Investment in Vintage Capital

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ABSTRACT

We study an economy in which firms use labor and various vintages of capital in a CES production function for the final good. We explicitly solve for the investment in capital of a given vintage as a function of its age, and for the resulting stocks of capital. We show that for reasonable parameter values, inverted-U-shaped dynamics of investment and S-shaped dynamics for capital arise in equilibrium.

1. Introduction

Vintage-capital models describe the optimal investment and replacement of technology-specific capital under technical progress. Early vintage-capital models of Johansen (1959), Solow (1960), and Arrow (1962) entailed no interaction among capital of different vintages. In models of that type, all investment flows into capital of the latest vintage, and none into any earlier vintage. A second generation of vintage-capital models Domar (1963), Jorgenson (1966), Greenwood, Hercowitz and Krusell (1997), Xepapadeas and Zeeuw (1999), and Goetz, Hritonenko and Yatsenko (2008), assumes that while capital of different vintages can participate in the same production process, the elasticity of substitution in production between capital of different vintages is infinite so that, again, all investment is in the latest-vintage capital where the efficiency of investment is highest.

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The implication that all investment should flow into capital of the latest vintage is in conflict with experience. Old structures are refurbished, old machines are repaired, old workers are retrained. In other words, firms invest in old capital while also investing in new, more efficient capital. Evidently, old and new capitals interact and the productivity of one depends on the quantity of the other. Prescott and Boyd (1987) and Chari and Hopenhayn (1991) model old and new capital as complements in production, and Boddy and Gort (1971) estimate a production function including equipment and structures the average ages of which differ.

In this paper we study an economy in which firms face a CES production function whose inputs are labor and different vintages of capital. We solve the model completely; in other words we solve analytically for each type of capital and for investment in that capital as a function of its age. Chari and Hopenhayn allow for a maximum of two adjacent vintages of (human) capital to enter the production function whereas we allow for an arbitrary number of vintages. We find that to obtain S-shaped diffusion of capital one must introduce a growth in the user-friendliness of technology or, rather, of capital as a function of its age. This point was established numerically by Cooley, Greenwood and Yorukoglu (1997) and Atkeson and Kehoe (2005). It remains to be seen whether similar results would emerge if learning depended on cumulative output as in Arrow (1962) or Klenow (1998).

We also find that an economy in which technological progress is faster and the elasticity of substitution in production is higher, will have an age-distribution of capital that first-order dominates the age-distribution of capital in other economies. These results are analytical and pertain to steady states.

The model stands in contrast to models in which capital combines with a fixed resource such as labor (Solow 1960, Arrow 1962, Jovanovic 1998), land and labor (Cooley Greenwood and Yorukoglu 1997) or management and labor (Atkeson and Kehoe 2005), and in which that resource can work with only one technology, one type of capital, or one vintage of plant. In such models new capital must necessarily drive out the old since only a limited number of its vintages can physically be in use at any time. Our model imposes no such limitation; an unlimited number of vintages can coexist in production. While the approach of Greenwood, Hercowicz

and Krusell (1997) also allows for such coexistence, it also imposes an infinite elasticity of substitution on the various vintages of capital with the result that all investment flows into the latest vintage.

Chari and Hopenhayn (1991), in a closely related paper, derive a coexistence of investment in capital of many vintages from a complementarity of old and new capital. In contrast to our model in which an arbitrary array of vintages take part in the same production function, their aggregate production function is a sum of production processes each of which combines at most two capital vintages. A second, less important difference is that in their model capital of a given vintage is created not with goods but with inexperienced labor and capital of the same vintage. We confirm analytically the numerical result in Figure 1 of Chari and Hopenhayn (1991), which is that a decrease in the elasticity of substitution in production between the various vintages of capital delays investment in old capital. In particular, we solve for the age of a vintage at which peak investment occurs, and it is a decreasing function of the elasticity of substitution between old capital and new capital. Closer to our paper is the model of Kredler (2008) extends this model by allowing a continuum of vintages and a continuum of human-capital inputs in each technological vintage.

2. The model

The economy consists of a unit measure of agents each endowed with a unit of labor and with preferences

$$U = \int_0^\infty e^{-\rho t} \frac{C_t^{1-\eta}}{1-\eta} dt ,$$

where C_t is consumption at date t. The agents receive income from wages and from the dividends of perfectly competitive firms that rent their labor and own their capital of various vintages.

Firms have access to constant-returns-to-scale production functions

$$y = K^{\alpha} N^{1-\alpha} \tag{1}$$

where y is output of the only consumption good, N denotes labor services rented, and K denotes an aggregate of capital stocks that the firms own. That aggregate is given by the CES form

$$K_t = \left(\int_{-\infty}^{t} A_{t-v} \left(z_v k_{v,t} \right)^{\beta} dv \right)^{1/\beta} \tag{2}$$

in which $k_{v,t}$ is the amount of capital at date t embodying technology of vintage v, z_v is the unit efficiency of the capital of vintage v, A_{t-v} is the age-dependent "learning curve" for the capital of vintage v, $t \in [0,\infty)$, $v \in (-\infty,t]$. The parameter $0 < \alpha \leq 1$ describes the return to scale and $\beta \in (0,1]$ indexes substitution possibilities yielding the elasticity of substitution $\sigma = 1/(1-\beta)$.[1] The case $0 < \beta < 1$, i.e., $\sigma > 1$ is realistic, because then no vintage of capital is essential: Positive output can be produced with any subset of vintages. We shall maintain this assumption throughout

Let x_t denote investment in new capital of vintage t and $u_{v,t}$ investment in old capital of vintage $v<t$ (of the age t-v). Let capital depreciate at the constant rate $\delta > 0$. Then the law of motion for capital of vintage v is $\frac{\partial k_{v,t}}{\partial t} = -\delta k_{v,t} + u_{v,t}$. This ordinary differential equation has the solution

$$k_{v,t} = e^{-\delta(t-v)} x_v + \int_{v}^{t} e^{-\delta(t-s)} u_{v,s} ds \tag{3}$$

for $v > 0$, and

$$k_{v,t} = e^{-\delta(t-v)} k_{v,0} + \int_{0}^{t} e^{-\delta(t-s)} u_{v,s} ds \tag{4}$$

for $-\infty < v < 0$, where $k_{v,0}$, $v \in (-\infty,0]$, is a distribution of capital over past vintages at date zero that is given.

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[1] The function in (2) generalizes the traditional vintage aggregation functions of the form $K_t = \int_{-\infty}^{t} A_{t-v} z_v k_{v,t} dv$, which occurs at $\beta = 1$ (i.e., $\sigma = \infty$) and assumes that vintages of capital are perfect substitutes as, for instance, in Arrow (1962). Models that feature A_t are Cooley *et al.* (1997) and Atkeson and Kehoe (2005) who assume that A grows exogenously in each plant as a function of its age. Estimates of A are provided by Bahk and Gort (1993).

The firm solves the following profit maximization problem

$$\max_{u,x,N} \int_0^{\infty} e^{-rt}\left(y_t - w_t N_t - x_t - \int_{-\infty}^{t} u_{v,t} dv \right) dt, \qquad x \geq 0, \quad u \geq 0, \tag{5}$$

subject to (1), (2), (3), and (4) and subject to the initial conditions. The decision variables in (5) are x_t, the investments in new capital, $u_{v,t}$, investments in the various vintages of old capital, and N_t. The unknown capital amount $k_{v,t}$ and product output y_t are determined from (2) and (1) respectively for $0<t<\infty$, $-\infty<v\leq t$.

Definition and standard properties of the equilibrium

The problem (1)-(4) entails no uncertainty so the solution of the model will be a perfect-foresight equilibrium for both firms and consumers. Wages must clear the labor market and interest rates must be such that representative consumer is happy to consume his wages and dividends at each date. The analysis simplifies if we assume that technological change occurs at a constant rate, so that efficiency of capital vintages is exponential:

$$z_v = e^{\gamma v}, \qquad \gamma > 0.$$

In this case, a constant-growth equilibrium has the following three well-known properties:

First, the rate of interest, r, is constant, and the isoelastic utility function U implies that consumption must grow at a constant rate which we shall denote by g. In the consumer's optimal savings problem, it is known (e.g., Rebelo 1991, p. 504) that the relation between the rate at which consumption grows ad the rate of interest is

$$g = \frac{r - \rho}{\eta}. \tag{6}$$

Second, the fraction of income invested is constant, and (3) implies that K grows at the rate $\gamma + g$. Taking logs of both sides of (1) and taking the time derivative yields the solution for the growth rate

$$g = \frac{\gamma\alpha}{1-\alpha}. \tag{7}$$

Third, the labor market is frictionless and all workers are always employed at a common wage. In other words, the markets for labor clear at the wage sequence (w_t). The equilibrium wage at each date equals the marginal product of labor evaluated at the equilibrium capital-labor ratio at that date:

$$w_t = (1-\alpha)y_t = (1-\alpha)K_t^{\alpha}. \tag{8}$$

This is because the representative firm sets $N_t = 1$ so that $y_t = K_t^{\alpha}$.

A firm's investment decision

The firm faces no convex adjustment costs of capital, only a constant cost of buying each type of capital. Therefore we would expect the optimal policy to result in the firm setting the marginal product of each type of its capital to its user cost. Since the price of each type of capital is normalized to unity, the various vintages of capital all have the same user cost of $r + \delta$ per unit. The Appendix shows that at an interior solution for investment in vintage v (i.e., at $u_{v,t} > 0$) the marginal product of $k_{v,t}$ must indeed equal its user cost; that is

$$\alpha z_v^{\beta} y_t^{(\alpha-\beta)/\alpha} A_{t-v} k_{v,t}^{\beta-1} = r + \delta, \qquad 0<v\leq t,\ \ 0< t<\infty. \tag{9}$$

If the left-hand side of (9) is less than its right-hand side, then $u_{v,t} = 0$, which means that at date t the firm does not invest in capital of vintage v.

The aim of this paper is to (i) characterize the evolution of each type of capital as it ages, and (ii) derive the optimal investment policy $u_{v,t}$. If we can assume that the solution is always interior (and this assumption we shall return to), then (i) and (ii) are straightforward. It will turn out, however, that when σ is sufficiently large (i.e., when β is close to unity), $u_{v,t}$ becomes zero after some point.

A diminishing rate of growth of A.--- If we allow arbitrary learning functions A then interiority will sometimes hold and sometimes not. However, we shall assume throughout that the growth of learning is a diminishing function of a vintage's age. That is, if we define $s = t\text{-}v$ to denote the age of vintage-v capital, we shall assume that $\frac{1}{A}\frac{\partial A}{\partial s}$ is a monotonically declining function, then the problem is well behaved in the sense that there is an interval of age of each capital stock on which investment is positive, and then when a critical age is reached, investment ceases and the capital is allowed to gradually depreciate to zero. That critical age may, however, be infinity in which case investment in each vintage continues for ever. Whether investment continues for ever or not depends on whether the parameter κ defined by

$$\kappa = \frac{\gamma(\sigma - 1) - \delta - g}{\sigma} = \gamma\beta - \delta(1-\beta) - \frac{\gamma\alpha}{1-\alpha}(1-\beta)$$

is positive or negative, and on whether $\frac{1}{A}\frac{\partial A}{\partial s}$ remains positive for ever, as it will be, for example, if $A(s) = 1 - e^{-\varphi s}$ or if $A(s) = s^{\varphi}$.

Proposition 1: Investment in a vintage continues (i.e., $u_{v,t} > 0$) up to the point where

$$\frac{1}{A}\frac{\partial A}{\partial s} = \kappa \tag{P1}$$

If κ < 0 so that the left-hand side of (P1) always exceeds the right-hand side, then investment in a vintage continues for ever. Capital $k_{v,t}$ peaks at the age s that satisfies

$$\frac{1}{A}\frac{\partial A}{\partial s} = \gamma \frac{\sigma - 1}{\sigma}. \tag{P2}$$

We shall prove this proposition in the process of deriving the optimal capital stocks and investment levels. Before proving it, we make a few remarks about it.

First, investment continues beyond the point where capital peaks. This is illustrated in Figure 1, which is drawn on the assumption that $\frac{A'(0)}{A(0)} > \kappa > 0$. Since investment in old capital eventually ends, presumably this is the realistic case. Then k peaks at date t_1 whereas investment ends at date $t_2 > t_1$. After investment ceases, capital simply depreciates to zero, hence $k_{v,t} = k_{v,v+t_2} e^{-\delta(t-t_2)}$ for $t > t_2$.

Second, in accord with the intuition contained in Figure 1 of Chari and Hopenhayn (1991), the higher the elasticity of substitution, the earlier investment stops, and the earlier the

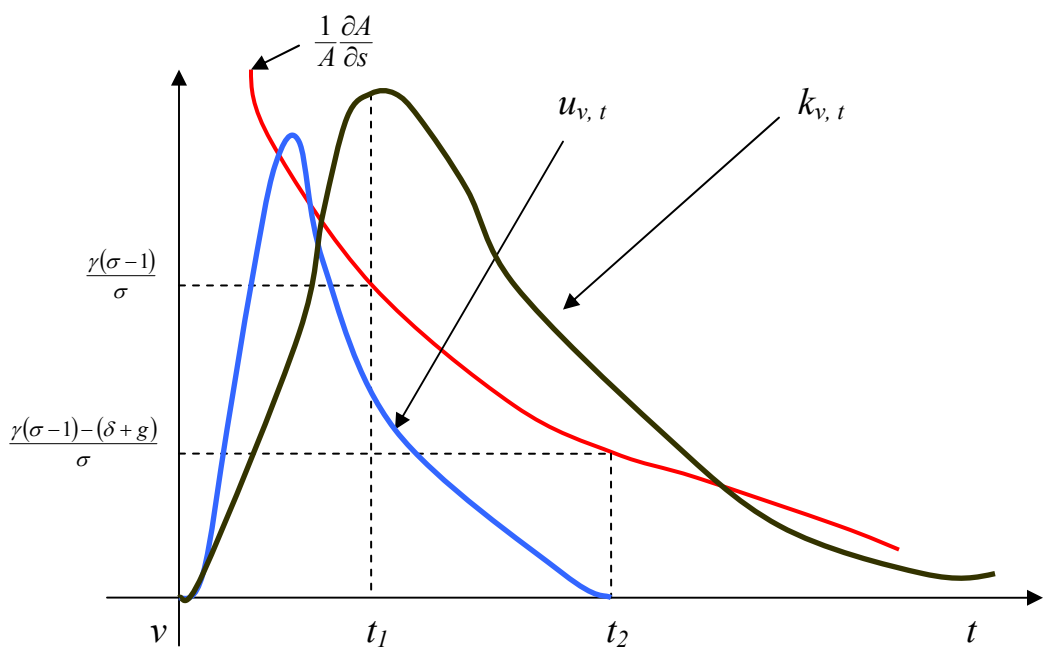

Figure 1: Investment and capital as a function of age when $\kappa > 0$.

capital stock peaks. This is because the right-hand-sides of (P1) and (P2) are both increasing in σ.

Third, as $\sigma \to \infty$, $\kappa \to \gamma$, and therefore t_1 and t_2 are both positive. Each generation of capital receives investment for a non-degenerate interval of time.

(i) *Optimal capital stocks*

Solving (9) we obtain

$$k_{v,t} = \left(\frac{\alpha}{r+\delta} z_v^{\beta} y_t^{(\alpha-\beta)/\alpha} A_{t-v} \right)^{1/(1-\beta)} \tag{10}$$

Now $y_t^{(\alpha-\beta)/\alpha} \propto \exp\left(\left(\frac{\alpha-\beta}{\alpha} \right) gt \right)$ and $z_v^{\beta} \propto \exp(\beta\gamma v) = \exp\left(\frac{\beta(1-\alpha)}{\alpha} gv \right)$. Substituting into (10) we see that since $\left(\frac{\alpha-\beta}{\alpha} \right) gt + \frac{\beta(1-\alpha)}{\alpha} gv = (1-\beta)gt - \frac{\beta(1-\alpha)}{\alpha} g(t-v) = (1-\beta)gt - \beta\gamma(t-v)$, (10) takes on the form

$$k_{v,t} = e^{gt} \chi_{t-v}, \tag{11}$$

where

$$\chi_s = \bar{k} e^{\frac{-\gamma\beta}{1-\beta} s} A_s^{\frac{1}{1-\beta}} \tag{12}$$

is a quasi-stationary component that depends on the capital age $s = t-v$ only, and where the constant can be shown to equal

$$\bar{k} = \left(\frac{\alpha}{r+\delta} \right)^{\frac{1}{1-\alpha}} \left(\int_0^{\infty} e^{-\frac{\gamma\beta}{1-\beta} s} A_s^{\frac{1}{1-\beta}} ds \right)^{\frac{\alpha-\beta}{\beta(1-\alpha)}} \tag{13}$$

Since output also grows at the rate *g,* the diffusion of capital relative to output is

$$\frac{k_{v,t}}{y_t} \propto \chi_{t-v}\,, \tag{14}$$

and it peaks at age $s = t\text{-}v$ at which $\partial\chi_{t-v}/\partial t = 0$, which implies that at the peak, $\frac{1}{A}\frac{\partial A}{\partial t} = \beta\gamma$. Since the elasticity of substitution is σ = 1/(1-β), the peak occurs at the point where

$$\frac{1}{A}\frac{\partial A}{\partial t} = \gamma\frac{\sigma-1}{\sigma}, \tag{15}$$

as claimed in (P2) of Proposition 1.

(ii) *Optimal investment*.

New capital.---Combining (11)-(13) with (3) at *v*=*t*, optimal investment in new capital is

$$x_t = k_{t,t} = \chi_0 e^{gt} = \bar{k}A_0^{\frac{1}{1-\beta}}e^{gt}\,. \tag{16}$$

Since this quantity grows as fast as income, the fraction of income invested in new capital is constant. If A_0=0 (as in the simulations below), there is no investment in new capital.

Old capital.---Since $\frac{\partial k_{v,t}}{\partial t} = -\delta k_{v,t} + u_{v,t}$, we differentiate (11) and use (12) to find that

$$u_{v,t} = e^{gt}\tilde{u}_{t-v}\,, \tag{17}$$

where $\tilde{u}_{t-v}$ depends on the capital age s=*t*–*v* only and is given by

$$\tilde{u}_s = \bar{k}e^{-\frac{\gamma\beta}{1-\beta}s}\left[\left(\delta + \frac{\gamma\alpha}{1-\alpha} - \frac{\gamma\beta}{1-\beta}\right)A_s^{\frac{1}{1-\beta}} + \frac{d}{ds}A_s^{\frac{1}{1-\beta}}\right]. \tag{18}$$

The exact condition for investment in the old, age-*s* capital to be positive is

$$\left(\delta+\frac{\gamma\alpha}{1-\alpha}-\frac{\gamma\beta}{1-\beta}\right)A_s^{\frac{1}{1-\beta}}\geq-\frac{d}{ds}A_s^{\frac{1}{1-\beta}}.$$

Letting $A_s^{\frac{1}{1-\beta}}\equiv m$, this condition is equivalent to $\frac{d\ln(m)}{ds}\geq\frac{\gamma\beta}{1-\beta}-\delta-\frac{\gamma\alpha}{1-\alpha}$. But $\frac{d\ln(m)}{ds}=\frac{1}{1-\beta}\frac{1}{A}\frac{\partial A}{\partial s}$, and so the investment-positivity condition reduces to $\frac{1}{A}\frac{\partial A}{\partial s}\geq\gamma\beta-\delta(1-\beta)-\frac{\gamma\alpha}{1-\alpha}(1-\beta)=\frac{\gamma(\sigma-1)-(\delta+g)}{\sigma}=\kappa,$ as claimed in (P1) of Proposition 1.

If learning is bounded, then the left-hand side converges to zero for large s, as drawn in Figure 1.

Age distributions of capital and Investment

The age distributions of capital and first-order dominance---In order to compare the implications of our model to Chari & Hopenhayn (1991) and specifically in terms of their first figure, we normalize the capital age-distribution in the simulation below as

$$\hat{k}_s=\frac{\chi_s}{\int_0^\infty\chi_u du}=\frac{e^{-\frac{\gamma\beta}{1-\beta}s}A_s^{\frac{1}{1-\beta}}}{\int_0^\infty e^{-\frac{\gamma\beta}{1-\beta}u}A_u^{\frac{1}{1-\beta}}du}. \tag{20}$$

The constant (13) is not presented in (20), so, $\hat{k}_s$ depends on parameters β and γ only.

In terms of equations (10) and (11), the term e^{gt} is common to technologies of all ages and does not affect the age distribution. Rather, what affects the rate at which the equilibrium capital stock declines with age are the parameters β and γ. It follows from (20) that the age profile $\hat{k}_s$ of capital becomes more pushed towards zero as γ rises. Indeed, the numerator of (20) at a given s is smaller for a larger γ. In addition, the denominator of (20) is larger for larger γ (and tends to the infinity as γ strives to zero). Since $A < 1$, a rise in β has the same effect on (20). This we have proved

Proposition 2: The age distributions of capital satisfy first-order stochastic dominance in γ, and when $A(s)\leq 1$ for all s (as in eq. (19) for example), in β as well. That is,

$$(\beta,\gamma) > (\beta',\gamma') \Rightarrow \hat{k}(s;\beta,\gamma) < \hat{k}(s;\beta',\gamma'). \tag{21}$$

This is quite intuitive – technological process makes old technologies obsolete, and this progress is measured by γ. Also, a higher β raises the elasticity of substitution and reduces the essentialness of old capital.

To keep capital and investment in the same units, we apply the same normalization to the age-profile of investment in old capital:

$$\hat{u}_s = \frac{\tilde{u}_s}{\int_0^\infty \chi_u du}.$$

Remarks.---In a related model without learning, Jovanovic (2009a) provides conditions for investment in old capital to be positive along a balanced-growth path. Such a characterization is quite limited for it says nothing about the investment profile.

An alternative formulation is to assume that there are many goods entering the utility function in the Spence-Dixid-Stiglitz tradition, but that each good is produced by a different vintage of capital, with new vintages coming on line over time, as in Jovanovic (2009b)

3. Evidence and simulation

Learning parameters.---We assume the learning function

$$A(s) = 1 - e^{-\varphi s}, \qquad \varphi > 0. \tag{19}$$

Bahk and Gort (1993, Sec. IV) find that most capital learning is over by the fifth or sixth year of the capital's age. Assuming that exactly 95 percent of the learning occurs by the end of the fifth year, the equation $1\text{-}e^{-5\varphi} = 0.95$ must hold, and this gives us the value $\varphi = 0.6$.

Share of capital and embodied technological progress.---Since real income per head tends to grow at roughly 1.5 percent per year, eq. (7) indicates that

$$\frac{\gamma\alpha}{1-\alpha} = 0.015 .$$

This is a joint restriction on α and γ. If we assume that K denotes a broad capital aggregate and that L denotes raw labor, then perhaps a value of $\alpha = 0.667$ is reasonable. In that case we must have $\gamma = 0.0075$.

The elasticity of substitution.---The parameter $\sigma = 1/(1-\beta)$ is not precisely estimated in the literature and so we shall follow Chari and Hopenhayn (1991) and entertain various values of β between zero and one (that imply values of σ between one and infinity).

The utility-function parameters.---The remaining parameters are the rate of utility discount, ρ, and the utility-curvature parameter η. They determine the rate of interest in (6). Since these parameters do not enter the other equations and the simulations, and since we are not sure what interest rate to apply to the discounting of the typical stream of incomes that investment in capital yields, we shall leave these two parameters unspecified.

The rate of depreciation is typically taken to be around 0.075, and so we shall assume this value for δ. Thus we may summarize our parameter-value choices as follows

α	β	γ	δ	φ
0.667	$0 < \beta < 1$	0.0075	0.075	0.6

Finally, the parameter κ ranges from -0.09 when $\beta = 0$ (so that $\sigma = 1$), to 0.0075 when $\beta = 1$ (so that $\sigma = \infty$).

Figure 2A plots the age distribution of capital for selected values of β between zero and 0.75.

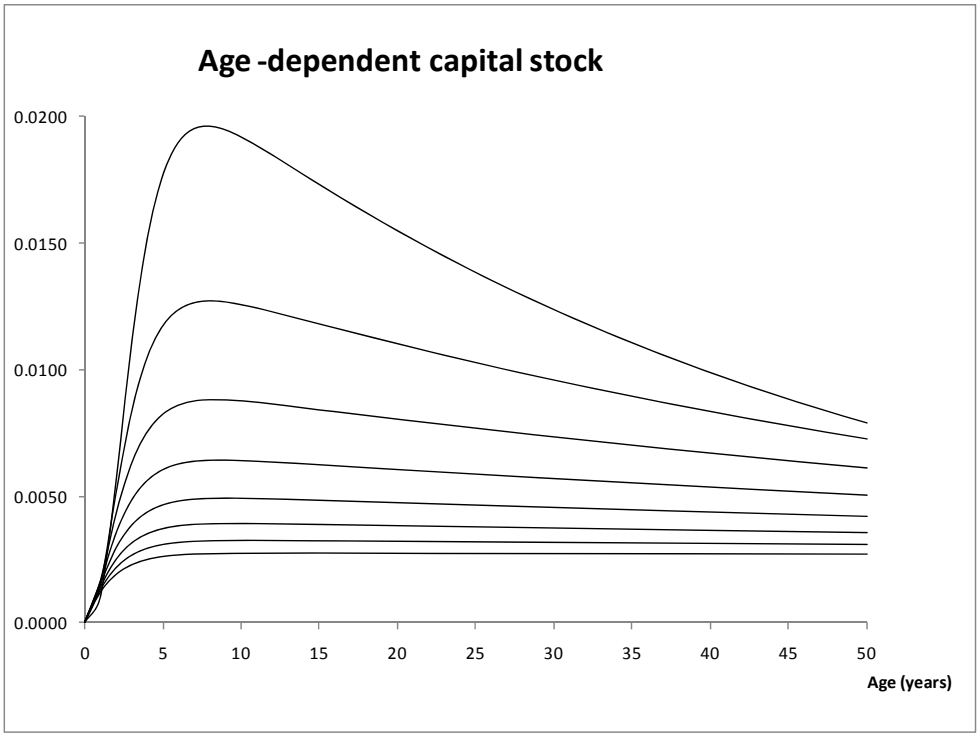

Figure 2A. Simulated age-distribution of capital for β = 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75

In Figure 2A, the curves must eventually cross since the area under them is the same. These crossings do not show up in Figure 2A because they occur after the technologies are 50 years old. The high-β densities peak early and then decay fairly rapidly. The low-β densities peak even earlier, but they do not decay as rapidly, and eventually overtake the densities pertaining to values of β that are higher.

Figure 2B plots the corresponding densities of investment which peak around the inflection points of the densities of capital.

Figure 2B. Investment into the aged capital for β = 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75

Figures 2C and 2D deal with values β quite close to unity and, consequently, with high elasticities of substitution. Here the crossing of the all the densities occurs within the first 50 years of the lives of the technologies.

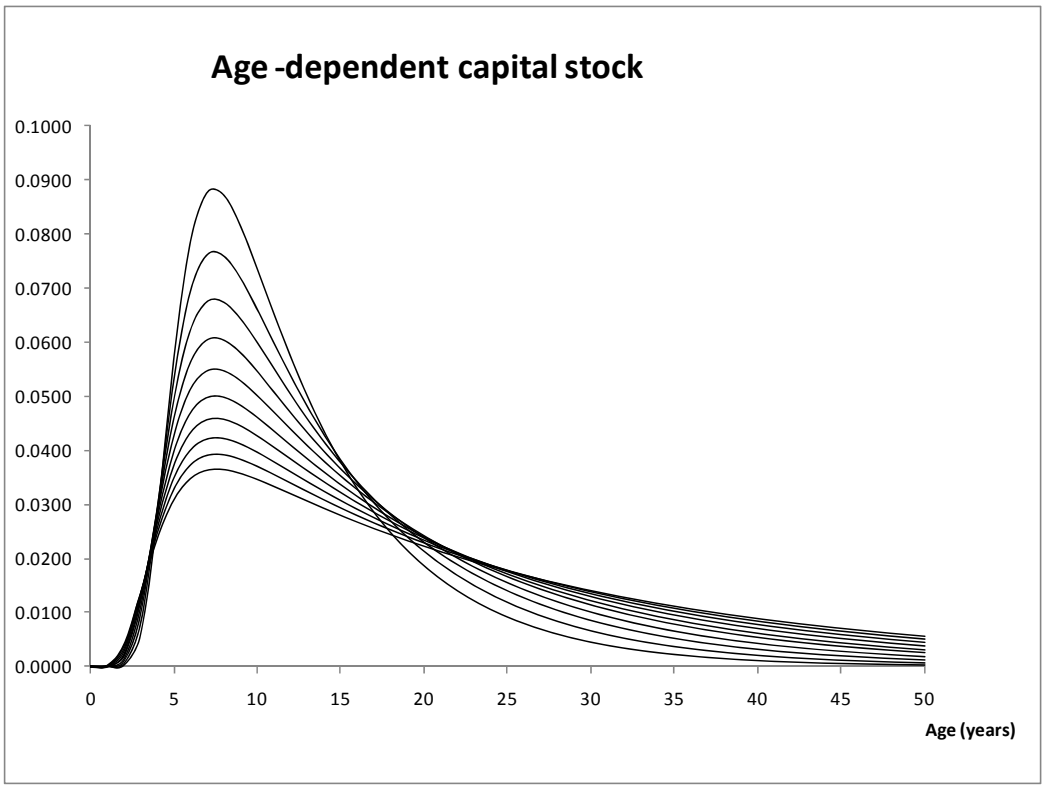

Figure 2C. Simulated age-distribution of capital for β=0.86, 0.87, 0.88, 0.89, 0.9, 0.91, 0.92, 0.93, 0.94, 0.95.

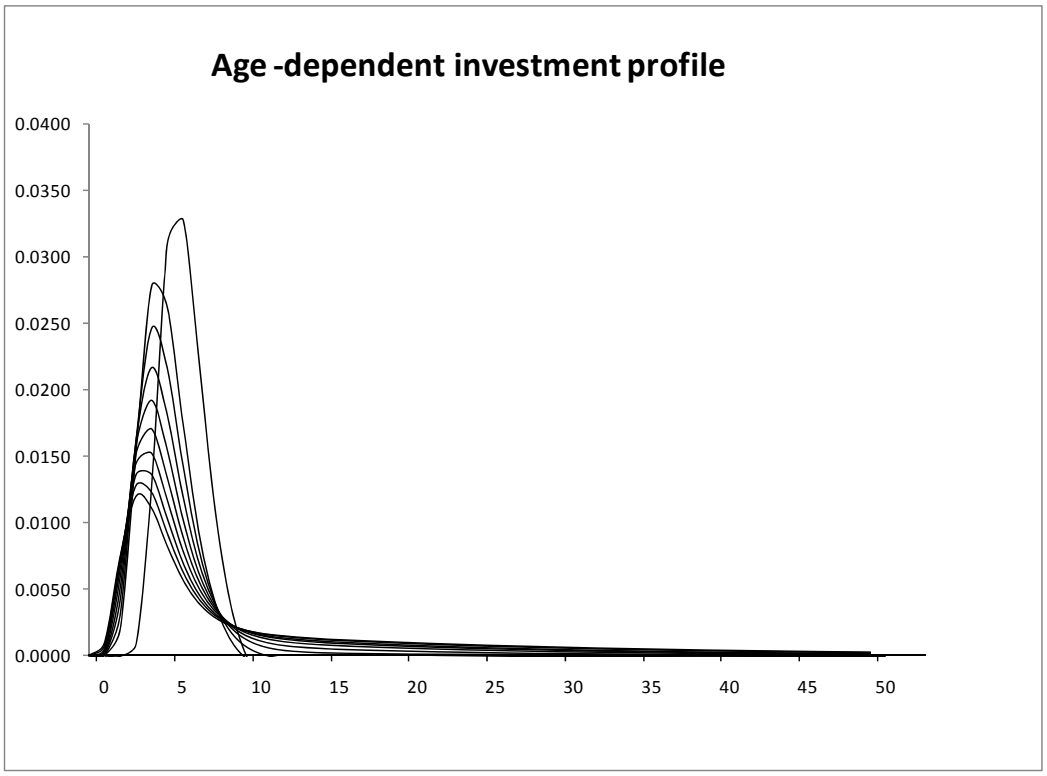

Figure 2D. Investment into capital for β = 0.86, 0.87, 0.88, 0.89, 0.9, 0.91, 0.92, 0.93, 0.94, 0.95.

In Figure 3 the black solid lines depict the case where $\beta = 0.95$. The lower bell-shaped curve is the investment rate, and the investment ceases when the technology is 9 years old. The peak of the capital-stock distribution occurs slightly before that. This is the case that corresponds most closely to the patterns depicted in Figure 1. The two gray bell-shaped curves represent the distributions of capital and investment when $\beta = 0.90$, and in this case investment continues for ever, but becomes quite small beyond age 15.

S-shaped diffusion of technology.---Both the capital stock and investment have an initial convex portion, and of course both distributions eventually reach a peak, so that both capital and investment have an initial S shape. This follows because our simulations assume that learning takes the form given by equation (19), which implies that $A_0 = 0$. This low initial level of a technology's productivity discourages investment in that technology early on in its life.

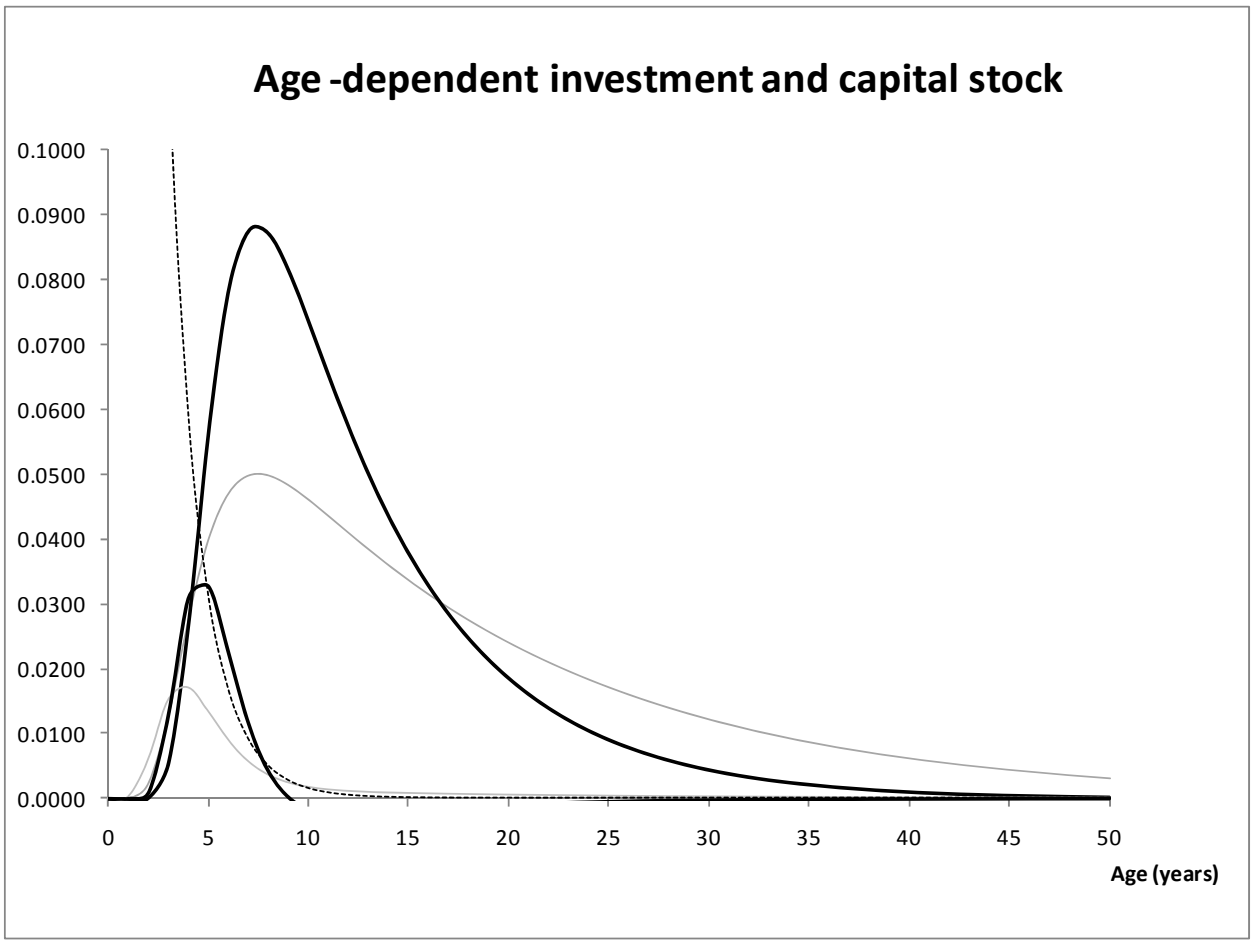

Figure 3. Age-distribution of capital and investment for β = 0.95 (κ=0.00262, black solid lines) and for β = 0.9 (κ = -0.00225, gray lines) . The dotted line is $\frac{1}{A}\frac{\partial A}{\partial t}$.

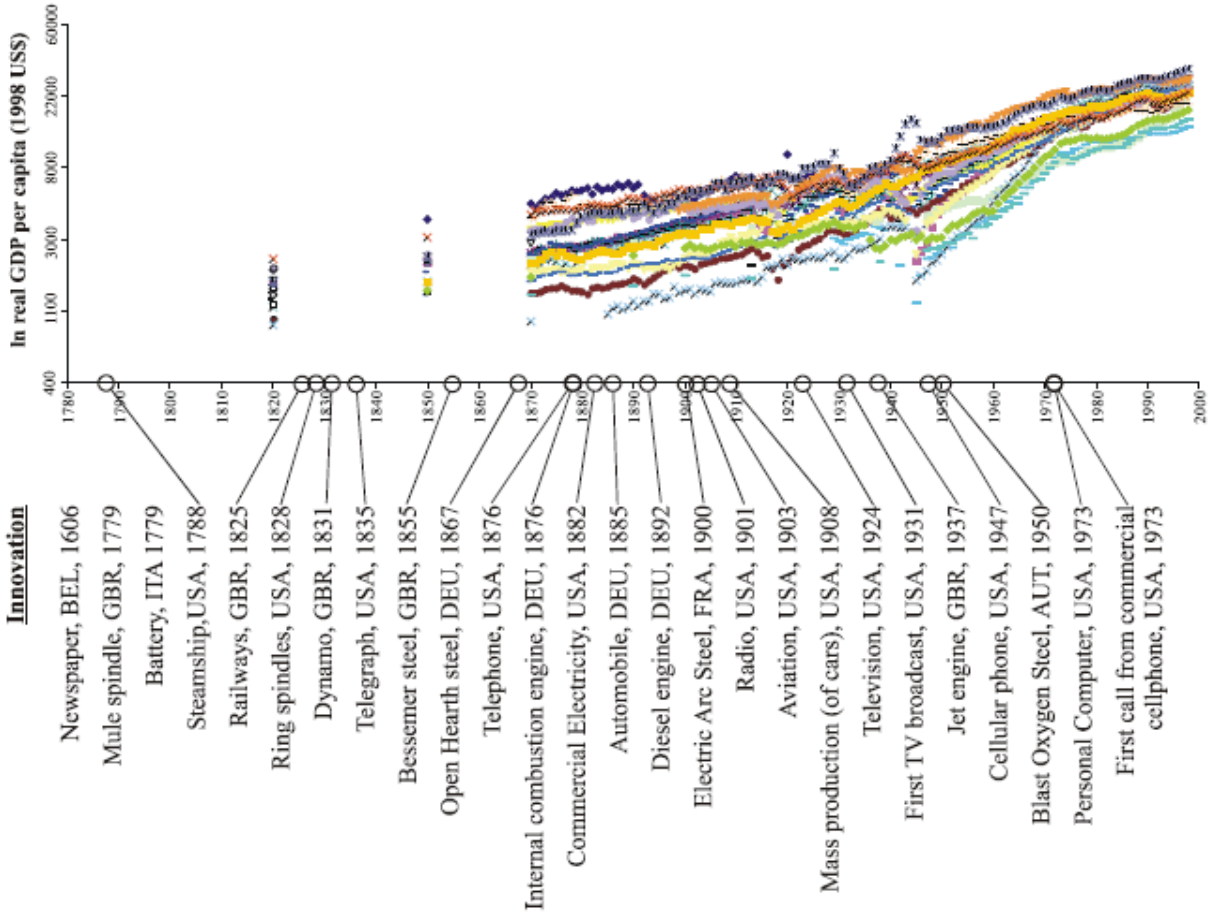

Figure 4: *The diffusion of some of the historically most important technologies*

4. Comparison to data

A contrast between the calibrated version and the facts is that broad technologies diffuse more slowly. Peak investment is predicted to be somewhere between year 5 and year 10 following a technology's introduction. Figure 4 is taken from Comin and Hobijn (2004). Presumably this discrepancy is because of the level of aggregation so that each of the Comin-Hobijn technologies is really a collection of technologies of different vintages so that as the use of each technology grows, earlier generations of that technology are being phased out by later generations.

At a lower level of aggregation we do see the pattern that Figures 2 and 3 dispolay, namely, waves of usages that rise and then fall. Figure 5, taken From Jovanovic and MacDonald (1994) shows the diffusion of various generations of DRAM chips.

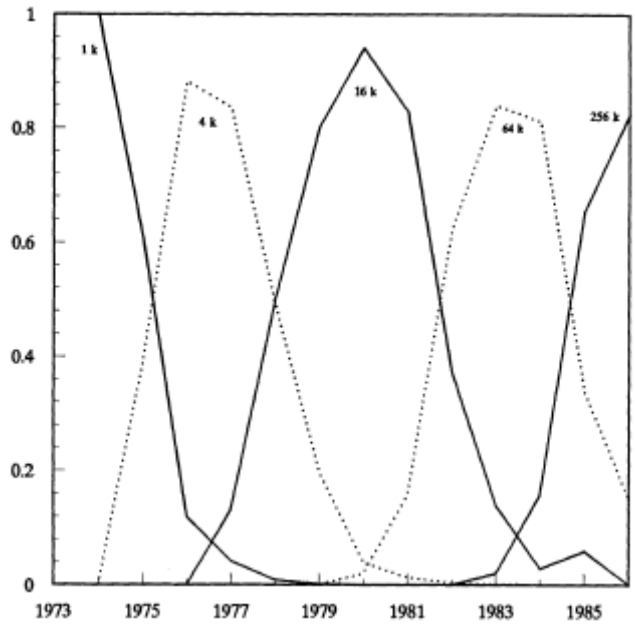

Figure 5: Diffusion of DRAM chips

Here the level of aggregation seems finer than Figures 2 and 3 would require, in that the peak adoption occurs when a technology is about four years old.

5. Conclusion

We have derived equilibrium paths of capital and investment of vintage technologies, and we have showed that economies in which technological progress is faster and the elasticity of substitution in production is higher, will have an age-distribution of capital that first-order dominates the age-distribution of capital in other economies.

References

1. Arrow, Kenneth (1962). “The Economic Implications of Learning by Doing.” *Review of Economic Studies* **29**, no. 3, 155-173.
2. Atkeson, Andrew and Patrick Kehoe (2005). “Modeling and Measuring Organization Capital.” *Journal of Political Economy* **113**, no. 5, 1026-1053.
3. Bahk, Byong-Hyong, and Michael Gort (1993). “Decomposing Learning by Doing in New Plants.” *J.P.E.* **101**, no. 4, 561–583.
4. Boddy, Raford and Michael Gort (1971). The Substitution of Capital for Capital. *Review of Economics and Statistics* **53**, No. 2, 179-188.
5. Boucekkine, R., F. del Rio and O. Licandro (1999). Endogenous vs. exogenously driven fluctuations in vintage capital models, *Journal of Economic Theory* **88**, 161-187.
6. Chari, V.V. and Hugo Hopenhayn (1991), "Vintage Human Capital." *Journal of Political Economy* **99**, no. 6, 1142-1165.
7. Comin, Diego and Bart Hobijn. (2004) “Cross-Country Technology Adoption: Making the Theories Face the Facts.” Journal of Monetary Economics 51, 39-53.
8. Cooley, Thomas, Jeremy Greenwood, and Mehmet Yorukoglu (1997). “The replacement problem.” Journal of Monetary Economics **40**, no. 3, 457-499.
9. Goetz, R., N. Hritonenko, and Y. Yatsenko (2008), The optimal economic lifetime of vintage capital in the presence of operating cost, technological progress, and learning, *Journal of Economic Dynamics and Control* **32**, 3032-3053.
10. Greenwood, Jeremy & Hercowitz, Zvi & Krusell, Per, 1997. "Long-Run Implications of Investment-Specific Technological Change," American Economic Review
11. Hritonenko, N. and Y. Yatsenko (1996), *Modeling and Optimization of the Lifetime of Technologies*, Dordrecht: Kluwer Academic Publishers.
12. Hritonenko, N. and Y. Yatsenko (2005), Turnpike and optimal trajectories in integral dynamic models with endogenous delay, *Journal of Optimization Theory and Applications* **127**, 109-127.
13. Hritonenko, N. and Y. Yatsenko (2007), The structure of optimal time- and age-dependent harvesting in the Lotka-McKendrik population model, *Mathematical Biosciences* **208**, 48-62.
14. Johansen, Leif (1959). “Substitution versus Fixed Proportions Coefficients in the Theory of Economic Growth: A Synthesis.” *Econometrica* **27**, 157-76.
15. Jovanovic, B. and G. MacDonald. “Competitive Diffusion.” (1994) *Journal of Political Economy* 102, no. 1, 24-52.
16. Jovanovic, B. (1998) Vintage capital and inequality, *Review of Economic Dynamics* **1**, 497-530.

17. Jovanovic, B. (2009a) When should firms invest in old capital, *International Journal of Economic Theory* **5**, 107-123.
18. Jovanovic, B. "The Technology Cycle and Inequality." (2009) *Review of Economic Studies* 76, no. 2, 707-729
19. Klenow, P. (1998) Learning Curves and the Cyclical Behavior of Manufacturing Industries. *Review of Economic Dynamics* **1**, 531–550.
20. Kredler, Matthias. "Experience vs. Obsolescence: A Vintage-Human-Capital Model." Univ. Carlos III Madrid June 2008.
21. Prescott, Edward and John Boyd (1987), "Dynamic Coalitions: Engines of Growth." *American Economic Review. Papers and Proceedings* **77**, 63-67.
22. Rebelo, Sergio. (1991). Long-Run Policy Analysis and Long-Run Growth. *Journal of Political Economy* **99**, No. 3, 500-521
23. Solow, R. (1960). "Investment and Technical Progress.', in *Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, CA.
24. Xepapadeas, A. and A. De Zeeuw (1999), Environmental policy and competitiveness: The Porter hypothesis and the composition of capital, *Journal of Environmental Economics and Management* **37**, 165-182.

5. Appendix

To derive necessary conditions for the firm optimization problem defined in (5), we apply the classic Lagrange multiplier method adjusted to vintage capital models as in Hritonenko and Yatsenko (1996, 2005). Below we use the standard notations $f(t) = f_t$ and $g(v,t) = g_{vt}$ for all involved functions.

We consider a representative firm and assume that its employment remains at its equilibrium level $N(t) = 1$ at all dates. This is an equilibrium constraint that the price-taking firm does not face; rather it treats the wage as exogenous to its choice of K. We concentrate on the firm investment decisions $x(t)$ and $u(v,t)$.

The Lagrangean of the firm problem (1)-(5) with taking the optimal equilibrium wage (8) into account can be written as

$$
\begin{aligned}
L = \int_0^\infty e^{-rt} \Big\{ & K^\alpha(t) - x(t) - \int_{-\infty}^t u(v,t)dv + \\
& + \int_0^t \lambda(v,t)\Big(e^{-\delta(t-v)} x(v) + \int_v^t e^{-\delta(t-s)} u(v,s)ds - k(v,t) \Big) dv + \\
& + \int_{-\infty}^0 \lambda(v,t)\Big(e^{-\delta(t-v)} k_0(v) + \int_0^t e^{-\delta(t-s)} u(v,s)ds - k(v,t) \Big) dv \ \Big\} dt,
\end{aligned}
$$

where $\lambda(v,t)$ is the unknown Lagrange multiplier function for the constraints (3) and (4). Using small variations $\delta u(v,t)$, $\delta x(t)$, $\delta y(t)$, and $\delta k(v,t)$ of the controls u, x, y, and k, we obtain that the corresponding increment δL of functional L is

$$
\begin{aligned}
\delta L \approx \int_0^\infty e^{-rt} \Big\{ & \delta y(t) - \delta x(t) - \int_{-\infty}^t \delta u(v,t)dv + \int_{-\infty}^0 \lambda(v,t)\Big(\int_0^t e^{-\delta(t-s)} \delta u(v,s)ds - \delta k(v,t) \Big) dv \\
& + \int_0^t \lambda(v,t)\Big(e^{-\delta(t-v)} \delta x(v) + \int_v^t e^{-\delta(t-s)} \delta u(v,s)ds - \delta k(v,t) \Big) dv \ \Big\} dt,
\end{aligned}
\tag{A1}
$$

where, by equations (1) and (2),

$$
\delta y(t) = \alpha \Big(\int_{-\infty}^t A(t-v) z^\beta(v) k^\beta(v,t) dv \Big)^{(\alpha-\beta)/\beta} \int_{-\infty}^t A(t-v) z^\beta(v) k^{\beta-1}(v,t) \delta k(v,t) dv \, .
$$

Next, interchanging limits of integration in the double integrals and making other routine transformations, we represent (A1) as:

$$
\begin{aligned}
\delta L \approx & \int_0^\infty \Big[\int_t^\infty e^{-rs-\delta(s-t)} \lambda(t,s)ds - e^{-rt} \Big] \delta x(t)dt + \\
& + \int_0^\infty \int_{-\infty}^t \Big[\int_t^\infty e^{-rs-\delta(s-t)} \lambda(v,s)ds - e^{-rt} \Big] \delta u(v,t)dvdt + \\
& + \int_0^\infty \int_{-\infty}^t e^{-rt} [\alpha y^{(\alpha-\beta)/\beta}(t) A(t-v) z^\beta(v) k^{\beta-1}(v,t) - \lambda(v,t)] \delta k(v,t)dvdt.
\end{aligned}
\tag{A2}
$$

Setting the coefficients at $\delta x(t)$, $\delta u(v,t)$, and $\delta k(v,t)$ in (A2) equal to zero and excluding the dual variable $\lambda(v,t)$, we obtain the first-order extremum condition for an interior solution $u>0$, $x>0$, namely

$$
1 = \alpha z^\beta(v) \int_t^\infty e^{-(r+\delta)(s-t)} y^{(\alpha-\beta)/\alpha}(s) A(s-v) k^{\beta-1}(v,s) ds
$$

or

$$\alpha z^{\beta}(v)\int_{t}^{\infty} e^{(r+\delta)s} y^{(\alpha-\beta)/\alpha}(s) A(s-v) k^{\beta-1}(v,s) ds = e^{-(r+\delta)t}, \quad 0<v\leq t, \ 0< t<\infty. \tag{A3}$$

Equality (A3) at $v<t$ delivers the extremum condition for interior $u(v,t)>0$, and the condition for interior $x(t)>0$ at $v=t$. Differentiation of (A3) with respect to t gives the necessary condition (9) represented in the text.