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Private Risk Premium and Aggregate Uncertainty in the Model of Uninsurable Investment Risk*

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Abstract

This paper studies cyclical properties of the private risk premium in a model where a continuum of heterogeneous entrepreneurs are subject to *aggregate* as well as idiosyncratic risks, both of which are assumed to be highly persistent. The calibrated model matches highly skewed wealth and income distributions of entrepreneurs found in the Survey of Consumer Finances. We provide an accurate numerical solution to the model even though the model is shown to exhibit serious nonlinearities that are absent in incomplete market models with idiosyncratic labor income risk. The model is able to generate the aggregate private risk premium of 2-3 percent and the low risk-free rate. However, it generates very little variation in these variables over the business cycle, suggesting that the model lacks the ability to amplify aggregate shocks.

JEL codes: E22, G11, M13
Keywords: Uninsurable investment risk, aggregate uncertainty, private risk premium

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1 Introduction

This paper examines quantitative properties of the general equilibrium model of uninsurable investment risk by extending the work by Angeletos (2007), Angeletos and Calvet (2006), and Covas (2006). The economy consists of a continuum of heterogeneous entrepreneurs who own and manage their own businesses. Entrepreneurs in our economy face *aggregate* as well as idiosyncratic investment risks, both of which are assumed to be highly persistent. The framework is designed to capture essential features of "financing frictions" facing privately held businesses.

As emphasized in Angeletos (2007), the size of privately held businesses is, by any measure, quite large in the US economy, and the roles played by entrepreneurs in growth and business cycles are no doubt important.[1] Reflecting their importance, the recent empirical finance literature has revealed a number of economically intriguing evidence on investment and portfolio decisions of entrepreneurs (e.g., Hamilton (2000), Moskowitz and Vissing-Jørgensen (2002), Heaton and Lucas (2000), Carroll (2002), and Gentry and Hubbard (2004)).

Although the model we study in this paper is arguably too stylized to address all of these empirical findings, this paper takes one of the initial steps to evaluate them from a macroeconomic perspective. Our primary focus is on the model's asset pricing implications. Specifically, we examine the model's ability to generate an empirically plausible size of the aggregate private risk premium and its variations over business cycles. In doing so, we first carefully calibrate the model by matching key cross-sectional characteristics of income and wealth distributions of entrepreneurs found in the Survey of Consumer Finances. The model in the present paper is closely related to the models by Angeletos (2007) and Covas (2006). These two papers, however, focus on whether the presence of uninsurable investment risk leads to under- or over-accumulation of capital stock and do not put much emphasis on cross-sectional heterogeneities of entrepreneurs. In this paper, on the other hand, we exert a great deal of effort to replicate the cross-sectional heterogeneities observed in the Survey of Consumer Finances (SCF). In particular, our calibrated model matches the highly skewed wealth and income distributions of entrepreneurs. Our calibration also implies that

[1] According to NIPA Table 1.13, the share of the noncorporate sector's value added amounts to 23 percent of the entire business sector's total value added in 2006. Davis et al. (2006) report that more than two-thirds of nonfarm business employment is accounted for by privately held firms. These numbers, of course, vary with definitions, but there is no doubt about the quantitative importance of private businesses in the US economy.

entrepreneurs face a large idiosyncratic risk, which is consistent with findings by Hamilton (2000) and Moskowitz and Vissing-Jørgensen (2002).

The most important deviation of our paper from the previous literature is that our model features aggregate risk as well as idiosyncratic risk. Augmenting the model with aggregate uncertainty allows us to examine business cycle dimensions of the model, such as variations of the private risk premium and the risk-free rate. As far as we know, this paper is the first attempt to assess the roles of aggregate uncertainty in this class of models.

While facing uninsurable investment risk of their private businesses, the entrepreneurs also have access to one-period riskless bonds, which they can use to self-insure against the idiosyncratic risk. Entrepreneurs are allowed to borrow from other entrepreneurs by short-selling the bonds. However, the borrowing is permitted only up to an exogenously specified amount. In this environment, the presence of aggregate uncertainty poses an important computational challenge. Since entrepreneurs differ in the size of their firms and their idiosyncratic productivities, some may choose to borrow as much as the borrowing constraint permits, while others may choose to invest all of their wealth in the safe asset. The general equilibrium of the model requires that the price of the safe asset clear the market every period and that the individual-level decision based on perceived evolution of the market-clearing bond price be accurate.

A similar computational difficulty is common in the literature of incomplete market models with aggregate and idiosyncratic labor income risk. It is found in that literature that the first moment of the wealth distribution accurately captures nearly all information regarding equilibrium prices. Consider the model studied by Krusell and Smith (1998) where workers are subject to the uninsurable risk of becoming unemployed. Workers can self-insure against this risk by accruing their wealth, i.e., more specifically, owning physical capital rented out to the representative firm. In this environment, solving the individual worker's problem requires a prediction of the next-period rental rate of capital, which in turn requires knowing the evolution of aggregate capital stock. However, the next-period aggregate capital stock is a function of the distribution of individual capital holdings, making it impossible to find the "true" equilibrium. However, in practice, Krusell and Smith (1998) and other studies, including Young (2005) show, that the next-period capital stock can be accurately predicted by a simple linear function of the current-period capital stock and the aggregate productivity state. This result is often called approximate aggregation. Krusell and Smith (2006) argue that the key to this finding is a linearity of the saving policy function of workers. That is, marginal propensities to save are approximately constant for a wide range of wealth levels. Consequently, workers with different wealth levels are simply the scaled-up or -down version of the average worker. In this environment, the information regarding the wealth distribution other than its mean simply do not have any implications for the equilibrium interest rate.

It is, however, unclear whether or not a similar statement can be made in our model. In our model, entrepreneurs operate production technology that is decreasing returns to scale in privately held capital. This implies that entrepreneurs who are "poor," measured by the size of their capital stock, have a better return-risk tradeoff. In particular, entrepreneurs who draw a good idiosyncratic shock (which is assumed to be highly persistent) have a strong incentive to borrow and invest in their private businesses. This incentive of borrowing, together with the presence of the borrowing constraint, generates a highly skewed wealth distribution populated by a large mass of "poor" entrepreneurs. These economic mechanisms are very different from those in the standard model of uninsurable labor income risk, studied by Krusell and Smith (1997, 1998), den Haan (1997), and Young (2005).

Despite such nonlinear features of the model, we are able to provide an accurate numerical solution. In our environment, solving for the "true" equilibrium requires us to know (i) the evolution of the entire wealth distribution and (ii) the mapping from the wealth distribution to the market-clearing bond price. In our approximate equilibrium, we assume that the entrepreneurs directly take the market-clearing bond price as a state variable and that they use a linear autoregressive rule (conditional on aggregate productivity states) to predict the next-period market-clearing price. We show that this autoregressive rule serves to obtain a very accurate numerical solution.

We find that the calibrated model generates the aggregate private risk premium of 2-3 percent. While this is smaller than the plausible figure reported by Moskowitz and Vissing-Jørgensen (2002), it is clearly large relative to the equity premium obtained in incomplete market models with labor income risk.[2] There are two channels through which risk premia are generated in our economy. The first is simply through idiosyncratic investment risk that is calibrated to be large. Risk-averse entrepreneurs demand the risk premium for investment in their risky businesses over a return from the safe asset. The second is through a wedge due to the borrowing constraint. As

[2]For instance, Krusell and Smith (1997) show that their model generated almost no equity premium.

we mentioned above, particularly small and productive entrepreneurs have a strong incentive to become larger (by investing more in their business). However, because the borrowing constraint limits their ability to expand their size, it creates risk premia over the safe asset. We find that this second mechanism is dominant.[3] We also show that the model is capable of generating the low risk-free rate. This result is generated as a consequence of heterogeneities in portfolio choice and productivities. As mentioned above, some entrepreneurs demand funds as much as the borrowing limit permits, whereas others choose to decide to save through the safe asset. The low risk-free rate arises as a result of the financial market reallocating funds from the rich and "not-so-productive" entrepreneurs to the productive and poorer entrepreneurs.

Although the model is able to generate the relatively large private risk premium and the low risk-free rate, it dramatically fails to generate an empirically plausible amount of volatilities of those aggregate variables. Their standard deviations are only about one-tenth of a percent. Note that in our model each entrepreneur could experience large changes in the return to his/her risky investment as implied by the large idiosyncratic risk. However, those variations of returns across entrepreneurs are washed out in the aggregate, thus causing little change in the average risk premium and the risk-free rate.

Our findings suggest that the "financial" frictions we introduce in the form of uninsurable idiosyncratic risk and the borrowing constraint are not enough to amplify the aggregate shock. Thus, augmenting the model with some "real" frictions such as capital adjustment costs or investment irreversibility appears to be an important and fruitful avenue for future work. However, an important but interesting challenge would be that in such an extended environment, there may be larger nonlinear variations in the wealth distribution, making it even harder to obtain an accurate numerical solution.

This paper is organized as follows. The next section lays out the model. Section 3 presents its calibration. Section 4 considers the properties of the steady state equilibrium. In this section we first emphasize the importance of nonlinear features evident in our environment and then explain in the context of the steady state the mechanism through which a relatively large private risk premium and the low risk-free rate arise. The model under

[3] Angeletos (2007) shows that his model is also able to generate the average risk premium amounting to about 1-8 percent, depending on calibrations, even under the absence of the borrowing constraint. However, he assumes that the idiosyncratic shock is i.i.d., whereas we assume that it is highly persistent, which appears empirically more appealing than the i.i.d. case.

aggregate uncertainty is solved in Section 5 where we first ensure that our approximate equilibrium is accurate and then examine the cyclical properties of the model. The last section concludes the paper by offering some ideas for future work. Two subsections in the Appendix are devoted to descriptions of the data construction from the SCF and the computational algorithm used to solve the model.

2 Entrepreneurial Economy

The economy is inhabited with measure one of infinitely lived entrepreneurs. Each entrepreneur has an ability to operate his/her own technology. This technology is subject to individual-specific shocks that are assumed to be uninsurable. This technology also faces aggregate uncertainty. Other features include that (i) entrepreneurs face an occasionally binding borrowing constraint and (ii) they are also allowed to trade one-period riskless bonds.

2.1 Environment

There is only one consumption good. The utility function of each entrepreneur, $U(\cdot)$, is strictly increasing, strictly concave, obeys the Inada conditions, and is twice continuously differentiable in consumption, c_{it}. Since there are idiosyncratic shocks, consumption will differ across agents. The entrepreneur's problem is to maximize the expected lifetime utility derived from consumption:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_{it}), \tag{1}$$

where $0 < \beta < 1$ is the discount factor.

There are two investment opportunities facing each entrepreneur: investment in his/her own business, or in one-period bonds that yield a sure return over the period. The idiosyncratic production risk cannot be insured by any direct insurance markets. Further, issuance of equity is not allowed, and thus the firm is "privately held" by the entrepreneur.[4] The entrepreneurs instead can borrow funds to finance both consumption and the

[4] This financing friction is one of the features that makes our model distinct from models in the lumpy investment literature. For example, in the model of Khan and Thomas (2007), production units are subject to idiosyncratic and aggregate uncertainty as in our model, but are allowed to issue shares that are held by the representative household. The focus of the lumpy investment literature is on micro- and aggregate-level implications of "real frictions" such as nonconvex adjustment costs and irreversibility of investment.

risky investment by investing a negative amount in the safe asset. However, there is a limit for borrowing denoted by $\underline{b}$. The limit is assumed common across the entrepreneurs.[5] Risky technology available to the entrepreneur i is represented by

$$y_{it} = \theta_t z_{it} f(k_{it}), \tag{2}$$

where θ_t and z_{it} respectively denote the aggregate and idiosyncratic components of an exogenously specified technology and k_{it} is entrepreneur i's capital stock in the risky investment. It is assumed that $f(\cdot)$ is continuously differentiable, strictly increasing, strictly concave with $f(0) = 0$, and satisfying the Inada conditions. The amount of labor input is fixed at a normalized value of one. The fixity of labor input is often adopted in the related literature (e.g, Angeletos and Calvet (2006), Cagetti and De Nardi (2006), and Caggese (2006)). As Krusell and Smith (2006) put it, it is consistent with interpretations that "the entrepreneur himself is an unsubstitutable input," or that "the labor input takes even longer to reallocate across production sites than does the capital input." Further, concavity of the production function can be interpreted as reflecting diminishing returns to "span of control" as in Lucas (1978). The decreasing returns to scale property is an important nonlinear feature of our model.[6] Quantitatively, it allows us to match the wealth distribution of entrepreneurs, which is highly skewed in the data. Furthermore, as we will discuss later, it produces an interesting interaction with the borrowing constraint $\underline{b}$.

Both idiosyncratic and aggregate uncertainties follow a first-order Markov process. Entrepreneurial capital depreciates at a fixed rate, δ, and the gross risky investment is given by:

$$i_{it} = k_{it+1} - (1 - \delta)k_{it}. \tag{3}$$

Let b_{it+1} denote the resources of the entrepreneur allocated to the risk-free asset, which delivers one unit of consumption good in the next period. The price of the asset is denoted as q_t. The rate of return is determined in equilibrium such that the bond market clears in each period. Furthermore, it varies over time under the presence of the aggregate shock θ_t. The entrepreneur's

[5] We do not attempt to endogenize the borrowing limit for simplicity. For such an attempt, see Meh and Quadrini (2006).

[6] This feature is absent in the model of Angeletos (2007), where technology exhibits constant returns to scale in labor input and the capital stock. In this case, capital income is linear in the capital stock. See Lemma 1 in Angeletos (2007).

budget constraint is written as follows:

$$c_{it} + k_{it+1} + q_t b_{it+1} = x_{it}, \tag{4}$$

$$x_{it+1} = \theta_{t+1} z_{it+1} f(k_{it+1}) + (1-\delta) k_{it+1} + b_{it+1}, \tag{5}$$

where x_{it} denotes the entrepreneur's period t wealth.

2.2 Individual Problem and the General Equilibrium

The set of decision-relevant state variables is the following four variables: (i) individual wealth x, (ii) the idiosyncratic technology shock z, (iii) the aggregate technology shock θ, and (iv) the measure of agents over (z, x), which we denote by Γ. Note that from here on we adopt the notational convention of dropping the time subscript t and using "prime" to denote one-period-ahead values. We can write the entrepreneur's dynamic decision problem in a recursive manner as follows:

$$\begin{aligned} V(z_i, x_i; \Gamma, \theta) &= \max_{c_i, k_i', b_i'} U(c_i) + \beta \mathbb{E} V(z_i', x_i'; \Gamma', \theta'), \\ &\text{subject to} \\ c_i + k_i' + q(\Gamma, \theta) b_i' &= x_i, \\ x_i' &= \theta' z_i' f(k_i') + (1-\delta) k_i' + b_i', \\ \Gamma' &= H(\Gamma, \theta, \theta'), \\ k_i' &\geqslant 0 \quad \text{and} \quad b_i' \geqslant \underline{b}, \end{aligned} \tag{6}$$

where $H(\cdot)$ is the equilibrium transition function for Γ and $\mathbb{E}$ is a conditional expectation operator with respect to z_i' and θ'.

From the properties of the utility and production functions of the entrepreneur, the optimal levels of consumption and the risky investment are always strictly positive. The only constraint that may be binding is the choice of b_i'. Taking first-order conditions of problem (6) and using the envelope condition, the first-order conditions of the problem are as follows:

$$U_c(c_i) = \beta(1+r) \mathbb{E} U_c(c_i') + \lambda_i, \tag{7}$$

$$U_c(c_i) = \beta \mathbb{E}\left[(\theta' z_i' f_k(k_i') + 1 - \delta) U_c(c_i')\right], \tag{8}$$

where λ_i is the Lagrange multiplier associated with the entrepreneur's borrowing constraint, $b_i' \geqslant \underline{b}$, and $r = 1/q - 1$ is the interest rate on the safe asset. The Lagrange multiplier is positive if the constraint is binding, and zero otherwise. Returns to safe and risky investments are related by:

$$\mathbb{E}\left[(\theta' z_i' f_k(k_i') + 1 - \delta) U_c(c_i')\right] = (1+r) \mathbb{E}\left[U_c(c_i')\right] + \lambda_i,$$

or equivalently,

$$r + \delta = \mathbb{E}\theta' z_i' f_k(k_i') + \frac{cov(\theta' z_i' f_k(k_i'), U_c(c_i')) - \lambda_i}{\mathbb{E}U_c(c_i')}. \tag{9}$$

The second term on the right-hand side corresponds to the private risk premium. Because the two terms $\theta' z_i' f_k(k_i')$ and $U_c(c_i')$ are negatively related, the covariance term in the second term on the right-hand side is negative, compensating for the investment risk taken by the risk-averse entrepreneur.[7] Although the investment risk alone creates the wedge between the marginal product of capital and the risk-free rate, the borrowing constraint serves to produce the additional wedge between the two, represented by λ_i.

The recursive competitive equilibrium is defined by the value function $V(z_i, x_i; \Gamma, \theta)$; the policy functions $\{k(z_i, x_i; \Gamma, \theta), b(z_i, x_i; \Gamma, \theta), c(z_i, x_i; \Gamma, \theta)\}$; the pricing function $q(\Gamma, \theta)$; and a law of motion for the distribution $\Gamma' = H(\Gamma, \theta, \theta')$ such that (i) given the aggregate states $\{\Gamma, \theta\}$, the bond price $q(\Gamma, \theta)$ and the law of motion for the distribution $\Gamma' = H(\Gamma, \theta, \theta')$, the entrepreneur's policy functions solve problem (6), (ii) entrepreneurial capital and bonds are given by

$$K = \int k(z_i, x_i; \Gamma, \theta) d\Gamma(z_i, x_i), \tag{10}$$

$$B = \int b(z_i, x_i; \Gamma, \theta) d\Gamma(z_i, x_i), \tag{11}$$

where all integrals are defined over the state space $Z \times \mathcal{X}$, (iii) the bond market clears $B = 0$, and (iv) H is generated by the optimal policy functions $\{k(z_i, x_i; \Gamma, \theta), b(z_i, x_i; \Gamma, \theta)\}$.

3 Parameterization and Calibration

The properties of the model can be evaluated only numerically. This section therefore assigns functional forms and parameter values to find the numerical solution of the model. We choose one period in the model economy to be one year. The discount factor, β, is thus set equal to 0.96. For the utility function, a constant relative risk-aversion (CRRA) specification is adopted:

$$U(c_i) = \frac{c_i^{1-\gamma}}{1-\gamma}, \tag{12}$$

[7] Angeletos and Calvet (2006) study a model similar to ours and derive the analytical solution corresponding to (9) under the conditions that (i) the idiosyncratic shock is i.i.d. normal, (ii) there is no aggregate uncertainty, (iii) agents have CARA preference, and (iv) there is no borrowing constraint.

where γ is the risk-aversion parameter. Since we lack evidence on γ for entrepreneurs, we simply set γ to 2, a number often used in the representative-agent RBC framework.

The entrepreneur's risky technology is specified by k_i^α. The curvature parameter α is determined later. The idiosyncratic productivity process is first-order Markov:

$$\ln(z_i') = \rho_z \ln(z_i) + \sigma_z(1-\rho_z)^{1/2}\epsilon', \tag{13}$$

where $\epsilon \sim N(0,1)$. While we have ample evidence on the idiosyncratic labor income process (e.g., Aiyagari (1994), Storesletten et al. (2004) among others), we lack direct information that would allow us to calibrate the process for entrepreneurs. It appears, however, relatively uncontroversial to assume a high degree of persistence in the process. We therefore choose the serial correlation parameter ρ_z at 0.90. Angeletos (2007) and Angeletos and Calvet (2006) assume an i.i.d. shock for analytical tractability, but our focus is on quantitative evaluations of the model and thus allowing for a high degree of persistence is very important to us. In fact, as we will see later, this assumption does make a difference in the results. For the given level of ρ_z, the unconditional standard deviation σ_z is determined together with the other two parameters α and $\underline{b}$ by using observable cross-sectional information available through the SCF (Survey of Consumer Finances). The procedure will be described shortly. Once we parameterize (13), we use the procedure suggested by Tauchen and Hussey (1991) to approximate it with a Markov chain with seven states.

The aggregate technology state switches between two states θ_1 and θ_2 with $\theta_2 > \theta_1$ according to a Markov chain as in Krusell and Smith (1997, 1998) and others. Accordingly, the transition matrix is specified as:

$$\begin{pmatrix} \pi_{1|1} & \pi_{1|2} \\ \pi_{2|1} & \pi_{2|2} \end{pmatrix}.$$

We set $\pi_{1|1} = \pi_{2|2} = 0.875$, so that the average duration of business cycles is 8 years in the model. The levels of aggregate states θ_1 and θ_2 are determined so that the volatility of aggregate output in our model economy matches its empirical counterpart, which is calculated as real output of the noncorporate private sector.[8] The standard deviation of HP-filtered aggregate output during the post-war period amounts to 2.8 percent. By setting $\theta_1 = 0.95$

[8] The data are available in NIPA Table 1.13.

and $\theta_2 = 1.05$ respectively, we are able to match the volatility of the observed data.[9]

It remains to determine the borrowing limit $\underline{b}$ together with the two other previously mentioned parameters α and σ_z. To pin down these three parameters, we match the steady-state implications of the model and the cross-sectional information available through the SCF. We look at the data in the five surveys between 1992 and 2004. We first define "entrepreneurs" in the SCF as those households that satisfy two qualifications: (i) own or share ownership in any privately held businesses, farms, professional practices or partnerships and (ii) have an active management role in any of these businesses. In our sample, entrepreneurial households amount to 8 percent of US households, on average, across the five surveys.[10] While entrepreneurs make up a small fraction of US households, they hold a large fraction of wealth, as established by many previous researchers, such as Quadrini (2000), Heaton and Lucas (2000), Carroll (2002), amd Gentry and Hubbard (2004),[11]

There is no readily available evidence for σ_z. We thus use information regarding entrepreneurs' income distribution. More specifically, we target the Gini coefficient of the income distribution to determine the level of idiosyncratic risk. This procedure yields $\sigma_z = 0.5$. Note that in models with uninsurable labor income risk, unconditional standard deviations of 20-40 percent are considered reasonable (e.g., Aiyagari (1994), Storesletten et al. (2004)). Therefore, our choice (unconditional standard deviation of 50 percent) is consistent with the idea emphasized by Hamilton (2000) and Moskowitz and Vissing-Jørgensen (2002) that idiosyncratic risk facing entrepreneurs is larger than idiosyncratic labor income risk. The span of control parameter α has strong influences on the shape of the firm size distribution and thus the wealth distribution. We choose the level of this parameter by matching the Gini coefficient of the wealth distribution of entrepreneurs in the SCF. The chosen value $\alpha = 0.75$ turns out to be similar to those used in the related papers. For example, Cagetti and De Nardi (2006) and Terajima (2006) set it to 0.88 and 0.70, respectively. Finally, to choose the level of $\underline{b}$, we target the average level of the ratio of debt to business net wealth. This

[9] The dispersion of 10 percent seems large in light of the volatility of aggregate productivity series. However, we choose this dispersion simply to match the volatility of output and this choice is innocuous for our conclusions.

[10] Our number is close to the corresponding number presented by Cagetti and De Nardi (2006), although they look at the 1989 SCF only. See Quadrini (2000) and Gentry and Hubbard (2004) for other definitions of entrepreneurs adopted in the literature.

[11] According to Table 1 in Cagetti and De Nardi (2006), entrepreneurs who qualify for our definition hold 33 percent of total wealth in the 1989 SCF. But if more broadly defined, entrepreneurs own up to 53 percent of total wealth in the US.

procedure yields $\underline{b} = -1200$. With this value, aggregate borrowing is 2.6 times as large as aggregate output in our model.[12]

Table 1 summarizes parameter values for the benchmark calibration discussed so far. Tables 2 and 3 compare cross-sectional statistics of the model and the data. Observe first that the model is able to match the two inequality measures and the debt to equity ratio fairly well.[13] In addition to the three statistics used to pin down the three parameters, we also consider wealth and income quintiles and the wealth-income ratio. The tables show that the model matches these overidentifying statistics reasonably well. Gini coefficients and quintiles for observed income and wealth distributions imply that entrepreneurs' wealth and income distributions are highly skewed. Our model is capable of replicating these important features.

4 Steady State Equilibrium

Before examining the dynamic properties of the model, this section looks at various aspects of the steady state equilibrium by suppressing aggregate uncertainty.

4.1 Saving Policy Rules and the Wealth Distribution

In our economy, the wealth distribution enters in agents' state variables. This poses an important challenge in solving the model when the aggregate shock is present. Because it is an infinite dimensional object, one needs to find a way to efficiently capture the information that is relevant for the equilibrium price. In our environment, the market-clearing bond price is a function of the wealth distribution. In the literature of heterogeneous agent models with uninsurable labor income risk, it is found that the first moment of the wealth distribution essentially captures all such information (e.g., Krusell and Smith (1997, 1998), Young (2005) among others). This result is often called approximate aggregation and is a key prerequisite for assessing the model's aggregate behavior under the presence of aggregate uncertainty.

Although we suppress aggregate uncertainty throughout this section, the discussion below provides useful insights for overcoming the above-mentioned challenge. A recent survey by Krusell and Smith (2006) lays

[12]The discussion in this paragraph relates one parameter to one statistic. However, they cannot be set independently.

[13]We are not able to achieve an exact match due to nonlinearities in the model.

out reasons why, in the economy with uninsurable labor income risk, information about the wealth distribution other than its mean has a very limited influence on workers' decisions and hence aggregate behavior of the model. By looking at the steady-state equilibrium of the economy, Krusell and Smith show that the saving decision rules are almost linear in wealth. More precisely, there is a small region in the wealth distribution, namely, the region of the low levels of wealth, where marginal propensities to save are not constant. However, since the region has a very thin mass and is populated with poor workers, the nonlinearity plays virtually no role in the aggregate.

To see if these conditions are met in our economy, we examine entrepreneurs' saving policy rules and the wealth distribution in the steady-state equilibrium. Figure 1 plots the saving policy functions and their slopes. The top panel plots total savings, $k'+qb'$, against initially available resources excluding current-period output, $(1-\delta)k+b$.[14] The red dashed line corresponds to the policy function for the entrepreneurs with the highest level of z (of all the seven possible values), whereas the blue line is for the entrepreneurs with the lowest level of z. While the upper panel does not clearly illustrate presence of nonlinearity, the lower panel, which plots their slopes, makes it clear that there is nonlinearity particularly among *productive* entrepreneurs.

Note that the nature of nonlinearity in our model is very different from that in the model with uninsurable labor income risk. For instance, consider the model of Krusell and Smith (1998) where workers face uninsurable risk of losing his/her job. In this environment, nonlinearity is most visible for those who are the poorest (measured by the wealth level) and are unemployed. The marginal propensities to save for those workers are less than those for the rest of population, since they have a necessity of cutting down their wealth to consume. In our environment, on the other hand, nonlinearity is most noticeable for the poorest and *most productive* entrepreneurs. This feature makes sense, given that the entrepreneurs with a low level of capital have a stronger incentive to borrow and invest the funds in their private businesses because of decreasing returns to scale in technology. The incentive is, of course, larger for those who draw better idiosyncratic shocks.

Another significant difference from the Krusell-Smith economy is the shape of the wealth distribution. Figure 2 presents the stationary wealth

[14]Recall that the state variable $x = zf(k) + (1-\delta)k + b$ includes current-period output. We, however, measure $(1-\delta)k+b$ on the horizontal axis. We do this because, this way, we can distinguish between those who draw a low z and high z, who had the same initial wealth $(1-\delta)k+b$ and thus the interpretation of the figure is more intuitive.

distribution x. By comparing Figure 2 with the lower panel of Figure 1, one can clearly see that a large mass of entrepreneurs lie in the region of the wealth distribution where saving policy functions are nonlinear (particularly for productive entrepreneurs). Further, the distribution is truncated on the left because a significant fraction of entrepreneurs are borrowing constrained.

Discussions so far suggest that properties of our economy are sufficiently different from those in heterogeneous-agent economies with uninsurable labor income risk scrutinized by many existing papers. But does this imply that approximate aggregation fails to hold in our economy? There remains an important issue to be addressed before answering this question: whether or not the distribution of marginal propensities to save varies over the business cycle. This point is emphasized by Young (2005), who studies the economy where the marginal propensity to save is heterogeneous as in our economy. He nevertheless finds that the approximate aggregation result still holds.[15] We will come back to this issue in a later section.

4.2 Private Risk Premia

We now examine how the model's properties discussed so far relate to private risk premia. Note that in our environment each entrepreneur's risky investment is directly subject to idiosyncratic risk that is calibrated to be very large. Therefore, it appears possible that risk-averse entrepreneurs demand a large private risk premium. Angeletos (2007), in fact, shows that his model is able to generate the average private risk premium of 2-8 percent (depending on calibrations). Our model is, however, different from his model in several important ways. First, the idiosyncratic shock in his model is assumed to be i.i.d., while it is highly persistent in our model. Second, entrepreneurs in our economy are subject to an exogenous borrowing constraint. As can be seen from Equation (9), a wedge associated with the borrowing constraint can play a significant role in generating risk premia in our model.

In our benchmark calibration, the economy generates the average risk premium of 2.9 percent in the steady state.[16] While it still seems lower than a realistic value reported by Moskowitz and Vissing-Jørgensen (2002), it is much larger than the equity premium generated from the incomplete market

[15] Young considers an extension to the model of Krusell and Smith (1998), where returns to saving differ for the rich and the poor due to progressive income taxation.

[16] Individual-level risk premia are averaged by using the size of the capital stock as weights.

models with labor income risk.[17]

To shed some light on the mechanism generating the relatively large average risk premium in our model, Table 4 presents private risk premia over wealth and idiosyncratic productivity levels. The table shows a clear pattern that more (less) productive entrepreneurs demand a higher (lower) risk premium, and smaller (larger) entrepreneurs demand a higher (lower) risk premium. To gauge the importance of the role played by the borrowing constraint, Table 5 further presents average Lagrange multipliers normalized by expected marginal utility of consumption $\lambda_i/\mathbb{E}\left[U_c(c_i')\right]$ over wealth and idiosyncratic productivity levels. It shows that the presence of the borrowing constraint plays a dominant role in generating private risk premia in our economy. The intuition is simple. As we emphasized before, smaller and more productive entrepreneurs have a stronger incentive to invest in their business, and this incentive interacts with the exogenously specified borrowing constraint. That is, smaller and more productive entrepreneurs are more likely to be constrained by the borrowing limit and thus the wedges created by the borrowing constraint are quantitatively more important for those entrepreneurs.

4.3 Portfolio Choice and the Risk-Free Rate

To gain further insights, Figure 3 looks at the portfolio choice of the entrepreneurs. The upper and lower panels, respectively, display the policy functions for risky and safe investments. The two panels show that the most and least productive entrepreneurs make entirely opposite portfolio choices. In our calibration, the former entrepreneurs always borrow as much as they can and put all funds in their private businesses, regardless of their wealth levels. On the other hand, the latter behave in an entirely opposite way. However, for those with medium-level productivity, it is optimal to borrow and invest in their businesses only up to a certain level of wealth, since the return from risky investment declines as the size of their projects increases. As the kinks in Figure 3 indicate, once entrepreneurs (with medium-level productivities) become rich enough, the borrowing constraint is no longer binding. Once entrepreneurs become rich enough, their optimal investment decision follows the principle that the expected returns from the risky business equals the risk-free rate. In the aggregate, our benchmark calibration

[17] It is difficult to obtain an off-the-shelf target for the empirical value for the average risk premium. However, Moskowitz and Vissing-Jørgensen (2002) report that the median of the capital gain distribution from investment in private businesses is 6.9 percent for the 1989 SCF.

implies 74.2 percent of the entrepreneurs are sellers of the safe asset. Further, almost 60 percent of the entrepreneurs are constrained by the borrowing limit (see Table 6).

The heterogeneous portfolio choices discussed above determine the market-clearing bond price. In particular, our model successfully generates a low risk-free rate that is 0.3 percent in the steady state. The low risk-free rate arises because, while there is a large mass of borrowers in the equilibrium as indicated by the high fraction of borrowers, supply of funds from those who are rich enough and have "medium-level" productivities is large enough to depress the risk-free rate.

Krusell and Smith (2006) study a two-period version of the model under the assumption that idiosyncratic shocks are i.i.d. They find that the two-period model is also able to generate a low risk-free rate as we find in our environment. The intuition behind their result, however, does not exactly conform with our result. They emphasize the feature in their setup that poor entrepreneurs have a strong motive to save through bonds, since they value insurance more (i.e., precautionary savings), which lowers the risk-free rate. The mechanism working in our setup is more subtle: in our environment and calibration, as we noted above, the least productive entrepreneurs are always savers, and similarly, the most productive entrepreneurs are always borrowers. This clear pattern appears because idiosyncratic risks are highly persistent (instead of i.i.d.) in our model and therefore the consideration of idiosyncratic productivity levels dominates the portfolio choices of these entrepreneurs. Further, as was shown above, the "average" entrepreneurs become savers as they become *richer* not poorer. While it is true (even in our model) that poor entrepreneurs have a stronger precautionary saving motive with everything else being constant, they also have higher marginal returns from investment in their businesses. This latter incentive counteracts the precautionary saving motive, producing the pattern that poorer entrepreneurs are actually borrowers.

4.4 Effects of Alternative Parameter Values

To see the sensitivity of our quantitative results, we consider alternative parameter values along two important dimensions, i.e., the size of the idiosyncratic investment risk σ_z and the borrowing constraint $\underline{b}$.

The results are summarized in Table 6. Consider first the case where the standard deviation of idiosyncratic risk is increased from 0.5 to 0.6. This change has a rather large impact on wealth and income distributions. The inequality measures are pushed upward to the levels that are considerably different from observed ones. The increased idiosyncratic risk raises the average private risk premium more than 1 percent. The following two mechanisms contribute to this result. First, increased risk obviously raises the private risk premium directly. The second effect comes from the borrowing constraint. As indicated by the larger fraction of constrained entrepreneurs (59.8 percent $\rightarrow$ 66.1 percent), higher idiosyncratic risk raises the chance of entrepreneurs being constrained by the borrowing limit, thereby raising the risk premia. Another implication of the larger idiosyncratic risk is the even lower risk-free rate (it decreases from 0.28 percent to -0.61 percent). This is because, all else being equal, the higher risk creates, on average, stronger precautionary saving motives, thus raising savings through the risk-free asset.

The last column of Table 6 summarizes the effects of the tighter borrowing constraint. There are two noticeable effects here. First, it increases the average risk premium. Again, this is because more entrepreneurs are constrained, as shown in the last row. Second, it depresses the risk-free rate because the amount of borrowing is reduced directly by the tighter constraint.

5 Dynamic Equilibrium

We now add aggregate risk to the economy. The first necessary step is to provide an accurate solution. Because the model's characteristics under aggregate uncertainty are largely unknown (relative to that of the models with uninsurable labor income risk), it requires an examination of the proposed solution method. Once we establish the accuracy of the solution, we examine the cyclical properties of the model.

5.1 Forecast Rules for the Market-Clearing Bond Price

To solve the model with aggregate uncertainty, we need to take a stand on how we capture information regarding the evolution of the wealth distribution Γ. Furthermore, we need to assume the way in which the information about the distribution is mapped into the market-clearing bond price.

We proceed by assuming that the agents take the market-clearing bond price directly as a state variable. This approach is more efficient than the conventional approach where one first postulates the evolution of a moment (moments) of the distribution and then relate it (them) to an equilibrium price.[18] We further presume a particular form in which the next-period market-clearing bond price is related to the current-period bond price. Specifically, we assume the agents forecast the next-period market-clearing bond price using the conditional autoregressive rule, written as

$$q' = \begin{cases} \beta_0(1) + \beta_1(1)q, & \text{if } \theta' = \bar{\theta}_1 \text{ and } \theta = \bar{\theta}_1 \\ \beta_0(2) + \beta_1(2)q, & \text{if } \theta' = \bar{\theta}_2 \text{ and } \theta = \bar{\theta}_1 \\ \beta_0(3) + \beta_1(3)q, & \text{if } \theta' = \bar{\theta}_1 \text{ and } \theta = \bar{\theta}_2 \\ \beta_0(4) + \beta_1(4)q, & \text{if } \theta' = \bar{\theta}_2 \text{ and } \theta = \bar{\theta}_2. \end{cases} \tag{14}$$

where the βs need to be determined in a way that is consistent with the rational expectation equilibrium. Observe that the forecast of the bond price can be conditioned not only on the current aggregate productivity state but also on the next-period aggregate productivity state, since both θ_i and q are realized simultaneously.

To obtain the numerical solution of the model, we solve the individual entrepreneur's problem taking the above law of motion for the bond price as given. When making their decisions in the current period, the agents form an expectation for the next-period market-clearing price. Using the obtained solution, we simulate a large panel data set that consists of 30,000 agents for 3,500 periods. In each period, we find a market-clearing bond price by using the bisection method. This simulation stage yields time series of the market-clearing bond price. After discarding the initial 500 observations, we run an OLS regression of the form (14) with the simulated bond prices. This process repeats until convergence of the coefficients is achieved. More details of the algorithm is presented in Appendix.

5.2 Approximate Equilibrium

Table 7 presents the converged coefficients and some accuracy measures. It shows that the R^2 of the regression is pretty high, although it is not as

[18] A similar approach is used in the literature in a different context (see, for example, Telmer and Zin (2002), Zhang (2005), and Gomes and Michaelides (2006)). This approach is different from the conventional approach, for example, taken by Krusell and Smith (1998). In their economy, the only equilibrium price to be determined is the rental rate of the aggregate capital stock. Krusell and Smith assume that the next-period aggregate capital stock is linearly related to the current-period capital stock conditional on the current-period aggregate productivity state. To the extent that this approximation is accurate, one can immediately find the correct equilibrium interest rate that equals the marginal product of capital. However, in our setup, the relationship between the market-clearing bond price and the wealth distribution is unclear.

high as that found in the literature on the Krusell-Smith economy. As den Haan (2007) points out, the R^2 may not necessarily be the best metric to assess the accuracy.[19] Thus, the table also considers a standard error of the regression and the maximum absolute error. The former statistic indicates that the agents make a forecasting error of 2 basis points on average, while the latter indicates that the maximum error over the 3,000 periods amounts to roughly 7 basis points.

These numbers look small, but the question is whether the agents miss important information that could have helped considerably improve the forecast of the equilibrium price. A simple way to assess this possibility is to examine the relationship between other observable variables and the forecast errors.[20] Specifically, we project the forecast errors on first through third moments of distributions of capital and total wealth. The regression result is shown in Table 8, where all of the coefficients are found to be statistically significant. However, the fit of the regression is poor. In fact, reduction of the standard error of the regression is fairly small (2.00-e04 → 1.91-e04). Similarly, reduction of the maximum absolute error is not very impressive. Based on these results, we conclude that our benchmark solution is "close" enough to the true solution. Furthermore, given that incorporating additional state variables into the solution algorithm is computationally very expensive, this does not seem to be a price worth paying for such a small improvement.[21]

It is, however, somewhat surprising that our simple forecasting rule works so well, given the serious nonlinearities in the model. As we discussed in subsection 4.1, an important remaining question was whether these nonlinear features vary over the business cycle, inducing sizable fluctuations of the wealth distribution. To shed some light on this issue, Figure 4 compares marginal propensities to save across the two aggregate states. It can be seen that the saving rate of the most productive entrepreneur increases in the good state, whereas that of the least productive entrepreneurs is indistinguishable between the two states. These asymmetric changes in marginal propensities to save do indicate that the nonlinearities we discussed in the previous section are an important element of our model. However, the re-

[19]The main reason is that an R^2 scales the error term by the variance of the dependent variable. The scaling of the dependent variable (e.g., level or first difference) can have a large impact on the level of R^2. See den Haan (2007) for more details.

[20]The idea was first proposed by den Haan and Marcet (1994).

[21]We have actually explored various cases where one of the higher-order moments is explicitly incorporated as an additional state variable. The improvement of accuracy measures is not very impressive while slowing down the algorithm considerably.

sults from our accuracy assessment imply that such nonlinearities do not play quantitatively important roles for predicting the market-clearing bond price.

5.3 Cyclical Properties of the Model

We now consider cyclical properties of the model. Our focus is on whether the model with aggregate uncertainty is able to account for the size and cyclical properties of the average private risk premium. While it is very difficult to come up with tight evidence on the average return and its cyclical properties, Moskowitz and Vissing-Jørgensen (2002) provide important suggestive evidence. According to them, an average return of a typical entrepreneur is similar to the return to investing in the index of publicly traded equity.

The first column of Table 9 presents the cyclical properties of our benchmark model. The table shows that the model with aggregate uncertainty generates the average risk premium of 2.5 percent and the risk-free rate of -0.2 percent.[22] The lower panel considers averages conditional on each aggregate productivity state. It shows that the average risk premium is procyclical. Given the properties of our model, this result is intuitive: in the good state, more entrepreneurs are trying to expand, but they are constrained by the borrowing limit, generating higher risk premia. Interestingly, however, the risk-free rate actually is lower in the good state in our economy. This is because while demand for funds rises in the good state, supply of funds also increases in the good state, since the entrepreneurs become richer on average in the boom period.[23]

The key problem of our model is the extremely small volatility of the financial variables. According to the evidence presented by Moskowitz and Vissing-Jørgensen (2002), volatility of the aggregate private equity return is as large as the return from investing in an aggregate index of publicly traded equities. Assuming that this observation is roughly plausible, fluctuations of the aggregate private risk premium in our economy are not even close to their empirical counterpart: the standard deviation of the aggregate private risk premium is only one-tenth of a percent. The volatility of the simulated risk free rate is also very small, relative to a reasonable range.[24]

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[22] Note that the average risk premium and the risk-free rate are somewhat lower than that found in the steady state. This is because aggregate uncertainty induces more precautionary savings.

[23] See the discussion in subsection 4.3.

[24] According to Guvenen and Kuruscu (2006), the size of the (public) equity risk pre-

To see if there is any quick way of fixing this problem, we consider a simple perturbation from the benchmark calibration, namely, a tighter borrowing constraint. In the previous section, we saw that it raises the level of the average risk premium and lowers the risk-free rate. We examine here whether it also has any magnification effects that raise the volatility of the financial market variables. Specifically, we change $\underline{b}$ from -1200 to -1000. With this tighter constraint, aggregate borrowing is 2.1 times as large as the value of aggregate output. The second column of Table 9 displays the results. As we expect from the steady state comparative static, the parameter change raises the average risk premium and lowers the risk free rate. However, it has no noticeable impact on the volatilities of the average risk premium and the risk-free rate (Table 9).

The finding here suggests that the most important problem of the model lies in the lack of an ability to magnify the aggregate shock. Note that in the model, each entrepreneur could experience large changes in the return to his/her risky investment, as is implied by large idiosyncratic risk. However, those variations of returns across entrepreneurs are washed out in the aggregate, thus causing little change in the average risk premium and the risk-free rate.

6 Conclusions and Future Research

This paper has focused on the quantitative characteristics of the model of uninsurable investment risk. The model is calibrated so that it matches key features of income and wealth distributions. We show that the calibrated model has many features that are different from incomplete market models with labor income risk, whose quantitative features have been extensively studied in the literature. In particular, we emphasize that the interactions between decreasing returns to production and the borrowing constraint play crucial roles in generating a sizable private risk premium and a highly skewed wealth distribution.

We have provided an accurate numerical solution to the model subject to aggregate uncertainty, which requires a computationally feasible way of accurately capturing the evolution of the market-clearing bond price. Using the numerical solution, we have examined the model's ability to generate empirically plausible cyclicality of the aggregate private risk premium and

mium is about 6 percent and its standard deviation is about 20 percent over 1890-1991. For the same period, the average risk-free rate is about 2 percent and its standard deviation is about 5.5 percent.

the risk-free rate. We find that an aggregate uncertainty of a reasonable magnitude induces little variation of those variables. We thus conclude that the model needs some mechanism through which the aggregate shock is magnified. There are a number of possible extensions we can consider in future work. For instance, augmenting the model with some "real" frictions, such as capital adjustment costs or irreversibility of investment, appears to be a fruitful avenue. An important challenge would be that such additional features may have large nonlinear variations in the wealth distribution, making it even harder to come up with an accurate numerical solution.

Finally, we would like to acknowledge that this paper abstracts away from a number of other important features. For example, we assumed that the economy consists of entrepreneurs only. This feature is certainly unrealistic, even though this segment of the population holds a large fraction of wealth in the economy. Relaxing this assumption immediately affects the market-clearing bond price, because other segments of the population can hold the safe asset. An important extension, for example, is to augment the model with the "corporate" sector, which issues publicly traded equity. Angeletos (2007) and Covas (2006) consider this extension by suppressing aggregate uncertainty. Adding aggregate uncertainty to this extended environment is challenging, but interesting, as entrepreneurs now have access to publicly traded equity, as well as the safe asset and their own business. Further, we completely ignore the extensive margin, i.e., entry into and exit from entrepreneurship as considered, for example, by Quadrini (2000) and Cagetti and De Nardi (2006). Incorporating the extensive margin into our framework is no doubt important.

7 Appendix

7.1 SCF data

The statistics used to calibrate the model are constructed using the Survey of Consumer Finances (SCF) for 1992, 1995, 1998, 2001, and 2004. "Entrepreneurs" are identified as households that satisfy the following two criteria: respondent or any member of the family living together (i) owns or shares ownership in any privately held businesses, farms, professional practices or partnerships and (ii) has an active management role in any of these businesses. The series used to calibrate the model are constructed as follows.

- Net Wealth = Assets − Debts
- Asset = Financial Assets + Nonfinancial Assets

- Financial Asset = Checking Accounts + Savings Accounts + MMDA + MMMF + Call Accounts + Certificates of Deposit + Savings Bonds + State and Local Tax-Exempt Bonds + Mortgage-Backed Securities + US Government Bonds + Corporate and Foreign Bonds + Stocks + Stock Mutual Funds + Tax-Free Bond Mutual Funds + Other Bond Mutual Funds + Combination and Other Mutual Funds + IRA Accounts + Thrift Plans + Future Pension Benefit Accounts + Cash Value of Life Insurance +Annuities, Trusts and Other Managed Assets + Other Financial Assets

- Nonfinancial Assets = Primary Residence + Other Residential Real Estate + Nonresidential Real Estate + Vehicles +Actively Managed Businesses + Nonactively Managed Businesses + Other Nonfinancial Assets

- Debts = Mortgages (Primary Residence and Other Residential Properties) + Debt Associated with Nonresidential Properties + Unsecured Lines of Credit + Installment Loans (Vehicles, Education, and Other) + Credit Card Debt + Other Debts

- Income = Wages and Salaries + Business Income + Other Interest Income + Dividends + Net Gains or Losses from the Sale of Stocks, Bonds, or Real Estate + Net Rent, Trusts, or Royalties + Unemployment Benefits + Child Support or Alimony + Welfare + Social Security or Other Pensions + Any Other Sources

- Wealth/Income Ratio = (Net Wealth)/Income

- Ratio of Debt to Business Net Wealth = Debt/(Net Wealth of Actively Managed Business)

Note that the model abstracts away from many of the components in the above data construction. Given that there is little guidance regarding how we associate the concepts in the model and the observed data, all balance-sheet and income components are included. To account for outliers in the micro data, we use a Stata module, Winsor, by which we replace the top and bottom 0.5 percent of observations by the next value, counting inward from the extremes. However, this treatment has little impact on the final results.

7.2 Solution Algorithm

In this subsection, we describe the numerical procedures used to compute the equilibrium in our economy with aggregate risk. This can be done in two steps. In the first step, the numerical procedure solves the individual's problem, taking the law of motion for the bond price as given. The forecasting rule of the bond price is to follow a first-order autoregressive process. The state variables of the problem are now $(z, x; \theta, q)$. The laws of motion can be written as follows.

$$q' = \begin{cases} \beta_0(1) + \beta_1(1)q, & \text{if } \theta = \bar{\theta}_1 \text{ and } \theta' = \bar{\theta}_1, \\ \beta_0(2) + \beta_1(2)q, & \text{if } \theta = \bar{\theta}_1 \text{ and } \theta' = \bar{\theta}_2, \\ \beta_0(3) + \beta_1(3)q, & \text{if } \theta = \bar{\theta}_2 \text{ and } \theta' = \bar{\theta}_1, \\ \beta_0(4) + \beta_1(4)q, & \text{if } \theta = \bar{\theta}_2 \text{ and } \theta' = \bar{\theta}_2. \end{cases} \tag{15}$$

Note that the law of motion for the next-period bond price is conditional on *both* the current-period and the next-period aggregate states. This is because the current-period bond price is known at the same time that the aggregate state is realized.

Given these laws of motion, the algorithm solves the entrepreneurial problem by finding a fixed point in the consumption function. The policy function $c(z, x; \theta, q)$ is approximated with a piecewise bilinear interpolant of the state variables, (x, q). The variable x is discretized in nonuniformly spaced grid points with 120 nodes. In particular, there are more grid points to lower values of wealth, because there is more curvature in the consumption function owing to the presence of borrowing constraints. The variable q is discretized in uniformly spaced grid points with 40 nodes. The idiosyncratic productivity process, z, is assumed to follow a Markov chain with seven states. The discretization of the exogenous stochastic process follows the numerical method proposed by Tauchen and Hussey (1991). The aggregate productivity process, θ, is assumed to follow a Markov chain with two states and the discretization is described in the main text.

Given an initial guess, c_0, use expressions (4), (7), and (8) with $\lambda = 0$ to find (c_1, k_1', b_1') at each grid point. After computing the solution at each grid point, check whether the choice of bond holdings violates the short-sales constraint. In cases where the short-sales constraint is violated—that is, $b_1' < \underline{b}$—let $b_1' = \underline{b}$ and use (4) and (8) to determine (c_1, k_1') at those grid points. Use c_1 as the new initial guess and iterate on this procedure until $\sup |\ln c_1 - \ln c_0|$ over all grid points is less than the convergence criterion,

$\epsilon_2 = 1.0e - 6$.

In the second step of the procedure the algorithm updates the coefficients of the laws of motion given by expression (15). It generates a large panel data set with 30,000 entrepreneurs for 3,500 periods, where the first 500 observations are discarded. At each point in time the procedure computes the equilibrium bond price, q_t, such that the aggregate bond holding is zero.

The price of the bond is determined using the bisection algorithm. The procedure first guesses the bond price q by assuming that the equilibrium bond price lies in the interval $[q_l, q_u]$. Given this interval, let the equilibrium bond price equal $\frac{1}{2}[q_l + q_u]$ and solve the entrepreneur's problem. Then, compute the sample average of bond holdings. If it is positive, then set $q_l = q$ and repeat the above steps. Otherwise, set $q_u = q$ and repeat until $|E(b')| < \epsilon_2$. Finally, it updates the coefficients of the aggregate laws of motion by running an ordinary least squares (OLS) regression and check whether the difference over all coefficients is less than the convergence criterion, $\epsilon_1 = 1.0e - 4$.

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Table 1: Parameter values for the benchmark economy

Discount factor	β	0.96
Risk aversion	γ	2
Curvature of production	α	0.75
Depreciation rate	δ	0.08
Serial correlation of productivity risk	ρ_z	0.90
Unconditional standard deviation of productivity risk	σ_z	0.50
Short-sales constraint on bonds	$\underline{b}$	−1200
Discretization of the state space		
Idiosyncratic productivity shock		
Number of states	n_z	7
Discrete states:		
Transition matrix:		
Wealth:	n_x	120
Aggregate productivity shock		
Number of states	n_θ	2
Discrete states:		
Transition matrix:		
Bond price:	n_q	40

Idiosyncratic productivity shock, discrete states:

$$\bar{z} = [0.41; 0.61; 0.76; 0.91; 1.09; 1.36; 2.00]$$

Transition matrix:

$$\Pi_z = \begin{bmatrix} 0.7186 & 0.2223 & 0.0499 & 0.0083 & 0.0009 & 0.0000 & 0.0000 \\ 0.2223 & 0.4099 & 0.2502 & 0.0938 & 0.0215 & 0.0022 & 0.0000 \\ 0.0499 & 0.2502 & 0.3324 & 0.2411 & 0.1040 & 0.0215 & 0.0009 \\ 0.0083 & 0.0938 & 0.2411 & 0.3136 & 0.2411 & 0.0938 & 0.0083 \\ 0.0009 & 0.0215 & 0.1040 & 0.2411 & 0.3324 & 0.2502 & 0.0499 \\ 0.0000 & 0.0022 & 0.0215 & 0.0938 & 0.2502 & 0.4099 & 0.2223 \\ 0.0000 & 0.0000 & 0.0009 & 0.0083 & 0.0499 & 0.2223 & 0.7186 \end{bmatrix}$$

Aggregate productivity shock, discrete states:

$$\bar{\theta} = [0.95; 1.05]$$

Transition matrix:

$$\Pi_\theta = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$$

The function $c(z, x; \theta, q)$ is defined over a continuum of wealth and the current-period bond price. Points outside the grid are found by piecewise bilinear interpolation

Table 2: Wealth and income distributions

	Quintiles					Gini
	0-20%	20-40%	40-60%	60-80%	80-100%	Index
Wealth						
SCF	0.634	0.209	0.102	0.044	0.012	**0.611**
Model	0.611	0.257	0.124	0.038	-0.030	**0.638**
Income						
SCF	0.635	0.214	0.101	0.043	0.008	**0.621**
Model	0.664	0.208	0.088	0.036	0.004	**0.642**

Notes: Statistics from the SCF are the average across five most recent surveys (1992, 1995, 1998, 2001, and 2004). See Appendix for the construction of income and wealth measures from the SCF.

Table 3: Wealth to income and debt to business wealth ratios

	Wealth to income ratio	Debt to business wealth ratio
SCF	8.20	**0.27**
Model	7.14	**0.27**

Notes: Statistics from the SCF are the average across the five most recent surveys (1992, 1995, 1998, 2001, and 2004). See Appendix for the construction of the wealth income ratio and the debt to business wealth ratio.

Table 4: Distribution of private risk premia

	Wealth Quintiles				
Productivity	0-20%	20-40%	40-60%	60-80%	80-100%
Low	0.007	0.004	0.004	0.004	0.004
Medium	0.082	0.047	0.029	0.016	0.006
High	0.171	0.125	0.097	0.075	0.049

Notes: Individual-level expected private risk premia are averaged for each productivity level and wealth quintile using the size of the capital stock as weights.

Table 5: Distribution of wedges associated with the borrowing constraint

	Wealth Quintiles				
Productivity	0-20%	20-40%	40-60%	60-80%	80-100%
Low	0.004	0.000	0.000	0.000	0.000
Medium	0.069	0.037	0.021	0.009	0.000
High	0.153	0.111	0.085	0.066	0.042

Notes: Individual-level Lagrange multipliers normalized by expected marginal utility of consumption $\lambda_i / \mathbb{E}\left[U_c(c'_i)\right]$ are averaged for each productivity level and wealth quintile using the size of the capital stock as weights.

Table 6: Effects of Alternative Parameter Values

	Benchmark	$\sigma_z = 0.6$	$\underline{b} = -1000$
Gini index (Wealth)	0.638	0.722	0.627
Gini index (Income)	0.642	0.750	0.647
Debt to business wealth ratio	0.27	0.26	0.24
Wealth to income ratio	7.14	7.17	7.16
Average private risk premium (%)	2.85	3.92	3.19
Risk-free rate (%)	0.28	−0.61	0.04
Fraction of borrowers	0.742	0.781	0.763
Fraction of the constrained	0.598	0.661	0.647

Table 7: Autoregressive Forecasting Rule

	β_0	β_1
$(\theta', \theta) = (\bar{\theta}_1, \bar{\theta}_1)$	0.0138	0.9860
	(0.0023)	(0.0023)
$(\theta', \theta) = (\bar{\theta}_2, \bar{\theta}_1)$	0.0223	0.9778
	(0.0063)	(0.0063)
$(\theta', \theta) = (\bar{\theta}_1, \bar{\theta}_2)$	0.0089	0.9910
	(0.0062)	(0.0062)
$(\theta', \theta) = (\bar{\theta}_2, \bar{\theta}_2)$	0.0185	0.9817
	(0.0024)	(0.0024)
R^2	0.993196	
s.e.	2.00e-04	
$\max_{\lvert\hat{\epsilon}\rvert}$	7.61e-04	

Notes: See Subsection 5.1 for the specific form of the forecasting rule. The rule is conditional on the current and next period's aggregate productivities θ and θ'.

Table 8: Residual Regression

	resid(q)
$\ln k$	0.004160
	(0.000549)
$\ln w$	-0.010291
	(0.000932)
$\ln \sigma_k$	0.011295
	(0.000746)
$\ln \sigma_w$	-0.004871
	(0.000361)
$\ln \gamma_k$	-0.002409
	(0.000284)
$\ln \gamma_w$	0.000954
	(0.000262)
R^2	0.092355
s.e.	1.91-e04
$\max_{\vert\epsilon\vert}$	6.48-e04

Note: Standard errors are in parenthesis. k and w represent the first moments of capital and wealth distributions, respectively. σ_x denotes the variance of the distribution of a variable x, and γ_x denotes the skewness of the distribution of a variable x.

Table 9: Time Series Properties of Asset Returns

	Benchmark	$\underline{b}_1 = -1000$
$E(R^e - r)$	0.025	0.027
$\sigma(R^e - r)$	0.001	0.001
$E(r)$	-0.002	-0.004
$\sigma(r)$	0.002	0.002
$E(R^e - r \mid \theta = \bar{\theta}_1)$	0.023	0.026
$E(R^e - r \mid \theta = \bar{\theta}_2)$	0.026	0.028
$E(r \mid \theta = \bar{\theta}_1)$	-0.001	-0.004
$E(r \mid \theta = \bar{\theta}_2)$	-0.002	-0.005

Notes: The term $E(R^e - r)$ denotes the value-weighted ex-ante entrepreneurial risk premium where R^e is the marginal product of entrepreneurial capital $\alpha\theta z k^{\alpha-1} - \delta$ and r is equal to $1/q$.

Figure 1: Savings decision rules

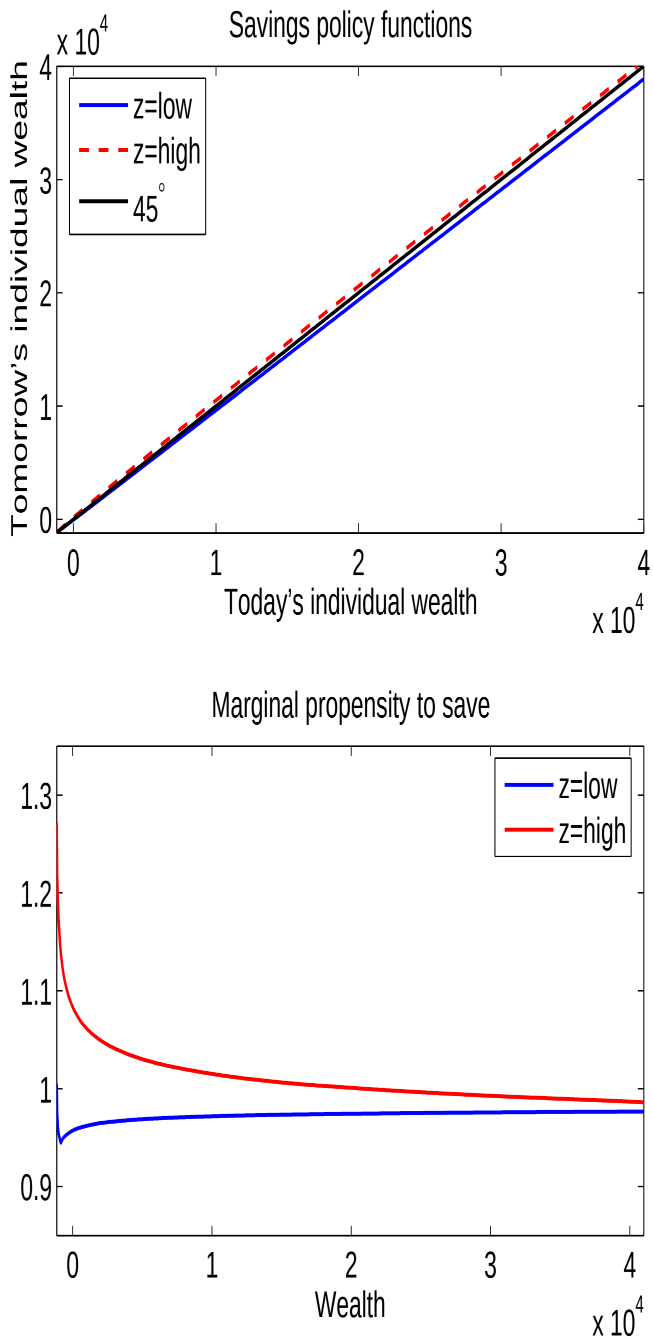


Figure 2: Cross-sectional distribution of wealth

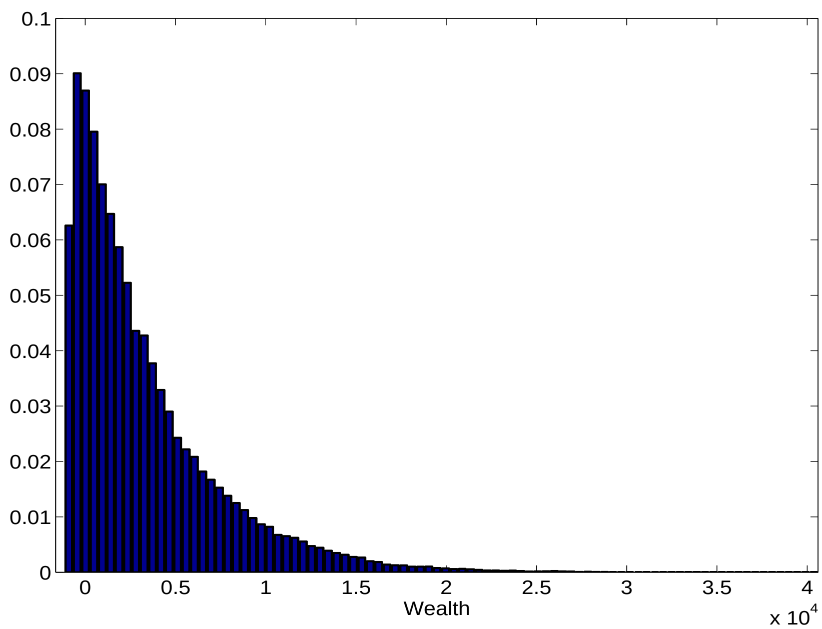
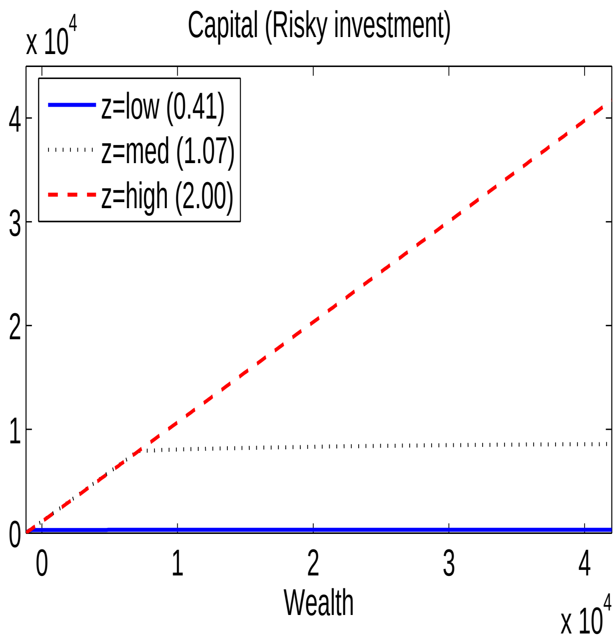


Figure 3: Portfolio choice of entrepreneurs



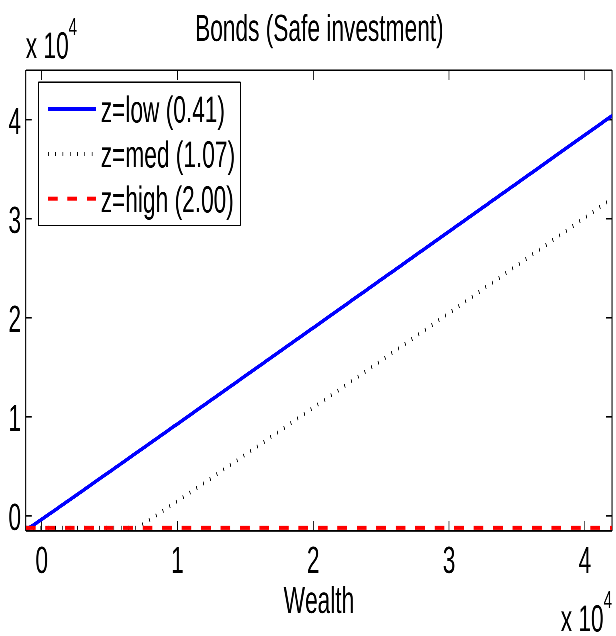


Figure 4: Marginal Propensities to Save across Aggregate States