Inflation and Unemployment in General Equilibrium*

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Abstract

When labor is indivisible, there exist efficient outcomes with some agents randomly unemployed (Rogerson 1988). We integrate this idea into the modern theory of monetary exchange, where some trade occurs in centralized markets and some in decentralized markets (as in Lagos and Wright 2006 e.g.). This delivers a general equilibrium model of unemployment and money, with explicit microeconomic foundations. We show the implied relation between inflation and unemployment can be positive or negative, depending on simple preference conditions. Our Phillips Curve provides a long-run, exploitable, trade off for monetary policy; it turns out, however, that the optimal policy is the Friedman rule.

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1 Introduction

The following has been well known at least since the work of Rogerson (1988): efficient allocations in economies with indivisible labor generally have some agents, chosen at random, unemployed, while others are employed, even though they are ex ante identical; and these allocations can be supported as competitive equilibria where agents trade lotteries. As we discuss in detail below, another interesting feature of these economies emphasized in Rocheteau et al. (2006) is that agents act as if they have quasi-linear utility. It turns out that one can use this result to construct a fairly general yet very tractable model of monetary exchange, using search theory, where some trades occur in centralized markets and some trades occur in decentralized markets, as in Lagos and Wright (2005).

To understand this, note that what makes the Lagos-Wright model tractable is the assumption of quasi-linear utility, since this implies that agents exiting the centralized market all hold the same amount of money, regardless of their histories (assuming interior solutions). Thus, if one is willing to assume quasi-linear utility, one can avoid having to track the distribution of money in the decentralized market as a state variable.[1] The observations in Rocheteau et al. (2006) allow one to dispense with quasi-linear utility: as long as we have indivisible labor, identical results concerning the distribution of money holdings can be derived for any monotone and con-

[1] For models that are much less tractable, precisely because one has to keep track of the relevant distribution, see Green and Zhou (1998), Zhou (1999), Camera and Corbae (1999), Zhu (2003,2005) and Molico (2006). Earlier search-based models, such as Kiyotaki and Wright (1989,1993), Aiyagari and Wallace (1991), Shi (1995), or Trejos and Wright (1995), were also simple, but only because they avoided the issue by assuming agents could only hold $m \in \{0, 1\}$ units of money. An alternative approach that uses large families instead of quasi-linear utility to achieve tractability is provided by Shi (1997).

cave utility function (again assuming interior solutions). However, there are some potential advantages to using indivisible labor instead of quasi-linear utility as a building block for monetary theory, including the fact that it generates unemployment.

In this paper we take seriously the implications of indivisible-labor models with money for the relationship between inflation and unemployment. In other words, we study the Phillips curve in general equilibrium. This seems to us to be a natural exercise. In addition to Rogerson (1988), many well-known papers adopt the indivisible labor model, including Hansen (1985), Cooley and Hansen (1989), Christiano and Eichenbaum (1992), Kydland (1994), Lungqvist and Sargent (2006), and Prescott, Rogerson and Wallenius (2006). Since all of these papers take seriously indivisible labor in models either without money, or with money but without microfoundations, we think it is reasonable to study the relation between unemployment and inflation in versions of the model with money based on microfoundations.

The goal here is to derive results showing how the relation between inflation and unemployment depends on primitives of the model, and in particular preferences. With additive separability between goods consumed in the centralized and decentralized markets, we show the Phillips curve is vertical (unemployment and inflation are independent). Then we show that with a more general specification the Phillips curve can have a positive or negative slope, depending on the utility function in a straightforward way. The intuition is simple. Inflation is a tax on economic activity in sectors that use cash relatively intensively. To the extent that goods produced in these sectors are substitutes for (complements with) goods produced using indivisible labor, by reducing consumption of the former, inflation will increase (reduce)

consumption of the latter, which will reduce (increase) unemployment.

Our inflation-unemployment trade-off does not depend on any complicated features of the model, like sticky wages or prices, irrational expectations, imperfect information, etc. Thus we conclude that one does not have to work very hard to generate an interesting relation between inflation and unemployment. Now, in this paper, we do not take a stand on what the actual relation is in the data – that is an entirely different project. Rather, we focus on getting a simple and interesting relation in theory. Note also that the trade-off here is a long-run trade-off, and it is exploitable by policy makers: under conditions that we make precise, it is *feasible* for monetary policy to permanently reduce unemployment by increasing inflation, as some Keynesians (used to?) think. We prove, however, that it is *optimal* to reduce inflation to a minimum, as Friedman prescribed.

2 Basic Assumptions

Time is discrete. There is a $[0, 1]$ continuum of agents who live forever. Following Lagos and Wright (2005), we assume there are two type of markets in which these agents interact. One is a frictionless centralized market, or CM; the other is a decentralized market, or DM, with two main frictions: a double-coincidence problem detailed below, and anonymity, which precludes private credit arrangements. These frictions make money essential.[2] The stock of money evolves according to $\hat{M} = (1 + \gamma)M$, where $\hat{z}$ indicates the value of any variable z next period. New money is injected (or withdrawn if

[2] See Kocherlakota (1998) and Wallace (2001) for formal discussions of essentiality, and especially the role of anonymity. Generally, these frictions make *some* medium of exchange essential; for now we assume that central bank money is the only storable tangible asset, and hence the only possible medium of exchange. In other words, this paper is not about the coexistence of money and other assets.

$\gamma < 0$) via lump sum transfers in the CM; it changes nothing if alternatively we assume new money injections pay for government consumption. We are interested in equilibria where money is valued – i.e. the price of M is positive in the CM at every date – and choose dollars to be the unit of account.

There is a vector of nonstorable consumption goods $\mathbf{x} \in \mathbb{R}^J_+$ produced by firms in the CM using labor h. Agents also have an endowment of CM goods $\mathbf{e}$, plus 1 unit of indivisible labor, which means the commodity space restricts $h \in \{0, 1\}$. As is standard in indivisible labor models, agents will trade randomized consumption bundles, or lotteries. In the DM, there is a different nonstorable good q that is not produced, but all agents have endowment $\bar{q}$. Utility in an interval consisting of one CM and one DM is $\upsilon^j(q, \mathbf{x}, h)$, where $(\mathbf{x}, h)$ come from the CM, q comes from the subsequent DM, and j indicates a preference shock realized at the start of the DM, after $\mathbf{x}$ and h are chosen. A special case is $\upsilon^j(q, \mathbf{x}, h) = U(\mathbf{x}, h) + u^j(q)$, as used in the related monetary literature. Although we actually start with this specification, for reasons that will become clear, we want to allow nonseparability between $(\mathbf{x}, h)$ and q.

The preference shock is modeled as follows: with probability $\sigma \leq 1/2$ the utility function is υ^H, and with probability $1 - \sigma$ it is υ^L, where to generate gains from trade we assume $\partial\upsilon^H(q, \mathbf{x}, h)/\partial q > \partial\upsilon^L(q, \mathbf{x}, h)/\partial q \ \forall (q, \mathbf{x}, h)$.[3] As we said, trade in the DM is bilateral. For simplicity we use a specific (but common) matching technology: every agent that draws υ^H is matched

[3] It is straightforward to allow more general preference or endowment shocks, but this simple specification suffices for our purposes. This way of modeling DM gains from trade differs from the related literature (but see Berentsen and Rocheteau 2002), although it would matter little for the typical application. The advantage for us is that our DM is a pure exchange market – there is no production – so employment is unambiguously given by hours worked in the CM.

with one that draws υ^L, while only a set with measure $\sigma/(1-\sigma)$ of those that draw υ^L are matched. We call an agent with υ^H a buyer and an agent with υ^L who happens to be matched a seller, since there is generally a deal to be done where the latter transfers some q to the former in exchange for some cash. For future reference, let $u(q, \mathbf{x}, h) \equiv \upsilon^H(\bar{q}+q, \mathbf{x}, h) - \upsilon^H(\bar{q}, \mathbf{x}, h)$ and $c(q, \mathbf{x}, h) \equiv \upsilon^L(\bar{q}, \mathbf{x}, h) - \upsilon^L(\bar{q}-q, \mathbf{x}, h)$ denote the instantaneous gain for the buyer and instantaneous cost for the seller from such a deal.

To streamline the presentation, standard curvature conditions are imposed on utility to guarantee consumption of all goods is strictly positive. With indivisible labor, one obviously cannot do something similar for h, and interiority of (the probability of) employment is an issue to which we return in detail. Assume agents discount between the DM and next CM at rate $\beta \in (0,1)$, but for simplicity, not between the CM and DM. Let $W(m)$ denote the CM value function, which depends only on money balances, since in all other respects agents are identical in this market.[4] Let $V(m, \mathbf{x}, h)$ denote the DM value function, which depends on money balances, plus choices from the previous CM, since in general these interact with current DM consumption in the utility function.

3 The CM

We begin with the special case $\upsilon^j(q, \mathbf{x}, h) = U(\mathbf{x}, h) + u^j(q)$, so that we can show how the indivisible labor model works without the complication of nonseparable utility. In this case, with a slight abuse of notation, we write $u(q, \mathbf{x}, h) = u^H(\bar{q}+q) - u^H(q)$ as $u(q)$, $c(q, \mathbf{x}, h) = u^L(\bar{q}) - u^L(q-\bar{q})$ as $c(q)$,

[4] It would be trivial to allow them to differ with respect to their endowment, taxes, or dividend income, too, since as will be clear below, these enter symmetrically with m.

and $V(q, x, h)$ as $V(m)$. Then in the CM an agent chooses a probability of employment ℓ, consumption of $\mathbf{x}$ if employed $\mathbf{x}_1$, consumption of $\mathbf{x}$ if unemployed $\mathbf{x}_0$, and money to take to the DM $\widetilde{m}$, to solve

$$W(m) = \max_{\mathbf{x}_1, \mathbf{x}_0, \ell, \widetilde{m}} \{\ell U(\mathbf{x}_1, 1) + (1-\ell) U(\mathbf{x}_0, 0) + V(\widetilde{m})\} \tag{1}$$

$$\text{s.t. } \ell \mathbf{p}\mathbf{x}_1 + (1-\ell)\mathbf{p}\mathbf{x}_0 + \widetilde{m} \leq w\ell + m + \mathbf{p}\mathbf{e} + \gamma M + \Delta,$$

where $\mathbf{p}$ is a price vector, w is the wage, γM is the lump sum money transfer, and Δ is dividend income, all measured in dollars.

Since problem (1) may look nonstandard, we make several comments. First, it is the natural extension of the static indivisible labor model in Rogerson (1988) to incorporate money, taking (temporarily) $V(\widetilde{m})$ as given. Generally, it is well known that consumption $\mathbf{x}_h$ ought to be contingent on employment status, although in the case where U is separable between $\mathbf{x}$ and h, $\mathbf{x}_0 = \mathbf{x}_1$. The same is true here, and it is precisely because h and $\widetilde{m}$ enter the objective function separatively that the latter is not contingent on employment status. Also, Rocheteau et al. actually derive problem (1) from a model with standard Arrow-Debreu markets, and no lotteries, where agents trade state-contingent commodity bundles $[\mathbf{x}(s), h(s), \widetilde{m}(s)]$ and s represents a sunspot.[5] Finally, although this problem generally does not have a quasi-concave objective function, under very mild conditions Rocheteau et al. show it has a unique solution and the second-order conditions hold.

[5] These results are based on Shell and Wright (1993), where it is shown how to support the relevant allocations in nonconvex economies as sunspot equilibria instead of lottery equilibria. Sunspot equilibria have some advantages over lottery equilibria, in general, but given our structure they are equivalent, and so for simplicity here we stick to the latter; see Garratt, Keister and Shell (2004) for more discussion.

The Lagrangian for problem (1) can be written

$$\begin{aligned} W(m) &= \ell U(\mathbf{x}_1, 1) + (1-\ell) U(\mathbf{x}_0, 0) + V(\widetilde{m}) \\ &\quad + \frac{\lambda}{w}\left[w\ell + m + \mathbf{pe} + \gamma M + \Delta - \ell \mathbf{p}\mathbf{x}_1 - (1-\ell)\mathbf{p}\mathbf{x}_0 - \widetilde{m}\right]. \end{aligned}$$

Assuming $\ell \in (0,1)$, the first-order conditions are:

$$x_{1j} \ : \ U_j(\mathbf{x}_1, 1) = \lambda p_j / w, \ j = 1, ...J \tag{2}$$

$$x_{0j} \ : \ U_j(\mathbf{x}_0, 0) = \lambda p_j / w, \ j = 1, ...J \tag{3}$$

$$\ell \ : \ U(\mathbf{x}_1, 1) - U(\mathbf{x}_0, 0) = \frac{\lambda}{w}(\mathbf{p}\mathbf{x}_1 - \mathbf{p}\mathbf{x}_0 - w) \tag{4}$$

$$\widetilde{m} \ : \ V'(\widetilde{m}) = \lambda / w \tag{5}$$

$$\lambda \ : \ w\ell + m + \mathbf{pe} + \gamma M + \Delta - \ell \mathbf{p}\mathbf{x}_1 - (1-\ell)\mathbf{p}\mathbf{x}_0 - \widetilde{m} = 0 \tag{6}$$

Notice (2)-(4) constitute $2J+1$ equations that can be solved for the $2J+1$ unknowns $(\mathbf{x}_1, \mathbf{x}_0, \lambda)$ under weak regularity conditions.[6] A key observation is that $(\mathbf{x}_1, \mathbf{x}_0, \lambda)$ is independent of $\widetilde{m}$ and ℓ, and may depend on prices $(\mathbf{p}, w)$ but not on m. Given this, (5) can be solved for $\widetilde{m}$ independently of ℓ, and it may also depend on $(\mathbf{p}, w)$ but not m. Then, given $(\mathbf{x}_1, \mathbf{x}_0, \lambda, \widetilde{m})$, (6) can be solved for ℓ.

The important result is that the choice set for $\widetilde{m}$ implied by (5) is the same for all agents, independent of their m; and as long as $V'' < 0$ this choice is unique, so they all take the same $\widetilde{m}$ out of the CM. This also holds in the standard Lagos-Wright model, except there it relies on quasi-linear utility. Another result that carries over from that model is that $W(m)$ is

[6] See Rocheteau et al. for details, but basically we need to rule out the case $w - \mathbf{p}\mathbf{x}_1 + \mathbf{p}\mathbf{x}_0 = 0$, or equivalently $U(\mathbf{x}_1, 1) = U(\mathbf{x}_0, 0)$, which implies a singularity in system (2)-(4). This case occurs only for special preferences. Actually, even in this case, the results in Lemma 1 go through, we just need a slightly different argument.

linear. To see this, notice

$$\frac{\partial W}{\partial m} = \left[U\left(\mathbf{x}_0, 0\right) - U\left(\mathbf{x}_1, 1\right) + \lambda\left(w + \mathbf{p}\mathbf{x}_0 - \mathbf{p}\mathbf{x}_1\right)\right]\frac{\partial \ell}{\partial m} + \frac{\lambda}{w}.$$

The first term vanishes by (4) (the envelope theorem). Therefore $\partial W/\partial m = \lambda/w$, and we have already established that λ is independent of m. Summarizing these results:

Lemma 1 *Assuming $\ell \in (0,1)$ for all agents, they all have the same optimal choice for $\widetilde{m}$ in the CM, and $W'(m) = \lambda/w$, independent of m.*

How can people who are otherwise identical but differ with respect to m choose the same $(\mathbf{x}_1, \mathbf{x}_0, \widetilde{m}, \lambda)$? Obviously the answer is that they choose different labor supply, which in this kind of model means that they work with different probabilities. To be precise, (6) implies

$$\ell = \ell(m) = \frac{\mathbf{p}\mathbf{x}_0 + \widetilde{m} - \mathbf{p}\mathbf{e} - \gamma M - \Delta - m}{w + \mathbf{p}\mathbf{x}_0 - \mathbf{p}\mathbf{x}_1}. \tag{7}$$

If we average (7) across households, using $\mathbb{E}m = M$ and $\widetilde{m} = M(1+\gamma)$, aggregate labor supply is

$$\bar{\ell} = \frac{\mathbf{p}\mathbf{x}_0 - \mathbf{p}\mathbf{e} - \Delta}{w + \mathbf{p}\mathbf{x}_0 - \mathbf{p}\mathbf{x}_1}. \tag{8}$$

On the other side of the market, labor demand comes from profit maximization by the representative firm producing x_j,

$$p_j f_j'(\bar{\ell}_j) = w, \tag{9}$$

where $\bar{\ell}_j$ is total employment at this firm, and household dividends are $\Delta = \sum_j p_j f_j(\bar{\ell}_j) - w\bar{\ell}_j$.[7]

[7] It is merely for notational convenience that we assume a representative firm produces each good x_j and that Δ is the same for all households.

We are ready for a preliminary but useful step. It is preliminary because, for now, we take $V(\widetilde{m})$ to be some exogenous function (it will be derived endogenously soon enough). Given the above findings, including the result that the only element of household choices that is contingent on money holdings is $\ell(m)$, and assuming an interior solution, we can describe an equilibrium in the CM as follows.[8]

Definition: A CM equilibrium is given by prices $(\mathbf{p}, w)$, a choice for each household $[\mathbf{x}_1, \mathbf{x}_0, \ell(m), \widetilde{m}, \lambda]$, and employment $\bar{\ell}_j$ at the representative firm producing good x_j, $j = 1, ...J$, such that the household choices solve (2)-(6), $\bar{\ell}_j$ solves (9), and both goods and labor markets clear:

$$x_{1j}\bar{\ell} + x_{0j}(1-\bar{\ell}) = f_j(\bar{\ell}_j) + \bar{x}_j,\ j = 1, ...J \tag{10}$$

$$\sum\nolimits_j \bar{\ell}_j = \bar{\ell} \tag{11}$$

To characterize CM equilibrium, first use (9) to eliminate w and (4) to eliminate λ from (2)-(3), and notice that prices drop out:

$$f_j'(\bar{\ell}_j)U_j(\mathbf{x}_1, 1) = \frac{U(\mathbf{x}_0, 0) - U(\mathbf{x}_1, 1)}{1 - \sum_i \frac{x_{1i} - x_{0i}}{f_i'(\bar{\ell}_i)}},\ j = 1, ...J \tag{12}$$

$$f_j'(\bar{\ell}_j)U_j(\mathbf{x}_0, 0) = \frac{U(\mathbf{x}_0, 0) - U(\mathbf{x}_1, 1)}{1 - \sum_i \frac{x_{1i} - x_{0i}}{f_i'(\bar{\ell}_i)}},\ j = 1, ...J \tag{13}$$

Then substitute (11) into (10) to get:

$$x_{1j}\sum\nolimits_i \bar{\ell}_i + x_{0j}\left(1 - \sum\nolimits_i \bar{\ell}_i\right) = f_j(\bar{\ell}_j) + \bar{x}_j,\ j = 1, ...J \tag{14}$$

Now (12)-(14) constitute $3J$ equations in the same number of unknowns, $(x_{1j}, x_{0j}, \bar{\ell}_j)_{j=1...J}$. Given a solution, we can recover relative prices $p_j/w =$

[8] In the following definition we impose goods and labor market clearing, but there is also a money market clearing condition $\widetilde{m} = (1+\gamma)M = \hat{M}$ that we do not impose explicitly, because it holds automatically when the others hold (Walras law).

$1/f_j'(\bar{\ell}_j)$ from (9), the multiplier $\lambda = \beta_1 U_j(\mathbf{x}_1, 1) f_j'(\bar{\ell}_j)$ from (2), the absolute price level $w = \lambda/V'(\hat{M})$ from (5), total employment $\bar{\ell}$ from (11), and household employment contingent on m from (7). This fully describes CM equilibrium.

Several comments are in order. First, the above analysis is predicated on $0 < \ell(m) < 1$ for all m in the support of the distribution $F(m)$ of money holdings. We give conditions below to guarantee that this is valid. Second, it almost goes without saying that we get equilibrium unemployment when $\ell(m) < 1$: some agents randomly get $h = 0$ while other fundamentally identical agents get $h = 1$. Third, one might recognize a version of the classical dichotomy: for this specification, (10)-(13) determine the real allocation, while inserting $\widetilde{m} = \hat{M}$ into (5) merely determines the price level.[9]

To close this section we sketch the following results. First, we claim the obvious symmetric planner problem for the CM allocation has a unique solution; see the Appendix for the full statement of the planner problem and the proof, which is not trivial because the objective function is not quasiconcave. Also, this allocation satisfies the same conditions as CM equilibrium given in (12)-(14). Hence, the following is automatic:

Lemma 2 *Given $V'(m)$, CM equilibrium exists, is unique, is efficient, and depends on M only in terms of the absolute price level.*

[9] As Sargent (1979) puts it, "A macroeconomic model is said to dichotomize if a subset of equations can determine the values of all real variables with the level of the money supply playing no role in determining the equilibrium value of any real variable. Given the equilibrium values of the real variables, the level of the money supply helps determine the equilibrium values of all nominal variables that are endogenous but cannot influence any real variable. In a system that dichotomizes the equilibrium values of all real variables are independent of the absolute price level." The Lagos-Wright model dichotomizes similarly (Aruoba and Wright 2003).

4 The DM

Consider a meeting where the buyer has m^b and the seller m^s dollars (even if $m^b = m^s = \widetilde{m} = \hat{M}$ in equilibrium, we want to know what happens more generally). Generally, the buyer gives d dollars to the seller for q units of the good. There are several ways to determine these terms of trade (q, d): the original Lagos-Wright model uses the generalized Nash bargaining solution; Rocheteau and Waller (2005) consider several alternative bargaining solutions; Rocheteau and Wright (2005) combine search with price taking (as in the Lucas-Prescott 1974 model of the labor market) and price posting (as Moen 1996 or Shimer 1995); and Galenianos and Kircher (2006) use auctions in a version of the model that allows for some multilateral matches.

It does not matter much for our purposes which solution concept we use, although auctions are a little more complicated due to multilateral matching. For the other mechanisms, in our environment, the outcome is generally $d = m^b$ – i.e. the buyer spends all his money – and q solves $g(q) = \beta_2 \lambda m^b / w$, where g is an endogenous function that is easy to characterize, although the exact details depend on the solution concept. While we could proceed abstractly and say nothing about g except that it satisfies certain properties, for concreteness we proceed using generalized Nash bargaining, with threat points given by the continuation values of not trading, which means consuming your endowment and going to the CM next period, with continuation value $\hat{W}(m)$.

Hence, the surpluses of the buyer and the seller are

$$u^H(\bar{q} + q) + \beta \hat{W}(m^b - d) - \left[u^H(\bar{q}) + \beta \hat{W}(m^b)\right]$$
$$u^L(\bar{q} - q) + \beta \hat{W}(m^s + d) - \left[u^L(\bar{q}) + \beta \hat{W}(m^s)\right].$$

By Lemma 1, $\hat{W}(m^b - d) - \hat{W}(m^b) = -d\hat{\lambda}/\hat{w}$ and $\hat{W}(m^s + d) - \hat{W}(m^s) = d\hat{\lambda}/\hat{w}$, where $\hat{\lambda}/\hat{w}$ is taken as given from the next CM, and the same for all agents. Then, since $u(q) = u^H(\bar{q} + q) - u^H(\bar{q})$ and $c(q) = u^L(\bar{q}) - u^L(\bar{q} - q)$, the generalized Nash bargaining problem reduces to

$$\max_{q,d} \left[u(q) - d\beta\hat{\lambda}/\hat{w}\right]^{\theta} \left[d\beta\hat{\lambda}/\hat{w} - c(q)\right]^{1-\theta} \tag{15}$$

where θ is the bargaining power of the buyer, and the maximization is subject to the constraint that the buyer cannot pay more that he has, $d \leq m^b$.

Even though our specification differs from the standard model in several respects – we have exchange but no production in the DM, we have preference shocks rather than random matching, etc. – (15) is identical (except for notation) to the bargaining problem in the standard Lagos-Wright model. Hence we can appeal to known results. First, in equilibrium, $m^b = m^s = \widetilde{m}$ and the constraint binds, so $d = \widetilde{m}$, which implies $q < q^*$ where q^* is the first-best outcome defined by $u'(q^*) = c'(q^*)$.[10] To say more, insert $d = \widetilde{m}$ into (15), take the first-order condition with respect to q, and rearrange to get the following:

Lemma 3 *Given $m^b = \widetilde{m}$, the DM bargaining solution is $d = \widetilde{m}$ and $q = q(\widetilde{m})$, where the function $q(\cdot)$ is defined as the solution to $\widetilde{m}\beta\hat{\lambda}/\hat{w} = g(q)$ with*

$$g(q) \equiv \frac{\theta c(q)u'(q) + (1-\theta)u(q)c'(q)}{\theta u'(q) + (1-\theta)c'(q)}.$$

[10] See Lagos and Wright (2005) for details, but the idea is as follows. It is easy to show that $\widetilde{m} \geq m^* \implies d = m^*$ and $q = q^*$, while $\widetilde{m} < m^* \implies d = \widetilde{m}$ and $q = g^{-1}\left(\widetilde{m}\beta\hat{\lambda}/\hat{w}\right)$, where $m^* \equiv g(q^*)\hat{w}/\hat{\lambda}\beta$ and $g(\cdot)$ is defined in Lemma 3 below. One can also show $\widetilde{m} < m^*$ in any equilibrium (assuming either the nominal interest rate is positive *or* $\theta < 1$). Since $g' > 0$ over the relevant range, this implies $q < q^*$. For now, we take $\widetilde{m} < m^*$ for granted, but it is easily verified in the full equilibrium. For example, in steady state, $\widetilde{m} < m^*$ and $q < q^*$ follow immediately from (19) in the next section.

For an agent in the current DM with some (arbitrary) $\widetilde{m}$ when other agents have $\hat{M}$, the payoff is[11]

$$\begin{aligned} V(\widetilde{m}) &= \sigma\left\{u^H\left[\bar{q}+q\left(\widetilde{m}\right)\right]+\beta\hat{W}\left(0\right)\right\} \\ &\quad +\sigma\left\{u^L\left[\bar{q}-q\left(\hat{M}\right)\right]+\beta\hat{W}(\widetilde{m}+\hat{M})\right\} \\ &\quad +(1-2\sigma)\left[u^L(\bar{q})+\beta\hat{W}(\widetilde{m})\right] \end{aligned}$$

Using our definitions of $u(\cdot)$ and $c(\cdot)$, this can be rewritten

$$\begin{aligned} V(\widetilde{m}) &= \sigma\left\{u\left[q\left(\widetilde{m}\right)\right]+\beta\hat{W}\left(0\right)\right\} \\ &\quad +\sigma\left\{-c\left[q\left(\hat{M}\right)\right]+\beta\hat{W}(\widetilde{m}+\hat{M})\right\} \\ &\quad +(1-2\sigma)\beta\hat{W}(\widetilde{m})+K, \end{aligned} \tag{16}$$

where $K \equiv \sigma u^H(\bar{q})+(1-\sigma)u^L(\bar{q})$. Differentiation implies

$$V'(\widetilde{m}) = \beta\frac{\hat{\lambda}}{\hat{w}}\left[1-\sigma+\sigma\frac{u'(q)}{g'(q)}\right]. \tag{17}$$

where it is understood that $q = q(\widetilde{m})$, and we used $\partial q/\partial\widetilde{m} = \beta\hat{\lambda}/\hat{w}g'(q)$ from Lemma 3, plus $\hat{W}'(m) = \hat{\lambda}/\hat{w}\ \forall m$ from Lemma 1.[12]

Definition: Taking as given $\hat{\lambda}/\hat{w}$ and the money holdings of the representative agent $\widetilde{m} = \hat{M}$, a DM equilibrium (with bargaining) is defined by the terms of trade as described above, $d = \widetilde{m}$ and $q = q(\widetilde{m})$, plus the marginal value of money $\hat{V}'(\widetilde{m})$ determined by (17).

[11] In words, with probability σ your utility is u^H and in this event you are matched with a seller for sure, while with probability $1-\sigma$ your utility is u^L and in this event you are matched with probability $\sigma/(1-\sigma)$ and unmatched with probability $(1-2\sigma)/(1-\sigma)$.

[12] Intuitively, with probability σ you find yourself a buyer, in which case the marginal value of money is $u'(q)q'(\widetilde{m}) = u'(q)\beta\hat{\lambda}/\hat{w}g'(q)$, and with probability $1-\sigma$ you do not, in which case you simply take your money to the next CM, so its marginal value is $\beta\hat{\lambda}/\hat{w}$.

5 Equilibrium in the Benchmark Model

To review the analysis to this point, we first described the CM by solving for the real allocation, relative prices, and multiplier λ, and then determining the absolute price level by solving for the nominal wage w in terms of V'. This gives us a simple model of unemployment, taking exogenously the marginal value of money. We then described the DM by solving for the terms of trade (q, d) and V' in terms of $\hat{\lambda}/\hat{w}$. We now combine the DM and CM.

Definition: General equilibrium is defined in terms of paths for $(\mathbf{p}, w)$, $[\mathbf{x}_1, \mathbf{x}_0, \ell(m), \widetilde{m}, \lambda]$, $\bar{\ell}_j$, (q, d) and $V'(\widetilde{m})$ such that at every date: (i) $(\mathbf{p}, w)$, $[\mathbf{x}_1, \mathbf{x}_0, \ell(m), \widetilde{m}, \lambda]$, and $\bar{\ell}_j$ yield a CM equilibrium taking $V'(\widetilde{m})$ as given; and (ii) (q, d) and $V'(\widetilde{m})$ yield a DM equilibrium taking $\hat{\lambda}/\hat{w}$ as given.

To characterize general equilibrium, combine (17) from the DM with (5) from the CM, to get

$$\frac{\lambda}{w} = \beta \frac{\hat{\lambda}}{\hat{w}} \left[1 - \sigma + \sigma \frac{u'(q)}{g'(q)} \right].$$

Using $\hat{\lambda}/\hat{w} = g(q)/\widetilde{m}\beta$, from Lemma 3, we arrive at

$$\frac{g(q_{-1})}{\widetilde{m}_{-1}} = \beta \frac{g(q)}{\widetilde{m}} \left[1 - \sigma + \sigma \frac{u'(q)}{g'(q)} \right] \tag{18}$$

where the subscript -1 indicates the previous period. Given any path for $\widetilde{m} = (1+\gamma)M$, which is determined by policy, to get general equilibrium we first find a (positive, bounded) path for q solving (18).[13] Given the path for q, we know the path for $w/\lambda = \widetilde{m}\beta/g(q)$, which we then feed into the CM to determine the rest of the equilibrium.

[13] This kind of difference equation is common in monetary economics, whether based on search, overlapping generations, cash in advance, or whatever, and typically there are many solution paths unless one restricts attention to stationary equilibria. See e.g. Lagos and Wright (2003).

For most of what follows, we assume that the growth rate of the money supply γ is constant, and focus on stationary equilibria (steady states) where all real variables, including q, are constant. This implies the inflation rate is γ. Also, the real interest rate is given by $1 + r = 1/\beta$, and the nominal rate by the Fisher equation $i = (1 + r)(1 + \gamma) - 1$.[14] Using these expressions, we can simplify (18) in steady state to

$$1 + \frac{i}{\sigma} = \frac{u'(q)}{g'(q)}. \tag{19}$$

Standard conditions imply (19) has a solution $q > 0$, and also $q < q^* \ \forall i > 0$ and $\partial q/\partial i < 0$ (the argument is the same as in the standard Lagos-Wright model).

This almost completes our description of the baseline model. However, we need to discuss the maintained assumption $\ell \in (0, 1)$, on which most of the analysis is based.[15] Note that in equilibrium all agents enter the CM with $m = 0$ if they were a buyer in the previous DM, $m = 2M$ if they were a seller in the previous DM, and $m = M$ if they were neither a buyer nor a seller. Therefore, from (7), $\ell(m)$ takes on one of three values:

$$\ell(m) = \begin{cases} \frac{\mathbf{p}(\mathbf{x}_0 - \mathbf{e}) - \Delta + M}{w + \mathbf{p}(\mathbf{x}_0 - \mathbf{x}_1)} & \text{if } m = 0 \\ \frac{\mathbf{p}(\mathbf{x}_0 - \mathbf{e}) - \Delta}{w + \mathbf{p}(\mathbf{x}_0 - \mathbf{x}_1)} & \text{if } m = M \\ \frac{\mathbf{p}(\mathbf{x}_0 - \mathbf{e}) - \Delta - M}{w + \mathbf{p}(\mathbf{x}_0 - \mathbf{x}_1)} & \text{if } m = 2M \end{cases} \tag{20}$$

Then it is not hard to put restrictions on primitives to guarantee $\ell(m) \in (0, 1)$ for $m \in \{0, M, 2M\}$.

[14] These can be interpreted as equilibrium rates of return, between two meetings of the CM, on real and nominal bonds that do not circulate in the DM, for whatever reason – say, because they are not tangible assets, but simply CM ledger entries. In equilibrium these assets are not actually traded in the CM, because W is linear in wealth, but we can still price them.

[15] It is not that there is anything wrong in principle with equilibria with $\ell(m) = 0$ or 1 for some values of m, just that the algebra becomes less tractable (see Chiu and Molico 2006).

Rocheteau at al. discuss this somewhat more generally, so we content ourselves with an example. Suppose $J = 1$, $e = 0$, $f(\ell) = \ell$, and $U(x, h) = v(x) - \psi(h)$ is separable, so that $x_0 = x_1 = x$. Given the technology, $w/p = 1$ and $\Delta = 0$. The binding constraints are $\ell(0) < 1$ and $\ell(2M) > 0$, which reduce to $1 > x + M/w$ and $x > M/w$. Inserting $M/w = g(q)/\beta\lambda = g(q)/\beta v'(x)$, this is equivalent to

$$g(q) < (1 - x)\beta v'(x) \text{ and } g(q) < \beta x v'(x). \tag{21}$$

From the CM, x^* is pinned down by $v'(x^*) = \psi(1) - \psi(0)$, and so as long as $\psi(1) - \psi(0) > v'(1)$ we know $x^* < 1$, and then we can satisfy (21) as long as $g(q)$ is not too big. Because $g(q) \leq g(q^*) = \theta u(q^*) + (1 - \theta)c(q^*)$, we and assumptions on primitives that guarantee q^* is not too big then implies $\ell(m) \in (0, 1)$ for all m in the relevant set.[16]

Given this, we take for granted that whatever assumptions are needed to guarantee $\ell(m) \in (0, 1)$ hold in what follows. Then the existence of a steady state equilibrium follows from the above analysis: we get q from (19), $w/\lambda = \widetilde{m}\beta/g(q)$ from Lemma 3 where $\widetilde{m} = (1 + \gamma)M$, $V'(\widetilde{m}) = \lambda/w$ from (5), and the rest of the equilibrium from Lemma 2. A sufficient condition for uniqueness of the solution to (19) is the monotonicity of $u'(q)/g'(q)$, which holds under some simple conditions, if not in general (see Lagos-Wright). Summarizing:

Proposition 1 *There exists a steady state equilibrium with valued money. It is unique if $u'(q)/g'(q)$ is monotone.*

[16] Intuitively, if q is too big then money is too valuable, and we either force some people to $\ell = 1$ (those with $m = 0$ trying to aquire the right $\widetilde{m}$), or forces some people to $\ell = 0$ (those with $m = 2M$ trying to spend down to $\widetilde{m}$).

Let us turn to efficiency by characterizing the first best – i.e. the optimal allocation when we are not constrained by anonymity in the DM, so that we do not have to use money. Then a planner maximizes the utility of the representative agent starting in the CM by solving

$$\mathcal{W} = \max_{\mathbf{x}_1, \mathbf{x}_0, \ell, q} \{\ell U(\mathbf{x}_1, 1) + (1-\ell) U(\mathbf{x}_0, 0) + \sigma [u(q) - c(q)] + K + \beta \mathcal{W}\}$$

subject to the obvious feasibility constraints, summarized by (10)-(11).[17] This can be solved as a sequence of static problems. Letting $\boldsymbol{\mu} \in \mathbb{R}^J$ be a vector of Lagrange multipliers, the first-order conditions for an interior solution are:

$$x_{1j} \quad : \quad U_j(\mathbf{x}_1, 1) = \mu_j, \; j = 1, ...J \tag{22}$$

$$x_{0j} \quad : \quad U_j(\mathbf{x}_0, 0) = \mu_j, \; j = 1, ...J \tag{23}$$

$$\ell \quad : \quad U(\mathbf{x}_1, 1) - U(\mathbf{x}_0, 0) = \boldsymbol{\mu}(\mathbf{x}_1 - \mathbf{x}_0) - \sum\nolimits_j \mu_j f_j'(\bar{\ell}_j) \tag{24}$$

$$q \quad : \quad u'(q) = c'(q) \tag{25}$$

It can be checked that (22)-(24) coincide with the CM equilibrium conditions (2)-(4) when we set $\mu_j = \lambda p_j / w$ and $\lambda = \sum_j \mu_j f_j'(\bar{\ell}_j)$. Comparing (25) and the DM equilibrium condition (19), $q = q^*$ and equilibrium is efficient iff $i = 0$ and $g(q) = c(q)$, where the latter is true iff $\theta = 1$ (see Lemma 3). In general, $q < q^*$ in equilibrium, and the optimal monetary policy makes q as big as possible, which means the Friedman rule $i = 0$; when $\theta < 1$ we do not get the first best at $i = 0$, however.

Proposition 2 *The Friedman rule $i = 0$ is optimal and achieves the first-best iff $\theta = 1$.*

[17] This problem is easily derived from primitive preferences u^H and u^L, with K the constant defined earlier. In principle, the planner could set q in a match as a function of (x, h) for both the buyer and the seller, but with the utility function used here it is easy to verify the solution entails the same q in all matches.

6 The Phillips Curve

The baseline model with $\upsilon^j(q, \mathbf{x}, h) = U(\mathbf{x}, h) + u^j(q)$ has some nice properties, including analytic tractability. However, the dichotomy to which we alluded has one very special implication: monetary policy affects q in the DM, but has no impact on the CM allocation and in particular no impact on unemployment $1 - \ell$. So the Phillips curve is vertical. Of course, if one believes that this is in the data, there is no problem. We prefer to be agnostic and have a theory capable of delivering more general predictions, and to this end we consider more general preferences. Then, in principle, careful data analysis (beyond the scope of this project) can pin down the parameterization – e.g. whether the actual relation between i and $1 - \ell$ is positive, negative or zero will reveal something about the underlying preferences that allow us to match the facts and model.[18]

It is intuitively clear that if q interacts with $(\mathbf{x}, h)$ in preferences then anything that changes q, including monetary policy, affects the CM allocation and hence $1 - \ell$. Unfortunately, nonseparable utility is somewhat less tractable – e.g. we cannot write the CM problem quite as neatly as (1) since agents do not know how much utility they get from their CM choice (x, h) until they get to the DM. Therefore we make a few special assumptions designed to facilitate the presentation, including $J = 1$, $e = 0$, and $f(\ell) = \ell$. Moreover, as long as $\upsilon^H(q, x, h)$ is nonseparable, we can maintain $\upsilon^L(q, x, h) = u^L(q) + U(x, h)$ and get all the main results with less clutter. Although this is not the most general case, it should be clear that the idea of a Phillips curve emerging naturally in general equilibrium is robust.

[18] There are other ways to break the dichotomy (e.g. Aruoba, Waller and Wright 2005 use generalized technologies in models with capital), but here we focus on preferences.

With $f(\ell) = \ell$, the real wage is 1, so we can replace w with p everywhere. Thus, Lemma 1 implies $W'(m) = \lambda/p$, and the Nash bargaining solution with the preferences in this section is

$$\max_{q,d} \left[u(q, x^b, h^b) - d\beta\hat{\lambda}/\hat{p}\right]^{\theta} \left[d\beta\hat{\lambda}/\hat{p} - c(q)\right]^{1-\theta}$$

subject to $d \leq m^b$.[19] The analog of Lemma 3 says that $d = m^b$ and $q = q(m^b, x^b, h^b)$, where $q(\cdot)$ is defined as the solution to $m^b\beta\hat{\lambda}/\hat{p} = g(q, x^b, h^b)$ with

$$g(q, x, h) \equiv \frac{\theta u_q(q, x, h)c(q) + (1-\theta)u(q, x, h)c'(q)}{\theta u_q(q, x, h) + (1-\theta)c'(q)}.$$

Generally, all agents have either $(\widetilde{m}_1, x_1, 1)$ or $(\widetilde{m}_0, x_0, 0)$ coming out of the CM, depending on whether they were employed and unemployed; hence there are two values of q observed in the DM, $q_1 = q(\widetilde{m}_1, x_1, 1)$ and $q_0 = q(\widetilde{m}_0, x_0, 0)$.

The analog of (16) is

$$\begin{aligned} V(m, x, h) \quad = \quad & \sigma\left\{u\left[q\left(m, x, h\right), x, h\right] + \beta\hat{W}(0)\right\} \\ & +\sigma\mathbb{E}\left\{-c[q(m^b, x^b, h^b)] + \beta\hat{W}(m + m^b)\right\} \\ & +(1-2\sigma)\beta\hat{W}(m) + K(x, h), \end{aligned} \tag{26}$$

where $K(x, h) \equiv \sigma\upsilon^H\left(\bar{q}, x, h\right) + (1-\sigma)\left[u^L(\bar{q}) + U(x, h)\right]$, and the expectation in (26) is over the buyer's state when you are a seller in the DM, which

[19] To derive this, in a match where a buyer has some arbitrary state (m^b, x^b, h^b) and a seller has (m^s, x^s, h^s), the surpluses associated with trade (q, d) are:

$$\upsilon^H(\bar{q} + q, x^b, h^b) + \beta\hat{W}(m^b - d) - \left[\upsilon^H(\bar{q}, x^b, h^b) + \beta\hat{W}(m^b)\right]$$
$$u^L(\bar{q} - q) + U(x^s, h^s) + \beta\hat{W}(m^s + d) - \left[u^L(\bar{q}) + U(x^s, h^s) + \beta\hat{W}(m^s)\right]$$

Using the definitions of u and c in Section 2, after some simple algebra, we get the expression in the text.

depends on his employment status in the previous CM. Thus,

$$V_m = \beta \frac{\hat{\lambda}}{\hat{p}} \left[1 - \sigma + \sigma \frac{u_q(q,x,h)}{g_q(q,x,h)} \right] \tag{27}$$

$$V_x = \sigma \left[u_x(q,x,h) - \frac{g_x(q,x,h) u_q(q,x,h)}{g_q(q,x,h)} \right] + K_x(x,h) \tag{28}$$

where $K_x(x,h) \equiv \sigma \upsilon_x^H(\bar{q},x,h) + (1-\sigma) U_x(x,h)$. These simplify a lot when $\theta = 1$, as we will eventually assume, although for now we stick to the general case.[20]

It is straightforward to write the Lagrangian for the CM problem and take the first-order conditions:

$$x_h \;:\; V_x(\widetilde{m}_h, x_h, h) = \lambda,\ h = 0,1 \tag{29}$$

$$\widetilde{m}_h \;:\; V_m(\widetilde{m}_h, x_h, h) = \lambda/p,\ h = 0,1 \tag{30}$$

$$\ell \;:\; V(\widetilde{m}_1, x_1, 1) - V(\widetilde{m}_0, x_0, 0) = \lambda \left(x_1 - x_0 - 1 + \frac{\widetilde{m}_1 - \widetilde{m}_0}{p} \right) \tag{31}$$

$$\lambda \;:\; p\ell + m + \gamma M - \ell p x_1 - (1-\ell) p x_0 - \ell \widetilde{m}_1 - (1-\ell)\widetilde{m}_0 = 0 \tag{32}$$

A key point to notice is that, exactly as in the separable model, m does not matter for any CM choice except ℓ (as always, assuming interior solutions). In particular, $\widetilde{m_h}$ is independent of m, although it now may depend on h. Hence, the distribution of money holdings in the CM is degenerate only after conditioning on CM employment status, but that is sufficient to keep the model tractable.

[20] The term $-g_x/g_q = \partial q/\partial x$ in (28) reflects how x generally affects the bargaining solution, just like the term $\beta\hat{\lambda}/\hat{p}g_q = \partial q/\partial m$ in (27) reflects how $\widetilde{m}$ generally affects it. These classic *holdup problems* go away when $\theta = 1$, which is one reason the analysis simplifies a lot in this case. To motivate it further, note that when search is directed, under certain assumptions about timing and commitment, the outcome is that buyers get all the surplus, and so in a sense $\theta = 1$ emerges endogenously (Corbae et al. 2003, Sec. 5). Alternatively, $\theta = 1$ implies very similar results to versions of the model with price taking (competitive equilibrium) or price posting (competitive search equilibrium), both of which also eliminate holdup problems.

Eliminating V and its derivatives V_x and V_m from (29)-(31) using (26)-(28) and imposing steady state, after some relatively routine algebra, we arrive at:

$$\lambda = \sigma\left[u_x(q_h,x_h,h) - \frac{g_x(q_h,x_h,h)u_q(q_h,x_h,h)}{g_q(q_h,x_h,h)}\right] + K_x(x,h),\ h=0,1 \tag{33}$$

$$0 = 1 + \frac{i}{\sigma} - \frac{u_q(q_h,x_h,h)}{g_q(q_h,x_h,h)},\ h=0,1 \tag{34}$$

$$\begin{aligned} 0 &= \sigma\left[u\left(q_1,x_1,1\right) - u\left(q_0,x_0,0\right)\right] + K(x_1,1) - K(x_0,0) \\ &\quad + \left[g(q_1,x_1,1) - g(q_0,x_0,0)\right](\sigma+i) - \lambda\left(x_1 - x_0 - 1\right) \end{aligned} \tag{35}$$

Given i, (33)-(35) constitute 5 equations determining $(x_1, x_0, q_1, q_0, \lambda)$. Then $\bar{\ell}x_1 + (1-\bar{\ell})x_0 = f(\bar{\ell}) = \bar{\ell}$ (32) yields aggregate employment

$$\bar{\ell} = \frac{x_0}{1 + x_0 - x_1}. \tag{36}$$

Finally, inserting $\beta\lambda\widetilde{m}_h/p = g(q_h,x_h,h)$ into $\bar{\ell}\widetilde{m}_1 + (1-\bar{\ell})\widetilde{m}_0 = (1+\gamma)M$, we get the nominal price level,

$$p = \frac{\beta\lambda(1+\gamma)M}{\bar{\ell}g(q_1,x_1,1) + (1-\bar{\ell})g(q_0,x_0,0)}. \tag{37}$$

This fully determines steady state equilibrium. It is clear that the dichotomy breaks down: there is no way to solve for the CM allocation $(x_1, x_0, \bar{\ell}, \lambda)$ independently of the DM allocation (q_1, q_0), and so if i affects the latter it affects the former. The system is somewhat complicated, however, for exactly this reason. Hence we analyze two subcases separately: one where there is interaction between (q, x) but these are separable from h; and one where (q, h) interact but these are separable from x.

Case 1: h is separable from (q, x). In this case $v^H(q,x,h) = u^H(q,x) - v(h)$ and $U(x,h) = U(x) - v(h)$, and we know that $x_1 = x_0 = x$, $\widetilde{m}_1 = \widetilde{m}_0 = \widetilde{m}$, and $q_1 = q_0 = q$. Also, with a slight abuse of notation, we can write

$u(q,x,h) = u(q,x)$ and $g(q,x,h) = g(q,x)$. Then (35) yields $\lambda = v(1) - v(0)$, and equilibrium is given by a pair (q,x) that solves the following versions of (33)-(34):

$$\begin{aligned}
\lambda &= \sigma\left[u_x(q,x) - \frac{g_x(q,x)u_q(q,x)}{g_q(q,x)}\right] + \sigma u_x^H(\bar{q},x) + (1-\sigma)U_x(x) \\
0 &= 1 + \frac{i}{\sigma} - \frac{u_q(q,x)}{g_q(q,x)}
\end{aligned}$$

Although one could proceed more generally, to make the essential point, at this stage we set $\theta = 1$ to reduce these to

$$\begin{aligned}
\lambda &= \sigma u_x^H(q,x) + (1-\sigma)U_x(x) \\
0 &= 1 + \frac{i}{\sigma} - \frac{u_q^H(q,x)}{c'(q)},
\end{aligned}$$

where we use $u_x(q,x) - u_x^H(\bar{q},x) = u_x^H(q,x)$ and $u_q(q,x) = u_q^H(q,x)$.

Differentiating, we get

$$\begin{aligned}
\frac{\partial q}{\partial i} &= \frac{-\left[\sigma u_{xx}^H(q,x) + (1-\sigma)U_{xx}(x)\right]c'(q)}{\sigma D} < 0 \\
\frac{\partial x}{\partial i} &= \frac{u_{qx}^H(q,x)c'(q)}{D} \backsimeq -u_{qx}^H
\end{aligned}$$

where $a \backsimeq b$ means a and b are equal in sign and

$$\begin{aligned}
D = \sigma\left[u_{xq}^H(q,x)^2 - u_{xx}^H(q,x)u_{qq}^H(q,x)\right] + \sigma u_{xx}^H(q,x)\left(1+\frac{i}{\sigma}\right)c''(q) \\
+(1-\sigma)U_{xx}(x)\left[\left(1+\frac{i}{\sigma}\right)c''(q) - u_{qq}^H(q,x)\right] < 0.
\end{aligned}$$

Hence, q unambiguously falls with i, while effect on x depends on complementarities between x and q. Also, since in the case under consideration $\bar{\ell} = x$,

$$\frac{\partial(1-\bar{\ell})}{\partial i} \backsimeq -\frac{\partial x}{\partial i} \backsimeq u_{xq}^H.$$

This is the desired result: the relation between inflation and unemployment.

Proposition 3 *When* $\upsilon^H(q,x,h) = u^H(q,x) - v(h)$*, given* $\theta = 1$*, we have* $\partial(1-\bar{\ell})/\partial i < 0$ *iff* $u^H_{xq} < 0$*.*

This result is very intuitive. Inflation is a direct tax on DM activity, and hence reduces q. If $u_{xq} > 0$ (q and x are complements) then inflation also reduces x and hence $\bar{\ell}$. But if $u^H_{xq} < 0$ (q and x are substitutes) then inflation increases x and $\bar{\ell}$. In the latter case, inflation causes people to substitute out of DM goods and into CM goods, increasing CM production and employment. We get a downward-sloping Phillips curve under simple and natural conditions; and an upward-sloping Phillips curve under alternative conditions.[21]

Case 2: x is separable from (q,h). The analysis here is different from the previous case, but similar enough that here we simply state the result and relegate details to the Appendix:

Proposition 4 *When* $\upsilon^H(q,x,h) = U(x) + u^H(q,h)$*, given* $\theta = 1$*, we have* $\partial(1-\bar{\ell})/\partial i < 0$ *iff* $u^H_{qh} < 0$*.*

This also very intuitive. Inflation reduces q. If $u_{qh} = u^H_{qh} > 0$ (q and h are complements, or q and leisure substitutes) this increases leisure and hence reduces $\bar{\ell}$. But if $u_{qh} = u^H_{qh} < 0$ (q and leisure are complements) then inflation reduces leisure and hence increases $\bar{\ell}$. Again, we get an upward- or downward-sloping Phillips curve under natural conditions.

These two propositions are robust, in the following sense. We set things up to get unambiguous results by assuming labor is not used to produce q

[21] It is perhaps surprising is that the results come out as clean as they do – e.g. why are there no generally ambiguous wealth and substitution effects? The reason is the same as the reason the model is so tractabe in general: with indivisible labor and lotteries, agents act as if they have quasi-linear utility.

(it comes from endowment $\bar{q}$). More generally, as long as x production is relatively labor intensive compared to q, it is possible to get $\bar{\ell}$ to increase if x goes up when q goes down in response to inflation. The exact conditions for an upward- or downward-sloping Phillips curve will change, of course, but the economic idea is robust on this dimension. Another virtue is that the results do not require anything tricky, like sticky wages or prices, or signal extraction problems, to generate a relation between inflation and employment (more generally, between inflation and economic activity). It is all based on rudimentary public finance considerations.

The other point to emphasize is that these considerations lead to a relation between inflation and employment that is stable, and *exploitable*, in the long run.[22] Given $u^H_{xq} < 0$ or $u^H_{qh} < 0$, Propositions 3 and 4 indicate that policymakers *can* achieve a permanently lower rate of unemployment by printing money at a faster rate. However, we now argue that this is not a good idea. To this end, consider the planner's problem:

$$\begin{aligned}\mathcal{W} \quad = \quad & \max_{x_1,x_0,\ell,q_1,q_0} \{\sigma\left[\ell\upsilon^H(\bar{q}+q_1,x_1,1)+(1-\ell)\upsilon^H(\bar{q}+q_0,x_0,0)\right] \\ & +\sigma\left[\ell\upsilon^L(\bar{q}-q_1,x_1,1)+(1-\ell)\upsilon^L(\bar{q}-q_0,x_0,0)\right] \\ & +(1-2\sigma)\left[\ell\upsilon^L(\bar{q},x_1,1)+(1-\ell)\upsilon^L(\bar{q},x_0,0)\right]+\beta\mathcal{W}\} \\ & \text{s.t. } \ell x_1+(1-\ell)x_0 \leq \bar{\ell}\end{aligned}$$

Note that q_h generally depends on the state (x_h, h) of the buyer in a match.[23] Also note that we can solve this as a sequence of static problems.

[22] Like any other model, some components we take to be structural are not necessarily so in the face of sufficiently big policy changes. If inflation gets high enough, e.g., agents might find some other way to trade in the DM – say, using foreign currency. But our trade off is stable in the sense that it does not depend on lagged responses by expectations or nominal prices or wages.

[23] Since the utility of the seller is additively separable in (x, h) and q, it is easy to check that q should not depend on the his state.

First-order conditions are

$$
\begin{aligned}
x_h &: \quad (1-\sigma)U_x(x_h,h) + \sigma \upsilon_j^H(\bar{q}+q_h, x_h, h) = \mu,\ h=0,1 \\
q_h &: \quad \upsilon_q^H(\bar{q}+q_h, x_h, h) = u_q^L(\bar{q}-q_h),\ h=0,1 \\
\ell &: \quad \mu\left(x_1 - x_0 - 1\right) = \sigma\left[\upsilon^H(\bar{q}+q_1, x_1, 1) - \upsilon^H(\bar{q}+q_0, x_0, 0)\right] \\
&\quad\ +\sigma\left[u^L(\bar{q}-q_1) - u^L(\bar{q}-q_0)\right] + (1-\sigma)\left[U(x_1,1) - U(x_0,0)\right]
\end{aligned}
$$

where μ is a multiplier. In the case $\theta = 1$, where Propositions 3 and 4 apply, it is easy to check that these conditions for efficiency coincide with the equilibrium conditions iff $i = 0$. Hence, we conclude that the efficient policy is the Friedman rule, regardless of whether the Phillips curve is upward-sloping, downward-sloping, or vertical.[24]

Proposition 5 *Given $\theta = 1$, the Friedman rule $i = 0$ achieves the solution to the planner's problem.*

7 Conclusion

In nonconvex economies, randomized allocations are desirable and can be supported as equilibria with lotteries. This not only generates unemployment, it is very convenient for monetary theory, because it provides an analytically tractable alternative to Lagos-Wright without quasi-linear utility. We used these ideas to construct a general equilibrium model of the Phillips curve. With separability between CM and DM goods, our Phillips

[24] The algebra is messy when $\theta < 1$, and it is harder to get clean results about even $\partial q/\partial i$, but there is no presumption that $i > 0$ could be efficient in this case: our conjecture is $q < q^*$ at $i = 0$ when $\theta < 1$, as in simpler versions of the model (Lagos-Wright), and hence $i = 0$ is still optimal even though it cannot achieve q^*. Again, $\theta = 1$ is the leading case not only because it is tractable, but because it makes the bargaining model similar to one with endogenous matching, to one with price taking and competetive equilibrium, or to one with price posting and competetive search equilibrium.

curve is vertical, but natural extensions to more general preferences allow it to slope either up or down. The idea is simple and plausible: inflation is a tax on any economic activity that is relatively cash intensive, and so if either labor intensive goods are substitutes for this activity, or leisure is a complement with this activity, inflation can reduce unemployment. This does not mean we should use inflation to unemployment. Our calculations indicate the optimal policy is the Friedman rule.

8 Appendix

8.1 CM Planner Problem

The first thing we do is discuss the CM planner problem used in Section 3 as the basis for Lemma 2. For ease of presentation, assume $J = 1$, but the argument in Rocheteau et al. (2006) can be used to extend the results to any number of goods. The problem is then

$$\max_{x_1,x_0,\ell} \left\{\ell U\left(x_1,1\right) + (1-\ell)U\left(x_0,0\right) + \mu\left[f(\ell) + e - \ell x_1 - (1-\ell)x_0\right]\right\},$$

where μ is a multiplier. Consider the case where the solution entails $\ell \in (0,1)$, which we can guarantee by simple assumptions such as $f(0) = \infty$ and $f(1) = 0$. First-order conditions are given by (22)-(24) and the discussion leading to Proposition 2 tells us that these are the same as the CM equilibrium conditions.

We now check the second-order conditions for a strict local maximum. We rule out the case $f'(\ell) - c_1 + c_0 \neq 0$ that is known to occur iff we have the very special utility function $U(x,h) = \phi[x + v(h)]$ (see e.g. Cooper 1987 or Rogerson and Wright 1988). The bordered Hessian evaluated where the first-order conditions are satisfied is

$$\mathbf{B} = \begin{bmatrix} 0 & -\ell & -(1-\ell) & f'(\ell) - c_1 + c_0 \\ -\ell & \ell U_{xx}\left(x_1,1\right) & 0 & 0 \\ -(1-\ell) & 0 & (1-\ell)U_{xx}\left(x_0,0\right) & 0 \\ f'(\ell) - c_1 + c_0 & 0 & 0 & 0 \end{bmatrix}$$

Computing the last two leading principal minors, we have

$$\begin{aligned} |\mathbf{B}_4| &= -\left[f'(\ell) - c_1 + c_0\right]^2 \ell(1-\ell)U_{xx}\left(x_1,1\right)U_{xx}\left(x_0,0\right) < 0 \\ |\mathbf{B}_3| &= -\ell^2(1-\ell)U_{xx}\left(x_0,0\right) - (1-\ell)^2\ell U_{xx}\left(x_1,1\right) > 0, \end{aligned}$$

which ensures that any solution to the first-order conditions is a strict local maximum.

We now claim that there is a unique solution to the first-order conditions and it constitutes the global maximum. We begin by breaking the problem in two. First define

$$\mathcal{W}(\ell) = \max_{x_1, x_0} \left[\ell U\left(x_1, 1\right) + (1-\ell) U\left(x_0, 0\right)\right]$$
$$\text{s.t. } \ell x_1 + (1-\ell) x_0 - f(\ell) - e \leq 0.$$

Since U is strictly concave, there is a unique solution $[x_0(\ell), x_1(\ell)]$. By the Theorem of the Maximum, $\mathcal{W}(\ell)$ is continuous and hence achieves a maximum over $\ell \in [0,1]$. Suppose by way of contradiction that there are two local maxima. Then, by continuity, $\mathcal{W}(\ell)$ has a local minimum at some $\ell^- \in (0,1)$. This means $[x_0\left(\ell^-\right), x_1\left(\ell^-\right), \ell^-]$ is a saddle point of the original problem, which contradicts the result that any solution to the first-order conditions is a local maximum. Hence there is a unique maximizer of $\mathcal{W}(\ell)$ say ℓ^*, and $[x_0(\ell^*), x_1(\ell^*), \ell^*]$ is the unique solution to the planner problem, as we claimed. ■

8.2 Proof of Proposition 4

We provide the analysis for Case 2 in Section 6. In this case, $v^H(q,x,h) = U(x) + u^H(q,h)$ and $U(x,h) = U(x) - v(h)$. Then we have $x_1 = x_0 = x$, although generally not $m_1 = m_0$ or $q_1 = q_0$. With slight abuse of notation, we write $u(q,x,h) = u(q,h)$ and $g(q,x,h) = g(q,h)$. Then (33)-(35) can be written

$$\begin{aligned}
\lambda &= U'(x) \\
0 &= 1 + \frac{i}{\sigma} - \frac{u_q(q_h,h)}{g_q(q_h,h)},\ h = 0,1 \\
0 &= (1-\sigma)\left[v(1) - v(0)\right] - \sigma\left[u\left(q_1,1\right) - u\left(q_0,0\right)\right] \\
&\quad + (\sigma + i)\left[g(q_1,1) - g(q_0,0)\right] - \lambda
\end{aligned}$$

Differentiating, we get

$$\frac{\partial q_h}{\partial i} = \frac{g_q(q_h, h)}{\sigma u_{qq}(q_h, h) - (i + \sigma)\, g_{qq}(q_h, h)} \tag{38}$$

$$\frac{\partial x}{\partial i} = \frac{g(q_1, 1) - g(q_0, 0)}{U''(x)}. \tag{39}$$

Again $\bar{\ell} = x$, and therefore

$$\frac{\partial(1 - \bar{\ell})}{\partial i} \backsimeq g(q_1, 1) - g(q_0, 0).$$

Hence unemployment decreases with inflation iff $g(q_1, 1) < g(q_0, 0)$.

Consider once again $\theta = 1$. Then $g(q, h) = c(q)$, and it is easy to show that $g(q_1, 1) < g(q_0, 0)$ iff $q_1 < q_0$ iff $u_{qh} < 0$. Then $\partial(1 - \bar{\ell})/\partial i \backsimeq u_{qh}$. Moreover, when $\theta = 1$,

$$\frac{\partial q_h}{\partial i} = \frac{c'(q_h)}{\sigma u_{qq}(q_h, h) - (\sigma + i)c'(q_h)} < 0.$$

This completes the argument. ■

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