Equilibrium Technology Diffusion, Trade, and Growth

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ABSTRACT

This paper studies the effects of opening to trade in a model where heterogeneous firms can acquire productivity-increasing technology from other firms in the economy. In the model, firms chose to upgrade their technology or to produce domestically and potentially for export with their existing technology. If firms choose to upgrade their technology, they receive a new technology from the distribution of producing firms' technologies. The distribution of technology is determined in equilibrium via the technology and production choices (both domestically and abroad) of individual firms. We solve for a balanced growth path equilibrium, allowing us to provide a theory of how trade interacts with equilibrium technology diffusion and in turn affects technological progress, growth, and the dynamic gains from trade.

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1. Introduction

This paper studies the effects of opening to trade in a model where technology diffusion is determined in equilibrium. Technology diffusion occurs as heterogeneous firms can acquire productivity-increasing technology from other firms producing in the economy. Firms who are not upgrading their technology produce domestically and potentially for export with their existing technology. The joint upgrading and production choice determines the distribution from which firms can acquire new productivity-increasing technologies and, hence, the rate of technological diffusion and growth. We show how opening to trade changes the relative value of upgrading versus producing, which in turn changes the equilibrium distribution of technology, the rate of technological diffusion, and growth.

We model firms as monopolistic competitors who are heterogeneous in their productivity/technology. The production decision is standard and each firm has the opportunity to export after paying a fixed cost.[1] This determines selection into exporting, with only the most productive firms exporting their goods to other markets. Thus, high productivity firms receive higher profits from the ability to sell their good domestically and abroad.

The technology/upgrading decision is relatively new and innovative. We build on Perla and Tonetti (2012), where firms chose to either upgrade their technology or continue to produce in order to maximize expected discounted profits for the infinite horizon. If a firm decides to upgrade its technology, it pays a fixed cost in return for a random productivity draw from the productivity distribution of producing firms in equilibrium. Thus, the key aggregate state variable for a firm is the distribution of firms producing at any instant. Economic growth is a result, as firms are continually able upgrade their technology by learning from other, better firms in the economy. Thus, this is a model of growth driven by endogenous technology diffusion.[2]

We solve for a balanced growth path equilibrium in this economy. There are essentially two steps to establishing the existence of a balanced growth path equilibrium. First, we characterize the evolution of the technology distribution over time, given the evolution of the firms' dynamic policy rule (i.e. upgrade or not). We show that the technology distribution evolves according to a repeated truncation of the time zero distribution. This result plus the assumption that the initial distribution is Pareto implies that every subsequent distribution of technology is Pareto itself. This allows us to completely characterize the path of the static-trade equilibrium as in Chaney (2008) or Eaton, Kortum, and Kramarz (2011) at every point in time. These results are independent of the balanced growth requirement which will allow us to study off balanced

[1] This setup follows the standard formulation heterogeneous monopolistic-competition frameworks of Melitz (2003), Chaney (2008), and Eaton, Kortum, and Kramarz (2011).

[2] This type of diffusion of ideas is closely related to the models of Lucas (2009) and Lucas and Moll (2012), who study knowledge/idea diffusion amongst individuals in a closed-economy. Kortum (1997) is an antecedent of these models where knowledge diffusion comes from an external source.

growth path dynamics in future versions. The second step solves for the firms control problem to upgrade its technology, given a perceived law of motion for the distribution.

The key equilibrium requirement (amongst others) is that the actual evolution of the technology distribution conforms with firms' perceived law of motion for the distribution of technology. On the balanced growth path we require that the distribution of technologies is stationary when appropriately scaled and that real GDP and aggregate sales grow at a constant rate.

We calibrate the model and perform several comparative dynamics, showing how changes in parameters affect growth rates on the balanced growth path. The main comparison focuses on how changes in iceberg trade costs affect growth rates. Changes in iceberg trade costs are interesting because they control how much each country trades with other countries and hence the degree of openness. We find that decreases in the iceberg trade costs can optimally increase or decrease the growth rate of the economy.

When technology adoption costs must be paid in output, growth rates increase. As the economy becomes more open, the relative value of being a firm with low productivity versus the value of being a firm with high productivity changes. Low productivity firms lose from increased trade, as they face more competition from foreign firms which reduces their profits and the value of the firm. High productivity firms are able to expand and export, increasing their profits and the value of the firm. Additionally, wages increase, especially decreasing the profits of domestic producers. The net effect of these forces is to push more low productivity firms to upgrade their technology sooner as the costs of searching in terms of forgone production are smaller and the potential benefits (i.e., becoming an exporter) are now larger. Because the amount and frequency of firms upgrading their technology is intimately tied to aggregate growth, the growth rate increases as the economy becomes more open.

When technology adoption costs must be paid by hiring labor, a general equilibrium affect dominates and growth rates decrease. Since the wage increases due to increased demand for labor to produce for export, the cost of technology adoption also rises. The increase in search costs dominate the increased convexity of the value function and growth declines as firms wait longer before upgrading their technology. Although the response of growth rates to openness depends on model parameters, reductions in iceberg trade costs always lead to increased welfare.

A second comparative static changes the "thickness" of the right tail of the initial productivity distribution, which is governed by the shape parameter in the Pareto distribution. We show that as the right tail of the initial productivity distribution becomes thicker, the elasticity of growth with respect to the degree of openness increases; the cost of autarky and benefit of frictionless trade (in terms of growth rates) both become larger. This result is distinct from, but related to the findings in other models of knowledge diffusion that growth increases with the thickness of

the right tail of the productivity distribution (see, e.g., Alvarez, Buera, and Lucas (2008), Lucas (2009), Perla and Tonetti (2012), Lucas and Moll (2012)). Moreover, this result suggest ways to use the cross-country evidence on the relationship between trade and growth to discipline this all important parameter.

The third comparative static focuses on scale effects. We keep the parameterization of the model the same, but double the number of countries to understand how the scale or size of the economy matters. We show that the relationship between growth and openness is unchanged when we increase the scale of the economy. Substantively, this result suggest that the key force behind the relationship between growth and openness in our model does not operate through scale effects per-se (i.e. firms upgrade faster because markets and profits are larger). Because scale seems to be absent, this result reinforces the idea that the driving force is how openness changes the relative value of firms across different productivity levels. Given the emphasis on scale effects in endogenous growth models (see, e.g., Jones (2005a) and the discussion in Ramondo, Rodriguez-Clare, and Saborio-Rodriguez (2012)) this result seems surprising. However, scale effects in newer models of knowledge diffusion are not well understood and we hope to explore them further in the future.

2. Model

2.1. Countries, Time, Consumers.

There are N countries with subscripts i denoting the identity of each country. Time is continuous and evolves for the infinite horizon. The representative consumer in country i is risk neutral with period utility function

$$U_i(t) = \int_t^\infty e^{-r(\tau-t)} C_i(\tau) d\tau \tag{1}$$

The utility function $U_i(t)$ is the discounted value of future consumption for the infinite future, where r is the exogenously given discount rate. Consumers supply labor to firms for the production of varieties, the fixed costs of production, and possibly for technology acquisition. Labor is supplied inelastically and the total units of labor in a country are L_i.

There exists a final good producer in each country that aggregates a bundle containing $q_i(v)$ quantity of variety v goods by a constant elasticity of substitution (CES) function. $Q_i(t)$ units of the final good are produced and purchased by the consumer, with total expenditure, $Y_i(t)$, defined by

$$Y_i(t) = P_i(t) Q_i(t), \tag{2}$$

where $P_i(t)$ is the CES aggregate price index. Consumers then divide how the aggregate good is divided between actual consumption and the goods component of "investment costs," which pay for the cost of firms upgrading their technology. Since the consumer supplies labor and owns all firms, income is the sum of total payments to labor and profits which are rebated back to the consumer. We abstract from borrowing or lending decisions, so consumers face the budget constraint

$$\underbrace{Y_i(t)}_{\textit{Expenditures}} = \underbrace{P_i(t)C_i(t)}_{\textit{Consumption}} + \underbrace{P_i(t)x_i(t)S_i(t)}_{\textit{"Investment" Cost}} = \underbrace{w_i(t)L_i + P_i\Pi_i(t)}_{\textit{Income}}. \tag{3}$$

These relationships are elaborated in detail below.

2.2. Firms

In each country there is a unit mass of infinitely lived, monopolistically competitive firms. Each firm alone can supply variety v.

The final good producer in each country is the purchaser of these varieties and solves

$$\max_{q_{ij}(v,t)} \quad P_i(t)Q_i(t) - \sum_{j=1}^{N} \int_{\Omega_{ij}(t)} q_{ij}(v,t)p_{ij}(v,t)dv$$
$$\text{s.t. } Q_i(t) = \left(\sum_{j=1}^{N} \int_{\Omega_{ij}(t)} q(v,t)^{\frac{\sigma-1}{\sigma}} dv \right)^{\frac{\sigma}{\sigma-1}}.$$

The the measure $\Omega_{ij}(t)$ defines the set of varieties consumed in country i from country j. The parameter σ controls the elasticity of substitution across varieties. The solution, $q_{ij}(v,t) = Q_i(t)\left(\frac{p_{ij}(v,t)}{P_i(t)}\right)^{-\sigma}$, provides the demand for a firms variety in each market.

Going forward, we drop the notation carrying around the variety identifier, as it is sufficient to identify each firm with its productivity level, z.

Firms are heterogeneous over their productivity, z, and hire labor, ℓ, to produce quantity q with linear production technology,

$$q = z\,\ell.$$

Productivity across firms, within a country, is described by the cumulative distribution function $F_i(z,t)$. We discuss this distribution in more detail later, however, it is important to recognize that $F_i(z,t)$ are the essential aggregate variables affecting firm decisions and they will be evolving over time endogenously as firms choose to upgrade their technology.

Each instant, firms face fixed costs to undertake two activities. First, all firms have the ability to pay a cost and search for a new technology, z. If they search, they will meet a firm that is actively producing, and thus the draw is from a transformation of the equilibrium distribution $F_i(z,t)$. In equilibrium, the optimal firm policy will be a threshold productivity, $h_i(t)$, such that all firms with productivity below $h_i(t)$ will search and will randomly meet a firm with productivity above $h_i(t)$. This meeting structure represents limited directed search towards more productive firms. Second, all firms have the ability to pay a fixed cost of hiring labor in market j, $w_j\kappa_j$, to export to foreign market j.

Finally, all firms face iceberg trade costs, $d_{ji} \geq 1$, to ship goods abroad from i to destination j.

Given this environment, firms must make several choices. These choices can be separated into problems that are static and dynamic. Below we first outline the dynamic problem of the firm, taking the profit functions and evolution of the productivity distribution as given, and then the static problem of the firm to derive the profit functions.

2.2.A. Firms Dynamic Problem

Given the static profit functions and a perceived law of motion for the productivity distribution which are described below, each firm has the choice to acquire a new technology, z, and also whether to export to market j or not. If a firm chooses to search and upgrade its technology, it will not produce any output in that instant, it will pay a search cost, and it will meet another firm that has chosen to produce and copy their productivity level. In other words, a firm is able to replicate (at a cost) the technology of another producer that is currently operating. Thus, the new productivity level is a random variable that's distribution is the equilibrium distribution of technology, conditional on the productivity being above the search threshold. This setup is very closely related to Perla and Tonetti (2012) and is similar to Lucas and Moll (2012).[3] The essential trade-off that a firm faces is between the benefits of operating its existing technology versus the expected net benefit of operating with a new technology. The firm's objective is to maximize the present discounted expected value of real profits, since it is owned by the consumers. With all profits ($\pi_{ji}(z,\tau)$) and costs ($x_i(\tau)$) in units of the aggregate consumption bundle when they are received or paid, $C_i(\tau)$, the firm problem is

$$V_i(z,t) = \max_{\substack{T_{ji}\geq t \\ T_i \geq t}} \left\{ \int_t^{T_i} e^{-r(\tau-t)}\pi_{ii}(z,\tau)d\tau + \sum_{j\neq i} \int_t^{T_{ji}} e^{-r(\tau-t)}\pi_{ji}(z,\tau)d\tau + e^{-r(T_i-t)}\left[W_i(T_i) - x_i(T_i)\right] \right\} \tag{4}$$

[3]This is also similar to Alvarez, Buera, and Lucas (2008) (with an internal source), with the important difference that we have a control problem that creates a recursion between the search decision of the firms and the endogenous evolution of the productivity distribution.

where

$$W_i(t) := \int V(\tilde{z}, t) dF_i(\tilde{z}, t | \tilde{z} > h(t)) \tag{5}$$

A firm chooses an absolute time, T_i, at which it will search for a new technology. For the waiting time before searching, $(T_i - t)$, the firm produces and earns profits from operating domestically. The firm also chooses an absolute time at which it will stop exporting to country j, T_{ji}. While $T_{ji} - t > 0$ the firm is an exporter to destination j and receives profits from this activity. Given the fixed cost of exporting, every exporter will also operate domestically, i.e., $T_{ji} \leq T$ and search occurs at time T_i when a firm is only operating domestically. When a firm chooses to search, it gets a new productivity draw with expected benefit $W_i(T_i)$ and it pays the fixed cost of searching $x_i(T_i)$.[4] By standard arguments, the solution to this problem can be shown to be reservation productivity functions, $h_i(t)$ and $\phi_{ji}(t)$. All firms with productivity less than or equal to $h_i(t)$ will search and all other firms will produce. All firms with productivity greater than or equal to $\phi_{ji}(t)$ will export to destination j and all firms with lower productivity will not export to j. Define the search and exporter thresholds as these indifference points:

$$h_i(t) := \max\{\ z \mid T_i(z, t) = t\ \} \tag{6}$$

$$\phi_{ji}(t) := \max\{\ z \mid T_{ji}(z, t) = t\ \} \tag{7}$$

The function $h_i(t)$ maps time into the largest productivity level such that the firm with that productivity level is upgrading its technology. Given this definition, the function $h_i^{-1}(z)$ defines the *time* at which a firm with productivity level z will draw a new technology. Then, since a draw comes from the equilibrium distribution of *producers*, the expected value of the new technology level, $W_i(T_i)$, is defined in (5). Notice that the value of the new technology is integrated with respect to the conditional productivity distribution $F_i(z, t | z > h_i(t))$ and hence is a function of the choices of the individual firms. We detail the evolution of this distribution—in equilibrium—in more detail below. This problem takes the profit functions as given, but they are the result of a static optimization problem.

2.2.B. Firms Static Problem

Below we describe a firm's static problem and suppress any explicit dependence upon time to ease notation. Given a firm's location, productivity level, aggregate prices, and final good producers' demand, the firm's static decision is to chose the amount of labor to hire, the price to set, and exporting decisions to each destination to maximize profits each instant. Total firm profits

[4] The effects of particular specifications of the search cost are detailed in Section 3.5.A. In particular, we examine the importance of the degree to which costs require hiring labor versus spending goods.

equal the sum of domestic profits plus the sum of exporting profits across export markets, i.e., $\pi(z) = \pi_{ii}(z) + \sum_{j \neq i} \pi_{ji}(z)$. Formally, the optimization problem is

$$P_i \pi_{ii}(z) = \max_{p_{ii}, \ell_{ii}} \; p_{ii}(z) z \ell_{ii} - w_i \ell_{ii}. \tag{8}$$

Using the demand function from the standard solution to the consumer's problem, the profit function satisfying (8) is

$$P_i \pi_{ii}(z) = \left(\frac{1}{\sigma}\right) \left(\frac{m \, w_i}{z}\right)^{1-\sigma} \frac{Y_i}{P_i^{1-\sigma}}, \quad \text{where} \quad m := \frac{\sigma}{\sigma - 1}. \tag{9}$$

where m is the standard markup over marginal cost, w_i is the wage rate in country i, and P_i is the aggregate price index in country i. The decision to export to market j is similar, but differs in that the firm faces variable iceberg trade costs and a fixed cost to sell in the foreign market, or

$$P_i \pi_{ji}(z) = \max_{p_{ji}, \ell_{ji}} \left\{ \, p_{ji}(z) d_{ji}^{-1} z \ell_{ji} - w_i \ell_{ji} - w_j \kappa_j \, , 0 \right\}. \tag{10}$$

Conditional on exporting, this implies that the profits from exporting to market j are

$$P_i \pi_{ji}(z) = \frac{1}{\sigma} \left(\frac{m \, d_{ji} \, w_i}{z}\right)^{1-\sigma} \frac{Y_j}{P_j^{1-\sigma}} - w_j \kappa_j \tag{11}$$

Given net profits described in (11), the productivity level ϕ_{ji}, which determines the cutoff productivity level above which firms from market i will export to market j, is

$$\phi_{ji} = k_1 \frac{w_i}{P_j} \left(\frac{w_j \kappa_j}{Y_j}\right)^{\frac{1}{\sigma-1}}, \quad \text{where} \quad k_1 := m \, d_{ji} \sigma^{\frac{1}{\sigma-1}}. \tag{12}$$

All firms in market i with productivity level greater than or equal to ϕ_{ji} will export to market j, earning positive profits. Recall that the exporter threshold, ϕ_{ji}, is directly related to T_{ji} in equation 7 of the dynamic problem.

3. Equilibrium

An equilibrium of the model economy consists of a set of initial productivity distributions and sequences of productivity distributions, firms' search and exporter thresholds, prices, and allocations, that solve firms' static and dynamic problems and satisfy market clearing and rational expectation conditions. Below, we describe key equilibrium relationships, which can be separated into static and dynamic equilibrium relationships. We then formally define a balanced

growth path equilibrium and state Proposition 2 which says that one exists and is proved by construction.

3.1. Dynamic Equilibrium Relationships

Now we will derive the law of motion for the distribution, which is a function of the amount of searchers and the evolution of the firm's dynamic control $h_i(t)$. The description in this section holds for regions of continuity in $h_i(t)$. $h_i(t)$ may not be continuous, particularly at "special times" that reset the economy like time 0 or potentially when a closed economy unexpectedly opens to foreign trade. Technical details surrounding discontinuities in $h(t)$ and more detailed derivations are provided in the appendix.

Derivation of Law of Motion and Searchers. Recall that $h_i(t)$ is defined as the reservation value below which agents search. As a tie-breaking rule, it is assumed that agents at the threshold search, and hence the function is right-continuous.[5]

The search technology of the environment is that searching firms only match with producing firms, as in Perla and Tonetti (2012). Therefore, agents searching at time t only meet agents strictly above $h_i(t)$. When thinking of the evolution of the distribution, $h(t)$ is an absorbing barrier sweeping through the distribution from below, and the infinimum of support of the productivity density is

$$\inf \text{support}\{F_i(\cdot, t)\} = h_i(t). \tag{13}$$

Thus, the distribution from which searching agents receive a draw is the existing distribution

$$f_i(z, t | z > h_i(t)) = f_i(z, t). \tag{14}$$

Law of Motion: Kolmogorov Forward Equation. A key determinant of the growth rate of the economy and of the evolution of the productivity distribution is the flow of searchers upgrading their technology, $S(t)$. There exists a flow of searchers during each infinitesimal time period, where the flow of searchers is the net flow of the probability current through the search threshold, $h(t)$. As is derived in the appendix,

$$S_i(t) = h_i'(t) f_i(h_i(t), t). \tag{15}$$

While $h(t)$ is an absorbing barrier removing mass from the system, the flow of searchers are a source that are redistributed back into the system. These agents who search have an equal

[5]This technical assumption does not alter any economic forces.

probability to draw any z in $f(z,t)$, as stated in equation 14. Hence, since the only time a firm's productivity changes is when it searches, the Kolmogorov forward equation (KFE) for $z > h_i(t)$ is simply the flow of searchers (source) times the density they draw from (redistribution density):

$$\frac{\partial f_i(z,t)}{\partial t} = S_i(t) f_i(z,t) \tag{16}$$

Using equation 15

$$\frac{\partial f_i(z,t)}{\partial t} = f_i(z,t) f_i(h_i(t),t) h_i'(t). \tag{17}$$

In words, this says that the search threshold is sweeping across the density at rate $h_i'(t)$ and as the search boundary sweeps across the density from below it collects $f_i(h_i(t),t)$ amount firms. Then $f_i(h_i(t),t)h_i'(t)$ is the flow of searchers to be distributed back into the distribution. Since the economic environment is such that searchers only meet existing producers above $h_i(t)$, but $h_i(t)$ is the infinimum of support of $f_i(z,t)$, then the searchers are redistributed across the entire support of $f_i(z,t)$. Since agents draw directly from the productivity density, they are redistributed throughout the distribution in proportion to the density and thus, the flow of searchers $f_i(h_i(t),t)h_i'(t)$ multiplies the density $f_i(z,t)$.

Solving the KFE.

Proposition 1. *$f_i(z,t)$ evolves according to repeated left truncations at $h_i(t)$ for **any** $h_i(t)$ and $F_i(0)$.*

A solution to the Kolmogorov forward equation 17 is

$$f_i(z,t) = \frac{f_i(z,0)}{1 - F_i(h_i(t),0)}. \tag{18}$$

That is, the distribution at date t is a truncation of the initial distribution at the minimum of support at time t, $h_i(t)$.

Solving the Firm Dynamic Problem. Solving (4) consists of jointly finding the optimal search policy function, $h_i(t)$ and the expected value of search, $W_i(t)$, given profit functions, a productivity distribution, $F_i(z,t)$, and it's law of motion. Below, we describe the general steps to finding this solution.

Recall that the equilibrium search threshold $h_i(t)$ is the minimum of the productivity distribution. Given parameter constraints (particularly a positive fixed cost of exporting) the exporter

productivity threshold is greater than the technology adoption search threshold. Thus, only non-exporters optimally choose to search, and the first order condition that determines the optimal search time is the derivative of the value function with respect to the search timing decision, where the discounted stream of export profits earned before searching is 0 with certainty:

$$V_i(z,t)_{|(\pi_{ji}=0)} = \max_{T_i \geq t} \left\{ \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau + e^{-r(T_i-t)} \left[W_i(T_i) - x_i(T_i) \right] \right\},$$

Taking the derivative of the value function of a non-exporting firm with respect to T_i yields

$$\frac{\partial V_i(z,t)_{|(\pi_{ji}=0)}}{\partial T_i} = \left[\frac{\partial \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau}{\partial T_i} + \frac{e^{-r(T_i-t)} \partial W_i(T_i)}{\partial T_i} - \frac{\partial e^{-r(T_i-t)} x_i(T_i)}{\partial T_i} \right] \tag{19}$$

$$= e^{-r(T_i-t)} \left[\pi_{ii}(z,T_i) - rW_i(T_i) + W_i^{'}(T_i) + rx_i(T_i) - x_i^{'}(T_i) \right] \tag{20}$$

Setting $T_i = t$, i.e., where the firm is just indifferent between switching technologies and producing, and recognizing that the productivity level of the indifferent firm is $z = h_i(t)$ by definition, we have the first order condition

$$0 = \pi_{ii}(h_i(t),t) - rW_i(t) + W_i^{'}(t) + rx_i(t) - x_i^{'}(t)$$

$$r(W_i(t) - x_i(t)) = \pi_{ii}(h_i(t),t) + W_i^{'}(t) - x_i^{'}(t) \tag{21}$$

To provide intuition, this FOC is analogous to the standard bellman equation in asset pricing, $rV(t) = \pi(t) + \frac{dV(t)}{dt}$, where the flow (net) value of an asset must equal its dividend plus capital gains. Since in our problem, this is the equity value of a firm, there is a natural arbitrage free pricing interpretation. If the LHS was larger than the RHS, then the current value of the firm would be larger than its dividend and resale value warrants, and an agent could make money by shorting the firm this instant and buying it an instant later.

Equation 21 is one equation in $h_i(t)$ and $W_i(t)$. We now want to find another equation in $h_i(t)$ and $W_i(t)$, providing two equations in two unknowns.

The second equation we focus on is the expected value of acquiring a new technology.

Since $h_i(t)$ is the minimum of support of $F_i(z,t)$ as stated in equation 13, we can rewrite equa-

tion 5 as

$$
\begin{aligned}
W_i(t) &= \int V_i(z,t) dF_i(z,t) \\
&= \int_{h_i(t)}^{\infty} \Bigg\{ \int_t^{h_i^{-1}(z)} e^{-r(\tau-t))} \pi_{ii}(z,\tau) d\tau + \sum_{j \neq i} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} \pi_{ji}(z,\tau) d\tau \\
&+ e^{-r(h_i^{-1}(z)-t)} \left[W_i(h_i^{-1}(z)) - x_i(h_i^{-1}(z)) \right] \Bigg\} dF_i(z,t) \qquad (22)
\end{aligned}
$$

The first integral in the inside bracket is the discounted value of domestic profits until the next change of technology, where the search time T_i has been replaced with the function $h_i^{-1}(z)$. The second integral in the inside bracket is the discounted value of profits from exporting. Similarly, the final exporting time, T_{ji}, has been replaced with the function $\phi_{ji}^{-1}(z)$, which is defined in (7). The function $\phi_{ji}(z)$ is the largest z such that a firm *stops* exporting to market j. Thus the inverse of this function defines the time when the firm stops exporting to market j. The final term in the inside bracket is the discounted value of the new technology net of search costs evaluated at the date $h_i^{-1}(z)$.

Outside the brackets, we then integrate over productivity levels with the existing (equilibrium) productivity distribution of producers, $F_i(z,t)$, since that is the distribution from which firms draw. This defines the expected value of acquiring a new technology.

Equations (21) and (22) give us two equations from which we can solve for the policy function, $h_i(t)$, and the expected value of a new productivity draw, $W_i(t)$, for a given a law of motion for the productivity distribution, $F_i(z,t)$.

3.2. The Pareto Distribution

The shape of the productivity distribution plays an important role, affecting both the dynamic technology acquisition decision of the firm and the firm's static production and export decisions. The parametric form of the initial productivity distributions across countries is an essential initial condition specified by the researcher. The Pareto distribution has a history in the growth (Kortum (1997); Jones (2005b); Perla and Tonetti (2012)), trade (Melitz (2003); Chaney (2008)), and industrial organization (Gabaix (2009)) literature as being both empirically motivated and particularly tractable. To maintain analytical tractability in the static firm problem and to allow for a balanced growth path, we will solve for the equilibrium of our baseline model under the assumption that the initial distributions are all Pareto with the same tail index.

Assumption 1. *The initial distributions of productivity are Pareto,* $F_i(z,0) = 1 - \left(\frac{h_i(0)}{z}\right)^{\theta} \quad \forall i$, *with densities,* $f_i(z,0) = \theta h_i(0)^{\theta} z^{-1-\theta}$.

Lemma 1. *Assumption 1 together with Proposition 1 implies*

$$f_i(z,t) = \theta h_i(t)^\theta z^{-1-\theta} \tag{23}$$

That is, if $F_i(z,0)$ is Pareto with tail index θ and minimum of support $h_i(0)$, then $F_i(z,t)$ remains Pareto with the same tail index θ and new minimum of support $h_i(t)$. This greatly simplifies the derivation of static equilibrium relationships and solving for the model along a balanced growth path. This assumption implies that the

3.3. Static Equilibrium Relationships

At every date t, there are essentially three aggregate equilibrium objects that determine the static allocation problem of production and trade across countries. These are the aggregate price index, trade shares, and aggregate sales. Below we describe each of these objects as an explicit function of time, with detailed derivations provided in the appendix.

Price Index. From standard CES arguments, the price index in market i is

$$P_i(t)^{1-\sigma} = \int_{h_i(t)}^{\infty} p_{ii}(z,t)^{1-\sigma} f_i(z,t)dz + \sum_{j\neq i} \int_{\phi_{ij}(t)}^{\infty} p_{ij}(z,t)^{1-\sigma} f_j(z,t)dz.$$

Given the assumption that initial productivity distributions are Pareto, the CES price index is

$$P_i(t)^{1-\sigma} = k_2(mw_i(t))^{1-\sigma} h_i(t)^{\sigma-1} + \sum_{j\neq i} k_3(md_{ji}w_j(t))^{-\theta} h_j(t)^\theta P_i(t)^{\theta+1-\sigma} \left(\frac{w_i(t)\kappa_i}{Y_i(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{24}$$

where $k_2 := \frac{\theta}{\theta+1-\sigma}$ and $k_3 := k_2\sigma^{\frac{\sigma-1-\theta}{\sigma-1}}$.

Note that this expression is similar—but not the same—as standard results for the CES price index in monopolistic competition models (e.g. Chaney (2008) or Eaton, Kortum, and Kramarz (2011)). The key differences regard the term for the home country effect on the left-hand side of (24). This term is not multiplied by the price index $P_i^{\theta+1-\sigma}$ as the foreign country terms are. The power term is $1-\sigma$ not θ as in the foreign country terms. Finally, the constant multiplying each of these terms are different as well (k_2 vs. k_3). The reason for this difference is that there is not a fixed cost of operating domestically as models such as Chaney (2008) or Eaton, Kortum, and Kramarz (2011) have.

Trade shares. Trade shares, λ_{ji}, equal the expenditure country j spends on goods from country

i relative to total expenditure in country j. Mathematically, the trade share is given by

$$\lambda_{ji}(t) = \int_{\phi_{ji}(t)}^{\infty} \frac{p_{ji}(z,t)q_{ji}(z,t)}{Y_j(t)} f_i(z,t)dz$$

Given the distributional assumptions, the optimal price and quantity rules for firms of productivity level z, and the price index in equation 24, the trade share is

$$\lambda_{ji}(t) = \frac{k_3(md_{ij}w_i(t))^{-\theta}h_i(t)^{\theta}P_j(t)^{\theta+1-\sigma}\left(\frac{w_j(t)\kappa_j}{Y_j(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2(mw_j(t))^{1-\sigma}h_j(t)^{\sigma-1} + \sum_{n\neq j} k_3(md_{nj}w_n(t))^{-\theta}h_n(t)^{\theta}P_j(t)^{\theta+1-\sigma}\left(\frac{w_n(t)\kappa_n}{Y_n(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \quad (25)$$

Note again, that this expression is similar—but not the same—as standard results for trade shares in monopolistic competition models. The key differences are the same issues arising in the price index discussed above.

From equation 25 we can derive a simple expression for the home trade share, $\lambda_{ii}(t)$, in terms of the real wage and technology parameters

$$\lambda_{ii}(t) = k_2 m^{1-\sigma}\left(\frac{w_i(t)}{P_i(t)}\right)^{1-\sigma} h_i(t)^{\sigma-1} \quad (26)$$

By inverting this expression, one can relate the wage to the home trade share in a way that is similar to the expression for the welfare gains from trade as discussed in Arkolakis, Costinot, and Rodriguez-Clare (2011), with a difference. The key difference is that the elasticity of the real wage with respect to the home trade share is not dictated by the shape parameter in the productivity distribution, θ, but by the preference parameter, σ.

Given the trade share formula, we want to express the profit functions in equations 9 and 11 in a more convenient format. Noting that domestic profits are a function of the real wage and the real wage's relationship to trade shares in (26), we have

$$P_i(t)\pi_{ii}(z,t) = k_4\lambda_{ii}(t)\left(\frac{z}{h_i(t)}\right)^{\sigma-1} Y_i(t), \quad \text{where} \quad k_4 = \frac{1}{k_2\sigma} \quad (27)$$

A similar formula for profits from a firm in market i exporting to market j is

$$P_i(t)\pi_{ji}(z,t) = k_4\lambda_{jj}(t)d_{ij}^{1-\sigma}\left(\frac{z}{h_j(t)}\right)^{\sigma-1}\left(\frac{w_i(t)}{w_j(t)}\right)^{1-\sigma} Y_j(t) - w_j(t)\kappa_j \quad (28)$$

Aggregate Sales. Total sales to country i, $Y_i(t)$, can be expressed as

$$Y_i(t) = w_i(t)L_i + \int_{h_i(t)}^{\infty} P_i(t)\pi_{ii}(z,t)f_i(z,t)dz + \sum_{j\neq i}\int_{\phi_{ji}(t)}^{\infty} P_i(t)\pi_{ji}(z,t)f_i(z,t)dz.$$

This simply says that total sales must equal all income earned from labor plus profits earned by firms and rebated to consumers. Substituting 26 into the profit functions and then integrating over productivity we have

$$\begin{aligned} Y_i(t) =& w_i(t)L_i + k_3k_4\lambda_{ii}(t)Y_i(t) \\ &+ \sum_{j\neq i}\left\{k_3k_4\lambda_{jj}(t)d_{ij}^{1-\sigma}\left(\frac{w_i(t)}{w_j(t)}\right)^{1-\sigma} Y_j(t)\left(\frac{\phi_{ji}(t)}{h_i(t)}\right)^{\sigma-1-\theta} - w_j(t)\kappa_j\left(\frac{\phi_{ji}(t)}{h_i(t)}\right)^{-\theta}\right\}. \end{aligned} \tag{29}$$

3.4. Market Clearing

Before constructing the market clearing conditions, we must specify a functional form for the search cost of upgrading technology, $x_i(t)$. The cost to draw a new productivity level is a convex combination of hiring domestic labor and spending final goods, given by $x(t) = \zeta\left[(1-\eta)\frac{w(t)}{P(t)} + \eta\mathbb{E}_t[z_i]\right]$. $\eta \in [0,1]$ controls the degree to which the cost of search requires labor as opposed to goods, while ζ affects the overall cost of upgrading technology. $\frac{w(t)}{P(t)}$ is the real cost of hiring a unit of labor, while $\mathbb{E}_t[z_i]$ is the amount of goods required to search. As is standard, the search cost in goods must grow with the economy or become irrelevant over time.[6]

Goods Market Clearing. Final goods are spent on either consumption or paying technology adoption costs. Since $\zeta\eta\mathbb{E}_t[z_i]$ final goods are spent per search and there is a flow of $S_i(t)$ searchers each instant, the goods market clearing condition is

$$\frac{Y_i(t)}{P_i(t)} = Q_i(t) = C_i(t) + \zeta\eta\mathbb{E}_t[z_i]S_i(t)$$

Labor Market Clearing. Wages, $w_i(t)$, are determined by the labor market clearing conditions. Aggregating the labor in market i used for domestic production, $L_{i,d}$, and for export production,

[6]This could alternatively be achieved by indexing the cost to the minimum productivity in the economy instead of the average or by making the cost a fixed fraction of output. Average productivity was chosen as the goods cost since it more closely corresponds with the benefit of searching, the expected value of a new productivity.

$L_{i,ex}$, yields

$$L_{i,d} = \int_{h_i(t)}^{\infty} \frac{p_{ii}(z,t)^{-\sigma} Y_i(t)}{z P_i(t)^{1-\sigma}} f_i(z,t) \quad \text{and} \quad L_{i,ex} = \sum_{j \neq i} \int_{\phi_{ji}(t)}^{\infty} \frac{d_{ji} p_{ji}(z,t)^{-\sigma} Y_j(t)}{z P_j(t)^{1-\sigma}} f_i(z,t). \tag{30}$$

Since the fixed cost of exporting from j to i requires units of i labor, the total amount of labor from i used in the production of fixed costs equals

$$L_{i,\kappa} = \sum_{j \neq i} \int_{\phi_{ij}(t)}^{\infty} \kappa_i f_j(z,t) dz = \kappa_i \sum_{j \neq i} \left(\frac{h_j(t)}{\phi_{ij}(t)} \right)^{\theta}. \tag{31}$$

Since the technology upgrade search cost is partially paid to hire labor, the search component of labor demand is

$$L_{i,x} = \zeta(1-\eta) S_i(t) \tag{32}$$

Equating aggregate labor supply, L_i, with aggregate labor demand yields the labor market clearing condition:

$$\begin{aligned} L_i &= L_{i,d} + L_{i,ex} + L_{i,\kappa} + L_{i,x} \\ L_i &= \int_{h_i(t)}^{\infty} \frac{p_{ii}(z,t)^{-\sigma} Y_i(t)}{z P_i(t)^{1-\sigma}} f_i(z,t) + \sum_{j \neq i} \int_{\phi_{ji}(t)}^{\infty} \frac{d_{ji} p_{ji}(z,t)^{-\sigma} Y_j(t)}{z P_j(t)^{1-\sigma}} f_i(z,t) + \kappa_i \sum_{j \neq i} \left(\frac{h_j(t)}{\phi_{ij}(t)} \right)^{\theta} + \zeta(1-\eta) S_i(t). \end{aligned} \tag{33}$$

3.5. A Balanced Growth Path Equilibrium

Definition 1. *A balanced growth path (BGP) equilibrium is a set of initial distributions $F_i(0)$ with support $[zmin_i, \infty)$, search and exporter thresholds $\{h_i(t), \phi_{ji}(t)\}_{t=0}^{\infty}$, firm price and labor policies $\{p_{ji}(z,t), \ell_{ji}(z,t)\}_{t=0}^{\infty}$, wages $\{w_i(t)\}_{t=0}^{\infty}$, aggregate trade shares and price indexes $\{\lambda_i(t), P_i(t)\}_{t=0}^{\infty}$, and a growth rate $g > 0$ such that for all countries i:*

- *Given aggregate prices and distributions*
  - *$h_i(t)$ is the optimal search threshold,*
  - *$\phi_{ji}(t)$ is the optimal export threshold to market j,*
  - *$p_{ji}(z,t)$ and $\ell_{ji}(z,t)$ solve the static optimization problem,*
- *Markets clear at each date t.*
- *Sales grow at a constant rate $Y(t) = Y_0 e^{gt}$,*

- *The distribution of productivities is stationary when re-scaled:*

$$f_i(z,t) = e^{-gt} f_i(ze^{-gt}, 0) \; \forall \, t, \; z \geq \; zmin_i e^{gt}$$

The initial distribution must have infinite right tailed support or the economy would not be able to grow indefinitely. Requiring sales to grow at a constant rate, the productivity distributions to be constant after rescaling, and trade shares and prices to be constant ensures that the BGP equilibrium features balanced growth. Restricting $g > 0$ ensures that the BGP equilibrium has growth.

3.5.A. Solving for a BGP

We will now prove existence of a BGP equilibrium by construction in a particular environment.

Assumption 2. *There are N symmetric countries.*

Assumption 3. *Preferences are such that $\sigma = 2$.*

Assumption 4. $L = 1$

We will guess that along the BGP, the search threshold, wages, and the value of search also grow at constant rate g. These guesses will be verified as part of the solution methodology.

Guess 1. The optimal search threshold grows at the same rate g as total sales: $h(t) = h_0 e^{gt}$.

Guess 2. Wages grow at the same constant rate g as total sales: $w(t) = w_0 e^{gt}$.

Guess 3. The optimal value of search grows at the same rate g as total sales: $W(t) = W_0 e^{gt}$.

Here we solve for the special case of N symmetric countries with CES substitution parameter $\sigma = 2$. Imposing the balanced growth path guess that h evolves according to $h(t) = e^{gt} h_0$, w evolves according to $w(t) = e^{gt} w_0$, the BGP restriction that total sales evolves with $Y(t) = e^{gt} Y_0$, and dropping the notation identifying the country, simplifies the profit functions from equations 27 and 28 to

$$P(t)\pi_d(z,t) = k_4 \lambda(t) \left(\frac{z}{h_0} \right) Y_0 \tag{34}$$

$$P(t)\pi_{ex}(z,t) = \pi_d(z,t) d^{-1} - e^{gt} w_0 \kappa. \tag{35}$$

with the explicit dependence on time now noted. Here the country identifier notation is changed such that $\pi_d(z,t)$ denotes the domestic profits of a firm and $\pi_{ex}(z,t)$ denotes the exporting profits to a single country.

Before proceeding we should note that the trade share, $\lambda(t)$, and price index, $P(t)$, potentially vary with time. We proceed to verify that they do not.

$\lambda(t)$ and $P(t)$ are constant if $h(t)$ and $w(t)$ grow at the same rate as sales, as stated in guess 1 and guess 2. Imposing symmetry, the trade share and associated price index are

$$P(t)^{1-\sigma} = k_2(mw(t))^{1-\sigma}h(t)^{\sigma-1} + (N-1)k_3(mdw(t))^{-\theta}h(t)^{\theta}P(t)^{\theta+1-\sigma}\left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{36}$$

$$\lambda(t) = \frac{k_3(mdw(t))^{-\theta}h(t)^{\theta}P(t)^{\theta+1-\sigma}\left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2(mw(t))^{1-\sigma}h(t)^{\sigma-1} + (N-1)k_3(mdw(t))^{-\theta}h(t)^{\theta}P(t)^{\theta+1-\sigma}\left(\frac{w(t)\kappa}{Y(t)}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \tag{37}$$

$$\tag{38}$$

Careful examination of equation (36) shows that only the ratio of wages, $w(t)$, to the search threshold, $h(t)$, affects the evolution of $P(t)$. Thus, if $h(t)$ and $w(t)$ both grow at constant rate g as stated in guess 1 and guess 2, the aggregate price index must be constant.

Next, careful examination of equation (37) shows that only the ratio of wages, $w(t)$, to either the minimum of support of the distribution, $h(t)$, or sales, $Y(t)$, affects the evolution of $\lambda(t)$. Thus, the trade share must be constant on a balanced growth path.

From equation 35, the cutoff value for exporting is

$$\phi(t) = \chi h(t) \quad \text{where} \quad \chi := \frac{d\kappa w_0}{k_4 Y_0 \lambda} \tag{39}$$

Sales are then

$$Y(t) = w(t)k_5 \quad \text{where} \quad k_5 = \frac{1-\kappa\chi^{-\theta}}{1-\left[k_3k_4\lambda\left(1+(N-1)d^{-1}\chi^{1-\theta}\right)\right]}. \tag{40}$$

Thus, guess 2, that wages are growing at the same rate as sales, is verified.

Summarizing, we have shown that given assumptions and guesses,

$$\lambda(t) = \lambda(0)\ \forall\, t, \quad P(t) = P(0)\ \forall\, t, \quad \text{and} \quad Y(t) = k_5 w_0 e^{gt}. \tag{41}$$

We have characterized completely how the profit functions are growing over time. The next step is to use this information in the firms dynamic problem and verify that the balanced growth path guesses solve the firm problem and satisfy the BGP equilibrium requirements.

The broad outline for the proof is to verify the economy is on a balanced growth path. A key aspect of this is to verify guess 3 that $W(t)$, the expected value of search, is growing at a constant rate, i.e., $W(t) = e^{gt}W_0$. To accomplish this, we will plug in our guesses above into the formula

for the value of search (equation 22) and verification comes if we can solve for a growth rate g and initial value of search W_0 that are independent of time.

Proposition 2. *Given assumptions 1-4, there exists a balanced growth path.*

Note, since P is constant, we can normalize the numeraire such that $P(t) = 1 \; \forall \; t$.

Proof

Given the balanced growth path guess, $h^{-1}(z)$ equals

$$h^{-1}(z) = \frac{\log\left(\frac{z}{h_0}\right)}{g}. \tag{42}$$

Using equation 39, $\phi^{-1}(z)$ equals

$$\phi_{ji}^{-1}(z) = \frac{\log\left(\frac{z}{\chi h_0}\right)}{g} \tag{43}$$

Recall from Lemma 1 that

$$f_i(z,t) = \theta h_i(t)^{\theta} z^{-1-\theta}. \tag{44}$$

Starting from equation 22, substituting the profit functions from equations 34 and 35 and using the guess $W(t) = W_0 e^{gt}$ we have

$$\begin{aligned} W_0 e^{gt} = & \int_{h(t)}^{\infty} \int_{t}^{h^{-1}(z)} e^{-r(\tau-t)} k_4 \lambda\left(\frac{z}{h_0}\right) Y_0 \, d\tau \, d\mathrm{F}(z,t) \\ & + \; (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} k_4 d^{-1} \lambda\left(\frac{z}{h_0}\right) Y_0 d\tau \, d\mathrm{F}(z,t) \\ & - \; (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} \kappa w_0 e^{g\tau} \, d\tau \, d\mathrm{F}(z,t) \\ & + \int_{h(t)}^{\infty} e^{-r(h^{-1}(z)-t)} \left[e^{gh^{-1}(z)} \left(W_0 - \zeta\left((1-\eta) w_0 + \eta \frac{\theta}{\theta-1} h_0 \right) \right) \right] \, dF(z,t). \end{aligned} \tag{45}$$

Computing the integrals and dividing by e^{gt} gives the initial value of search,

$$W_0 = \frac{(N-1)w_0\kappa\chi^{-\theta} + \theta\left(g\left(W_0 - \zeta\left[(1-\eta)w_0 - \eta\frac{\theta}{\theta-1}h_0\right]\right)(\theta-1) + k_4 Y_0\lambda\right)}{(r+g(\theta-1))(\theta-1)}. \tag{46}$$

Note that W_0 is not explicitly a function of time, a key step in verifying the guess that g is constant and $W(t) = W_0 e^{gt}$.

Given that the initial wage, w_0, and initial sales, Y_0, are solved from the sales and labor market clearing equations, equation 46 is one equation in two unknowns, g and W_0. We now turn to the FOC of the dynamic firm problem to derive a second equation in g and W_0.

Given the assumptions, the value function is

$$V(z,t) = \max_{T\geq t}\left\{\int_t^T e^{-r(\tau-t)}\pi_d(z,\tau)d\tau + (N-1)\int_t^T e^{-r(\tau-t)}\pi_{ex}(z,\tau)d\tau \right. \tag{47}$$

$$\left. + e^{-r(T-t)}\left[W(T) - \zeta\left((1-\eta)w(T) + \eta\frac{\theta}{\theta-1}h(T)\right)\right]\right\}, \tag{48}$$

Following the solution to the firm problem in Section 3.1, the first order condition is

$$\left.\frac{\partial V(z,t)_{|(\pi_{ex}=0)}}{\partial T}\right|_{(T=t,z=h(t))} = (g-r)\left[W_0 - \zeta\left((1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right)\right] + k_4 Y_0\lambda = 0. \tag{49}$$

Now we have two equations (46) and (49) in W_0 and g for which we can solve for the value of search and the growth rate on the BGP. As t has dropped out of equations 46 and 49, W_0 and g are not functions of time, confirming the guess that the value of search grows geometrically along the balanced growth path at constant rate g ($W(t) = W_0 e^{gt}$).

■

The addition of fixed exporting costs introduces complications that prevent much analytical analysis of the affect parameters have on growth rates. In the following section, we demonstrate the mechanisms at work in the model by using a calibrated model to explore the link between openness to trade, growth, and welfare.

4. Comparative Statics

In this section, we calibrate the model and solve for the balanced growth path. We then perform several comparative statics to illustrate the workings of the model.

Table 1 outlines the parameterization of the model. One set of parameters we calibrate based on

Table 1: Parameterization

Parameter	Source or Target
$\sigma = 2$	Assumption 3 (consistent with Broad and Weinstein (2006))
θ = 4	Simonovska and Waugh (2012)
$r = 0.10$	—
$N = 10$	—
$\eta = 1$	search cost all in goods
Search cost, ζ	match 2 percent growth rate
Fixed export cost, κ	match 10 percent of firms exporting
Iceberg trade cost, d	match 80 percent home trade share

previous work or introspection. These are described in the top panel. The curvature parameter σ is pinned down by Assumption 3. However, we should note that this assumption is not inconsistent with the best available evidence. Estimates of this CES parameter from Broda and Weinstein (2006) find a median estimate of σ near two. Inferences from high-frequency changes in trade flows and relative prices support a value of two as well, see, e.g., the discussion in Ruhl (2008).

The θ parameter is set equal to four as a baseline. There are various ways to get at this parameter, i.e. by looking at the distribution of sales or sizes across firms or from how trade flows respond to various shocks. The specific value of four is from Simonovska and Waugh (2012), who use price and trade flow data to estimate the heterogeneity parameter in the Eaton and Kortum (2002) trade model. In our comparative statics, we illustrate how θ affects the response of growth to changes in trade flows.

We picked the interest rate to equal ten percent and set the number of countries equal to ten. There is nothing deep about these choices, though we do explore scale effects and how the number of countries in the economy affects the growth rate.

The bottom panel of Table 1 outlines the remaining parameters are the cost to search for a new technology, ζ, the fixed cost to export, κ, and the iceberg trade cost, d. We jointly pick these parameters to match a two percent growth rate, ten percent of all firms exporting, and an

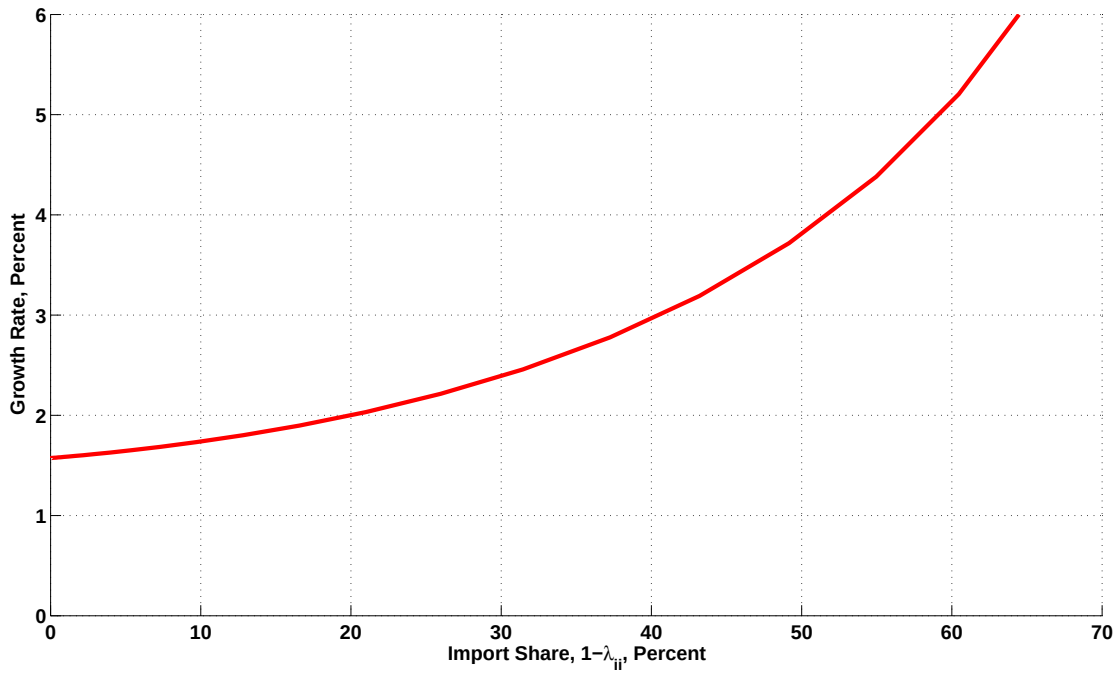

Figure 1: Openness Increases Growth: Growth Rate vs. Imports

80 percent home trade share. These targets are roughly consistent with properties of the U.S. economy.

To illustrate how growth depends on openness, we started from the baseline calibration and varied the iceberg trade costs to trace out how growth responds on the balanced growth path. Figure 1 plots the results. The vertical axis reports the growth rate in percent. The horizontal axis reports the import share for a country, i.e., $1 - \lambda_{ii}$, which grows as trade costs decrease. As an orientation device, note that when the import share equals 20 percent, the growth rate is 2 percent as calibrated.

Figure 1 shows that the growth rate of the economy increases as countries trade more and become open. For example, when trade costs are lowered such that the trade share increases from 20 percent to 40 percent, the growth rate increases from 2 percent to 3 percent. At the other extreme, when the economy is closed and countries do not trade with each other, the growth rate is about 1.6 percent.

What drives this result is that reductions in trade costs change the relative value of being a firm with productivity level z. This in turn changes the incentives for a firm to draw a new productivity, which in turn changes the growth rate of the economy. While these forces are complex, there are essentially two forces at work changing the relative value of a firm. First, all domestic firms face more competition from foreign firms which reduces the market share for domestic firms and reduces their profits. Second, high z firms are able to expand and export

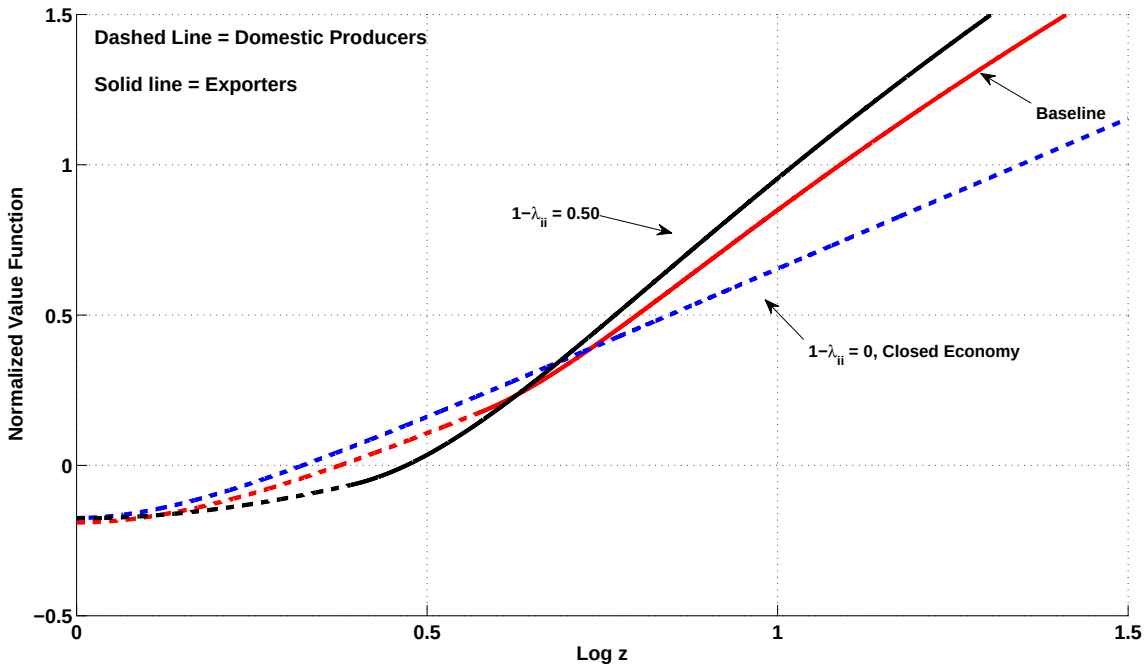

Figure 2: Value Functions and Openness

increasing profits for high z firms. The net effect of these two forces is to change the relative value of a high z firm versus a low z firm. This provides an incentive for a low z firm to draw a new technology term soon than later. Since growth is generated by search, as can be seen in the BGP relationship $S(t) = \theta g$, greater incentives to search generates higher growth.

Figure 2 illustrates this by plotting the value function of a firm (normalized by the average value) versus the log of its productivity level under different levels of openness. The blue line plots the value functions when the economy is closed. The red and black line plot the value function when the economy is open. Notice that as the economy opens up, the value functions as a function of z begin to rotate counterclockwise, becoming increasingly convex. The value of having a low z firm is becoming worse relative to a closed economy. As trade barriers decrease, foreign competition increases. Additionally, increased labor demand by high productivity firms to increase exports causes domestic wages to rise. Ultimately, lower trade costs increase the value of having a high z relative to having a low z.

This change in the relative profitability of firms is best illustrated in Figure 3, which plots the static profits of a firm on the vertical axis and the log of firm productivity on the horizontal axis, for different values of the iceberg trade costs. The profits from exporting are more than offsetting any loss in domestic profits from foreign competition and increased labor costs. This then provides an incentive for the low z firm to draw a new technology level sooner rather than later.

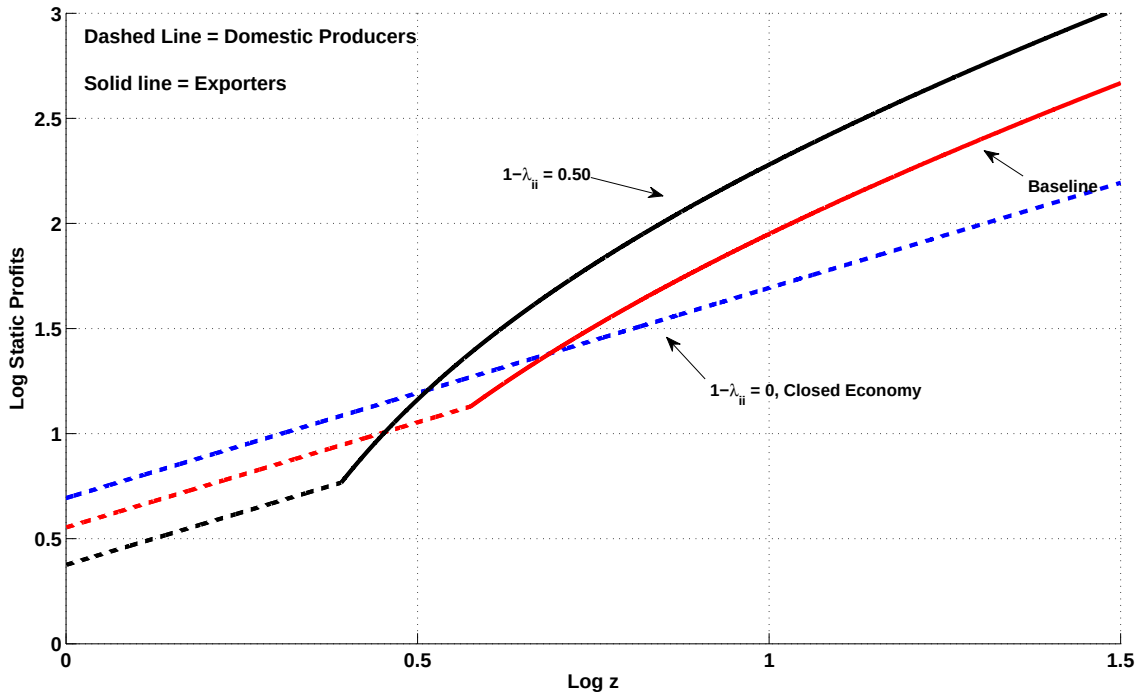

Figure 3: Static Profits and Openness

The positive relationship between growth and openness—in the absence of cross-country idea diffusion—is a unique feature of the model. Alvarez, Buera, and Lucas (2012) generate an increase in growth from openness in a model with equilibrium technology diffusion because opening to trade changes the number of ideas/productivity that (exogenously) an agent has an opportunity to meet in a given instant. In our model, the number of ideas sampled per instant is fixed. Independent of whether the country is open or closed, a firm has the opportunity to draw one new productivity from the distribution of domestic firms, once the cost is paid. The critical force is the dynamic, forward looking nature of firms in our model. Firms *choose* to draw a new productivity more frequently in response to opening to trade as higher productivity levels associated with exporting have become relatively more valuable.

A critical parameter in this model is θ. Figure 4 plots the relationship between growth and openness under several parameterizations of θ. In all the parameterizations of θ, we recalibrate the other parameters to match the same targets discussed above. As Figure 4 illustrates, the relationship between growth and openness becomes steeper. For example, when θ is three, a move to a 40 percent trade share increases growth to about 3.5 percent (compared to 3 when θ is two). Similarly, a move to autarky decreases growth to about 1.25 percent versus 1.6 percent when θ equals four.

This observation is related to the standard role θ plays in idea flow models (i.e. Alvarez, Buera,

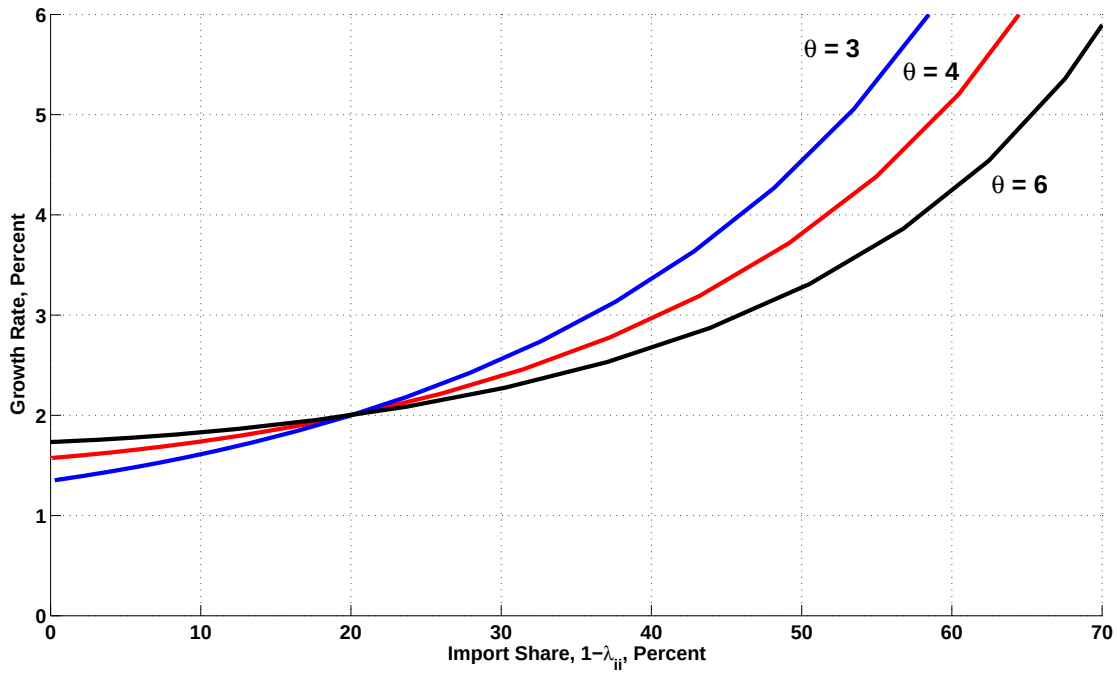

Figure 4: Higher θ, More Elastic Growth

and Lucas (2008), Lucas (2009), Perla and Tonetti (2012), Lucas and Moll (2012), Alvarez, Buera, and Lucas (2012). A lower θ is associated with a thicker right tail of the idea distribution, meaning draws from the idea distribution lead to larger jumps in productivity and, in equilibrium, faster growth rates. The same force is present here. Holding everything else constant, a smaller θ will lead to a faster growth rate. However, Figure 4 is saying something more. The response of growth to a change in trade costs is more sensitive the thicker the tail of the underlying idea distribution.

Figure 5 plots how the results depend upon the number of countries. Here we did not recalibrate or change the parameters. We kept all parameters from the baseline parameterization the same and doubled the number of countries from ten to 20. As Figure 5 shows, doubling the number of countries does not change the relationship between the scale of the economy and the growth rate.

This result is suggestive about the workings of the model. First, this result suggests that the key mechanism behind the relationship between growth and openness in our model is not coming through a scale effect per-se. What we mean by the previous sentence is the intuition that firms simply upgrade faster because markets and profits are larger. If this were true, then we would expect to see a relationship between the scale of the economy and growth. Because scale seems to be absent, this result reinforces the idea that the driving force is how openness changes the relative value of firms across different productivity levels.

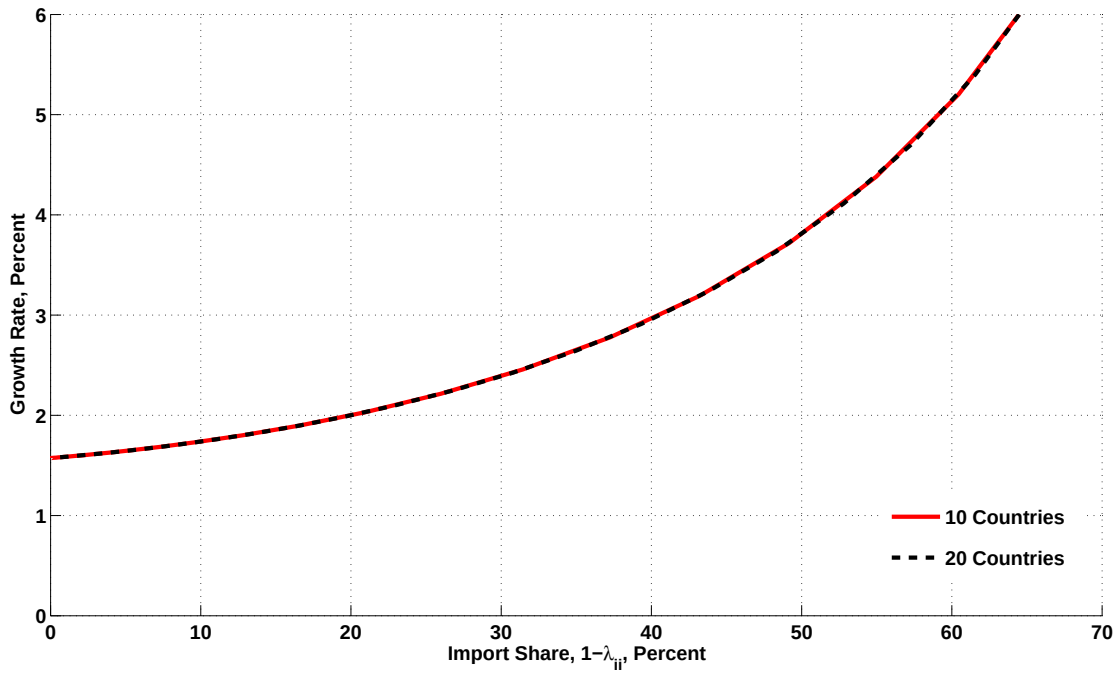

Figure 5: More Countries, No Change in Growth Rates

This result differs substantially from Alvarez, Buera, and Lucas (2012). Scale effects play a prominent role with the number of ideas a country has access to depending upon the number of countries. The growth rate on a balanced growth path with symmetric countries is linear in the number of countries in the economy. The critical difference is that our economy does not have cross-country idea diffusion as the Alvarez, Buera, and Lucas (2012) economy does.

Even without cross-country idea diffusion, the absence of growth scale effects does seem surprising. In fact, endogenous growth models have previously emphasized scale effects and openness as a way to increase scale and hence growth (see, e.g., the discussion of scale effects in Jones (2005a) and Ramondo, Rodriguez-Clare, and Saborio-Rodriguez (2012)). We should note that scale effects in newer models of knowledge diffusion are not well understood and we hope to explore more in the future.

One of the most important determinants of the effect of openness on growth is whether the process of technology adoption requires more labor or more goods. Figure 6 repeats the exercise of reducing iceberg trade costs as featured in Figure 1, but now varies the composition of labor and goods in the search cost.

As η decreases, labor becomes a larger component of the cost of technology adoption. For small η, the growth rate actually decreases as iceberg trade costs fall. Thus, opening to trade does not always increase growth rates. This is the result of strong general equilibrium effects on the wage rate. Decreased trade costs lead to increased demand for labor, as exporting firms want

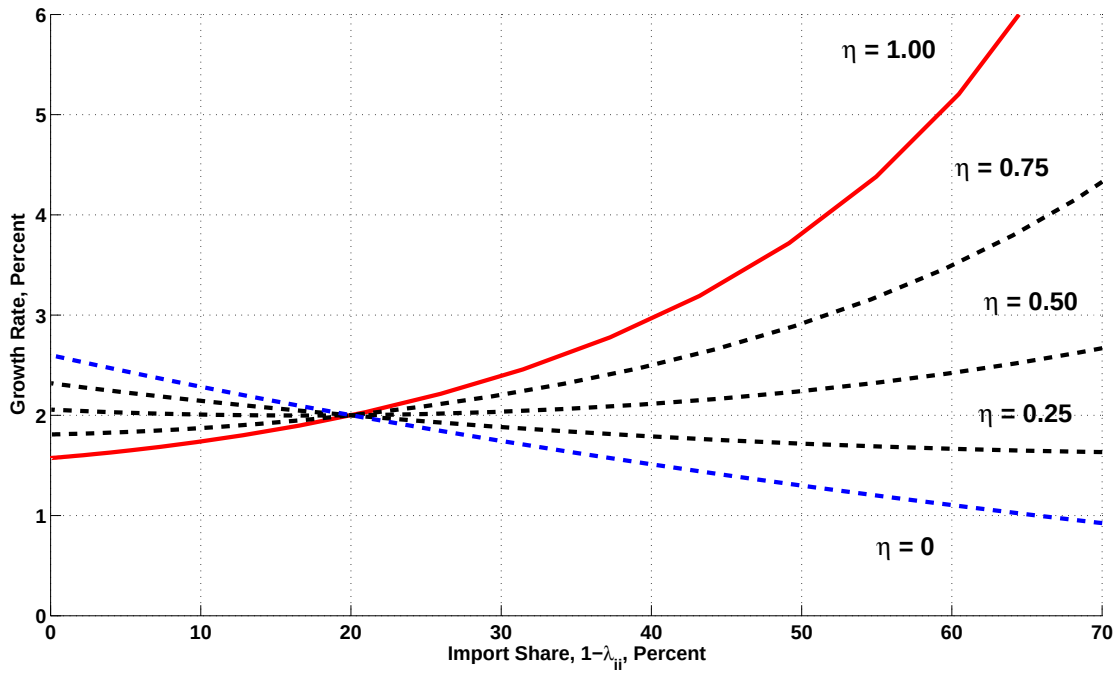

Figure 6: Lower η, Slower Growth

to produce more to sell abroad and more firms become exporters. Wages increase in response to the increased labor demand, and thus the larger the labor component in the search cost the larger the increase in the cost of upgrading technology. The economy continues to grow as trade costs fall, but the economy grows more slowly for low η, as the cost of search increases more rapidly. Given that the empirical literature on the relationship between growth and trade has found mixed evidence, this theory suggests future research into the costs of technology adoption and technological progress across countries may prove insightful.

Using the representative consumer's utility function, we can analyze the welfare implications of these growth patterns. Although growth rates can increase or decrease in response to reduced trade costs, there exist welfare gains from trade even if growth rates decline. Table 4 presents the welfare costs of autarky for an economy where the cost of search is in goods ($\eta = 1$). In the open economy, welfare is 1.13 times higher than under autarky. Moreover, the change in welfare can be decomposed into two components, static and dynamic. The static component captures the time zero change in the level of consumption while the dynamic component accounts for the change in the growth rate. The static gain of 1.07 results from the more standard increase in varieties exported and increase in quantities produced for a given variety as in Chaney (2008). Real GDP is 9 percent higher with international trade, with most of the increased production going to increase consumption, and some going to lay for higher technology adoption costs as firms upgrade more frequently. The dynamic gains of 1.05 come from the

Table 2: Welfare Cost of Autarky, $\eta = 1$ (cost of search in goods)

	Open	Autarky	Ratio
Welfare	13.45	11.86	1.13
Dynamic Component	12.50	11.86	1.05
Static Component	1.07	1.00	1.07
Imports/GDP	0.20	0	—
Real GDP	1.09	1.00	1.09
Growth rate	2.00	1.57	1.27

increased growth rate under openness and multiplies the static gains of 1.07 to generate total welfare improvements of 1.13.

Table 4 documents the welfare gains from openness when the cost of search is in labor ($\eta = 0$). Even though growth rates increase in autarky from 2 percent to 2.6 percent, there are still welfare gains of 1.04 percent from openness. Initially, real GDP is 12 percent larger in the open economy due to increased production and exports, which more than offsets the drag small growth rates to increase overall welfare.

Table 3: Welfare Cost of Autarky, $\eta = 0$ (cost of search in labor)

	Open	Autarky	Ratio
Welfare	14.05	13.51	1.04
Dynamic Component	12.50	13.51	0.93
Static Component	1.12	1.00	1.12
Imports/GDP	0.20	0	—
Real GDP	1.12	1.00	1.12
Growth rate	2.00	2.60	0.77

5. Conclusion

This paper contributes a novel dynamic model of growth and international trade, driven by technology diffusion based on Perla and Tonetti (2012). Firms choose to upgrade their productivity through technology adoption to remain competitive and profitable, with the incentives to upgrade dependent on the shape of the endogenously determined productivity distribution. Highly productive firms benefit from a decline in trade costs, as they are the exporters who can take advantage of increased sales abroad. Low productivity firms only sell domestically and are hurt by the increased competition from foreign firms and by increased wages. Under most calibrations, in equilibrium this leads lower productivity firms to upgrade their technology more frequently, which increases aggregate growth. The increased pace of technology adoption has

aggregate benefits beyond those to the individual firm, since in the future upgrading firms may adopt its improved technology. However, aggregate growth rates do not always increase in response to reduced trade costs, since the growth response to increased openness depends on the cost of technological improvement and the strength of general equilibrium wage effects. Nonetheless, while the gains and losses from reduced trade barriers are not distributed evenly across firms, the representative consumer who owns all firms benefits from openness.

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A. Appendix

1.1. Dynamic Problem

Let the aggregate price index, P_i, be the numeraire. Then the firm solves the following dynamic programming problem, that's solution is a sequence of search thresholds, $h_i(t)$, and export thresholds for each destination, $\phi_{ji}(t)\ \forall j \neq i$.

$$V_i(z,t) = \max_{\substack{T_{ji} \geq t \\ T_i \geq t}} \left\{ \int_t^{T_i} e^{-r(\tau - t)} \pi_{ii}(z,\tau) d\tau + \sum_{j \neq i} \int_t^{T_{ji}} e^{-r(\tau - t)} \pi_{ji}(z,\tau) d\tau + e^{-r(T_i - t)} \left[W_i(T_i) - x_i(T_i) \right] \right\} \tag{50}$$

where

$$W_i(t) := \int V_i(z,t) dF_i(z,t | z > h_i(t)) \tag{51}$$

Define the search and exporter thresholds as these indifference points:

$$h_i(t) := \max\{\ z \mid T_i(z,t) = t\ \} \tag{52}$$

$$\phi_{ji}(t) := \max\{\ z \mid T_{ji}(z,t) = t\ \} \tag{53}$$

The function $h_i(t)$ maps time into the largest productivity level such that the firm with that productivity level is upgrading its technology. Given this definition, the function $h_i^{-1}(z)$ defines the *time* at which a firm with productivity level z will draw a new technology. Then, since a draw comes from the equilibrium distribution of *producers*, the expected value of the new technology level, $W_i(T)$, is defined in (5). Notice that the value of the new technology is integrated with respect to the conditional productivity distribution $F_i(z,t|z > h_i(t))$ and hence is a function of the choices of the individual firms.

1.1.A. Derivation of Law of Motion and Searchers

Now we will derive the law of motion for the distribution, which is a function of the mass of searchers and the evolution of the firms dynamic control $h_i(t)$. One key issue, that may be particularly relevant at the beginning of time, is whether $h_i(t)$ is continuous. The law of motion for $F_i(z,t)$ is written for continuous and discontinuous regions. Recall that $h_i(t)$ is defined as the reservation value below which agents search. As a tie-breaking rule, it is assumed that agents at the threshold search, and hence the function is right-continuous. Define $\mathbb{S}_i(t)$ as the *mass* of searchers at time t. At points of continuity in $h_i(t)$ it should be equal to 0. Define

$S_i(t)$ as the *flow* of searchers at time t, which is not defined at points of discontinuity of $h_i(t)$. It is important to recognize that $h_i(t)$ may not be continuous, particularly at "special times," that reset the economy like time 0 or potentially when a closed economy unexpectedly opens to foreign trade. A discontinuity can be introduced by unexpectedly discretely changing the value of search or cost of search, i.e., a sudden change in iceberg trade costs.

The search technology of the environment is that agents only match other agents in the producing region, as in Perla and Tonetti (2012). Therefore, agents searching at time t only meet agents strictly above $h_i(t)$, when thinking of the evolution of the distribution $h_i(t)$ is an absorbing barrier, and the minimum of support of the productivity density is

$$\lim_{\Delta \to 0} \inf \text{support}\{F_i(\cdot, t+\Delta)\} = h_i(t) \tag{54}$$

$$\inf \text{support}\{F_i(\cdot, t)\} = h_i(t), \quad \text{at points of continuity} \tag{55}$$

Thus $f_i(z, t+\Delta)$ equals $f_i(z,t)$ at points of continuity in $h_i(t)$.

1.1.B. Mass of Searchers

The mass of agents searching at time t are agents below the $h_i(t)$ threshold

$$\mathbb{S}_i(t) := F_i(h_i(t), t) \tag{56}$$

Note from equation 55, at points of continuity,

$$\mathbb{S}_i(t) = F_i(\inf \text{support}(\cdot, t), t) = 0$$

1.1.C. Flow of Searchers

A key determinant of the growth rate of the economy and of the evolution of the productivity distribution is the flow of searchers upgrading their technology. At points of continuity of $h_i(t)$, there exists a flow of searchers during each infinitesimal time period. The flow of searchers is net flow of the probability current through the search threshold, $h_i(t)$.[7] To derive this, at time t fix the reference frame of the probability distribution of z with respect to the search threshold $h_i(t)$. That is, consider the change of variables $\tilde{z} := z - h_i(t)$. Let the transformed probability distribution function be $\tilde{f}_i(\tilde{z}, t)$. Note that the only time a firms productivity changes is when they search, i.e., z does not change in the continuation region, and hence the process is $dZ = 0 \cdot dt$ while the firm is producing. Using a special case of Ito's lemma with no diffusion (or more basic methods with a Taylor series), the process for $\tilde{Z}$ is $d\tilde{Z} = -h_i'(t)dt$. Equation 5.1.3 in Gardiner

[7] Equivalently, this is the flux of a vector field through the barrier (of 1 dimension, z in this case).

(2009) gives the more general case of the probability current with diffusion at $\tilde{z}, t$:

$$J_i(\tilde{z}, t) = -h_i'(t)\tilde{f}_i(\tilde{z}, t)$$

We are interested in the flow of searchers at the search threshold $h_i(t)$, and thus the flow at $\tilde{z} = 0$. This flow of searchers is given in equation 5.1.13 of Gardiner (2009). Since this is a simple 1 dimensional problem, the surface integral is trivial and the normal is $n = -1$. Note that since this is just a translation of z, the probability current at $\tilde{z} = 0$ and $z = h_i(t)$ are identical

$$S_i(t) = -1 \times J_i(0, t) \tag{57}$$

$$= h_i'(t)\tilde{f}_i(0, t) \tag{58}$$

$$= h_i'(t)f_i(h_i(t), t) \tag{59}$$

1.1.D. Law of Motion at Points of Discontinuity

At points of discontinuity in $h_i(t)$, a mass $\mathbb{S}_i(t)$ "exit" and draw from $\lim_{\Delta \to 0} F_i(\cdot, t + \Delta)$. The law of motion at $t+ := \lim_{\Delta \to 0} t + \Delta$ is therefore

$$F_i(z, t+) = \underbrace{F_i(z, t)}_{\text{Was below } z} - \underbrace{\mathbb{S}_i(t)}_{\text{Searched}} + \underbrace{\mathbb{S}_i(t)F_i(z, t+)}_{\text{Searched and drew} \leq z}, \quad \text{for } z \geq h_i(t+) \tag{60}$$

$$F_i(z, t+) - F_i(z, t) = -(1 - F_i(z, t+))F_i(h_i(t), t) \tag{61}$$

1.1.E. Law of Motion at Points of Continuity

The flow of "exiting" firms is defined by equation 59. These agents have an equal probability to draw any z in $f_i(z, t + \Delta)$, which equals $f_i(z, t)$ at points of continuity in $h_i(t)$. Hence, since $dZ = 0 \cdot dt$, the Kolmogorov forward equation for $z > h_i(t)$ is simply the flow of searchers who draw z:

$$\frac{\partial f_i(z, t)}{\partial t} = S_i(t)f_i(z, t) \tag{62}$$

Using equation 59

$$\frac{\partial f_i(z, t)}{\partial t} = f_i(z, t)f_i(h_i(t), t)h_i'(t) \tag{63}$$

In words, this says that the absorbing barrier (i.e., the search threshold) is sweeping across the density at rate $h_i'(t)$ and as the barrier sweeps across the density from below, it collects $f_i(h_i(t), t)$ amount firms. Then $f_i(h_i(t), t)h_i'(t)$ is the flow of searchers to be distributed back into the distribution. Since the economic environment is such that searchers only meet existing producers

above $h_i(t)$, but $h_i(t)$ is the minimum of support of $f_i(z,t)$, then the searchers are redistributed across the entire support of $f_i(z,t)$. Since agents draw directly from the productivity density, they are redistributed throughout the distribution in proportion to the density and thus, the flow of searchers $f_i(h_i(t),t)h_i'(t)$ multiplies the density $f_i(z,t)$.

1.1.F. Solving the KFE

A solution to equation (63) is a truncation for **any** $h_i(t)$ and $F_i(0)$

$$f_i(z,t) = \frac{f_i(z,0)}{1 - F_i(h_i(t),0)}. \tag{64}$$

That is the distribution at date t is a truncation of the initial distribution.

Note by properties of the Pareto distribution, given that the distribution evolves by repeated truncations, if $F_i(z,0)$ is Pareto with tail index θ and minimum of support $h_i(0)$, then $F_i(z,t)$ remains Pareto with tail index θ and minimum of support $h_i(t)$.

1.1.G. Solving the Firm Dynamic Problem

The steps to solving (4) consists of jointly finding an optimal waiting policy function, $h_i(t)$, and the expected value of search, $W_i(t)$, given a productivity distribution $F_i(z,t)$ and it's law of motion. Below, we describe the general steps to finding this solution.

Recall that the equilibrium search threshold $h_i(t)$ is the minimum of the productivity distribution. Given parameter constraints (particularly the fixed cost of exporting) such that not all firms export, the exporter threshold is greater than the search threshold. Thus, only non-exporters would want to search, and the FOC that determines the optimal search time is the derivative of the value function with respect to the search timing decision, where the discounted stream of export profits earned before searching is 0 with certainty:

$$V_i(z,t)_{|(\pi_{ji}=0)} = \max_{T_i \geq t} \left\{ \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau + e^{-r(T_i-t)} \left[W_i(T_i) - x_i(T_i) \right] \right\},$$

Taking the derivative of the value function of a non-exporting firm with respect to T yields

$$\frac{\partial V_i(z,t)_{|(\pi_{ji}=0)}}{\partial T_i} = \frac{\partial \int_t^{T_i} e^{-r(\tau-t)} \pi_{ii}(z,\tau) d\tau}{\partial T_i} + \frac{e^{-r(T_i-t)} \partial W_i(T_i)}{\partial T_i} - \frac{\partial e^{-r(T_i-t)} x_i(T_i)}{\partial T_i} \tag{65}$$

$$= e^{-r(T_i-t)} \left[\pi_{ii}(z,T_i) - rW_i(T_i) + W_i'(T_i) + rx_i(T_i) - x_i'(T_i) \right] \tag{66}$$

Setting $T_i = t$, i.e., where the firm is just indifferent between switching technologies and producing, and recognizing that the productivity level of the indifferent firm is $z = h_i(t)$ by definition, we have the first order condition

$$0 = \pi_{ii}(h_i(t), t) - rW_i(t) + W_i'(t) + rx_i(t) - x_i'(t) \tag{67}$$

Now equation (67) gives us one equation in $h_i(t)$ and $W_i(t)$. We now want to find another equation in $h_i(t)$ and $W_i(t)$ essentially giving us two equations in two unknowns.

The second equation we focus on is the expected value of acquiring a new technology:

Since $h_i(t)$ is the minimum of support of $F_i(z,t)$ as stated in equation 55, we can rewrite equation 51 as

$$\begin{aligned} W_i(t) &= \int V_i(z,t) dF_i(z,t) \\ &= \int_{h_i(t)}^{\infty} \Bigg\{ \int_t^{h_i^{-1}(z)} e^{-r(\tau-t))} \pi_{ii}(z,\tau) d\tau + \sum_{j \neq i} \int_t^{\phi_{ji}^{-1}(z)} e^{-r(\tau-t)} \pi_{ji}(z,\tau) d\tau \\ &+ e^{-r(h_i^{-1}(z)-t)} \left[W_i(h_i^{-1}(z)) - x_i(h_i^{-1}(z)) \right] \Bigg\} dF_i(z,t) \end{aligned} \tag{68}$$

The first integral in the inside bracket is the discounted value of domestic profits until the next change of technology. Here we substituted for the time T_i with the function $h_i^{-1}(z)$. The second integral in the inside bracket is the discounted value of profits from exporting. Again, we substituted for the final exporting time, T_{ji}, for the function $\phi_{ji}(z)$, which is defined in (53). The function $\phi_{ji}(z)$ is the largest z such that a firm stops exporting to market j. Thus the inverse of this function defines the time when the firm stops exporting to market j. The final term in the inside bracket is the discounted value of the new technology net of search costs evaluated at the date $h_i^{-1}(z)$.

Outside the brackets, we then integrate over productivity levels with the existing (equilibrium) productivity distribution $F_i(z,t)$. This defines the expected value of acquiring a new technology.

Equations (67) and (68) give us two equations from which we can solve for the policy function $h_i(t)$ and the value of search $W_i(t)$ for a given a law of motion for the productivity distribution $F_i(z,t)$.

1.2. Static Problem

This derivation is for the specification where the fixed cost of exporting is $w_j \kappa_j$.

We suppress dependence on time for clarity of notation.

1.2.A. Prices, Quantities, and Profits

$$p_{ii}(z) = \frac{m w_i}{z} \tag{69}$$

$$p_{ij}(z) = \frac{m d_{ji} w_i}{z} \tag{70}$$

$$q_{ii}(z) = \frac{p_{ii}(z)^{-\sigma} Y_i}{P_i^{1-\sigma}} = z \ell_{ii}(z) \tag{71}$$

$$q_{ji}(z) = \frac{p_{ji}(z)^{-\sigma} Y_j}{P_j^{1-\sigma}} = \frac{z}{d_{ji}} \ell_{ji}(z) \tag{72}$$

$$P_i \pi_{ii}(z) = \frac{1}{\sigma} \left(\frac{m\ w_i}{z} \right)^{1-\sigma} \frac{Y_i}{P_i^{1-\sigma}} \tag{73}$$

$$P_i \pi_{ji}(z) = \max\{ \frac{1}{\sigma} \left(\frac{m\ d_{ji}\ w_i}{z} \right)^{1-\sigma} \frac{Y_j}{P_j^{1-\sigma}} - w_j \kappa_j, 0 \} \tag{74}$$

1.2.B. Exporter Threshold

ϕ_{ji} solves $\pi_{ji}(z) = 0$.

$$\begin{aligned}
\left(\frac{m\ d_{ji}\ w_i}{\phi_{ji}} \right)^{1-\sigma} &= \sigma w_j \kappa_j \frac{P_j^{1-\sigma}}{Y_j} \\
\frac{m\ d_{ji}\ w_i}{\phi_{ji}} &= \left(\frac{\sigma w_j \kappa_j}{Y_j} \right)^{\frac{1}{1-\sigma}} P_j \\
\phi_{ji} &= \frac{m\ d_{ji}\ w_i}{P_j} \left(\frac{\sigma w_j \kappa_j}{Y_j} \right)^{\frac{1}{\sigma-1}}
\end{aligned}$$

Define $k_1 := m\ d_{ji} \sigma^{\frac{1}{\sigma-1}}$

$$\phi_{ji} = k_1 \frac{w_i}{P_j} \left(\frac{w_j \kappa_j}{Y_j} \right)^{\frac{1}{\sigma-1}} \tag{75}$$

1.2.C. Aggregate Price Index

$$P_i^{1-\sigma} := \int_{h_i}^{\infty} p_{ii}(z)^{1-\sigma} f_i(z)\, dz + \sum_{j \neq i} \int_{\phi_{ij}}^{\infty} p_{ij}(z)^{1-\sigma} f_j(z)\, dz \tag{76}$$

Let $f_i(z) = \theta h_i^{\theta} z^{-1-\theta}$. Recall, $p_{ii}(z) = \frac{m w_i}{z}$ and $p_{ij}(z) = \frac{m d_{ji} w_i}{z}$.

$$P_i^{1-\sigma} = (m w_i)^{1-\sigma} \theta h_i^{\theta} \int_{h_i}^{\infty} \frac{z^{-1-\theta}}{z^{1-\sigma}}\, dz + \sum_{j \neq i} (m d_{ji} w_j)^{1-\sigma} \theta h_j^{\theta} \int_{\phi_{ij}}^{\infty} \frac{z^{-1-\theta}}{z^{1-\sigma}}\, dz$$

Define $k_2 := \frac{\theta}{\theta+1-\sigma}$.

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\theta} h_i^{\sigma-1-\theta} + \sum_{j \neq i} k_2 (m d_{ji} w_j)^{1-\sigma} h_j^{\theta} \phi_{ij}^{\sigma-1-\theta}$$

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\sigma-1} + \sum_{j \neq i} k_2 (m d_{ji})^{1-\sigma} w_j^{1-\sigma} h_j^{\theta} (m\, d_{ji} \sigma^{\frac{1}{\sigma-1}})^{\sigma-1-\theta} \left(\frac{w_j}{P_i}\right)^{\sigma-1-\theta} \left(\frac{w_i \kappa_i}{Y_i}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}$$

Define $k_3 := k_2 \sigma^{\frac{\sigma-1-\theta}{\sigma-1}}$.

$$P_i^{1-\sigma} = k_2 (m w_i)^{1-\sigma} h_i^{\sigma-1} + \sum_{j \neq i} k_3 (m d_{ji} w_j)^{-\theta} h_j^{\theta} P_i^{\theta+1-\sigma} \left(\frac{w_i \kappa_i}{Y_i}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{77}$$

1.2.D. Aggregate Trade Shares

$$\lambda_{ji} := \int_{\phi_{ji}}^{\infty} \frac{p_{ji}(z) q_{ji}(z)}{Y_j} f_i(z) dz \tag{78}$$

Recall $q_{ji}(z) = \frac{p_{ji}(z)^{-\sigma} Y_j}{P_j^{1-\sigma}}$.

$$\lambda_{ji} = \int_{\phi_{ji}}^{\infty} \frac{p_{ji}(z)^{1-\sigma}}{P_j^{1-\sigma}} f_i(z) dz$$

$$\lambda_{ji} = \frac{1}{P_j^{1-\sigma}} \int_{\phi_{ji}}^{\infty} p_{ji}(z)^{1-\sigma} f_i(z) dz$$

Recall from the derivation of the Aggregate Price Index,

$$\int_{\phi_{ij}}^{\infty} p_{ij}(z)^{1-\sigma} f_j(z)dz = k_3(md_{ji}w_j)^{-\theta} h_j^{\theta} P_i^{\theta+1-\sigma} \left(\frac{w_i\kappa_i}{Y_i}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{79}$$

so,

$$\lambda_{ji} = \frac{1}{P_j^{1-\sigma}} \left[k_3(md_{ij}w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j\kappa_j}{Y_j}\right)^{\frac{\sigma-1-\theta}{\sigma-1}} \right] \tag{80}$$

substituting the aggregate price index from equation 77,

$$\lambda_{ji} = \frac{k_3(md_{ij}w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j\kappa_j}{Y_j}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1} + \sum_{n\neq j} k_3(md_{nj}w_n)^{-\theta} h_n^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_n\kappa_n}{Y_n}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}} \tag{81}$$

Deriving λ_{ii}.

Note $\sum_i \lambda_{ji} = 1$. That is, $\sum_{i\neq j} \lambda_{ji} + \lambda_{ii} = 1$.

$$\sum_{i\neq j} \lambda_{ji} = \frac{\sum_{i\neq j} k_3(md_{ij}w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_j\kappa_j}{Y_j}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1} + \sum_{n\neq j} k_3(md_{nj}w_n)^{-\theta} h_n^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_n\kappa_n}{Y_n}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}}$$

Define $\Gamma_{ji} := \sum_{i\neq j} k_3(md_{ij}w_i)^{-\theta} h_i^{\theta} P_j^{\theta+1-\sigma} \left(\frac{w_i\kappa_i}{Y_i}\right)^{\frac{\sigma-1-\theta}{\sigma-1}}$. Then,

$$\sum_{i\neq j} \lambda_{ji} = \frac{\Gamma_{ji}}{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1} + \Gamma_{ji}}$$

Since $\sum_{i\neq j} \lambda_{ji} + \lambda_{ii} = 1$

$$\lambda_{ii} = \frac{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1}}{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1} + \Gamma_{ji}}$$

By definition of P_j and Γ_{ji},

$$\lambda_{ii} = \frac{k_2(mw_j)^{1-\sigma} h_j^{\sigma-1}}{P_j^{1-\sigma}} \tag{82}$$

Deriving π_{ii} and π_{ji} in terms of trade shares.

Using $\pi_{ii}(z)$ derived in equation 73,

$$\pi_{ii}(z) = k_4 \lambda_{ii} \left(\frac{z}{h_i}\right)^{\sigma-1} Y_i, \quad \text{where} \quad k_4 = \frac{1}{k_2 \sigma} \tag{83}$$

Similarly, using $\pi_{ji}(z)$ derived in equation 74,

$$\pi_{ji}(z) = k_4 \lambda_{jj} d_{ij}^{1-\sigma} \left(\frac{z}{h_j}\right)^{\sigma-1} \left(\frac{w_i}{w_j}\right)^{1-\sigma} Y_j - w_j \kappa_j \tag{84}$$

1.2.E. Aggregate Sales

The cost to draw a new productivity level is a convex combination of hiring domestic labor and spending final goods, given by

$$x(t) = \zeta \left[(1-\eta)\frac{w(t)}{P(t)} + \eta \mathbb{E}_t[z_i]\right]. \tag{85}$$

$$Y_i = w_i L_i + \int_{h_i}^{\infty} P_i \pi_{ii}(z) f_i(z) dz + \sum_{j \neq i} \int_{\phi_{ji}}^{\infty} P_i \pi_{ji}(z) f_i(z) dz \tag{86}$$

$$\begin{aligned} Y_i = w_i L_i &+ k_4 \lambda_{ii} h_i^{1-\sigma} Y_i \int_{h_i}^{\infty} z^{\sigma-1} f_i(z) dz \\ &+ \sum_{j \neq i} \left\{ k_4 \lambda_{jj} d_{ij}^{1-\sigma} h_j^{1-\sigma} \left(\frac{w_i}{w_j}\right)^{1-\sigma} Y_j \int_{\phi_{ji}}^{\infty} z^{\sigma-1} f_i(z) dz - w_j \kappa_j \int_{\phi_{ji}}^{\infty} f_i(z) dz \right\} \end{aligned}$$

$$Y_i = w_i L_i + k_3 k_4 \lambda_{ii} Y_i + \sum_{j \neq i} \left\{ k_3 k_4 \lambda_{jj} d_{ij}^{1-\sigma} \left(\frac{w_i}{w_j}\right)^{1-\sigma} Y_j \left(\frac{\phi_{ji}}{h_i}\right)^{\sigma-1-\theta} - w_j \kappa_j \left(\frac{\phi_{ji}}{h_i}\right)^{-\theta} \right\} \tag{87}$$

1.2.F. Goods Market Clearing

$$\frac{Y_i}{P_i} = Q_i = C_i + \zeta \eta \mathbb{E}_t[z_i] \tag{88}$$

1.2.G. Wages: Labor Market Clearing

For each producer, the quality of labor demanded to produce a good in country i for destination i is

$$\ell_{ii}(z) = \frac{p_{ii}(z)^{-\sigma} Y_i}{z P_i^{1-\sigma}} \tag{89}$$

Aggregating the labor in market i used for domestic production

$$L_{i,d} = \int_{h_i}^{\infty} \frac{p_{ii}(z)^{-\sigma} Y_i}{z P_i^{1-\sigma}} f_i(z) dz \tag{90}$$

For each producer, the quality of labor demanded to produce a good in country i for destination j is

$$\ell_{ji}(z) = \frac{d_{ji} p_{ji}(z)^{-\sigma} Y_j}{z P_j^{1-\sigma}} \tag{91}$$

Aggregating the labor in market i used for export production

$$L_{i,ex} = \sum_{j \neq i} \int_{\phi_{ji}}^{\infty} \frac{d_{ji} p_{ji}(z)^{-\sigma} Y_j}{z P_j^{1-\sigma}} f_i(z) dz \tag{92}$$

Since the fixed cost of exporting from j to i requires units of i labor, the total amount of labor from i used in the production of fixed costs equals

$$L_{i,\kappa} = \sum_{j \neq i} \int_{\phi_{ij}}^{\infty} \kappa_i f_j(z) dz$$

$$L_{i,\kappa} = \kappa_i \sum_{j \neq i} \left(\frac{h_j}{\phi_{ij}} \right)^{\theta} \tag{93}$$

Since the technology upgrade search decision is partially paid to hire labor, the search component of labor demand is

$$L_{i,x} = \zeta(1 - \eta) S_i(t) \tag{94}$$

Equating aggregate labor supply, L_i, with aggregate labor demand yields the labor market clearing condition:

$$L_i = L_{i,d} + L_{i,ex} + L_{i,\kappa} + L_{i,S}$$
$$L_i = \int_{h_i}^{\infty} \frac{p_{ii}(z)^{-\sigma} Y_i}{z P_i^{1-\sigma}} f_i(z) dz + \sum_{j \neq i} \int_{\phi_{ji}}^{\infty} \frac{d_{ji} p_{ji}(z)^{-\sigma} Y_j}{z P_j^{1-\sigma}} f_i(z) dz + \kappa_i \sum_{j \neq i} \left(\frac{h_j}{\phi_{ij}} \right)^{\theta} + L_{i,x} \quad = \zeta(1-\eta) S_i(t) \tag{95}$$

1.3. BGP with Symmetry

Solving for a BGP given Assumptions 2, 3, 4 and Guesses 1, 2, and 3.

$$\begin{aligned} P(t)\pi_d(z,t) &= k_4 \lambda(t) \left(\frac{z}{h(t)} \right) Y(t) \\ &= k_4 \lambda(t) \left(\frac{z}{h_0} \right) Y_0 \end{aligned} \tag{96}$$

$$\begin{aligned} P(t)\pi_{ex}(z,t) &= k_4 \lambda(t) d^{-1} \left(\frac{z}{h(t)} \right) Y(t) - w(t)\kappa \\ &= \pi_d(z,t) d^{-1} - e^{gt} w_0 \kappa. \end{aligned} \tag{97}$$

Imposing symmetry, the price index and trade share are

$$P(t)^{1-\sigma} = k_2 (m w(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1) k_3 (m d w(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}} \tag{98}$$

$$\lambda(t) = \frac{k_3 (m d w(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w_(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}{k_2 (m w(t))^{1-\sigma} h(t)^{\sigma-1} + (N-1) k_3 (m d w(t))^{-\theta} h(t)^{\theta} P(t)^{\theta+1-\sigma} \left(\frac{w(t)\kappa}{Y(t)} \right)^{\frac{\sigma-1-\theta}{\sigma-1}}}. \tag{99}$$

$$\tag{100}$$

Given Guesses 1 and 2, the price index and trade share must be constant on a balanced growth path.

BGP total sales is

$$Y(t) = w(t) + k_3 k_4 \lambda Y(t) + (N-1) \left[k_3 k_4 \lambda d^{-1} Y(t) \left(\frac{\phi(t)}{h(t)} \right)^{1-\theta} - \kappa w(t) \left(\frac{\phi(t)}{h(t)} \right)^{-\theta} \right] \tag{101}$$

From equation 97, the cutoff value for exporting is

$$\phi(t) = \chi h(t) \quad \text{where} \quad \chi := \frac{d\kappa w_0}{k_4 Y_0 \lambda} \tag{102}$$

Thus,

$$Y(t) = w(t) + k_3 k_4 \lambda Y(t) + (N-1)\left[k_3 k_4 \lambda d^{-1} Y(t) \chi^{1-\theta} - \kappa w(t) \chi^{-\theta}\right] \tag{103}$$

Then rearranging and collecting terms we have

$$Y(t) = w(t) k_5, \quad \text{where} \quad k_5 = \frac{1 - \kappa \chi^{-\theta}}{1 - \left[k_3 k_4 \lambda \left(1 + (N-1) d^{-1} \chi^{1-\theta}\right)\right]}. \tag{104}$$

Thus, Guess 2 is verified that wages are growing at the same rate as sales.

Summarizing,

$$\lambda(t) = \lambda(0) \; \forall \, t, \quad P(t) = P(0) \; \forall \, t, \quad \text{and} \quad Y(t) = k_5 w_0 e^{gt}. \tag{105}$$

1.3.A. Proof of Proposition 2

Normalize the numeraire such that $P(t) = 1 \; \forall \, t$.

Proof

First, note that given the balanced growth path guess, $h^{-1}(z)$ equals

$$h^{-1}(z) = \frac{\log\left(\frac{z}{h_0}\right)}{g}. \tag{106}$$

Second, using equation 102, $\phi^{-1}(z)$ equals

$$\phi_{ji}^{-1}(z) = \frac{\log\left(\frac{z}{\chi h_0}\right)}{g} \tag{107}$$

Finally, recall Assumption 1 that the initial distribution is Pareto and hence, by the solution to the Kolmogorov forward equation from equation 18, the pdf at time t is

$$f_i(z,t) = \theta h_i(t)^{\theta} z^{-1-\theta}. \tag{108}$$

Starting from equation 68, substituting the profit functions from equations 96 and 97, using the

search cost in equation 85, and using the guess $W(t) = W_0 e^{gt}$ we have

$$W_0 e^{gt} = \int_{h(t)}^{\infty} \int_{t}^{h^{-1}(z)} e^{-r(\tau - t)} k_4 \lambda \left(\frac{z}{h_0} \right) Y_0 \, d\tau \, d\mathrm{F}(z,t) \tag{109}$$

$$+ \; (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau - t)} k_4 d^{-1} \lambda \left(\frac{z}{h_0} \right) Y_0 d\tau \, d\mathrm{F}(z,t)$$

$$- \; (N-1) \int_{\phi_{ji}(t)}^{\infty} \int_{t}^{\phi_{ji}^{-1}(z)} e^{-r(\tau - t)} \kappa w_0 e^{g\tau} \, d\tau \, d\mathrm{F}(z,t)$$

$$+ \int_{h(t)}^{\infty} e^{-r(h^{-1}(z) - t)} \left[W_0 e^{gh^{-1}(z)} - \zeta((1-\eta) w_0 + \eta \frac{\theta}{\theta - 1} h_0) e^{gh^{-1}(z)} \right] dF(z,t).$$

Computing integrals yields

$$W_0 = \frac{k_4 Y_0 \theta \lambda}{(r + g(\theta - 1))(\theta - 1)} + \frac{(N-1) w_0 \theta \kappa \left(\frac{\kappa d w_0}{k_4 \lambda Y_0} \right)^{-\theta}}{(r + g(\theta - 1))(\theta - 1)} \tag{110}$$

$$- \frac{(N-1) w_0 \kappa \left(\frac{\kappa d w_0}{k_4 \lambda Y_0} \right)^{-\theta}}{r + g(\theta - 1)} + \frac{g W_0 \theta}{r + g(\theta - 1)} - \frac{g \zeta((1-\eta) w_0 + \eta \frac{\theta}{\theta - 1} h_0) \theta}{r + g(\theta - 1)}.$$

Combining terms and dividing by e^{gt} finally gives the initial value of search,

$$W_0 = \frac{(N-1) w_0 \kappa \chi^{-\theta} + \theta (g(W_0 - \zeta((1-\eta) w_0 - \eta \frac{\theta}{\theta - 1} h_0))(\theta - 1) + k_4 Y_0 \lambda)}{(r + g(\theta - 1))(\theta - 1)}. \tag{111}$$

Note that W_0 is not explicitly a function of time, a key step in verifying the guess that g is constant and $W(t) = W_0 e^{gt}$.

Given that the initial wage, w_0, and initial sales, Y_0, are solved from the sales and labor market clearing equations we now have one equation in two unknowns, g and W_0.

The relevant value function for a marginal searcher is

$$V(z,t)_{|(\pi_{ex}=0)} = \max_{T\geq t}\left\{\int_t^T e^{-r(\tau-t)}\pi_d(z,\tau)d\tau + e^{-r(T-t)}\left[W(T) - \zeta\left[(1-\eta)w(T) + \eta\frac{\theta}{\theta-1}h_T\right]\right]\right\}, \tag{112}$$

Taking the derivative of the value function of a non-exporting firm with respect to T yields

$$\frac{\partial V(z,t)_{|(\pi_{ex}=0)}}{\partial T} = e^{gT-r(T-t)}(g-r)W_0 - e^{gT-r(T-t)}(g-r)\zeta\left[(1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right] + \frac{e^{-r(T-t)}k_4Y_0z\lambda}{h_0}$$

Setting $T = t$, i.e., where the firm is just indifferent between switching technologies and producing, we have

$$0 = e^{gt}(g-r)(W_0 - \zeta\left[(1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right]) + \frac{k_4Y_0z\lambda}{h_0}$$

Finally recall that the productivity level of the indifferent firm, where $T = t$, is $z = h(t)$ by definition. Substituting $z = h(t) = h_0e^{gt}$ gives

$$0 = (g-r)(W_0 - \zeta\left[(1-\eta)w_0 + \eta\frac{\theta}{\theta-1}h_0\right]) + k_4Y_0\lambda. \tag{113}$$

Now we have two equations (46) and (49) in W_0 and g for which we can solve for the value of search and the growth rate on the BGP. As t has dropped out of equations 46 and 49, W_0 and g are not functions of time, confirming the guess that the value of search grows geometrically along the balanced growth path at constant rate g ($W(t) = W_0e^{gt}$). ■