some conceptual tools that will be helpful in deriving our results on the internal structure of cities. First, we introduce a land rent schedule for which, given the wage schedule, firms' profits are exactly 0. Using (3), we obtain

$$q_F(s; w(s)) = \left(\alpha A z(s)^\gamma w(s)^{-\alpha}\right)^{\frac{1}{1-\alpha}} (1/\alpha - 1) \,. \tag{6}$$

Analogously, there is a land rent schedule for which, given the income schedule, households get utility level U. Using (1), we obtain

$$q_H(s; W(s), U) = \beta^{\frac{\beta}{1-\beta}} (1-\beta) \left(\frac{W(s)}{U}\right)^{\frac{1}{1-\beta}} \,. \tag{7}$$

It is customary in urban economics to refer to these functions as the *bid rent functions* of firms and workers, respectively (Alonso (1964), Fujita (1989)). The former is the maximum rent a competitive firm can pay for location s and still earn nonnegative profits. The latter is the maximum rent a worker can pay for location s and still get utility U. To conserve on notation, we will use $q_F^*(s)$ to denote $q_F(s; w^*(s))$ and $q_H^*(s)$ to denote $q_H(s; W^*(s), U^*)$.

The following lemma is self-evident:

Lemma 3 *If $\theta_F^*(s) > 0$ then $q_F^*(s) \geq q_H^*(s)$ and if $\theta_H^*(s) > 0$ then $q_H^*(s) \geq q_F^*(s)$.*

The next two lemmas draw out the implications of optimal commuting.

Lemma 4 *If $\theta_F^*(s) > 0$, then $W^*(s) = w^*(s)$.*

If there are firms in location s, the income that a resident can receive by residing in that location must equal wages offered in that location. The fact that it cannot be any less is obvious. If it were more, a person working at s would do better by commuting to locations that give higher income, contradicting the presence of firms at s.

Lemma 5 $W^*(s) \geq W^*(r)\exp(-\kappa\,|r-s|)$ *for any* r *and* s.

Income available at any location s cannot be less than what a resident of that location would earn by first commuting to some other location r and following the commuting pattern from location r.

We are now ready to prove our key proposition that there is a simple condition on fundamentals that determines whether workers commute in equilibrium or not. Furthermore, we can show that if there is commuting in equilibrium, then it must be toward the center.

Proposition 1 (Internal Structure I) *If there is commuting in equilibrium, then* $\kappa \leq [\delta\gamma(1-\beta)]/[1-\alpha\beta]$ *and commuting is always toward the city center.*

Proof. Assume two distinct locations r and s such that $m^*(s,r) > 0$ (i.e., there is commuting from s to r). Given that $\theta_F^*(r) > 0$, $q_F(r, w^*(r)) \geq q_H(r; W^*(r)) = q_H(r, w^*(r))$, where the first inequality follows from Lemma 3 and the second equality follows from Lemma 4. Using the definitions of $q_F(\cdot)$, $q_H(\cdot)$, and $z(r) = z(0)\exp(-\delta\, r)$, we obtain an upper bound on $w^*(r)$:

$$w^*(r) \leq K(z^*(0), U^*)\exp\left(-\frac{\delta\gamma(1-\beta)}{(1-\alpha\beta)}r\right),$$

where $K(z^*(0), U^*)$ is a constant, given $z^*(0)$ and U^*.

Next, given that $\theta_H^*(s) > 0$, we have $q_F(s; W^*(s)) \leq q_F(s; w^*(s)) \leq q_H(s, W^*(s))$, where the first inequality follows from the fact that $q_F(s; w(s))$ is decreasing in $w(s)$ and $W^*(s) \geq w^*(s)$ and the second inequality follows from Lemma 3. Since $m^*(s,r) > 0$, $W^*(s) = w^*(r)\exp(-\kappa|r-s|)$. This implies a lower bound on $w^*(r)$:

$$w^*(r)\exp(-\kappa|r-s|) \geq K(z^*(0), U^*)\exp\left(-\frac{\delta\gamma(1-\beta)}{(1-\alpha\beta)}s\right).$$

The two bounds together imply

$$\exp(-\kappa|r-s|) \geq \exp\left(-\frac{\delta\gamma(1-\beta)}{(1-\alpha\beta)}(s-r)\right).$$

This condition can be satisfied only if commuting is toward the city center; i.e., $s > r$, and $\kappa \leq [\delta\gamma(1-\beta)]/[1-\alpha\beta]$. ■

Corollary 1 *If $\kappa > [\delta\gamma(1-\beta)]/[1-\alpha\beta]$, in equilibrium firms colocate with their workers, since there cannot be any commuting.*

The intuition for the proposition and corollary is simple. Because of the exponential decay in $z(\cdot)$, firms are more productive the closer they are to the center. The inequality $\kappa \leq [\delta\gamma(1-\beta)]/[(1-\alpha\beta)]$ ensures that the rise in productivity toward the city center is large enough for businesses to be able to compensate for the commuting costs of workers and be able to pay at least as much for land as workers would be willing to pay to live there (this point is explained in more detail in Section 5.1). If this inequality is violated, commuting cannot be supported in equilibrium and leads to a city that has mixed (business and residential) use of land everywhere; i.e, firms colocate with their workers.

The next proposition shows that if commuting can be supported in equilibrium, the city will have a central business district and a surrounding residential ring. The proof uses the fact that the presence of firms in a location puts a cap on equilibrium wages in locations closer to the city center (so that workers do not strictly prefer to commute further in toward the center). Given this cap on wages, firms' capacity to pay for land closer toward the center increases at a faster rate than workers' willingness to pay for land closer to the center when $\kappa < [\delta\gamma(1-\beta)][(1-\alpha\beta)]$. So, when a location is used for business purposes, all locations closer to the center end up in business use as well. From boundedness of the city, we can find an S_F that is the boundary of the business district. Because of commuting costs, income of workers declines exponentially as one goes further out from S_F. Given this, there will be a solid residential ring between S_F and the boundary S of the city.

Proposition 2 (Internal Structure II: Monocentric City) *If* $\kappa < [\delta\gamma(1-\beta)][(1-\alpha\beta)]$, *then (i) there is an* $S_F > 0$ *such that for* $s < S_F$, $\theta_F^*(s) = 1$ *and* $s > S_F$, $\theta_F^*(s) = 0$ *and (ii) there is an* $S > S_F$ *such that for* $S_F < s < S$, $\theta_H^*(s) = 1$ *and* $\theta_H^*(s) = 0$ *elsewhere (except possibly at* S_F *and* S*).*

Proof. (i): In any equilibrium with $U^* > 0$, there will be some s such that $\theta_F^*(s) > 0$. We now show that for any $r < s$, $\theta_F^*(r) = 1$. We have that $W^*(r)\exp(-\kappa(s-r)) \leq W^*(s) = w^*(s)$ (the inequality follows from Lemma 5 and the equality follows from Lemma 4). In addition, $q_F(s; w^*(s)) \geq q_H(s; W^*(s)) = q_H(s; w^*(s))$ (the inequality follows from Lemma 3 and the equality from Lemma 4). Combining these and using the definitions of $q_F(\cdot)$, $q_H(\cdot)$, and $z(r) = z(0)\exp(-\delta r)$ yields

$$K(z^*(0), U^*)\exp\left(-\frac{\delta\gamma(1-\beta)}{(1-\alpha\beta)}s\right)\exp(\kappa(s-r)) \geq W^*(r).$$

Next, we want to show that $q_F(r; W^*(r)) > q_H(r; W^*(r))$, which is equivalent to

$$K(z^*(0), U^*)\exp\left(-\frac{\delta\gamma(1-\beta)}{(1-\alpha\beta)}r\right) > W^*(r).$$

Since $\kappa < [\delta\gamma(1-\beta)]/[1-\alpha\beta]$, the above inequality follows from the previous one. Since $W^*(r) \geq w^*(r)$, we obtain $q_F(r; w^*(r)) > q_H(r; W^*(r))$. This proves that locations closer to the city center than s cannot be in residential use.

To prove that the locations will be in business use, we must also establish that $q_F(r, w^*(r))$ is at least as large as d. Since $q_F(s; w^*(s)) \geq d$ (or else $\theta_H^*(s)$ cannot be positive), it is sufficient to show that $q_F(r, w^*(r)) > q_F(s, w^*(s))$. From the definition of $q_F(\cdot)$, $q_F^*(r) > q_F^*(s)$ if and only if

$$w^*(s) > \exp\left(-\left(\frac{\gamma\delta}{\alpha}\right)(s-r)\right)w^*(r)$$

Since $\kappa < [\delta\gamma(1-\beta)]/[(1-\alpha\beta)]$ implies $\kappa < \gamma\delta/\alpha$, and $w^*(r)\exp(-\kappa(s-r)) \le w^*(s)$, the above inequality is satisfied and $q_F^*(r) > q_F^*(s) \ge d$. Therefore, for all $r < s$, $\theta_F^*(r) = 1$. Let $\mathcal{F} = \{r \ge 0 : \theta_F^*(r) > 0\}$. By assumption $\mathcal{F}$ is nonempty and by Lemma 2 it is bounded above. Hence, it possesses a least upper-bound of S_F. Then, for all $r < S_F$, $\theta_F^*(r) = 1$ and for $r > S_F$, $\theta_F^*(r) = 0$.

(ii): Given the existence of the CBD, $w^*(r) = w^*(0)\exp(-\kappa r)$ for $r \in [0, S_F)$ and workers are indifferent between working in any location within the CBD. Therefore, for any s, $W^*(s) = w^*(0)\exp(-\kappa s)$. The city boundary will be at S such that $U^* = w^*(0)\exp(-\kappa S)/d^{1-\beta}$. Thus, there is an $S > S_F$ such that for $S_F < s < S$, $\theta_H^*(s) = 1$ and $\theta_H^*(s) = 0$ elsewhere (except possibly at S_F and S).

■

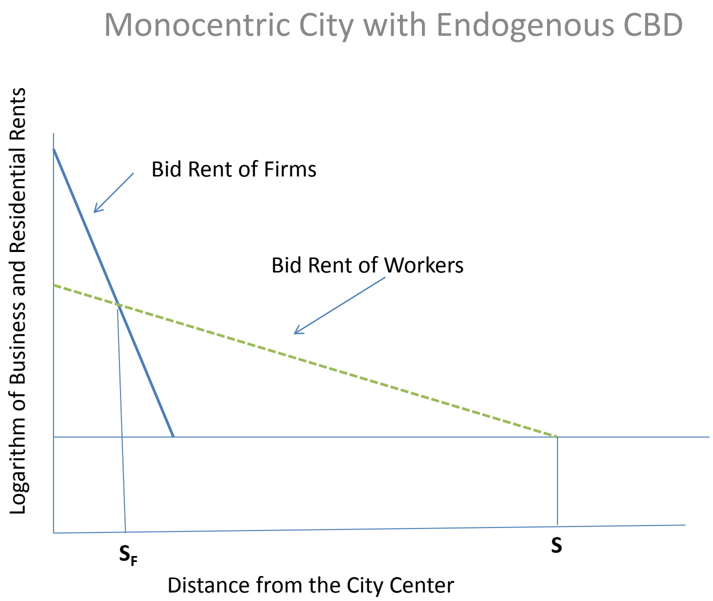
In the body of the paper, we focus on the CBD case as that seems more relevant to the U.S. The details for the case where $\kappa > [\delta\gamma(1-\beta)]/[1-\alpha\beta]$ (the decentralized employment case) is given in Appendix C. When $\kappa = [\delta\gamma(1-\beta)]/[1-\alpha\beta]$, the internal structure of the city is indeterminate, as both forms are consistent with equilibrium. Except for this "razor's edge" case, equilibrium internal structure is unique.

4 Existence and Uniqueness of Equilibrium

In this section, we establish the existence of equilibrium for the CBD case shown in Figure 1. The proof is by construction. The construction will show that $q^*(s)$ and U^* are uniquely determined, and $\theta_F^*(s)$ and $\theta_H^*(s)$ are uniquely determined (except at S_F^* and S^*). In addition, if $\theta_F^*(s) > 0$, $n^*(s)$ and $w^*(s)$ are uniquely determined, and if $\theta_H^*(s) > 0$, $c^*(s)$ and $l^*(s)$ are uniquely determined.

Outside of the CBD, there are no transactions for labor and, consequently, wages are not uniquely pinned down. In what follows, we will assume that $w^*(s) = w^*(0)\exp(-\kappa s)$ for all $s > S_F^*$ (for $s > S_F^*$, the wage cannot be any higher as workers would then prefer to

Figure 1



work at their home location; but the wage could be lower provided firms continue to make nonpositive profits). This schedule has the advantage that $W^*(s) = w^*(s)$ for all s and, so, we can use $W^*(s)$ and $w^*(s)$ interchangeably.

Given $w^*(\cdot)$, the land rent schedule that is consistent with zero profits is

$$q_F^*(s) = q_F^*(0) \exp\left(-\frac{\delta\gamma - \kappa\alpha}{1-\alpha} s\right) \text{ for } s \geq 0, \tag{8}$$

where $q_F^*(0) = (1-\alpha) A z^*(0)^\gamma n^*(0)^\alpha$, and, given $W^*(\cdot)$, the land rent schedule that gives the same utility to the household everywhere is given by

$$q_H^*(s) = q_H^*(0) \exp\left(-\frac{\kappa}{1-\beta} s\right) \text{ for all } s \geq 0, \tag{9}$$

where $q_H^*(0) = \beta^{\frac{\beta}{1-\beta}} (1-\beta) \left(w^*(0)/U^*\right)^{\frac{1}{1-\beta}}$. Observe that when $\kappa < [\gamma\delta(1-\beta)]/[1-\beta\alpha]$, $q_F^*(\cdot)$ has a steeper (negative) slope than $q_H^*(\cdot)$ (as shown in Proposition 2) and they intersect each other at only one point corresponding to S_F^* (the boundary of the CBD). This implies

$$\frac{q_H^*(0)}{q_F^*(0)} = \exp\left(\frac{\kappa(1-\alpha\beta) - \delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)} S_F^*\right). \tag{10}$$

Now that we have assessed the functional forms for rents, we can derive a compact expression for the set of labor market clearance conditions given in equilibrium condition (7). An implication of Proposition 2 is that there isn't a unique schedule for $m^*(\cdot)$ because workers are indifferent to commuting anywhere in the CBD. However, for any equilibrium commuting pattern, the total supply and the total demand for labor hours must be equal to each other at the border of the business district. The total supply of labor time available at the border of the CBD, taking into account the time lost in commuting, is $\int_{S_F}^{S} [2\pi r/l(r)] \exp(-\kappa(r-S_F))dr$, where $1/l(r)$ gives the number of people living per unit land in location r. If the employment density at a CBD location r is $n(r)$, the labor time needed at the border of the commercial district to fulfill this demand is $\exp(\kappa(S_F - r))n(r)$. Therefore, the total time needed at the border of the CBD to satisfy the total labor demand inside the commercial district is $\int_0^{S_F} 2\pi r n(r) \exp(\kappa(S_F - r))dr$. Equality of labor demand and supply then requires

$$\int_0^{S_F^*} 2\pi r n^*(r) \exp\left(\kappa(S_F^* - r)\right) dr = \int_{S_F^*}^{S^*} \frac{2\pi r}{l^*(r)} \exp\left(-\kappa(r - S_F^*)\right) dr. \tag{11}$$

Substituting in the expressions for $n^*(r)$ and $l^*(r)$, given (8) and (9), and using (10), the above equality can be written as

$$\left[\int_{S_F^*}^{S^*} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right] =$$

$$\frac{(1-\beta)}{(1-\alpha)}\alpha \left[\int_0^{S_F^*} r \exp\left(-\frac{\gamma\delta - \alpha\kappa}{1-\alpha} r\right) dr\right] \exp\left(-\frac{\kappa(1-\alpha\beta) - \delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)} S_F^*\right). \tag{12}$$

Then, we have the following proposition:

Proposition 3 (CBD Boundary) *For $S^* > 0$, there is a (i) unique $S_F^* = S_F(S^*) \in (0, S^*)$ that satisfies (12), (ii) $S_F(S^*)$ is strictly increasing in S^* and (iii) $\lim_{S^* \to 0} S_F(S^*)) = 0$ and $\lim_{S^* \to \infty} S_F(S^*) = \bar{S}_F > 0$.*

Proof. (i): Given any $S^* > 0$, $\kappa < [\delta\gamma(1-\beta)]/[1-\alpha\beta]$ implies that the r.h.s. of (12) is increasing in S_F^*. The l.h.s. of (12) is clearly decreasing in S_F^*. Furthermore, the r.h.s. is 0 for $S_F^* = 0$ while the l.h.s. is strictly positive, and the r.h.s. is strictly positive for $S_F^* = S^*$ while the l.h.s. is 0. Therefore, for each $S^* > 0$, there is a unique $S_F^* \in (0, S^*)$ that ensures (12) is satisfied. (ii): As S^* goes up and S_F^* does not change, the integral on the l.h.s. goes up. Since the r.h.s. is increasing in S_F^*, the equilibrium S_F^* must be strictly higher. Thus, $S_F(S^*)$ is strictly increasing in S^*. (iii): Since $S_F(S^*) < S^*$ for all S, it follows that $\lim_{S^* \to 0} S_F(S^*) = 0$. To prove the other limiting result, recall that in any equilibrium S_F is bounded above (Proposition 2). Since $S_F(S)$ is strictly increasing, $\lim_{S \to \infty} S_F(S)$ must converge to some number $\bar{S}_F > 0$.

■

It is intuitive that a more spread-out city will have a more spread-out CBD, i.e., the CBD boundary will "chase" the city boundary. However, this effect weakens rapidly, which implies an upper bound on the CBD boundary. The reason for this is that workers living at the edge of the city consume a lot of land as compensation for their long commute, which means that population density at the edge of the city is quite low. Consequently, as the city boundary recedes farther from the center, the measure of people moving out to the edge becomes very small. Because of the small numbers involved, this movement has virtually no effect on equilibrium quantities in the city center. So, in the limit, there is no change in the CBD boundary.[9]

Turning next to the requirement that a population of P needs to live in the city (equi-

[9]The upper bound on the CBD boundary derived in Lemma 2 depends on d; in contrast, d does not appear directly in equation (12).

librium condition (8)), we have that

$$P = \int_{S_F^*}^{S^*} \frac{2\pi r}{l^*(r)} dr. \tag{13}$$

From $w^*(.)$, $z(\cdot)$, and equations (10) and (12) we obtain

$$n^*(r) = n^*(0) \exp\left(-\frac{\delta\gamma - \kappa}{1-\alpha} r\right) \text{ for all } r \in [0, S_F^*).$$

Next, from the definition of $z(0)$, we have

$$z^*(0) = n^*(0) \left[\int_0^{S_F^*} 2\pi r \exp\left(-\left[\frac{\delta\gamma - \kappa}{1-\alpha} + \delta\right] r\right) dr \right]. \tag{14}$$

Using these expressions, we find that (13) implies

$$n^*(0) = \frac{P}{2\pi} \frac{1}{\left[\int_0^{S_F(S^*)} r \exp\left(-\frac{\gamma\delta - \alpha\kappa}{1-\alpha} r\right) dr\right]} \frac{\left[\int_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{15}$$

Thus, given S^*, both S_F^* and $n^*(0)$ are uniquely determined. Hence, the rest of the equilibrium values, such as $z^*(.), w^*(.), q_F^*(.), q_H^*(.), l^*(.)$, and U^* are also uniquely determined.

The final equilibrium condition is the determination of the city boundary. S^* is determined by the condition that land rents at the boundary must be d:

$$q_H^*(S^*) = d. \tag{16}$$

We have the following proposition:

Proposition 4 (City Boundary) *Given P, there is a unique $S^* \in (0, \infty)$ that satisfies $q_H^*(S^*) = d$. Furthermore, S^* is strictly increasing and unbounded in P.*

Proof. See Appendix B.

The proof is somewhat lengthy and is thus given in the appendix. The proposition itself has a simple intuition. Given the necessity to host a measure P of people in the city, as S^* collapses toward zero, the marginal value of land, for both firms and residents, diverges to infinity, which drives land rents toward infinity as well. As S^* increases, the marginal value of land and, so, land rents, decline monotonically. As S^* diverges to ∞, the CBD boundary S_F^* converges to $\bar{S}_F$ (Proposition 3). The proof shows that the CBD land rents converge, and land rents at the city boundary go to zero due to the costs of commuting. These properties imply that there is a unique S^* where prices at the boundary are equal to d. The fact that S^* is increasing in P is, of course, intuitive: If there are more people living in the city, land rents in the city will rise, which will lead to more land being diverted from nonurban to urban use and, hence, a larger city boundary.

5 Some Implications

In this section, we present some of the key equilibrium implications of the model.

5.1 Emergence and Size of the Business Core

The condition for the emergence of a CBD (Proposition 1) can be written as

$$\alpha\kappa + (1-\alpha)\frac{\kappa}{1-\beta} < \gamma\delta. \tag{17}$$

The r.h.s. is the percentage increase in business revenue resulting from a unit decrease in the distance from the city center: Being closer to the center makes firms more productive. The l.h.s. gives the percentage increase in total costs from moving a unit of distance closer to the city center. It is composed of two parts. The first term is the percentage increase in wages (κ) needed to compensate workers for the longer commute weighted by the share of wages in production costs. In the second term, $\kappa/(1-\beta)$ is the minimum percentage increase in rents needed to outbid workers for land closer to the city center: Worker wages rise by the percentage κ and, since workers spend only $(1-\beta)$ of their income on land consumption, they are willing to pay $\kappa/(1-\beta)$ more for this land. This term, weighted by the share of land in production costs, gives the percentage increase in costs due to higher land prices. Thus, the CBD will emerge only if the percentage increase in revenue is large enough to outbid workers for land and to pay higher wages.

We would expect a CBD to emerge if industry scale is important to production; namely, if γ is high, or if the benefits of industry scale decline very rapidly when the firm moves away from other firms, i.e., if δ is high. This is confirmed by the condition. However, the condition also highlights the importance of other factors.

First, a business core is more likely to emerge when less important land is in production, i.e., the closer α is to 1. In the limit, when α is 1, the condition for a business core boils down to $\kappa < \gamma\delta$. This condition is intuitive: A business core will emerge if the output cost of making workers travel to the city center is lower than the output cost of moving production away from the city center.

Second, when α is less than 1, workers' need for land also becomes relevant. Now, $\kappa < \gamma\delta$ is a necessary condition for a CBD to emerge, but it is not sufficient. If workers care little about land, i.e., β is sufficiently close to 1, the condition for a CBD would be violated. Basically, firms would be unable to outbid workers for land closer to the city center.

Given the existence of a business core, an interesting property of the model is that the CBD radius depends on parameters and city radius S^* only. This is evident in equation (12), where level variables like A, P, and d do not enter the equation directly. In other words, these variables affect the CBD radius only through their affects on S^*. Thus, if we know how these level variables affect S^*, we would know how they effect the CBD radius. The effect of changes in parameters that do appear in (12) on the CBD size is more complex because such changes affect the CBD size directly, given S^*, and through induced equilibrium changes in S^*.

5.2 Increasing Returns and Downtown Rent Gradient

The rent gradients around the city center are informative about the importance of external increasing returns in production, namely, γ. From (8), the log of the rent gradient in the business district is given by

$$\ln(q_F^*(s)) = \ln(q_F^*(0)) - \left[\frac{\delta\gamma - \kappa}{1-\alpha} + \kappa\right] s, \text{ for } s \in [0, S_F^*). \tag{18}$$

It is indeed the case that the strength of the external increasing returns in production, γ, affects the magnitude of the slope positively. However, the expression for the slope coefficient also indicates the presence of other forces. In particular, if proximity is important to take advantage of increasing returns, captured in a high δ, firms will pay more to be close to the city center. Also, the magnitude of the slope is inversely related to $1 - \alpha$ (the condition for a CBD implies $\gamma\delta > \kappa$): If firms do not need much land to produce, they can afford high downtown rents and competition will force them to pay those high rents. A similar force is at work with regard to κ: When κ is low, wages rise only moderately toward the city center and firms closer to the center can afford to pay higher rents (they are more productive but their labor costs are not much higher). The bottom line from (18) is the steep rent gradients seen around city centers are indeed indicative of γ being strictly positive, but the actual gradient is also affected by "internal structure" parameters, δ and κ, and by the land share parameter, $1 - \alpha$.

The model also makes predictions regarding downtown rents and city population size. It is of interest to look at these predictions because the behavior of downtown commercial rents with respect to city population size can be revealing about the strength of the externality parameter γ. From (4) and (6), $q_F^*(0) = [(1-\alpha)/\alpha]w^*(0)n^*(0)$, which, using (14) and (15), implies

$$\ln q_F^*(0) = K + J(S^*) + \ln A + (\gamma + \alpha) \ln P. \tag{19}$$

Here, K is a constant that depends on model parameters and $J(\cdot)$ is a nonlinear function of the city boundary. All else held constant, the elasticity of city center rents with respect to city population is $\gamma + \alpha$. This elasticity is unaffected by "internal structure" parameters δ and κ. We are not aware of an attempt to infer $\gamma + \alpha$ from "peak" (downtown) rents across cities, but wage regressions discussed in the next section give us some idea about these parameters.

5.3 Increasing Returns and City Wages

A common way of estimating the strength of local increasing returns is by estimating the elasticity of wages with respect to city population. For city center wages, the model implies a relationship similar to that seen in (19):

$$\ln w^*(0) = K' + J'(S^*) + \ln A + (\gamma - [1 - \alpha]) \ln P. \tag{20}$$

In this context, a finding of a positive elasticity of wages with respect to population, all else held constant, would be an indication of local increasing returns.

Many studies have attempted to estimate this elasticity, with average wage or average labor productivity in place of $w^*(0)$ and with various proxies to control for A. Such attempts often also control for the area of the city (therefore, in effect, for S^*). These studies generally find a positive elasticity of wages and labor productivity with respect to city population. In a metastudy, Melo, Graham, and Noland (2009) (Table 2, p. 335) report (across various types of data sets and methodologies) median estimates of 0.038 and 0.032 for the elasticity of labor productivity and the elasticity of wages with respect to population, respectively.

This evidence suggests that $\gamma - [1 - \alpha]$ is positive but does not directly identify γ. To get an estimate of γ, we need an estimate for $1-\alpha$. One estimate comes from Brinkman (2013) who uses data on commercial land prices and quantities for Columbus, OH, and estimates $1-\alpha$ to be 0.015. Ciccone (2002) also suggests 0.015 as a reasonable estimate for (non-farm) business land share. Finally, Rappaport (2008) uses 0.016, citing unpublished results by Jorgenson, Ho, and Stiroh (2005). Combining this evidence with the median U.S. estimate for elasticity of wages with respect to population suggests a value for γ around 0.05.

6 Welfare, Population, and Stability

So far, we have taken population as exogenous. We now turn to the determination of equilibrium population and start by characterizing how the equilibrium utility level U^* varies with P. For this, we first show that U^* can be expressed as a function of S^* and parameters only. Given that equilibrium S^* is strictly increasing in P, this function can be used to determine how P and U^* are related.

To express U^* in terms of S^* and parameters only, we will use two equilibrium conditions. One condition is that land rent at the boundary is d, i.e., $d = q_H^*(0)\exp\left(-\kappa/(1-\beta)S^*\right)$. From (10), $q_H^*(0)$ can be replaced by $q_F^*(0)\exp\left(\frac{\kappa(1-\beta\alpha)-\delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)}S_F^*(S^*)\right)$, and the condition can then be expressed in terms of $n^*(0)$ and S^*. The other condition is that workers living at S^* get utility U^*, i.e., $U^* = \beta^\beta(1-\beta)^{1-\beta}d^{-(1-\beta)}w^*(0)\exp(-\kappa S^*)$. Again, we can express this condition in terms of $n^*(0)$ and S^*. These two conditions then can be used to eliminate $n^*(0)$, which leaves an expression for U^* that depends on S^* and parameters only. Taking ln of U^* we have

$$\ln U^* = \frac{\gamma}{\alpha+\gamma}\ln\left[\int_0^{S_F(S^*)} r\exp\left(-\frac{\delta(\gamma+1-\alpha)-\kappa}{1-\alpha}r\right)dr\right] + \qquad (21)$$
$$-\kappa\frac{1-\beta(\alpha+\gamma)}{(1-\beta)(\gamma+\alpha)}S^* + \left(\frac{\gamma+\alpha-1}{\gamma+\alpha}\right)\frac{-\kappa(1-\beta\alpha)+\delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)}S_F(S^*) + K$$

where K is a constant that depends on parameters.

Proposition 5 *If* $1 - \beta(\alpha + \gamma) > 0$*,* $\lim_{P \to 0} U^* = \lim_{P \to \infty} U^* = 0$*. In addition,* U^* *is eventually declining in* P*.*

Proof. We first show that $S^* \to 0$ implies $\ln(U^*)$ diverges to $-\infty$. From Proposition 3, $\lim_{S^* \to 0} S_F(S^*) = 0$. As $S^* \to 0$, the first term on the r.h.s. of (21) diverges to $-\infty$, while the second and third terms approach zero, hence $\lim_{S^* \to 0} \ln(U^*) = -\infty$. Next, from Proposition 3 again, $\lim_{S^* \to \infty} S_F(S^*) = \bar{S}_F$. As $S^* \to \infty$, the first and third terms on the r.h.s. of (21) will approach a finite number, while the second term will approach $-\infty$ as long as $1 - \beta(\alpha + \gamma) > 0$, and so $\lim_{S^* \to \infty} \ln(U^*) = -\infty$. When S^* is large, the behavior of $\ln U^*$ is dominated by the term involving S^*, since S_F^* converges to a constant. Hence, $\ln U^*$ is eventually declining in S^*. Because S^* is strictly increasing in P, these properties hold with respect to P as well.

■

The condition $1 - \beta(\alpha + \gamma) > 0$ is our analog of what Fujita, Krugman, and Venables (1999) call the "no-black-hole condition." If this condition is violated, then, as is evident from the expression of $\ln U^*$, utility deliverable by the city would be increasing in S^*. Since S^* is strictly increasing in P, utility deliverable by the city would be strictly increasing in P. The model would then imply that the entire population of an economy would tend to gravitate to one giant city — the "black hole," so to speak. A necessary condition for this inequality to be violated is $\gamma > 1 - \alpha$. As already mentioned, empirical evidence is supportive of this inequality. Thus, for the economy to not collapse into a black hole, land must be of sufficient importance in the worker's utility function, or, equivalently, β must be sufficiently low.

To understand why this condition ensures that utility is eventually declining in P, consider a city that is large enough that S_F^* is close to its upper bound. In this case, with more population, employment density $n^*(0)$ goes up with almost no expansion of the CBD. Consequently, wages respond as $n^*(0)^{\gamma+\alpha-1}$ (when S_F^* does not change, we can express $z^*(0)$ as a multiple of $n^*(0)$) and land rents in the CBD respond as $n^*(0)^{\gamma+\alpha}$, including at S_F^*.

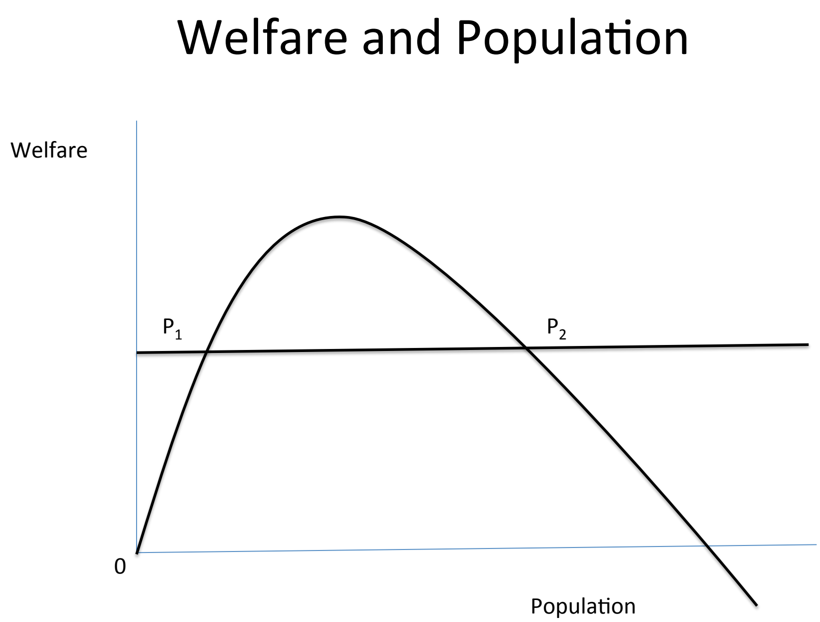
Hence, utility of workers at S_F^* (which could be devoted either to business or residential use) responds as $w^*(0)/q_F^*(0)^{1-\beta} = n^*(0)^{(\gamma+\alpha-1)-(\gamma+\alpha)(1-\beta)} = n^*(0)^{\beta(\alpha+\gamma)-1}$. Thus, as the city gets larger and the CBD size increases very slowly, a necessary condition for utility to decline is $\beta(\alpha+\gamma)-1<0$. Since $\gamma > [1-\alpha]$, utility will decline if land is sufficiently important in utility, i.e., if β is sufficiently low.

To understand why utility is initially increasing in population, consider a small city. For such a city, we can ignore commuting costs, so a worker at the boundary of the city pays d for housing and gets utility $w^*(0)/d^{1-\beta} = Kz^*(0)^{\gamma}n^*(0)^{\alpha-1}$ (where K is a constant). As more people move in, the CBD expands to use more land in production and the increase in population is accompanied by a modest increase in $n^*(0)$. This means that workers benefit from higher industry scale (a higher $z^*(0)$) without running into diminishing returns with respect to labor in production. Although workers at the boundary might commute more because of the expansion of the city, this negative effect of expansion is small compared to the positive effect of higher industry scale on wages and, thus, utility is increasing in population.

The inverted-U shape hinges on the value of β. Davis and Heathcote (2007) estimate that land accounts for 36 percent of the value of aggregate housing stock. Given that households spend about 25 percent of their budgets on housing (which includes the services from structures), a plausible value of $(1-\beta)$ is around $0.25 \times 0.36 \approx 0.10$ and, hence, $\beta = 0.90$. For this value, the upper bound on $(\gamma+\alpha)$ for the "no-black-hole" condition to be satisfied is 1.11. Since the estimates for the $\gamma - [1-\alpha]$ are generally well below 0.11, we conclude that the no-black-hole condition is satisfied and $U^*(P)$ resembles an inverted-U.

The inverted-U has implications for equilibrium population levels for a city. Assume that workers can obtain utility $\bar{U}$ outside the city. Given the inverted-U shape, there will be three possible equilibrium population outcomes. One stable equilibrium is at $P=0$: There is no production, utility delivered by the city is 0 and, thus, workers have no incentive to move there. As shown in Figure 2, if peak utility in the city is higher than $\bar{U}$, two more equilibria with positive population levels are possible: the points marked P_1 and P_2. At these population levels, the city delivers $\bar{U}$ to its residents so no resident has an incentive to leave nor is there any incentive for a nonresident to move in.

Figure 2



When there are three equilibria, the middle one is unstable in the following sense: If the population is slightly positively perturbed, utility delivered by the city will exceed $\bar{U}$ and nonresidents will move in and the city will grow until P reaches P_2. On the other hand, if population is slightly negatively perturbed, utility delivered by the city will be lower than $\bar{U}$ and, thus, will induce emigration. The city will shrink until population is 0. Such sensitivity to perturbations is not true for the equilibria at 0 and P_2. If population were to be positively perturbed at $P = 0$, utility will remain below $\bar{U}$, the city will lose population, and P will fall back to zero. Similarly, a positive or negative perturbation around P_2 will move utility in a direction that will induce population growth or population decline back to P_2. Thus, 0 and P_2 are two stable equilibria for the city.

7 Urban Growth Boundary and Rents

In this section, we use the model to explore the effects of an increase in the demand for urban land on city wages and rents when there are constraints on the supply of new urban land. The standard demand-supply analysis suggests that, all else the same, cities that cannot expand easily will experience a larger increase in rents as demand for urban land increases. We show that this intuition needs to be modified when there are increasing returns to industry scale: A city that can expand easily benefits more from increasing returns and, in the long run, will see a *larger* increase in rents.[10]

In the model presented thus far, land (at varying distances from the city center) is the only fixed factor of production. In this model, for empirically relevant parameter values, restrictions on the supply of urban land will be a depressive force for urban rents: Cities that cannot expand physically will see a smaller rise in rents than cities that can expand easily. This result flows from the fact that, in the long run, wages and city population are positively related, as found in the empirical literature cited in Section 5.3. But in the short run, when productive factors such as buildings and houses are given, there is diminishing returns to population and cities that physically expand and absorb more workers will see a smaller rise in real wages and, thus, in residential land rents.

To develop these points, we consider two cities that are in full spatial equilibrium with land rent at the boundary equal to d. The cities have identical primitives and are identical in terms of size and population. We assume that both cities are at the stable equilibrium with positive population, in which utility is declining with respect to population.

We consider a common increase in the TFP in the two cities, which draws in workers from the rest of the economy. Following this increase in the demand for urban land, we

[10] There is a small body of theoretical literature on the effects of urban land-use restrictions on city wages and rents. Brueckner (1990), Ding, Knaap, and Hopkins (1999), and Brueckner (2007) study the impact of urban growth boundaries in the context of the standard monocentric city model with a (negative) congestion externality, while Bertaud and Brueckner (2005) examine the impact of building height restrictions, again in the standard monocentric city model. However, none of these studies allow for production externalities and, therefore, miss the production-side effects of such controls.

assume that one of these two cities can expand its boundary (the nonurban use-value of the land is held fixed at some level d), while the other city cannot physically expand at all. We call the former the *Unrestricted City* (Un) and the latter the *Restricted City* (R).

Figure 3

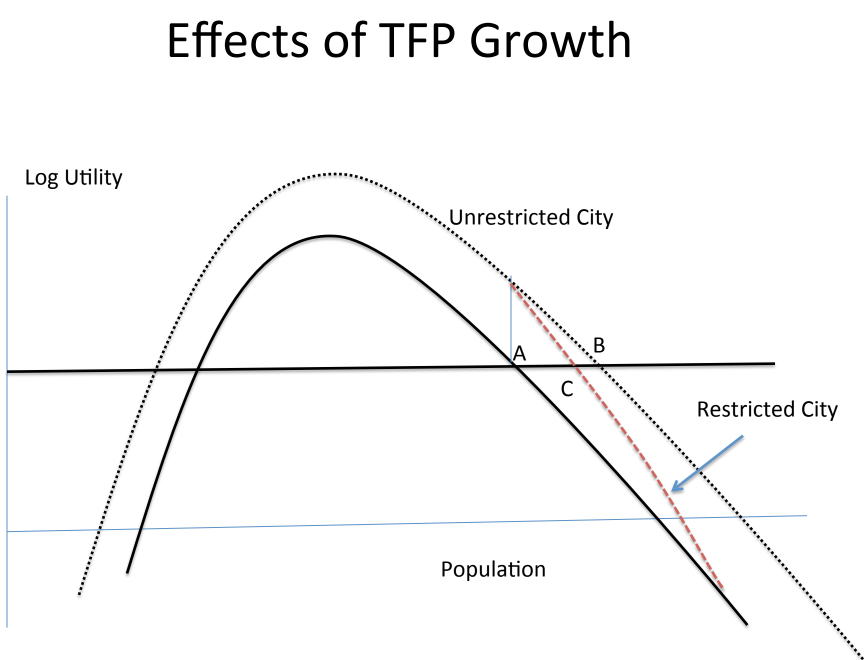


Figure 3 shows what happens in these two cities. The solid line is the $U(P)$ curve for the two cities before the increase in A. The increase in the TFP shifts the utility achievable for any population in both cities upward. For the Unrestricted City, the new curve is denoted by the dotted line. The city draws in population until it reaches point B. The Restricted City also draws in more people but utility falls faster there than in the Unrestricted City and it grows up to the point C. Proposition 4 implies that following the shock, the Unrestricted City will be physically larger than the Restricted City. From the figure, it is clear that the population will increase more in the Unrestricted City. Of course, in the new equilibrium, both cities will deliver the same utility to workers residing there.

In what follows, we analyze the impact on employment density, wages, and land rents in the two cities. From the expression for $n^*(0)$ in (15), we see that it is not immediately possible to tell how $n^*(0)$ compares across restricted and unrestricted cities: The unrestricted city has higher P and larger S relative to the restricted city. However, when we use the fact that both cities deliver the same utility to workers, it becomes possible to compare employment densities. Specifically, in both the restricted and the unrestricted city, the firm's bid rent and the worker's bid rent coincide at S_F. This implies

$$KAz_i^*(0)^\gamma n_i^*(0)^\alpha \exp\left(\frac{\kappa(1-\alpha\beta)-\delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)} S_F(S_i^*)\right) = (w_i^*(0)/U^*)^{\frac{1}{1-\beta}}, i \in \{Un, R\},$$

where K is a constant and U^* is the common utility delivered by the two cities. Using the fact that both $w_i^*(0)$ and $z_i^*(0)$ can be expressed in terms of $n_i^*(0)$, S_i^*, and parameters, we can express $n_i^*(0)$ in terms of U^*, S_i^*, and parameters:

$$n_i^*(0) = KA^{\frac{\beta}{1-\beta(\alpha+\gamma)}} U^{*\frac{-1}{1-\beta(\alpha+\gamma)}} \exp\left(-\frac{\kappa(1-\alpha\beta)-\delta\gamma(1-\beta)}{(1-\alpha)[1-\beta(\alpha+\gamma)]} S_F(S_i^*)\right) \times \left[\int_0^{S_F(S_i^*)} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right] r\right) dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}} i \in \{Un, R\},$$

where K is a positive number that depends on parameters. By virtue of the no-black-hole condition $1-\beta(\alpha+\gamma) > 0$ and the upper bound on κ, $n_i^*(0)$ is increasing in S_F, holding U^* fixed. Since S_F is strictly increasing in S_i^*, city center employment density in the unrestricted city must exceed that in the restricted city in the new equilibrium.

At first blush, this result — which is quite general — appears counterintuitive. Why would employment density in the city-center be *higher* in the city that can expand? The key point is that at an unchanged level of employment density, productivity would be higher in the expanding city because $z(0)$ would be higher: Employment density is the same, but there is more employment since the CBD is bigger. Since a higher $z(0)$ is similar to a higher A, the Unrestricted City will have higher employment density.

Now, we consider what happens to wages offered by firms locating at the city center, namely, $w_i^*(0)$. In both cities, $w_i^*(0) = \alpha Az_i^*(0)^\gamma n_i^*(0)^{\alpha-1}$. Using the relationship between

$z_i^*(0)$ and $n_i^*(0)$,

$$w_i^*(0) = \alpha A \left[2\pi \int_0^{S_F(S_i^*)} r \exp\left(- \left[\frac{\delta\gamma - \kappa}{1-\alpha} + \delta \right] r \right) \right]^\gamma n_i^*(0)^{\gamma - [1-\alpha]}, i \in \{Un, R\}.$$

Since the term in square brackets is increasing in S_F, higher employment density in the unrestricted city means that it will have higher wages in the city center than the restricted city does, provided $\gamma - [1-\alpha] \geq 0$.

Finally, we can turn to the effects on city rents. It is helpful to break up the discussion in terms of how demand shocks affect business rents and how they affect residential rents. The bid rent for a firm at the city center is $(1-\alpha)Az_i^*(0)^\gamma n_i^*(0)^\alpha$. Using the relationship between $z_i^*(0)$ and $n_i^*(0)$,

$$q_{i\,F}^*(0) = (1-\alpha) A \left[2\pi \int_0^{S_F(S_i^*)} r \exp\left(- \left[\frac{\delta\gamma - \kappa}{1-\alpha} + \delta \right] r \right) \right]^\gamma n_i^*(0)^{\gamma+\alpha}, i \in \{Un, R\}.$$

We already know that, in the new equilibrium, the unrestricted city will be larger in size and that it will have a higher employment density. Therefore, business rents at the center of the Unrestricted City will be higher.

For residential rents, we can proceed by considering bid rent for residential space at the center of the two cities. We have

$$q_{i\,H}^*(0) = \beta^{\beta/(1-\beta)}(1-\beta) w_i(0)^{*(\frac{1}{1-\beta})} U^{*(\frac{-1}{1-\beta})}, i \in \{Un, R\}. \tag{22}$$

Since U is the same for both cities, the ordering of workers' bid rent for space at the center of the city depends on the ordering of wages at the center of the city. Therefore, the conditions that govern the ranking of $w_i^*(0)$ also govern the ranking of $q_{i\,H}^*(0)$. If wages at the city center are higher in the Unrestricted City, workers in the Unrestricted City will be willing to bid more for land at their city center than workers in the Restricted City. Then, since rent in any location is $\max\{q_{iF}^*(s), q_{i\,H}^*(s)\}$, rents will be higher in *every* comparable location (in terms of distance from the city center) in the Unrestricted City.

We summarize these findings in the following proposition:

Proposition 6 *If two cities have the same fundamentals and one of the cities has an urban growth boundary that is just binding, an increase in the demand for urban land (i) will cause rents paid by businesses in the Unrestricted City to exceed the rents paid by businesses in the Restricted City, and (ii) when $\gamma \geq 1-\alpha$, will cause rent paid by workers in the Unrestricted City to exceed those paid by workers in the Restricted City.*

Proposition 6 is a statement about the *long-run* effects of land supply restrictions. In the long run, the only fixed factor is the quantity of land at various distances from the city center. Empirically, the strength of increasing returns, γ, appears large enough to overcome the diminishing returns in production due to the fixity of land.

In the short run, when productive factors other than land are also fixed, residential rents will rise *less* in the expanding city if the combined force of diminishing returns, because of the totality of fixed factors, overwhelms γ. To develop this point formally, consider the following modification of the production function:

$$Az(r)^{\gamma}n(r)^{\eta}b(r)^{\psi}, \gamma \in [0,1), \eta, \psi > 0, \text{ and } \eta + \psi \in (0,1). \tag{23}$$

Here, $b(r)$ is building density and the exponent to land in the production function is now $1-(\eta+\psi)$. Correspondingly, let the utility function be redefined to explicitly include housing such that:

$$U(r) = c(r)^{\varsigma}h(r)^{\vartheta}l(r)^{1-\varsigma}, \vartheta, \varsigma > 0, (\vartheta + \varsigma) \in (0,1). \tag{24}$$

Here, $h(r)$ is housing space density (amount of housing space per unit of land) at location r. The exponent to land in the utility function is, therefore, $1-(\vartheta+\varsigma)$.

Assume that in the long run, building density can be rented at the flow cost of $\omega w(r)$ per unit density and housing space can be rented at the flow cost of p units of the consumption good (all structures are owned by the same entities that own all the land). Then, the building-to-employment ratio and the consumption-to-housing space ratio is optimally constant across business and residential locations and the production and utility functions reduce to $[A\psi/\eta\omega]^{\psi}z(r)^{\gamma}n(r)^{\eta+\psi}$ and $[\vartheta/\varsigma p]^{\vartheta}c(r)^{\varsigma+\vartheta}l(r)^{1-(\varsigma+\vartheta)}$, respectively. Aside from ad-justments to multiplicative constants, these functions are isomorphic to the production and utility functions assumed up to now, where $\alpha = \eta + \psi$and $\beta = \varsigma + \vartheta$. Thus, if building and house space densities can be varied at will at the given prices, the equilibrium outcomes will follow those displayed up to now.

Our interest is in how residential land rents in the two cities respond in the short run to a common change in the TFP. We take the short run to mean that $b(r)$ and $h(r)$ schedules are fixed for locations that are built up, but new structures can be built on land that is currently in nonurban use. We also assume that a structure built for residential purposes cannot be used for commercial purposes and vice versa, so the fixity of building and housing space densities implies that S_F is fixed in the short run as well. These assumptions reflect the notion that once a structure is in place, various rigidities make it costly to change its size or purpose.

Given that both cities are identical to start with, the commercial and residential schedules are identical and correspond to the initial long-run equilibrium schedules. We will denote these fixed schedules by $\bar{b}(r)$ and $\bar{h}(r)$, and the common CBD and city boundaries by $\bar{S}_F$ and $\bar{S}$. Now, consider a common rise in the TFP in the two cities. The increase in the TFP will result in an increase in utility deliverable by the cities and, thus, create incentives for people to move to these cities from elsewhere. The city boundary of the Unrestricted City will expand, so that boundary rents fall back to the nonurban use value of land (all new housing constructions will occur beyond $\bar{S}$ in this city). As before, we would expect utility to decline more rapidly in the Restricted City in response to higher population. Thus, in the new short-run equilibrium, population will rise more in the Unrestricted City.

In the new short-run equilibrium, firms in both cities can choose to locate at any $r \in [0, \bar{S}_F]$. Each location comes "pre-equipped" with building density $\bar{b}(r)$, which, since it is the initial equilibrium building density, declines at a common exponential rate from the city center. This implies that $n_i^{*SR}(r)$, $i \in \{Un, R\}$ decline exponentially at a common rate from the city center. Hence,

$$w_i^{*SR}(0) = K[n_i^{*SR}(0)]^{\gamma+\eta-1}, i \in \{Un, R\}, \tag{25}$$

where K is a common constant. For city center wages to rise with employment density, γ must exceed $1-\eta$, or, equivalently, it must exceed $[1-\alpha] + \psi$: Industry increasing returns must be strong enough to overcome the force of diminishing returns from both land and structures. Empirical evidence suggests, however, that this is unlikely. Valentinyi and Herrendorf (2008) report that the cost share of structures in U.S. manufacturing is 0.09, which exceeds the estimates of $\gamma - [1-\alpha]$ (around 0.036) by a large margin.[11] Thus, in the short run, wages are negatively related to employment density.

Next, consider workers who choose to reside in some common intermediate location $r \in (\bar{S}_F, \bar{S})$ in the two cities. Workers choose $c(r)$, $h(r)$, and $l(r)$, given housing space price $p_i^{*SR}(r)$, land rents $q_i^{*SR}(r)$, and income $W_i^{*SR}(r)$. Then,

$$p_i^{*SR}(r)h_i^{*SR}(r)l_i^{*SR}(r) = \vartheta W_i^{*SR} \text{ and } q_i^{*SR}(r)l_i^{*SR}(r) = (1-\varsigma-\vartheta)W_i^{*SR}, i \in \{Un, R\}.$$

In equilibrium, housing space density must equal $\bar{h}(r)$, which implies

$$p_i^{*SR}(r) = \left[\frac{\vartheta}{(1-\vartheta-\varsigma)\bar{h}(r)}\right] q_i^{*SR}(r), i \in \{Un, R\}.$$

Substituting these expressions into the utility function, and recognizing that $W_i^{*SR}(r) = w_i^{*SR}(0)\exp(-\kappa r)$, gives

$$q_i^{*SR}(r) = K(r, U^*)w_i^{*SR}(0)^{\frac{1}{1-\varsigma}}, i \in \{Un, R\}, \tag{26}$$

[11]It is likely that the cost share of structures in service industries is even higher, but we do not have information on service industries. Valentinyi and Herrendorf's definition of services includes housing services. Since housing services tend to be more intensive in structures than business services, their estimate of the share of structures (0.15) is probably too high for our purposes.

and

$$l_i^{*SR}(r) = J(r, U^*) n_i^{*SR}(0)^{\frac{-\varsigma(\gamma+\eta-1)}{1-\varsigma}}, i \in \{Un, R\}, \tag{27}$$

where U^* is the common equilibrium utility enjoyed by workers in the two cities, and $K(r, U^*)$ and $J(r, U^*)$ are constant terms that depend on r, U^*, and parameters and, hence, are common across the two cities.

Given that $\gamma < (1 - \eta)$, we can show that $n_{Un}^{*SR}(0) > n_R^{*SR}(0)$. Suppose $n_{Un}^{*SR}(0) \leq n_R^{*SR}(0)$. Then (27) implies $l_{Un}^{*SR}(r) \leq l_R^{*SR}(r)$, which implies that residential density in the expanding city must be at least as high as in the restricted city. Since there are also people living beyond $\bar{S}$ in the expanding city, this is inconsistent with employment density being the same or lower in the Unrestricted City. If employment density is higher in the Unrestricted City, wages must rise less in that city and so must residential land rents.

The opposite predictions regarding the long-run and short-run effects of constraints on the supply of urban land may have support in the data. As mentioned in Section 5.3, there is strong evidence that current productivity is positively affected by current population, all else held constant. We interpret this as evidence that, in the long run, cities that can accommodate more people will be more productive and, therefore, have higher land rents. In contrast, studies that focus on the recent boom in residential house prices find that cities with exogenously higher supply restrictions experienced smaller population growth and bigger house price increases (Glaeser, Gyourko, and Saiz (2008) and Huang and Tang (2013)). Our model is consistent with both findings: Without any frictions in adjustment of factors of production, local increasing returns imply that productivity and land rents are increasing in population. With frictions, productivity of workers and rents could be decreasing in population.

8 Conclusion

The concentration of businesses in city centers is a ubiquitous feature of the urban landscape and indicates the presence of local increasing returns. In this paper, we presented a tractable city model with external increasing returns to industry scale in which the spatial concentration of businesses emerges endogenously. We used the model to shed light on several empirical phenomena, such as the steep rent gradients seen around the city center, the positive elasticity of wages with respect to city population, and the fact that modern economies organize themselves into a system of large cities. The model is tractable enough to serve as a "laboratory" for analyzing spatial policies with macroeconomic effects. We used the model to study the implications of urban growth boundaries in the context of rising demand for urban land.

Glossary

NOTATION	PARAMETERS OF MAIN MODEL
$1-\alpha$	Share of land in the production function
$1-\beta$	Share of land in the utility function
κ	Commuting technology for workers
γ	Importance of proximity to other workers in production
δ	Decay in proximity with distance
A	TFP term common to all firms
d	Value of land in nonurban use
P	Population in the city
S_F	Boundary between the CBD and residential ring
S	Boundary of the city
$z(\cdot)$	Index of proximity to other workers
$\theta_F(\cdot)$	Fraction of land devoted to business use in a location
$\theta_H(\cdot)$	Fraction of land devoted to residential use in a location
$q(\cdot)$	Land rent schedule
$W(\cdot)$	Maximum income schedule
$n(\cdot)$	Intensity of labor per unit of land hired by firms
$l(\cdot)$	Land consumption schedule per worker
$m(\cdot,\cdot)$	Commuting pattern
$u(\cdot)$	Utility schedule for workers
$\pi(\cdot)$	Profit schedule of firms
$q_H(\cdot)$	Bid rent curve for land by workers
$q_F(\cdot)$	Bid rent curve for land by firms
	ADDITIONAL PARAMETERS FOR MODEL
ψ	WITH STRUCTURES
ϑ	Share of structures in the production function
	Share of structures in the utility function

References

Abdel-Rahman, H. M., and A. Anas (2004): "Theories of System of Cities," in *Cities and Geography: Handbook of Regional and Urban Economics, Volume 4*, ed. by J. V. Henderson, and J.-F. Thisse, pp. 2293–2339. Elsevier: Amsterdam.

Alonso, W. (1964): *Location and Land Use: Toward a General Theory of Land Rent.* Harvard University Press: Cambridge, MA.

Anas, A., R. J. Arnott, and K. A. Small (2000): "The Panexponential Monocentric City Model," *Journal of Urban Economics*, 47, 165–179.

Arrow, K. J. (1962): "The Economic Implications of Learning by Doing," *Review of Economic Studies*, 29, 155–173.

Bertaud, A., and J. K. Brueckner (2005): "Analyzing Building-Height Restrictions: Predicted Impacts and Welfare Costs," *Regional Science and Urban Economics*, 35, 109–125.

Brinkman, J. C. (2013): "Congestion, Agglomeration, and the Structure of Cities," Working Paper No. 13-25, Federal Reserve Bank of Philadelphia.

Brueckner, J. K. (1987): "The Structure of Urban Equilibria: A Unified Treatment of the Muth-Mills Model," in *Urban Economics: Handbook of Regional and Urban Economics, Volume 2*, ed. by E. S. Mills, pp. 821–845. Elsevier: Amsterdam.

——— (1990): "Growth Controls and Land Values in an Open City," *Land Economics*, 66(3), 237–248.

——— (2007): "Urban Growth Boundaries: An Effective Second-Best Remedy for Unpriced Traffic Congestion?" *Journal of Housing Economics*, 16(3-4), 263–273.

Ciccone, A. (2002): "Agglomeration Effects in Europe," *European Economic Review*, 46, 213–227.

Davis, M. A., and J. Heathcote (2007): "The Price and Quantity of Residential Land in the United States," *Journal of Monetary Economics*, 54, 2595–2620.

Ding, C., G. J. Knaap, and L. D. Hopkins (1999): "Managing Urban Growth with Urban Growth Boundaries: A Theoretical Analysis," *Journal of Urban Economics*, 46(1), 53–68.

Dong, X., and S. L. Ross (forthcoming): "Accuracy and efficiency in simulating equilibrium land use patterns for self-organizing cities," *Journal of Economic Geography*.

Duranton, G., and D. Puga (2014): "Urban Land Use," in *Handbook of Regional and Urban Economics, Volume 5*, ed. by G. Duranton, J. V. Henderson, and W. C. Strange, pp. 467–560. North Holland: Amsterdam.

Fujita, M. (1989): *Urban Economic Theory: Land Use and City Size*. Cambridge University Press: Cambridge, UK.

Fujita, M., P. Krugman, and A. J. Venables (1999): *The Spatial Economy: Cities, Regions, and International Trade*. The MIT Press: Cambridge, MA.

Fujita, M., and H. Ogawa (1982): "Multiple Equilibria and Structural Transition of Non-Monocentric Urban Configurations," *Regional Science and Urban Economics*, 12, 161–196.

Glaeser, E. L., J. Gyourko, and A. Saiz (2008): "Housing Supply and Housing Bubbles," *Journal of Urban Economics*, 64, 198–217.

Huang, H., and Y. Tang (2013): "Residential Land Use Regulation and the US Housing Price Cycle Between 2000 and 2009," *Journal of Urban Economics*, 71(1), 93–99.

Jorgenson, D. W., M. S. Ho, and K. J. Stiroh (2005): "Growth of U.S. Industries and Investments in Information Technology and Higher Education," in *Measuring Capital in the New Economy*, ed. by C. Corrado, J. Haltiwanger, and D. Sichel, pp. 403–478. University of Chicago Press: Chicago.

Lucas, Jr., R. E. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3–42.

Lucas, Jr., R. E. (2001): "Externalities and Cities," *Review of Economic Dynamics*, 4, 245–274.

Lucas Jr., R. E., and E. Rossi-Hansberg (2002): "On the Internal Structure of Cities," *Econometrica*, 70(4), 1445–1476.

Marshall, A. (1890): *Principles of Economics, 8th edition.* Macmillan and Company: London.

Melo, P., D. Graham, and R. Noland (2009): "A Meta-Analysis of Estimates of Urban Agglomeration Economies," *Regional Science and Urban Economics*, 39(3), 332–342.

Pendall, R., J. Martin, and W. Fulton (2002): "Holding the Line: Urban Containment in the United States," Discussion paper, Brookings Institution Center on Urban and Metropolitan Policy.

Rappaport, J. (2008): "Consumption Amenities and City Population," *Regional Science and Urban Economics*, 38, 533–552.

Romer, P. (1986): "Increasing Returns and Long-Run Growth," *Journal of Political Economy*, 94, 1002–1037.

Valentinyi, A., and B. Herrendorf (2008): "Measuring Factor Income Shares at the Sectoral Level," *Review of Economic Dynamics*, 11(4), 820–835.

Appendix A: Proofs of Lemmas

Lemma 1 In any equilibrium $z^*(0) \leq P$.

Proof. From equilibrium condition (9), we get

$$\begin{aligned} z^*(0) &= \int_0^\infty \theta_F^*(r) n^*(r) 2\pi r \exp(-\delta r) dr \\ &\leq \int_0^\infty \theta_F^*(r) n^*(r) 2\pi r dr \\ &\leq P. \end{aligned}$$

■

Lemma 2 In any equilibrium with $U^* > 0$, there is $S(U^*) > 0$ such that for all $s > S(U^*)$, $\theta_F^*(s) = \theta_H^*(s) = 0$.

Proof. First we find an upper bound on the wage firms can pay workers at any location. From (3), $\pi(s) = (\alpha A z(s)^\gamma w(s)^{-\alpha})^{\frac{1}{1-\alpha}} (1/\alpha - 1) - q(s)$. Since $z^*(s) = z^*(0)\exp(-\delta s)$ and $q^*(s) \geq d$, Lemma 1 implies that the wage a firm can pay at s and still earn nonnegative profits is bounded above by

$$\overline{w}(s) = [(1/\alpha - 1)/d]^{\frac{1-\alpha}{\alpha}} (\alpha A P^\gamma \exp(-\gamma \delta s))^{\frac{1}{\alpha}}.$$

Second, the minimum wage a firm at location s must pay to attract workers is bounded below by

$$\underline{w} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} U^* d^{(1-\alpha)}.$$

$U^* > 0$ implies $\overline{w}(0) > \underline{w}$, otherwise firms will be unable to afford any workers and utility obtained by all workers would be 0. This, in turn, implies that there will be some distance $\hat{s} > 0$ beyond which firms will be unable to pay $\underline{w}$ and, so, there will be no firms beyond $\hat{s}$.

Next, we determine an upper bound on the income of any worker who locates beyond $\hat{s}$. Since such a worker must, at a minimum, commute to $\hat{s}$ and since the highest wage any business can pay is $\overline{w}(0)$, in any equilibrium $W^*(s)$ is bounded above by

$$\overline{W}(s) = \overline{w}(0) \exp\left(-\kappa(s - \hat{s})\right) \text{ for all } s \geq \hat{s}.$$

Since $\overline{w}(0) > \underline{w}$, there exists $\bar{s} > \hat{s}$ such that $\overline{W}(s) < \underline{w}$ for all $s > \bar{s}$. Hence, there will be no workers living beyond $\bar{s}$. It follows that $\bar{s} = S(U^*)$.

■

Lemma 3 If $\theta_F^*(s) > 0$, then $q_F^*(s) \geq q_H^*(s)$, and if $\theta_H^*(s) > 0$, then $q_H^*(s) \geq q_F^*(s)$.

Proof. From equilibrium condition (2), $\pi^*(s) = 0$, which implies that $q^*(s) = q_F^*(s)$. In equilibrium, $u^*(s) \leq U^*$ for all s, which implies that $q^*(s) \geq q_H^*(s)$ for all s. Together, these inequalities imply that $q_F^*(s) \geq q_H^*(s)$. An analogous argument establishes the second part.

■

Lemma 4 If $\theta_F^*(s) > 0$, then $W^*(s) = w^*(s)$.

Proof. By optimization, $W^*(s) \geq w^*(s)$. Suppose that $w^*(s) < W^*(s)$. Then, there exists some location o for which $W^*(s) = [w^*(o) \exp(-\kappa |s - o|)] > w^*(s)$. Consider some other location k from which workers commute to s. These workers earn $w^*(s) \exp(-\kappa |s - k|)$. However, if these workers commuted to location o they can earn more because $w^*(o) \exp(-\kappa |k - o|) \geq w^*(o) \exp(-\kappa |s - o|) \exp(-\kappa |s - k|) = W^*(s) \exp(-\kappa |s - k|) > w^*(s) \exp(-\kappa |s - k|)$. This implies that $m^*(k, s) = 0$ for all k and hence (by equilibrium condition (7)) implies $\theta_F^*(s) = 0$. Thus, if $\theta_F^*(s) > 0$ then $w^*(s) = W^*(s)$.

■

Lemma 5 $W^*(s) \geq W^*(r) \exp\left(-\kappa |r - s|\right)$ for any r and s.

Proof. Let o be such that $W^*(r) = w^*(o)\exp(-\kappa|r-o|)$. If households at s commuted to o, they would get $w^*(o)\exp(-\kappa|o-s|) \geq w^*(o)\exp(-\kappa|r-o|)\exp(-\kappa|r-s|) = W^*(r)\exp(-\kappa|r-s|)$. Therefore, $W^*(s) \geq W^*(r)\exp(-\kappa|r-s|)$.

■

Appendix B: Proof of Proposition 4

Lemma 6 *Let* $0 \leq s_L < s_U$. *Let* $\Lambda(s_L, s_U) = [\int_{s_L}^{s_U} se^{k_2 s}ds]/[\int_{s_L}^{s_U} se^{k_1 s}ds]$. *Then,* $\Lambda(s_L, s_U)$ *is increasing (decreasing) in both* S_U *and* S_L *if* $k_1 < (>) k_2$.

Proof. We will first establish the following two sets of inequalities:

$$e^{(k_2-k_1)s_L} < \frac{\int_{s_L}^{s_U} se^{k_2 s}ds}{\int_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_U}, \quad k_1 < k_2, \tag{28}$$

and

$$e^{(k_2-k_1)s_U} < \frac{\int_{s_L}^{s_U} se^{k_2 s}ds}{\int_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_L}, \quad k_2 < k_1. \tag{29}$$

Turning first to the l.h.s. inequality in (28), note that $se^{k_1 s} = se^{s_L k_1 + (s-s_L)k_1}$. Multiplying both sides of this equation by $e^{(k_2-k_1)s_L}$ yields $e^{(k_2-k_1)s_L}se^{k_1 s} = se^{s_L k_2 + (s-s_L)k_1} \leq se^{s_L k_2 + (s-s_L)k_2} = se^{k_2 s}$, where the inequality follows because $k_2 > k_1$ and $s - s_L \geq 0$. Furthermore, the inequality is strict for all $s \in (s_L, s_U]$. Therefore, integrating the first and last expressions in the chain with respect to s, we have

$$e^{(k_2-k_1)s_L}\int_{s_L}^{s_U} se^{k_1 s}ds < \int_{s_L}^{s_U} se^{k_2 s}ds.$$

Turning to the r.h.s. of the inequality, note that $se^{k_1 s} = se^{s_U k_1 + (s-s_U)k_1}$. Multiplying both sides of this equation by $e^{(k_2-k_1)s_U}$ yields $e^{(k_2-k_1)s_U} se^{k_1 s} = se^{k_2 s_U + (s-s_U)k_1} \geq se^{s_U k_2 + (s-s_U)k_2} = se^{k_2 s}$, where the inequality follows since $k_2 > k_1$ and $s - s_U \leq 0$. Furthermore, the inequality is strict for all $s \in [s_L, s_U)$. Therefore, integrating the first and last terms in the chain with respect to s, we have

$$e^{(k_2-k_1)s_U} \int_{s_L}^{s_U} se^{k_1 s} ds > \int_{s_L}^{s_U} se^{k_2 s} ds.$$

The proof of (29) is entirely analogous.

We now turn to the proof of the lemma for $k_1 < k_2$. Observe that

$$\operatorname{sgn}\left(\frac{\partial \ln(\Lambda(s_L, s_U))}{\partial s_U}\right) = \operatorname{sgn}\left(\frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} se^{k_2 s} ds} - \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} se^{k_1 s} ds}\right).$$

Suppose, to get a contradiction, that $\partial \Lambda(s_L, s_U)/\partial s_U \leq 0$. Then, we must have

$$\frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} se^{k_2 s} ds} \leq \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} se^{k_1 s} ds}.$$

Or, given that all elements are positive, we have

$$\exp([k_2 - k_1] s_U) = \frac{s_U \exp(k_2 s_U)}{s_U \exp(k_1 s_U)} \leq \frac{\int_{s_L}^{s_U} se^{k_2 s} ds}{\int_{s_L}^{s_U} se^{k_1 s} ds}.$$

But this contradicts the r.h.s. inequality in (28). Therefore, $\partial \Lambda(s_L, s_U)/\partial s_U > 0$.

Next,

$$\operatorname{sgn}\left(\frac{\partial \ln(\Lambda(s_L, s_U))}{\partial s_L}\right) = \operatorname{sgn}\left(\frac{s_L \exp(k_1 s_L)}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds} - \frac{s_L \exp(k_2 s_L)}{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds}\right).$$

Suppose, to get a contradiction, that $\partial\Lambda(s_L, s_U)/\partial s_L \leq 0$. Then, we must have

$$\frac{s_L \exp(k_1 s_L)}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds} \leq \frac{s_L \exp(k_2 s_L)}{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds}.$$

Or, given that all elements are positive, we have

$$\frac{\int\limits_{s_L}^{s_U} s e^{k_2 s} ds}{\int\limits_{s_L}^{s_U} s e^{k_1 s} ds} \leq \frac{s_L \exp(k_2 s_L)}{s_L \exp(k_1 s_L)} = \exp([k_2 - k_1] s_L).$$

But this contradicts the l.h.s. inequality in (28). Therefore, $\partial\Lambda(s_L, s_U)/\partial s_L > 0$.

The proof for $k_2 < k_1$ is analogous.

■

Lemma 7 *Let* $I(s_U, s_L, k) = \int\limits_{s_L}^{s_U} s \exp(-ks)\, ds$. *Then, (i)* $\lim_{s_U, s_L \to \infty} I(s_U, s_L, k) = 0$ *and (ii)* $\lim_{s_U \to \infty, s_L \to \underline{s}} I(s_U, s_L, k) = \bar{I} > 0$.

Proof. Observe that

$$\int\limits_{s_L}^{s_U} s e^{-ks} ds = \frac{s_U e^{-ks_U} - s_L e^{-ks_L}}{-k} - \frac{e^{-ks_U} - e^{-ks_L}}{k^2}.$$

To prove (i), we notice that, as s_U and s_L go to ∞, the second term goes to 0, and the first term (on an application of l'Hopital's rule to s/e^{ks}) also goes to 0. To prove (ii), we observe that if s_U goes to ∞ and s_L converges to $\underline{s}$, then $I(s_U, s_L, k)$ converges to

$$\frac{-\underline{s}e^{-k\underline{s}}}{-k} + \frac{e^{-k\underline{s}}}{k^2} > 0.$$

■

Proof of Proposition

(i) $q_H^*(S^*)$ is declining in S^*.

Note that $q_H^*(S^*) = q_H^*(0)e^{-\frac{\kappa}{(1-\beta)}S^*}$. Since $e^{-\frac{\kappa}{(1-\beta)}S^*}$ is decreasing in S^*, it is sufficient to show that, $q_H^*(0)$ is decreasing in S^*.

Using (10) and $q_F^*(0) = (1-\alpha)\left(z^*(0)\right)^\gamma \left(n^*(0)\right)^\alpha$, we get

$$q_H^*(0) = (1-\alpha)\left(z^*(0)\right)^\gamma \left(n^*(0)\right)^\alpha \exp\left(\frac{\kappa(1-\alpha\beta) - \delta\gamma(1-\beta)}{(1-\alpha)(1-\beta)} S_F(S^*)\right).$$

Since $\kappa < [\delta\gamma(1-\beta)]/[(1-\alpha\beta)]$ and $S_F(S^*)$ is increasing in S^*, the exponential term is decreasing in S^*.

Next, we will show that $z^*(0)$ and $n^*(0)$ are decreasing in S^* also, which would imply that $q_H^*(0)$ is declining in S^*. Since $S_F(S^*)$ is increasing in S^*, by Lemma 6, the ratio of the integrals in the expression for $n^*(0)$ in (15) is decreasing in S^*. The remaining fractional term is clearly decreasing in S^*, and so employment density at the city center is decreasing in $S\ ^*$.

Below is the expression for $z^*(0)$:

$$z^*(0) = P\frac{\left[\int\limits_0^{S_F(S^*)} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha} + \delta\right] r\right) dr\right]}{\left[\int\limits_0^{S_F(S^*)} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha} + \kappa\right] r\right) dr\right]} \frac{\left[\int\limits_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{30}$$

The condition $\kappa < [\delta\gamma(1-\beta)]/[(1-\alpha\beta)]$ and $\gamma \in (0,1]$ implies $\delta > \kappa$. Then, by Lemma 6 again, the first of the two ratios of integrals in (14) is decreasing in S^*. And, since $\beta < 1$, the second ratio of integrals is also decreasing in S^*. Hence, $z^*(0)$ is decreasing in S^*.

Hence, $q_H^*(0)$ is decreasing in S^*.

(ii) $\lim_{S^* \to \infty} q_H(S^*) = 0$.

Again, using $q_H(S^*) = q_H^*(0)e^{-\frac{\kappa}{(1-\beta)}S^*}$ and using (10), (15), (12), and (14), one can get an expression for $q_H^*(0)$ as

$$q_H^*(0) = \tag{31}$$

$$K(1-\alpha)\left(\frac{P}{\left[\int\limits_{S_F(S^*)}^{S^*} r\exp\left(-\frac{\kappa\beta}{1-\beta}r\right)dr\right]}\right)^{\alpha+\gamma}\left[\int\limits_0^{S_F(S^*)} r\exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right]r\right)dr\right]^{\gamma}\times$$
$$\left(\exp\frac{(-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma)(\gamma+\alpha-1)}{(1-\alpha)(1-\beta)}S_F^*\right),$$

where K is a positive constant. Given that $\lim_{S^* \to \infty} S_F(S^*) = \bar{S}_F$, the last two terms approach finite numbers. And, by Lemma 7, $\int\limits_{S_F(S^*)}^{S^*} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds$ approaches a strictly positive finite number. Thus, $S^* \to \infty$ implies $q_H^*(0)$ approaches a finite number. Therefore, the limiting behavior of $q_H^*(S^*)$ is governed by the limiting behavior of $\exp(-\kappa S^*)$.

Hence, $\lim_{S^* \to \infty} q_H^*(S^*) = 0$.

(iii) $\lim_{S^* \to 0} q_H^*(S^*) = \infty$.

Since $S^* > S_F(S^*), S^* \to 0$ implies $S_F(S^*) \to 0$. We will show that $n^*(0)$ diverges to ∞ while $z^*(0)$ approaches a constant, implying $q_H^*(0) \to \infty$.

In equation (14), from Lemma 6 and $\delta > \kappa$, the first of the two ratios is bounded by

$$e^{-(\delta-\kappa)S_F(S^*)} \leq \frac{\left[\int\limits_0^{S_F(S^*)} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right]r\right)dr\right]}{\left[\int\limits_0^{S_F(S^*)} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\kappa\right]r\right)dr\right]} \leq 1,$$

and as $S_F(S^*) \to 0$ the ratio inside converges to 1. Similarly, by Lemma 6, the second ratio is also bounded by

$$\exp\left(-\kappa S^*\right) \leq \frac{\left[\int\limits_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa}{1-\beta}r\right)dr\right]}{\left[\int\limits_{S_F(S^*)}^{S^*} r \exp\left(-\frac{\kappa\beta}{1-\beta}r\right)dr\right]} \leq \exp\left(-\kappa S_F(S^*)\right), \tag{32}$$

and as S and S_F converge to 0 (and so both $\exp(\kappa S_F)$ and $\exp(\kappa S)$ converge to 1), the ratio inside also converges to 1. So $z^*(0)$ converges to a finite number.

For $n^*(0)$ in (15), the denominator in the the first ratio, $\int\limits_0^{S_F(S^*)} s \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds \to 0$ as $S_F \to 0$ and so the first ratio diverges to ∞. The second ratio in $n^*(0)$ is identical to the ratio in (32) and so approaches 1 as $S^* \to 0$. So, $n^*(0) \to \infty$ as $S^* \to 0$.

Hence, $q_H^*(0) \to \infty$ as $S^* \to 0$.

Given (i)–(iii), there is a unique value of S^* that satisfies $q_H^*(S^*) = d$.

(iv) S^* is strictly increasing in P and $\lim_{P\to\infty} S^* = \infty$.

We have

$$q_H^*(0)e^{-\frac{\kappa}{(1-\beta)}S^*} = d. \tag{33}$$

From (31), it follows that $q_H^*(0)$ is increasing in P and so the l.h.s. of this equation is increasing in P. By (i) the l.h.s. is decreasing in S^*. Therefore, S^* is strictly increasing in P.

To prove that S^* increases unboundedly in P, suppose to the contrary that there exists $\bar{S}$ such that $S^* < \bar{S}$ for all P. Then, from equation (31), $q_H^*(0)$ increases unboundedly in P. Therefore, there exists P sufficiently large for which $q^*(S^*(P)) = q_H^*(0)e^{-\frac{\kappa}{(1-\beta)}S^*(P)} > q_H^*(0)e^{-\frac{\kappa}{(1-\beta)}\bar{S}} > d$, which contradicts the definition of S^*.

■

Appendix C: Decentralized City

Proposition 7 (Internal Structure III: Decentralized City) *If $\kappa > [\delta\gamma(1-\beta)]/[1-\alpha\beta]$, there is an $S > 0$ such that for $s < S$, $\theta_F^*(s) > 0$ and $\theta_H^*(s) > 0$, and for $s > S$, $\theta_F^*(s) = 0$ and $\theta_H^*(s) = 0$.*

Proof. In any equilibrium with $U^* > 0$, there will be some s such that $\theta_F^*(s) > 0$. By Proposition 1, there is no commuting. So, by labor market clearance condition $\theta_H^*(s) > 0$. We claim that for $r < s$, $\theta_F^*(r) > 0$ and $\theta_H^*(r) > 0$. Suppose, to get a contradiction, that $\theta_F^*(r) = 0$ and $\theta_H^*(r) = 0$. By the equilibrium condition (6), $q^*(r) = d \leq q^*(s)$. Given that $z^*(r) > z^*(s)$, and $\pi^*(s) = 0$ (because $\theta_F^*(s) > 0$), $w^*(r) > w^*(s)$ (i.e., if firms at r are more productive than at s, and land prices at r are not any higher, wages at r must be strictly higher for profits to be nonpositive). But this contradicts $u^*(r) \leq U^*$, since strictly higher wages at r with $q^*(r) \leq q^*(s)$ would give strictly higher utility at r than at s. Hence, either $\theta_F^*(r)$ or $\theta_H^*(r)$ must be greater than 0, which means both $\theta_F^*(r)$ and $\theta_H^*(r)$ must be greater than 0. Let $\mathcal{F} = \{r \geq 0 : \theta_F^*(r) > 0\}$. By assumption, $\mathcal{F}$ is nonempty and by Proposition 2 it is bounded above. Hence, it possesses a least upper bound S. Hence, for all $r < S$, $\theta_F^*(r) > 0$ and $\theta_H^*(r) > 0$.

■

Next, we construct the equilibrium. The construction will show that $q^*(.)$, and U^* are uniquely determined. $\theta_F^*(.)$ and $\theta_H^*(.)$ are uniquely determined (except, possibly, at S^*).

In addition, where $\theta_F^*(s) > 0$ and $\theta_H^*(s) > 0$, $n^*(s)$, $w^*(s)$, $c^*(s)$, and $l^*(s)$ are uniquely determined.

By Proposition 7, the city is contained within a circle of finite radius S^*. By Lemma 3, $q_F^*(s) = q_H^*(s)$, and by Lemma 4, $W^*(s) = w^*(s)$. Then,

$$w^*(r) = w^*(0) \exp\left(-\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} r\right), \quad r \leq S^*, \tag{34}$$

and

$$q^*(r) = q^*(0) \exp\left(-\frac{\delta\gamma}{1-\alpha\beta} r\right), \quad r \leq S^*. \tag{35}$$

Given that there is no commuting in equilibrium, the labor market equilibrium condition collapses to

$$n^*(r)\theta_F^*(r) = \theta_H^*(r)/l^*(r) \text{ and } \theta_F^*(r) + \theta_H^*(r) = 1, \quad r \leq S^* \tag{36}$$

Given that $n^*(r) = [\alpha q^*(r)]/[(1-\alpha)w^*(r)]$ (from firm's maximization problem) and $l^*(r) = (1 -\beta)w^*(r)/q^*(r)$ (from worker's maximization problem), we get $\theta^*{}_F(r) = [1-\beta]/[1-\alpha\beta]$ and $\theta_H^*(r) = \beta[1-\alpha]/[1-\alpha\beta]$ for $r \leq S^*$. Thus, the proportion of land devoted to production is constant across all locations in the city.

The requirement that population P lives in the city collapses to the condition

$$P = \int_0^S 2\pi r \theta_H^*(r)/l^*(r) dr. \tag{37}$$

The final equilibrium condition is the determination of the city boundary S^*. Since $q^*(r)$ (from equation (35)) is declining at an exponential rate, it will fall to d at some point. Hence, S^* is determined by the condition

$$q^*(S^*) = d. \tag{38}$$

Now that we have given all the equilibrium conditions, we can simplify them further to show that the equilibrium is unique. Using equations (37), $1/l^*(r) = q^*(r)/(1-\beta)w^*(r) = (1-\alpha)n^*(r)/(1-\beta)\alpha$, and $\theta_H^*(r) = \beta[1-\alpha]/[1-\alpha\beta]$, one can solve for $n^*(0)$ in terms of S^* and parameters:

$$n^*(0) = \frac{P[1-\alpha\beta]}{2\pi[1-\beta]\int_0^{S^*} r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right) dr}. \tag{39}$$

Thus, it is clear that for any S^*, $n^*(0)$ is uniquely determined, which then implies that the rest of the equilibrium values, such as $z^*(.), w^*(.), q^*(.), l^*(.)$, and U^*, are also uniquely determined. Finally, similar to the CBD case, we have the following proposition that proves the existence and uniqueness of S^*.

Proposition 8 *There is a unique $S^* \in (0,\infty)$ that satisfies $q^*(S^*) = d$. Furthermore, S^* is strictly increasing and unbounded in P.*

Proof.

(i): $q^*(S^*)$ is decreasing in S^*:

$$q^*(S^*) = q^*(0) \exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S^*\right). \tag{40}$$

Since the exponential term is decreasing in S^*, it is sufficient to show that $q^*(0)$ is decreasing in S^*. Using equations (39) and $q^*(0) = (1-\alpha)A\left(z^*(0)\right)^\gamma n^*(0)^\alpha$ and expressing $z^*(0)$ in terms of $n^*(0)$, we get

$$\begin{aligned} q^*(0) \;&=\; (1-\alpha)AP^{(\alpha+\gamma)} \times \left[\frac{\int_0^{S^*} r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right) dr}{\int_0^{S^*} r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right) dr}\right]^\gamma \times \\ &\left[\frac{2\pi[1-\beta]}{[1-\alpha\beta]}\int_0^{S^*} r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right) dr\right]^{-\alpha}. \end{aligned} \tag{41}$$

By Lemma 6, the ratio of integrals in the second term is decreasing in S^*, and last term is also decreasing in S^*. Thus, $q^*(S^*)$ is decreasing in S^*.

(ii) $\lim_{S^*\to\infty} q^*(S^*) = 0$.

As S^* approaches ∞, by Lemma 7, all integrals in (41) approach positive constants, so, $q^*(0)$ approaches a constant. Hence, $\lim_{S^*\to\infty} q^*(S^*) = 0$.

(iii) $\lim_{S^*\to 0} q^*(S^*) = \infty$.

As S^* approaches 0, the second term in (41) approaches 1 (this follows from an application of L'Hospital's Rule) and the last term approaches ∞. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S^*\right)$ approaches 1, $\lim_{S^*\to 0} q^*(S^*) = \infty$.

(i)–(iii) imply that there is a unique $S^* \in (0,\infty)$, such that $q^*(S^*) = d$.

(iv) S^* is strictly increasing and unbounded in P.

From (41), $q^*(S^*)$ is strictly increasing in P. By (i) $q^*(S^*)$ is decreasing in S^*. Hence S^* is strictly increasing in P. To prove that S^* is unbounded in P, assume to the contrary that there is $\bar{S}$ such that $S^* < \bar{S}$ for all P. Then, equation (41) implies that $q^*(0)$ is unboundedly increasing in P. Hence, there exists P sufficiently large for which $q^*(S^*(P)) = q^*(0)\exp^{-\frac{\gamma\delta}{1-\alpha\beta}} > d$, which contradicts the definition of $S^*(P)$.

■

We now characterize how the equilibirum utility level U^* varies with population inside the city. Just like in the CBD case, equilibrium conditions in the previous section can be used to express U^* as a function of S^* and parameters only. We use $d = q^*(0)\left(-\frac{\delta\gamma}{1-\alpha\beta}S^*\right)$ and $U^* = \beta^\beta(1-\beta)^{1-\beta}d^{-(1-\beta)}w^*(0)\exp\left(-\frac{\delta\gamma(1-\beta)}{1-\alpha\beta}S^*\right)$. One can express these equations in terms of $n^*(0)$, U^*, S^* and other parameters. Using these two equations, we can eliminate $n^*(0)$ and express U^* in terms of S^* and parameters. Taking ln of U^* yields

$$\ln U^* = \frac{\gamma}{\gamma+\alpha}\left\{\ln\left[\int_0^{S^*} r\exp\left(-\frac{\delta\left(1-\beta\alpha+\gamma\beta\right)}{1-\beta\alpha}r\right)dr\right] - \frac{\delta\left(1-\beta(\alpha+\gamma)\right)}{(1-\beta\alpha)}S^*\right\} + K$$

where K is a constant that depends on parameters. Thus, on the logarithmic scale, $\ln U^*$ has a component that starts at 0 and declines linearly with S^*, provided the no-black-hole condition $1-\beta(\alpha+\gamma)>0$ is satisfied, and a component that starts at $-\infty$ and rises at most logarithmically with S^*. Since the rate of change of $\ln(x)$ is infinite at $x=0$, $\ln U^*$ must be increasing at $S^*=0$. Furthermore, since the derivative of the ln term declines monotonically to 0 with S^*, there is some $\hat{S}>0$ at which $\ln U^*$ peaks and then declines monotonically, asymptoting to $-\infty$. Then U^* is hump-shaped, with $\lim_{S^*\to 0} U^* = \lim_{S^*\to\infty} U^* = 0$. Since S^* is strictly increasing and unbounded in P, we have

Proposition 9 *Assume* $1-\beta(\alpha+\gamma)>0$. *Then,* $\lim_{P\to 0} U^* = \lim_{P\to\infty} U^* = 0$ *and* U^* *is single-peaked.*

■