

# Liquidity and Spending Dynamics\*

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## Abstract

How do financial frictions affect the response of an economy to aggregate shocks? In this paper, we address this question, focusing on liquidity constraints and uninsurable idiosyncratic risk. We consider a search model where agents use a liquid asset to smooth individual income shocks. We show that the response of this economy to aggregate shocks depends on the rate of return on liquid assets. When liquid assets pay a low return, agents hold smaller liquidity reserves and the response of the economy tends to be larger. In this case, agents expect to be liquidity constrained and, due to a precautionary motive, their consumption decisions respond more to changes in expected income. On top of this, there is a general equilibrium effect that magnifies the economy response. After a positive aggregate shock, agents' consumption increases and this raises income expectations, further reducing the precautionary demand for liquid assets.

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# 1 Introduction

During times of financial instability the demand for liquid assets typically increases. Facing increasing uncertainty, agents tend to shift to cash, government bonds, gold or other safe assets. In recent emerging market crises this phenomenon has been described as “flight to quality.” In this paper, we explore a feed-back mechanism, based on the demand for liquidity, that may explain why small changes in fundamentals can have large effects on real activity. As agents are more uncertain about their future income, they self-insure by building up precautionary reserves and reducing their spending. This reduces the demand for goods and leads to an output contraction. The perceived level of aggregate risk increases further, pushing agents even more towards liquid assets. Therefore, the initial flight to liquidity is amplified through the endogenous response of agents’ spending and output.

While this intuition goes back to the idea of the “multiplier,” of Hume and Keynes, in this paper we explore it from a different point of view. We leave aside sticky prices and other sources of nominal rigidity, and we focus instead on a flexible price model with a limited supply of liquid assets. In our model the demand for liquidity is fully microfunded and related to risk aversion and individual income uncertainty.

Within this approach, we address the following question: how does the supply of liquidity in the economy affects the response of output to aggregate shocks? We consider an economy where liquid assets are supplied by the government and a low supply of liquidity simply corresponds to a low rate of return on these assets. Our main result is that a low liquidity supply tends to magnify the response of output to aggregate shocks. When liquid assets pay a low rate of return, liquidity is costly and agents hold small precautionary reserves. It follows that the volatility of their income in the short run has a bigger impact on their spending decisions.

We consider a model of decentralized exchange in the tradition of search models with money. Agents are anonymous and, thus, credit arrangements are not feasible. Transactions are financed using a government-supplied asset, which we call money. There is a large number of households composed of a consumer and a producer. We introduce individual income uncertainty assuming that the producer is exposed to an idiosyncratic productivity shock. The consumer has to make his consumption decision before knowing the realization of this shock. Therefore, consumption is determined both by his endowment of real money and by his income expectations. In this environment we look at the effect of an aggregate shock to productivity

on aggregate output.

On the financial side, we assume that the government pays each period a fixed rate of return on money balances and uses taxation to keep the stock of money constant. In this context, there is an increasing mapping between the real rate of return on money and the level of real taxation. We describe a monetary regime directly in terms of the choice of the rate of return and we focus on two monetary regimes. First, we look at the case where the rate of return reaches its maximum feasible level, equal to the inverse of the agents' discount rate. This is a "Friedman rule" type of regime and in this case the economy achieves the first-best allocation. Second, we look at the case where the rate of return is so low that agents are liquidity constrained for any realization of the productivity shock. We will refer to the first case as an "unconstrained economy" and to the second one as a "constrained economy." Then, we compare the effect of an aggregate shock in the constrained and in the unconstrained economies. We show that, in the first case, there is an amplification effect, capturing the precautionary motive discussed above. This effect is further reinforced by a general equilibrium mechanism that generates a feed-back between consumption decisions and income expectations. These effects are absent in the unconstrained economy.

The approach in this paper is closely related to the large literature on money in models with search, going back to Diamond (1984) and Kiyotaki and Wright (1989). The search model in Diamond (1981) has a built-in amplification mechanism, due to the assumption of increasing returns in the matching function. Our model shares his focus on coordination motives in decentralized trading, but we look at a different mechanism, which works through risk aversion and the precautionary behavior of agents. From a methodological point of view, our model uses quasi-linear preferences as in Lagos and Wright (2005) to simplify the analysis of the cross-sectional distribution of money balances.

The paper is also related to the literature on the public supply of liquid assets, including Woodford (1990) and Holmstrom and Tirole (1998). In this paper we share their view of government-supplied liquid assets as a necessary ingredient to finance private transactions. However, our argument can also be made in an environment where liquid assets are supplied by the private sector, as long as their supply is limited due to some financial market imperfection.

Finally, the paper is related to the large literature exploring the relation between financial frictions and aggregate volatility, including Bernanke and Gertler (1989), Acemoglu and Zilibotti (1997), Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), Kiyotaki and

Moore (1997). To the best of our knowledge, this is the first paper to address this issue from the point of view of limited liquidity supply.

The rest of the paper is organized as follows. In Section 2 we introduce our environment and solve for the first-best allocation of resources. In Section 3 we define and characterize the competitive equilibrium. In particular, we analyze separately the unconstrained economy and the constrained one. Section 4 addresses the main question of the paper, that is, how the two economies react to an aggregate shock. Section 5 concludes. Finally, the appendix contains all the proofs that are not presented in the text.

## 2 The Model

Consider an economy with a continuum of infinitely-lived households, composed of two agents, a consumer and a producer. Time is discrete and each period agents produce and consume a single, perishable consumption good. The economy has a simple periodic structure, that is, each time period  $t$  is divided in three sub-periods,  $s = 1, 2, 3$ . To simplify the exposition, we will call them “periods,” whenever there is no risk of confusion. There is an exogenous supply of fiat money, which will be used as the medium of exchange. The government pays a constant nominal rate of return  $R$  on money balances, which is financed with taxation.

In periods 1 and 2, the consumer and the producer from each household travel to spatially separated markets, or islands, where they interact with consumers and producers from other households. On each island there is a competitive market, as in Lucas and Prescott (1974). Each market is characterized by anonymity, therefore the only type of trades that are feasible are spot exchanges of money for goods. There is a continuum of islands and each island receives the same mass of consumers and producers every period. We assume that the allocation of agents to islands satisfies a law of large numbers, so that each island receives a representative sample of consumers and producers. The consumer and the producer do not communicate during periods 1 and 2, however, at the end of both periods, they meet and they share money holdings and information. In period 3 consumers and producers trade in a centralized market.

Each period 1 the producer has access to the linear technology

$$y_1 = \theta n$$

where  $y_1$  is output,  $n$  is labor effort by the producer, and  $\theta$  is a random productivity parameter, which is the same for all the producers in a given island. The productivity shock is realized

after producers and consumers have been assigned to their islands, and is revealed to all the agents in the island. The distribution of productivity shocks across islands is given by a continuous distribution with bounded support  $[\underline{\theta}, \bar{\theta}]$  and cumulative distribution function  $F(\theta)$ . In periods 2 and 3 the producers have fixed endowments of consumption goods,  $e_2$  and  $e_3$ . We will assume that the value of  $e_3$  is large, so as to ensure that the consumption  $c_3$  is non-negative in all the equilibria we study.

The shock  $\theta$  is the source of idiosyncratic income volatility in our economy. The consumers in period 1 has to make his spending decisions before knowing the shock faced by his producer. This introduces a precautionary motive in his behavior. The strength of this precautionary motive will depend on the ability of households to self-insure against income shocks. This, in turns, will depend on the nominal rate of return on money  $R$  fixed by the government.

The preferences of the household are represented in recursive form by the utility function:

$$V_t = \mathbb{E}_t [c_{1,t} + \beta u(c_{2,t} - v(n_t)) + \beta^2 c_{3,t} + \beta^3 V_{t+1}],$$

where  $c_{s,t}$  is consumption in period  $(s, t)$ ,  $n_t$  is labor effort and  $V_t$  is the expected utility of a consumer at date  $t$ . The function  $u$  is increasing and strictly concave, and the function  $v$ , representing the cost of effort, is increasing and convex. The assumption that the cost of effort takes the form of a “monetary” cost at date 2, is made mostly for analytical convenience. Given that we want to focus on the demand side of the economy, this allows to simplify greatly the supply side. A natural interpretation of this assumption is to interpret  $n_t$  as a capacity choice by the firm owned by the household, and to interpret  $v(n_t)$  as a maintenance cost, incurred at date 2, which is increasing in the capacity employed. We also make some useful technical assumptions on  $u$  and  $v$ . Namely, assume  $u$  satisfies the standard Inada condition  $\lim_{x \rightarrow 0} u'(x) = \infty$ . The function  $v$  satisfies the conditions  $v(0) = v'(0) = 0$  and the elasticity  $v''n/v'$  is bounded above.

The discount factor  $\beta \in (0, 1)$  is constant between consecutive sub-periods. Assuming that in period 3 producers and consumers are located in the same island and have linear utility, is essential to simplify the analysis of the equilibrium. This assumption allows us to obtain an analytical characterization of the steady state distribution of money balances, as in Lagos and Wright (2005).<sup>1</sup> In equilibrium, agents in period  $(3, t)$  will reallocate money balances among

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<sup>1</sup>The extension of the Lagos and Wright (2005) environment to a 3-period setup is also pursued in Berentsen et al. (2005), which use it to study the distributional effects of monetary policy.

themselves, so that in the following period,  $(1, t + 1)$ , they will all start with the same stock of money.

Finally, we need to specify the way in which the government determines money supply. Let  $M$  denote the money stock in period 1. The government injects money in periods 2 and 3 to cover for the interest payments. In period 3 the government retires money, so as to keep the money stock constant. This requires a lump sum tax,  $T$ , that satisfies the following condition:<sup>2</sup>

$$R(R^2M - T) = M. \tag{1}$$

Monetary policy is fixed and is characterized by the two parameters  $R$  and  $M$ . Since we want to focus on steady state equilibria with stationary nominal prices, we need to restrict  $R$  in the interval  $[0, 1/\beta]$ . We will see that given choices for  $R$  and  $M$ , will determine the real value of the nominal tax  $T$ .

Notice that we allow for  $R \stackrel{\leq}{\geq} 1$ . In the case  $R > 1$  the asset in our economy resembles a liquid nominal bond, while in the case  $R < 1$  it is like fiat money subject to a positive inflation tax.<sup>3</sup> The assumption of interest-paying money balances is a general way of introducing a government-supplied liquid asset. Since we will have stationary nominal prices,  $R$  will correspond to the real rate of return on money in this economy. In turns, higher levels of  $R$  will correspond to a larger real supply of public liquidity.

## 2.1 First-best allocation

In this section we describe the first-best allocation, as a benchmark for our economy. Consider a social planner who allocates consumption to households and decides the labor effort of the producers. Given that agents do not make any investment decision, there is no intertemporal link between time  $t$  and  $t + 1$ . This allows us to simplify the welfare analysis, by focusing on the “static” planner problem which only includes periods  $s = 1, 2, 3$ .

Each household is characterized by a pair  $(\theta, \hat{\theta})$ , where  $\theta$  denotes the productivity shock of the producer and  $\hat{\theta}$  the productivity shock in the island visited by the consumer. An allocation is given by consumption functions  $\{c_s(\theta, \hat{\theta})\}_s$  and an effort function  $n(\theta, \hat{\theta})$ . The planner

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<sup>2</sup>Here we are assuming that the government retires  $T$  at the end of period  $(3, t)$  and, at the beginning of period  $(1, t + 1)$ , it pays the interest  $R$  on the remaining money balances  $R^2M - T$ .

<sup>3</sup>The model can be rewritten with a zero nominal interest rate, and a positive or negative growth rate of the money stock. It is easy to show that our steady state with  $R < 1 (> 1)$  and stationary prices, corresponds to the steady state of an economy with zero nominal rate, positive (negative) money growth, and positive (negative) inflation.

chooses an allocation that maximizes the ex-ante utility of the representative household

$$\int \int \left[ c_1(\theta, \hat{\theta}) + \beta u(c_2(\theta, \hat{\theta}) - v(n(\theta, \hat{\theta}))) + \beta^2 c_3(\theta, \hat{\theta}) \right] dF(\hat{\theta}) dF(\theta).$$

The planner is constrained only by the resource constraints of the economy. In period 1 total consumption in an island with shock  $\hat{\theta}$  cannot exceed production by producers located in the same island, that is,<sup>4</sup>

$$\int c_1(\theta, \hat{\theta}) dF(\theta) \leq \int \hat{\theta} n(\hat{\theta}, \theta) dF(\theta).$$

In periods 2 and 3, average consumption cannot exceed the given endowment, that is,

$$\begin{aligned} \int \int c_2(\theta, \hat{\theta}) dF(\theta) dF(\hat{\theta}) &\leq e_2; \\ \int \int c_3(\theta, \hat{\theta}) dF(\theta) dF(\hat{\theta}) &\leq e_3. \end{aligned} \tag{2}$$

The following Proposition characterizes the optimal allocation.<sup>5</sup>

**Proposition 1** *The optimal allocation is characterized by*

$$\begin{aligned} c_1(\theta, \hat{\theta}) &= c_1^{FB}(\hat{\theta}) = \hat{\theta} n^{FB}(\hat{\theta}), \\ n(\theta, \hat{\theta}) &= n^{FB}(\theta), \\ c_2(\theta, \hat{\theta}) &= c_2^{FB}(\theta), \end{aligned}$$

for each pair  $(\theta, \hat{\theta})$ , where  $n^{FB}(\theta)$  and  $c_2^{FB}(\theta)$  satisfy

$$\frac{\theta}{v'(n^{FB}(\theta))} = \mu \text{ for all } \theta, \tag{3}$$

$$\beta u'(c_2^{FB}(\theta) - v(n^{FB}(\theta))) = \mu \text{ for all } \theta, \tag{4}$$

$\mu$  is a positive constant, and

$$c_2^{FB}(\theta) - v(n^{FB}(\theta)) = e_2 - \int v(n^{FB}(\tilde{\theta})) dF(\tilde{\theta}) \text{ for all } \theta. \tag{5}$$

The variable  $\mu$  in the proposition corresponds to the Lagrange multiplier of the resource constraint (2). This proposition highlights two interesting features of the optimal allocation. First, the optimal effort of the producer of household  $(\theta, \hat{\theta})$  depends exclusively on his own

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<sup>4</sup>Here, we are making the assumption that the distribution of  $\theta$  for the consumers (producers) related to the producers (consumers) in island  $\hat{\theta}$ , is independent of  $\hat{\theta}$  and equal to  $F(\theta)$ . This assumption simplifies notation, but can be easily dispensed with.

<sup>5</sup>In the appendix we will give a full characterization of the optimum.

type  $\theta$ . Second and more important, at the optimum  $c_2 - v(n)$  is constant across households. This guarantees that the marginal utility of all households in period 2 is equalized and shows that the first-best allocation displays full insurance. Agents consumption in period 2 is set exactly to compensate for the effort level determined by the idiosyncratic shock in period 1. Finally, notice that, given linearity, the consumption levels at date 3 are indeterminate.

So far we have assumed that the planner can overcome the informational problem faced by the households, that is, the planner knows the shocks faced by the consumer and the producer from each household, when choosing the allocation. However, Proposition (1) shows that this requirement is not necessary, given that the optimal allocation only uses information about the local shock  $\theta$  to determine the consumption and production of the agents located in that island.

### 3 Equilibrium

We turn now to the definition and characterization of the competitive equilibrium. We begin by characterizing optimal individual behavior for given prices. The function  $p_{1,t}(\theta)$  denotes the nominal price in period 1 in the island with shock  $\theta$ , and  $p_{2,t}$  and  $p_{3,t}$  are the nominal prices in periods 2 and 3. Consider a household with an initial stock of money  $m$  in period  $t$ . The consumer chooses  $c_1(\hat{\theta})$ , only based on the productivity shock observed in the island where he is located,  $\hat{\theta}$ , while the producer chooses his labor effort  $n(\theta)$  only based on the productivity in his island,  $\theta$ .<sup>6</sup>

Given that all exchanges are anonymous, agents have to use cash to finance their purchases, and, moreover, cash holdings are restricted to be non-negative. In period 1 the consumer budget constraint and the liquidity constraint are then:

$$m_1(\hat{\theta}) + p_{1,t}(\hat{\theta})c_1(\hat{\theta}) \leq m, \tag{6}$$

$$m_1(\theta) \geq 0. \tag{7}$$

At the end of period 1, the consumer and the producer get back together, therefore the cash available to consumers at the beginning of period 2 includes the producer's revenue from the previous period, including any interest income. The budget constraint and the liquidity

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<sup>6</sup>Given that we use a recursive formulation for the household problem, we drop the time subscript  $t$  for the household-specific shocks,  $\theta$  and  $\hat{\theta}$ , and for the household choice variables.

constraint are now:

$$m_2(\theta, \hat{\theta}) + p_{2,t}c_2(\theta, \hat{\theta}) \leq R \left( m_1(\hat{\theta}) + p_{1,t}(\theta) y_1(\theta) \right), \quad (8)$$

$$m_2(\theta, \hat{\theta}) \geq 0. \quad (9)$$

Finally, in period 3, consumer and producer are located in the same island and they only need to finance the net expenditure  $c_3 - e_3$ . The constraints are now:

$$m_3(\theta, \hat{\theta}) + p_{3,t}c_3(\theta, \hat{\theta}) \leq p_{3,t}y_3 + R(m_2 + p_{2,t}y_2) - T, \quad (10)$$

$$m_3(\theta, \hat{\theta}) \geq 0. \quad (11)$$

Let  $V_t(m)$  denote the expected utility at the beginning of period  $t$  of a household with initial nominal balances  $m$ . The household problem is characterized by the Bellman equation

$$V_t(m) = \max_{\{c_s\}, \{m_s\}, n} \mathbb{E} \left[ c_1 + \beta u(c_2 - v(n)) + \beta^2 c_3 + \beta^3 V_{t+1}(Rm_3) \right], \quad (12)$$

subject to the constraints (6)-(11) for each pair of shocks  $\theta$  and  $\hat{\theta}$ , and the technological constraints

$$y_1(\theta) = \theta n(\theta), \quad y_2 = e_2, \quad y_3 = e_3.$$

Note that  $\{c_s\}$ ,  $\{m_s\}$  and  $n$  are functions of the shocks  $\theta$  and  $\hat{\theta}$  in the manner described above.

We are now in a position to define a competitive equilibrium. Let  $H_t$  denote the cross-sectional distribution of cash balances at the beginning of period  $(t, 1)$ , with support  $\mathcal{M}_t$ .

**Definition 1** *A competitive equilibrium is given by a sequence of prices  $\{\{p_{1,t}(\theta)\}_\theta, p_{2,t}, p_{3,t}\}$ , of allocations  $\{\{n_t(\theta, m)\}_{\theta, m \in \mathcal{M}_t}, \{c_{1,t}(\hat{\theta}, m)\}_{\hat{\theta}, m \in \mathcal{M}_t}, \{c_{2,t}(\theta, \hat{\theta}, m), c_{3,t}(\theta, \hat{\theta}, m)\}_{\theta, \hat{\theta}, m \in \mathcal{M}_t}\}$  and of money distributions  $\{H_t\}$ , such that:*

1. *The allocations are solutions to (12) for each  $t$  and  $m \in \mathcal{M}_t$ .*
2. *Market clears*

$$\begin{aligned} \int m dH_t &= M, \\ \int c_{1,t}(\hat{\theta}, m) dH_t &= \int \hat{\theta} n_t(\hat{\theta}, m) dH_t \quad \forall \hat{\theta}, \\ \int \int \int c_{s,t}(\theta, \hat{\theta}, m) dF dF dH_t &= e_s, \quad s = 1, 2. \end{aligned}$$

3. *The sequence  $\{H_t\}$  is consistent with the transition probability for money holdings derived from individual behavior.*

In this definition, we omit the money balances from the allocation and we omit the money market equilibrium conditions in periods 1 and 2. Market clearing in the goods markets ensures that the money market clears in these two periods.<sup>7</sup> The final money balances  $m_3$  can be derived from the consumer budget constraints and are used, implicitly, to check condition 3.

From now on, we focus on steady states where nominal prices and allocations are constant over time and where the cross sectional distribution of money holdings is degenerate, i.e., all agents begin each period  $(1, t)$  with  $m = M$ . As in Lagos and Wright (2005), competitive equilibria of this simple form exist because agents have linear utility in period 3, while the value function  $V(m)$  is concave. In equilibrium, all agents will adjust their consumption in period 3, so as to reach the same level of  $m_3$ , irrespective of the history of their shocks. For any initial distribution  $H_0$  of money holdings, the economy will converge in one period to this steady state.<sup>8</sup>

### 3.1 Unconstrained equilibrium

Now we consider two cases which capture the effect of the liquidity constraints on the equilibrium determination. First, we look at the case where the government sets a high enough rate of return on liquid assets that the economy is able to achieve the first-best allocation. This case corresponds to  $R = 1/\beta$ , that is, to a monetary policy that follows the Friedman rule. Second, we look at the case where the rate of return is so low that the allocation in periods 1 and 2 looks like the allocation in a static economy with uninsurable income shocks. We show that this case arises whenever  $R \leq \hat{R}$ , for a given cutoff  $\hat{R} \in (0, 1/\beta)$ .

The following proposition shows that, under the monetary policy  $R = 1/\beta$ , there exists an equilibrium that achieves the efficient allocation of resources.<sup>9</sup>

**Proposition 2** *Under the monetary policy  $R = 1/\beta$  and  $M > 0$ , there exists an unconstrained equilibrium that implements the first-best allocation.*

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<sup>7</sup>This is just Walras' law, period by period.

<sup>8</sup>The assumption of a large  $e_3$  guarantees that this transition is feasible for any  $H_0$ .

<sup>9</sup>This efficiency result is related to a result in Rocheteau and Wright (2005), which extends Lagos and Wright (2005) environment to a setting with competitive markets.

**Proof.** We proceed by guessing and verifying that the economy can achieve an equilibrium that implements the first-best. First, conjecture that agents are never liquidity constrained, that is,  $m_s > 0$  for each  $s$  and each realization of  $\theta$  and  $\hat{\theta}$ . We will call this type of equilibrium “unconstrained.” Moreover, conjecture the following values for the nominal prices:

$$p_1(\theta) = \kappa \text{ for all } \theta, \quad (13a)$$

$$p_2 = \kappa u'(c_2^{FB}(\theta) - v(n^{FB}(\theta))), \quad (13b)$$

$$p_3 = \kappa. \quad (13c)$$

Up to the scaling factor  $\kappa$ , these are equal to the shadow prices derived from the planner problem in 2.1. The factor  $\kappa$  is endogenous and is derived below.

The household optimality conditions can be rearranged to give the labor supply equation,

$$n(\theta) = v'^{-1} \left( \theta R \frac{p_1(\theta)}{p_2} \right) \text{ for all } \theta, \quad (14)$$

and, the consumer Euler equations,

$$\begin{aligned} 1 &= \beta R \frac{p_1(\hat{\theta})}{p_2} \mathbb{E} \left[ u' \left( c_2(\theta, \hat{\theta}) - v(n(\theta)) \right) \mid \hat{\theta} \right] \text{ for all } \hat{\theta}, \quad (15) \\ u' \left( c_2(\theta, \hat{\theta}) - v(n(\theta)) \right) &= \beta R \frac{p_2}{p_3} \text{ for all } \theta, \hat{\theta}, \\ 1 &= \beta^3 R \frac{p_3}{p_1}, \end{aligned}$$

which are derived under the conjecture that  $m_s > 0$ .<sup>10</sup> With  $R = 1/\beta$ , it is easy to check that these optimality conditions are satisfied if we substitute the first-best allocation for  $\{c_s\}$  and  $n$ .

Next, we check that the consumers are, indeed, not liquidity constrained. To do that, substitute the first best allocations in the consumer’s budget constraints and use market clearing to obtain:

$$m_1(\hat{\theta}) = M - p_1 c_1^{FB}(\hat{\theta}), \quad (16)$$

$$m_2(\theta, \hat{\theta}) = R(m_1(\hat{\theta}) + p_1 c_1^{FB}(\theta)) - p_2 c_2^{FB}(\theta), \quad (17)$$

$$m_3(\theta, \hat{\theta}) = R(m_2(\theta, \hat{\theta}) + p_2 e_2) + p_3 [e_3 - c_3(\theta, \hat{\theta})] - T, \quad (18)$$

where  $T$  satisfies (1). To have a stationary, degenerate distribution of money balances, we set  $m_3(\theta, \hat{\theta}) = M/R$ . Choosing a small enough value for  $\kappa$  it is possible to show that (16) and (17)

<sup>10</sup>See the Appendix for the derivation of these optimality conditions, in the general case  $m_s \geq 0$ .

give  $m_1 > 0$  and  $m_2 > 0$  for any realization of  $\theta$  and  $\hat{\theta}$ . Finally, we can use equation (18) to obtain  $e_3 - c_3(\theta, \hat{\theta})$  as a residual. Our assumption that  $e_3$  is large ensures that  $c_3$  will always be non-negative. ■

Notice that, under this monetary policy, the price level is indeterminate, since we can choose any scaling value  $\kappa > 0$  smaller than a given cutoff  $\hat{\kappa}$ , which depends on  $M$ . To pin down the price level, one can fix a value for the real tax  $\tau \equiv T/p_3$  and then obtain  $p_3 = T/\tau$ . This shows that an alternative statement of Proposition 2 is that if  $\tau$  is larger than a cutoff  $\hat{\tau}^U$  then there exists an equilibrium with  $R = 1/\beta$ .<sup>11</sup> A high level of real taxation corresponds to a high real value of the liquid asset  $M$  circulating in the economy. To sustain the first best level of risk sharing the government has to commit enough fiscal resources to ensure that public liquidity pays the real rate of return  $R = 1/\beta$ .

A further remark on the equilibrium distribution of nominal balances. If  $R = 1/\beta$  and  $\kappa < \hat{\kappa}$ , this distribution is not uniquely pinned down. In this case the value function  $V(m)$  is locally linear at  $m = M$ , and agents are indifferent between consuming a bit less (more) in period 3 and increasing (decreasing) their money balances  $m_3$ .<sup>12</sup> Therefore, there are also equilibria where agents choose different values for  $m_3$  and the distribution of money balances is non-degenerate. These equilibria are identical to the one described in terms of consumption in periods 1 and 2, and in terms of labor effort. They only differ for the distribution of  $c_3$ , and therefore are ex-ante equivalent in terms of welfare.

### 3.2 Constrained equilibrium

We now turn to the case where agents are constrained. In particular, for tractability, we focus on the case where the constraint is always binding in period 2, and never in period 1. The following proposition shows that, with an assumption on the distribution  $F(\theta)$ , this type of equilibrium arises when  $R$  is sufficiently low.

**Proposition 3** *Suppose the lower support of  $\theta$  is 0. There exists a cutoff  $\hat{R} \in (0, 1/\beta)$  such that, if  $R \leq \hat{R}$ , then there is an equilibrium with:*

(i)  $m_1(\hat{\theta}) > 0$  for all  $\hat{\theta}$ .

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<sup>11</sup>The monetary regime can be defined either in terms of the pair  $(R, M)$ , with  $R \in (0, 1/\beta]$  and  $M > 0$ , or in terms of the pair  $(\tau, M)$  with  $\tau$  in some interval  $[\underline{\tau}, \bar{\tau}]$  and  $M > 0$ . This two ways of defining the monetary regime are interchangeable, as long as we are only concerned with stationary price paths.

<sup>12</sup>They are not indifferent to large changes in  $m_3$ , because their liquidity constraints can be binding in the following periods 1 or 2.

(ii)  $m_2(\theta, \hat{\theta}) = 0$  for all  $\theta, \hat{\theta}$ .

From now on, for brevity, we will call this type of equilibrium a “constrained equilibrium.” The reason why agents are never against the constraint in period 1 is that the lower bound for producers’ income is zero and  $u(\cdot)$  satisfies the Inada condition  $\lim_{x \rightarrow 0} u(x) = \infty$ . Therefore, consumers always keep some extra cash at the end of period 1 to make sure that they can consume a positive amount in period 2. It follows that, under these assumptions, the constraint  $m_1 \geq 0$  will never bind for any level of  $R$ .

Given that  $m_2 = 0$ , the consumer budget constraints in periods 1 and 2 can be aggregated to give

$$p_2 c_2 = p_2 e_2 - R p_1(\hat{\theta}) c_1 + R p_1(\theta) \theta n. \quad (19)$$

Notice the presence of  $e_2$  on the right-hand side. Since the liquidity constraint is binding for all agents at date 2, a simple “quantity-theory equation” determines the price level  $p_2$ , that is,

$$RM = p_2 e_2,$$

and the real value of the money balances is identical to the endowment  $e_2$ . Since all agents begin period 1 with  $m = M$ , (19) follows.

The equilibrium in periods 1 and 2 is formally equivalent to the equilibrium of a two-period economy where agents face uninsurable income shocks. The household problem reduces to choosing  $c_1(\hat{\theta})$  and  $n(\theta)$  to maximize

$$\mathbb{E} \left[ c_1(\hat{\theta}) + \beta u(c_2 - v(n)) \right]$$

subject to (19) for all  $\theta$  and  $\hat{\theta}$ . In particular, the consumer needs to choose  $c_1(\hat{\theta})$  before knowing the realization of the productivity shock  $\theta$ , which will determine his available income in period 2. The consumer Euler equation in period 1 can now be written as:

$$1 = \beta R \frac{p_1(\hat{\theta})}{p_2} \int_{\underline{\theta}}^{\bar{\theta}} u' \left( e_2 - R \frac{p_1(\hat{\theta})}{p_2} c_1(\hat{\theta}) + R \frac{p_1(\theta)}{p_2} \theta n(\theta) - v(n(\theta)) \right) dF(\theta). \quad (20)$$

On the other hand, optimal labor supply is still given by (14), given our assumption that labor effort enters in quasi-linear form in period 2. In fact, the main role of this assumption is to simplify the treatment of labor supply, so that we can focus on consumer choice. Substituting the market clearing condition  $c_1(\hat{\theta}) = \hat{\theta} n(\hat{\theta})$  and the labor supply equation (14), equation 20

defines a functional equation for  $p_1(\cdot)$ . The proof of Proposition 3 shows that this functional equation has a unique solution.

To ensure that the allocations derived for periods 1 and 2 are part of a full dynamic equilibrium, we need to ensure that the constraint  $m_2 \geq 0$  is indeed binding for all pairs of shocks  $\theta$  and  $\hat{\theta}$ . The proof of Proposition 3 shows that when  $R \leq \hat{R}$  this condition is met. An interesting feature of this equilibrium is that, as long as  $R \in (0, \hat{R})$ , the choice of  $R$  has no effects on the consumption allocation. The only real variables affected by the choice of  $R$  are the value of real balances in period 3 and the value of the real tax  $\tau$ , required to support the equilibrium. As in the case of Proposition 2, also Proposition 3 can be restated in fiscal terms. Namely, there is a cutoff  $\hat{\tau}^C$ , such that if  $\tau \leq \hat{\tau}^C$ , then there is an equilibrium with  $R \leq \hat{R}$  and binding liquidity constraints. If the government commits limited fiscal resources to sustain the value of public liquidity, then agents have limited liquid assets to insure against temporary income shocks.

The crucial difference between this case and the unconstrained case in Proposition 2 is that now consumers in period 1 are concerned about the value of their income  $p_1(\theta)\theta n(\theta)$ , when making their spending decisions. The income risk, due to the volatility of  $\theta$ , induces a precautionary behavior on the consumers' side. This has two consequences. First, if a consumer expects higher realizations of  $\theta$ , he will increase consumption in period 1, since he is less worried about having low liquidity in period 2. This will increase the price  $p_1(\hat{\theta})$ , in the market where the consumer is located, and increase output. Second, there is a general equilibrium feed-back involving the spending decisions of consumers. If consumers in all other markets are increasing their demand, the prices  $p_1(\theta)$  increase and the output of the producers increases. Hence, the consumer in island  $\hat{\theta}$  will expect a higher income and he will increase his own demand. Through this channel the initial increase in demand is amplified. Both effects will play a crucial role in the following section.

We conclude this section with a characterization of the equilibrium price function  $p_1(\theta)$ .

**Lemma 1** *The price  $p_1(\theta)$  is a monotone decreasing function of  $\theta$ . The function  $R\theta p_1(\theta)/p_2$  is monotone increasing in  $\theta$ .*

This Lemma shows that the price  $p_1$  has to be lower for higher values of  $\theta$ , in order to encourage consumers to increase spending when they are in islands with high productivity. In the case of the unconstrained equilibrium, this is not necessary, because consumers can spread

their increased spending over a longer horizon (i.e. until period 3) and they are happy to increase their spending at the constant price  $p_1 = 1$ .

## 4 Liquidity Constraints and the Response to Aggregate Shocks

The main question we want to address is how an economy with binding liquidity constraints responds to an aggregate productivity shock, and whether it responds more or less than an unconstrained economy. Let us focus on the effects of a shift in the distribution of the productivity shocks  $\theta$ , which can be interpreted as an unexpected aggregate shock.<sup>13</sup> Specifically, we consider a shift that reduces the probability of low shocks and increases the probability of high shocks, i.e. an increase in the sense of first-order stochastic dominance. This has two direct effects on output. First, there is an *own-productivity effect*. Since a mass of producers ends up in islands with higher productivity shocks, there is a mechanical increase in total output. This effect is positive both in an unconstrained economy and in a constrained economy. Second, there is an effect that works through income expectations. The behavior of a consumer in a given island, with a given local productivity shock, changes because he expects the producer of his household to receive, on average, higher productivity shocks in other islands. This means that the household will expect higher income. In the unconstrained economy this will have no effect on consumption decisions, given that the household is, *de facto*, fully insured against income shocks. On the contrary, in the constrained economy this will have a positive effect on consumption expenditures. This *expected income effect* tends to make the output response of an economy with binding liquidity constraints larger relative to the unconstrained benchmark. On top of these two direct effects, we also analyze a *general equilibrium effect*. Interestingly, this effect magnifies the response in the liquidity constrained economy, and dampens it in the unconstrained case.

### 4.1 Expected income and general equilibrium

Suppose that the distribution of the productivity shocks depends continuously on the aggregate parameter  $\zeta \in \mathbb{R}$ . In particular, assume that a distribution with a higher  $\zeta$  first-order stochastically dominates a distribution with lower  $\zeta$ . From now on, we write the cumulative

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<sup>13</sup>At the expense of further notation, the results in this section can be easily extended to a model with an explicit treatment of aggregate shocks.

distribution function of  $\theta$  as  $F(\theta; \zeta)$ .<sup>14</sup> Define aggregate output as

$$Y \equiv \int_{\underline{\theta}}^{\bar{\theta}} \theta n(\theta; \zeta) dF(\theta; \zeta) = \int_{\underline{\theta}}^{\bar{\theta}} c_1(\theta; \zeta) dF(\theta; \zeta).$$

We use  $d \ln Y / d\zeta$  as a measure of output response, that is, we focus on the proportional response of output. This normalization will help in comparing different monetary regimes, since the average level of output will be in general different. This measure can be decomposed as follows

$$\frac{d \ln Y}{d\zeta} = \frac{\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial f(\theta; \zeta)}{\partial \zeta} \theta n(\theta; \zeta) d\theta}{Y} + \frac{\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial c_1(\theta; \zeta)}{\partial \zeta} dF(\theta; \zeta)}{Y}. \quad (21)$$

The first member on the right-hand side represents the own-productivity effect we discussed above. This shock increases the probability of higher values of  $\theta$ , and thus increases aggregate output in both monetary regimes. The second member captures the endogenous response of consumption for each given level of  $\theta$ . This captures both the expected income and the general equilibrium effects. We focus now our attention on this term and show that it has opposite signs in the two regimes.

From now on, we use  $U$  to denote the unconstrained equilibrium derived in Section 3.1 and  $C$  for the constrained equilibrium in Section 3.2.

First, consider the case of an unconstrained economy, under the monetary regime  $R = 1/\beta$ . In this case the second term on the right-hand side of equation (21) represents a pure general equilibrium effect. Consider market  $\hat{\theta}$ , that is, the market in period 1 in an island with productivity shock  $\hat{\theta}$ . The demand and the supply sides of the market  $\hat{\theta}$  are respectively fully determined by

$$1 = \beta R \frac{p_1(\hat{\theta}; \zeta)}{p_2(\zeta)} \int u'(c_2(\hat{\theta}, \theta; \zeta) - v(n(\theta; \zeta))) dF(\theta; \zeta) \quad (22)$$

and

$$n(\hat{\theta}; \zeta) = v'^{-1} \left( \hat{\theta} R \frac{p_1(\hat{\theta}; \zeta)}{p_2(\zeta)} \right). \quad (23)$$

Consider first the following partial equilibrium exercise. Fix  $\bar{\zeta}$  and keep constant  $p_1(\theta; \bar{\zeta})$  for all  $\theta \neq \hat{\theta}$ ,  $p_2(\bar{\zeta})$  and  $p_3(\bar{\zeta})$ . If the consumers on market  $\hat{\theta}$  face these prices, they expect  $u'(c_2 - v(n))$  to be constant and equal to  $\beta p_2(\bar{\zeta}) / p_3(\bar{\zeta})$ , by the Euler equation between periods 2 and 3. Then, equation (22) shows that the price  $p_1(\hat{\theta}; \zeta)$  will be unchanged, and (23) shows that labor effort,  $n(\hat{\theta}; \zeta)$ , will be constant. However, once we look at the general equilibrium,

<sup>14</sup> Assume that the support is  $[\underline{\theta}, \bar{\theta}]$  and is independent of  $\zeta$ .

we realize that prices must change in response to an aggregate shock. In particular, when  $\zeta$  increases, more households have to exert a high labor effort. Since households want to keep  $u'(c_2 - v(n))$  constant, this tends to increase the demand for consumption in period 2. Given that the endowment of period 2 goods is fixed this cannot be an equilibrium, and prices in period 2 must increase. Hence, by (23), labor effort and consumption in period 1 will decrease in any island  $\theta$ . The following proposition formalizes this mechanism.

**Proposition 4** *Consider an unconstrained economy with  $R = 1/\beta$ . The consumption function  $c_1^U(\theta; \zeta)$  is decreasing in  $\zeta$  for all  $\theta$ .*

It follows that the general equilibrium effect is negative and dampens the output response to the aggregate shock.

Next, consider the case of a constrained economy, assuming  $R < \hat{R}$ .<sup>15</sup> Proposition 5 shows that in this case the second term on the right-hand side of equation (21) is positive, as, for any given  $\theta$ , consumption in period 1 increases with  $\zeta$ . We can decompose this effect into three different elements. First, there is a partial equilibrium effect driven by income expectations. Consider the same partial equilibrium exercise we have performed in the unconstrained case. Demand and supply in market  $\hat{\theta}$  are still given by (22) and (23). However, now  $u'(c_2 - v(n))$  is no longer constant, as agents are liquidity constrained and

$$c_2(\hat{\theta}, \theta; \zeta) = e_2 - R \frac{p_1(\hat{\theta}; \zeta)}{p_2(\zeta)} c_1(\hat{\theta}; \zeta) + R \frac{p_1(\theta; \zeta)}{p_2(\zeta)} \theta n(\theta; \zeta).$$

Recall that Lemma 1 shows that  $\theta n(\theta; \zeta) R p_1(\theta; \zeta) / p_2(\zeta) - v(n(\theta; \zeta))$  is increasing in  $\theta$ , so the marginal utility  $u'(c_2 - v(n))$  is decreasing in  $\theta$ . It follows that, now, when  $\zeta$  increases the integral on the right-hand side of (22) becomes smaller and relative prices in island  $\hat{\theta}$  tend to increase.<sup>16</sup> The intuition behind this result is that when a liquidity constrained consumer expects higher income, his marginal value of money decreases. Then, he reduces his precautionary reserves and increases consumption in period 1. On the top of this channel, there is a general equilibrium feed-back effect. When an agent expects higher income and decides to spend more in period 1, the prices of the goods produced in period 1 increase in all islands, increasing labor supply and output. This increases further expected income in period 2 and leads to a further increase in consumption spending in period 1.

<sup>15</sup>Notice that  $\hat{R}$  depends on  $\zeta$  continuously. For small changes in  $\zeta$ ,  $R < \hat{R}$  ensures that the economy remains in a constrained equilibrium.

<sup>16</sup>The fact that the integral decreases follows from the fact that an increase in  $\zeta$  leads to a shift of the distribution of  $\theta$  in the sense of first order stochastic dominance.

**Proposition 5** Consider a constrained economy, with  $R < \hat{R}$ . The consumption function  $c_1^C(\theta; \zeta)$  is increasing in  $\zeta$ , for all  $\theta$ .

This proposition shows that the expected output effect together with the general equilibrium mechanism tend to magnify the output response to aggregate shocks in the constrained economy.

The question that remains to be addressed is whether, overall, a constrained economy reacts more or less to an aggregate shock than an economy with full insurance. Going back to equation (21), Propositions 4 and 5 show that the second term of the decomposition is negative in the unconstrained case and positive in the constrained one. However, we do not know the relative magnitude of the first term in the decomposition, which we know is positive in both cases. In order to compare the total output response of the two economies, we turn to some examples.

## 4.2 An example with a binary shock

Assume that  $u(x) = x^{1-\sigma}/(1-\sigma)$ ,  $v(x) = x^2/2$  and  $\theta \in \{0, \bar{\theta}\}$ . Denote by  $\pi$  the probability of  $\theta = \bar{\theta}$ .<sup>17</sup> In this case,  $\pi$  takes the place of  $\zeta$ , since a higher  $\pi$  corresponds to first-order stochastically dominant distribution.

**Proposition 6** Suppose  $v(x) = x^2/2$  and  $\theta \in \{0, \bar{\theta}\}$ , then  $d \ln Y / d\pi$  is bigger when  $R \leq \hat{R}$  than when  $R = 1/\beta$ .

**Proof.** When  $R = 1/\beta$ , the unconstrained equilibrium is characterized by

$$c^U(\theta; \pi) = c^{FB}(\theta; \pi) = \frac{\theta^2}{\mu(\pi)},$$

where, from (3)-(5),  $\mu$  is implicitly defined by

$$\left(\frac{\beta}{\mu(\pi)}\right)^{\frac{1}{\sigma}} + \frac{\pi}{2} \left(\frac{\bar{\theta}}{\mu(\pi)}\right)^2 = e_2. \quad (24)$$

Then, notice that aggregate output can be written as

$$Y^U = \frac{\pi \bar{\theta}^2}{\mu(\pi)}.$$

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<sup>17</sup>All the results obtained for continuous distributions can be easily generalized to discrete distributions.

Consider an increase in  $\pi$ . Equation (24) implies that  $\mu'(\pi) > 0$ . Then

$$\frac{dY^U}{d\pi} = \frac{\bar{\theta}^2}{\mu(\pi)} - \frac{\pi\bar{\theta}^2}{\mu(\pi)^2} \frac{d\mu(\pi)}{d\pi},$$

and

$$\frac{d \ln Y^U}{d\pi} = \frac{1}{\pi} - \frac{1}{\mu} \frac{d\mu(\pi)}{d\pi} \leq \frac{1}{\pi}.$$

When, instead,  $R \leq \hat{R}$ , then the economy achieves a fully constrained equilibrium allocation.

Then, from (??) we get

$$c^C(\bar{\theta}) = \bar{r},$$

and substituting in (??) gives a functional equation in  $\bar{r}$  given by

$$\bar{\theta} = \beta\bar{r} \left( \pi u' \left( e_2 - \frac{1}{2}\bar{r}^2 \right) + (1 - \pi) u' (e_2 - \bar{r}^2) \right). \quad (25)$$

The total output is now

$$Y^C = \pi\bar{r}\bar{\theta}$$

which implies

$$\frac{dY^C}{d\pi} = \bar{r}\bar{\theta} + \pi\bar{\theta} \frac{d\bar{r}}{d\pi}$$

Given that equation (25) shows that  $d\bar{r}/d\pi > 0$ , it follows that

$$\frac{d \ln Y^{LC}}{d\pi} = \frac{1}{\pi} + \frac{1}{\bar{r}} \frac{d\bar{r}}{d\pi} > \frac{1}{\pi},$$

completing the proof. ■

This proposition shows that, in the example considered, the output response to a positive aggregate shock is always higher when the liquidity constraint of the agents in the economy is binding. In particular, with these specific functional forms, the own-productivity effect, adjusted for the output levels, is identical in the two economies, so that what matters is only the second term in equation (21).

### 4.3 Liquidity and output volatility

Now we turn to numerical examples that show how, also for more general productivity distributions, the constrained economy can be more responsive to aggregate shocks. These examples also allow us to explore the monetary regimes with  $R \in (\hat{R}, 1/\beta)$ , where the liquidity constraints are occasionally binding.

We keep CRRA utility and quadratic effort cost, as in the previous example, and we set  $\sigma = 1$ . We assume  $\theta$  uniformly distributed in  $[0, \bar{\theta}]$ . We look at a shift in the distribution of  $\theta$  that reduces the mass for  $[0, \epsilon]$  by 0.1 and increases the mass for  $[\bar{\theta} - \epsilon, \bar{\theta}]$  by the same amount, for  $\epsilon = \bar{\theta}/5$ . This represents a simple increase in the sense of first order stochastic dominance. Figure 1 shows the output response to this shift for different levels of monetary policy  $R$ .

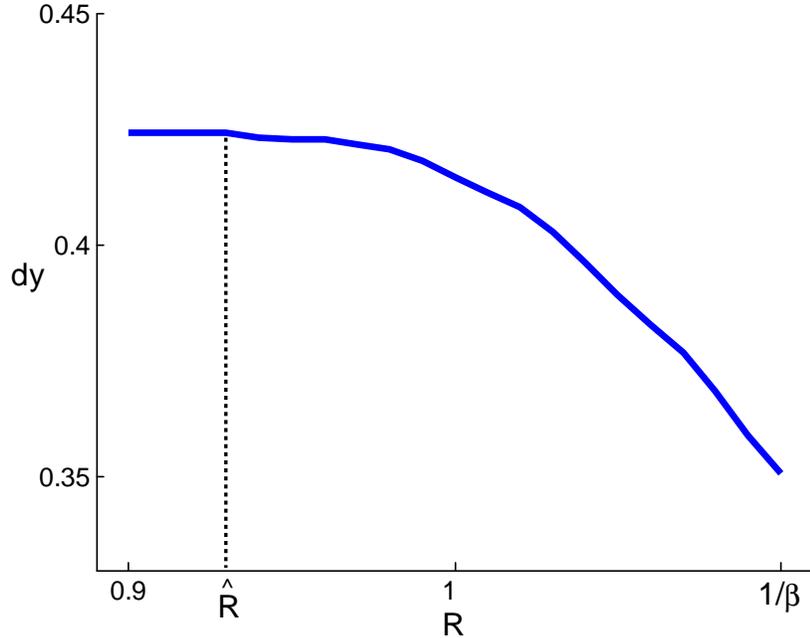


Figure 1. Monetary Policy and Output Response

Notice that the output response to aggregate shocks is significantly higher in the constrained economy, that is, when  $R \leq \hat{R}$ , than in the unconstrained one, that is, when  $R = 1/\beta$ . When  $R \in (\hat{R}, 1/\beta)$  there is an equilibrium where some agents are constrained and some are not. In the figure shown the response is monotone in  $R$ . We will see in the following that this is not always the case.

The next figure illustrates the decomposition exercise analyzed above. The green line represents the own-productivity effect, i.e. the first term of (21), and the blue line the total effect. The difference between the two represents the second term in the decomposition (21), capturing both the expected income effect and the general equilibrium effect.

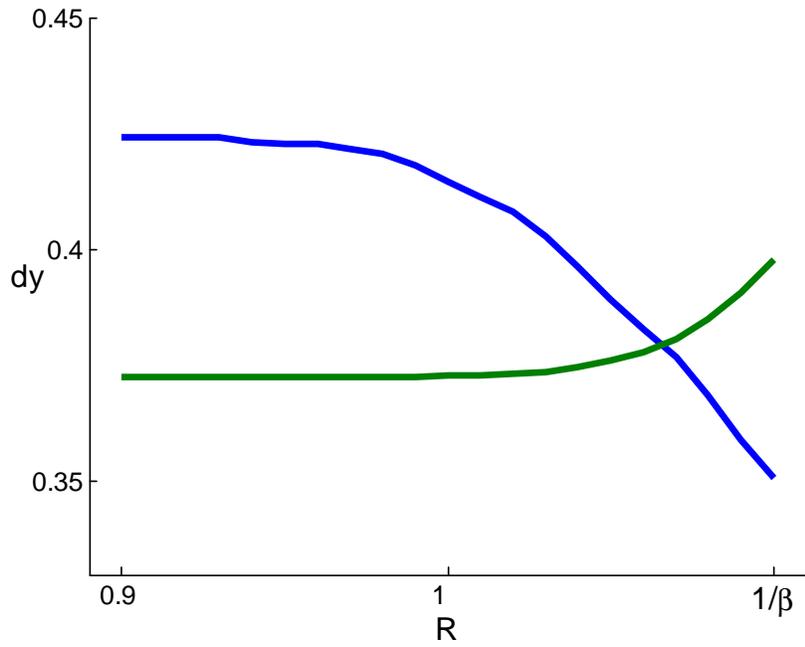


Figure 2. Decomposition of the Output Response

Finally, next figure, shows the output response of an economy with lower risk aversion, i.e.  $\sigma = 0.85$ , represented by the red line. As intuition suggests, the gap in the output response between the constrained and the unconstrained economy tends to be higher when agents are more risk averse.

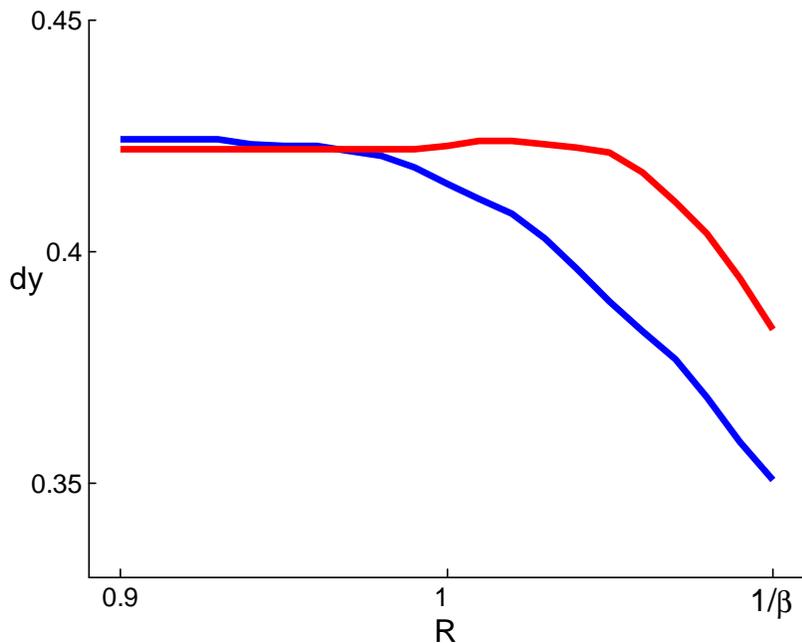


Figure 3. Output Response and Risk Aversion

Finally, it is worth noticing that when risk aversion is lower the output response of the economy is non-monotone in  $R$  in the region where liquidity constraints are occasionally binding.

[To be completed.]

## 5 Conclusions

In this paper we have analyzed how different monetary regimes can affect the response of an economy to aggregate shocks. When agents are liquidity constrained, the response of the economy tends to be magnified. In particular, when a positive aggregate shock hits the economy, consumers have higher income expectations, they need to build up less precautionary reserves and increase consumption spending. This feeds-back into higher income expectations of other agents and amplifies further the spending response.

Our mechanism is driven by the combination of decentralized trade, risk aversion and idiosyncratic uncertainty. All these three ingredients are necessary. In fact, the amplification effect described would disappear in a representative agent version of the model with no idiosyncratic risk, even if we keep an anonymous setup with decentralized trading.

To keep the model analytically tractable, we have assumed linear preferences in period 3. This allows agents, in the unconstrained regime, to fully insure against negative income shocks in period 2 by adjusting their period 3 consumption. In a model with concave utility in all periods, this type of adjustment would only be possible if shocks are temporary. In that case, the consumer would be able to smooth a negative income shock in period 2, by lowering consumption by a small amount in all future periods, so as to go back to the initial money balances. Our simplifying assumptions allows agents to adjust in one shot. However, it would be interesting to study a more general environment, and to explore the full dynamics of our mechanism, with different degrees of persistence for the shocks.

# Appendix

## Preliminary Results for Proposition 3

In order to prove Proposition 3 it is useful to perform a change of variables and to prove some preliminary lemmas. These results will also be useful also to prove Propositions 5.

Define the real wage in terms of period 2 consumption as:

$$w(\theta) \equiv \theta R \frac{p_1(\theta)}{p_2}, \quad (26)$$

and define the value of consumption at date 1, in island  $\theta$ , in terms of period 2 consumption:

$$x(\theta) \equiv w(\theta) v'^{-1}(w(\theta)). \quad (27)$$

Let the function  $W(x)$  be the inverse of  $wv'^{-1}(w)$ , defined implicitly by

$$x = W(x) v'^{-1}(W(x)). \quad (28)$$

Given that  $wv'^{-1}(w)$  is strictly monotone for  $w \in [0, \infty)$ , and  $\lim_w wv'^{-1}(w) = \infty$ , this inverse is well defined for  $x \in [0, \infty)$ .

Also define the function

$$I(x) \equiv e_2 + x - v(v'^{-1}(W(x))). \quad (29)$$

Finally, define the upper bound on the inverse elasticity of labor supply:

$$\bar{\eta} = \sup_n \frac{nv''(n)}{v'(n)}.$$

**Lemma 2** *The functions  $W(x)$  and  $I(x)$  satisfy the following properties:*

$$W'(x) = \frac{v''(n)}{nv''(n) + v'(n)},$$

$$I'(x) = \frac{nv''(n)}{nv''(n) + v'(n)} \in [0, \delta],$$

where  $n = v'^{-1}(W(x))$  and

$$\delta = \frac{\bar{\eta}}{1 + \bar{\eta}} < 1.$$

**Proof.** From (28), differentiating both sides and substituting gives:

$$1 = nW'(x) + \frac{v'(n)}{v''(n)}W'(x)$$

where  $n$  is defined in the statement of the lemma, and we obtain the first result.

Differentiating (29) gives:

$$I'(x) = 1 - \frac{v'(n)}{v''(n)}W'(x)$$

and, substituting  $W'(x)$  the second result follows. ■

Now, we can rewrite the functional equation (20) in terms of the function  $x(\cdot)$ , as

$$\hat{\theta} - \beta W(x(\hat{\theta})) \int u'(I(x(\theta)) - x(\hat{\theta})) dF(\theta) = 0. \quad (30)$$

Define the mapping  $x = T(x_0)$ , where  $x(\hat{\theta})$  solves the following equation for each  $\hat{\theta}$ :

$$\hat{\theta} - \beta W(x(\hat{\theta})) \int u'(I(x^0(\theta)) - x(\hat{\theta})) dF(\theta) = 0. \quad (31)$$

Let  $\bar{x}$  be the unique solution to:

$$\bar{\theta} = \beta W(\bar{x}) u'(I(\bar{x}) - \bar{x}).$$

**Lemma 3** *If  $x^0(\theta) \in [0, \bar{x}] \forall \theta$ , equation (31) has a unique solution  $x(\hat{\theta}) \in [0, \bar{x}]$  for each  $\hat{\theta} \in [\underline{\theta}, \bar{\theta}]$ . This solution is increasing if  $x^0(\theta)$  increases on a set of positive measure.*

**Proof.** A unique solution exists in  $[0, \infty)$ , because as  $x(\hat{\theta})$  varies from 0 to  $\min_{\theta} I(x^0(\theta))$ , the left-hand side of (31) is a continuous decreasing function, going from  $\hat{\theta}$  to  $-\infty$ .

To show that the solution is bounded by  $\bar{x}$ , suppose, instead, that  $x(\hat{\theta}) > \bar{x}$ . Consider the following chain of inequalities

$$\begin{aligned} \hat{\theta} &= \beta W(x(\hat{\theta})) \int u'(I(x^0(\theta)) - x(\hat{\theta})) dF(\theta) \geq \\ &\geq \beta W(x(\hat{\theta})) u'(I(\bar{x}) - x(\hat{\theta})) > \\ &> \beta W(\bar{x}) u'(I(\bar{x}) - \bar{x}) = \bar{\theta}, \end{aligned}$$

where the first inequality follows from the fact that  $I$  is increasing and  $u'$  decreasing, and the second from the fact that  $W$  is increasing and from our hypothesis. Since  $\hat{\theta} \leq \bar{\theta}$ , we have a contradiction.

From the implicit function theorem, and from the fact that  $I$  is increasing, it follows that the solution  $x(\hat{\theta})$  is increasing if  $x^0(\theta)$  increases on a set of positive measure. ■

**Lemma 4** *Fix an  $a > 0$ . Let  $x(\hat{\theta}) = T(x^0)(\hat{\theta})$  and  $x^a(\hat{\theta}) = T(x^0 + a)(\hat{\theta})$ . Then the following inequality applies*

$$x^a(\hat{\theta}) - x(\hat{\theta}) \leq \delta a,$$

where  $\delta$  is defined as in Lemma 2.

**Proof.** Notice that  $x^a(\hat{\theta})$  satisfies

$$\hat{\theta} - \beta W(x^a(\hat{\theta})) \int u'(I(x^0(\theta) + a) - x^a(\hat{\theta})) dF(\theta) = 0,$$

by definition. Suppose, by contradiction, that  $x^a(\hat{\theta}) > x(\hat{\theta}) + \delta a$ . Given that  $I'(x) \leq \delta$  we have

$$I(x^0(\theta) + a) - I(x^0(\theta)) \leq \delta a \forall \theta,$$

and this implies

$$I(x^0(\theta) + a) - x^a(\hat{\theta}) < I(x^0(\theta)) - x(\hat{\theta}) \forall \theta.$$

Then

$$\begin{aligned} \hat{\theta} &= \beta W(x^a(\hat{\theta})) \int u'(I(x^0(\theta) + a) - x^a(\hat{\theta})) dF(\theta) > \\ &> \beta W(x(\hat{\theta})) \int u'(I(x^0(\theta)) - x(\hat{\theta})) dF(\theta) = \hat{\theta}, \end{aligned}$$

a contradiction. ■

### Proof of Proposition 3

The discussion in the text shows that, if we conjecture an equilibrium with  $m_2 = 0$  for all  $\theta, \hat{\theta}$ , then (i) the price level at date 2 is given by:

$$p_2 = \frac{RM}{e_2},$$

and (ii) the prices at date 1, can be solved by solving the functional equation (20). This functional equation has been restated above as a functional equation in terms of the function  $x(\cdot)$  defined in (27) (see the functional equation (31)).

Lemmas 3 and 4, show that the map  $T$ , defined in (31), is a map from the space of bounded functions in  $[0, \bar{x}]$ , to the same space, and satisfy the assumptions of Blackwell's Theorem. Namely, Lemma 3 proves monotonicity and Lemma 4 proves the discounting property, for the discount factor  $\delta$ . Therefore a solution exists and is unique.

This step also gives us the real allocation in periods 1 and 2. In particular, this allocation be characterized by the following  $c_2(\theta, \hat{\theta})$  and  $n(\theta)$ :

$$\begin{aligned} n(\theta) &= v'^{-1}(W(x(\theta))), \\ c_2(\theta, \hat{\theta}) &= I(x(\theta)) - x(\hat{\theta}), \end{aligned}$$

(see (28) and (29)).

To check that  $m_2 = 0$  is optimal, we need to verify that

$$u'(c_2(\theta, \hat{\theta}) - v(n(\hat{\theta}))) \geq \beta R \frac{p_2}{p_3} \text{ for all } \theta, \hat{\theta}. \quad (32)$$

The results in Lemma 1 can be used to show that the allocation derived for periods 1 and 2 satisfies the following condition:  $c_2(\theta, \hat{\theta}) - v(n(\theta))$  is increasing in  $\theta$  and decreasing in  $\hat{\theta}$ . Therefore, a necessary and sufficient condition for 32 is:

$$u'(c_2(\bar{\theta}, \underline{\theta}) - v(n(\bar{\theta}))) \geq \beta R \frac{p_2}{p_3}.$$

To check this condition we need to derive the equilibrium value of  $p_3$ . The consumer Euler equation in period 3 is

$$\frac{1}{p_3} = \beta R \int \frac{1}{p_1(\hat{\theta})} dF(\hat{\theta}),$$

while the consumer Euler equation in period 1 can be rewritten as:

$$\frac{1}{p_1(\hat{\theta})} = \beta \frac{R}{p_2} \int u'(c_2(\theta, \hat{\theta}) - v(n(\theta))) dF(\theta).$$

Substituting, we obtain:

$$\frac{1}{p_3} = \beta^2 \frac{R^2}{p_2} \int \int u'(c_2(\theta, \hat{\theta}) - v(n(\theta))) dF(\theta) dF(\hat{\theta}), \quad (33)$$

which gives us the equilibrium value of  $p_3$ . Substituting in (32) we get:

$$u'(c_2(\bar{\theta}, \underline{\theta}) - v(n(\bar{\theta}))) \geq \beta^3 R^3 \int \int u'(c_2(\theta, \hat{\theta}) - v(n(\theta))) dF(\theta) dF(\hat{\theta}).$$

This gives us the upper bound  $\hat{R}$

$$\hat{R} \equiv \frac{1}{\beta} \left( \frac{u'(c_2(\bar{\theta}, \underline{\theta}) - v(n(\bar{\theta})))}{\int \int u'(c_2(\theta, \hat{\theta}) - v(n_1(\hat{\theta}))) dF(\hat{\theta}) dF(\theta)} \right)^{\frac{1}{3}}.$$

To derive an upper bound on real taxes at date 3 note that, in a constrained equilibrium, real balances at date 3 are:

$$\frac{M}{p_3} = \xi R$$

where

$$\xi \equiv \beta^2 e_2 \int \int u'(c_2(\theta, \hat{\theta}) - v(n(\theta))) dF(\theta) dF(\hat{\theta}).$$

This implies that the map between  $R$  and real taxation is the following, and is independent of  $M$ ,

$$\begin{aligned} \tau &= \xi R \left( R^2 - \frac{1}{R} \right), \\ &= \xi (R^3 - 1). \end{aligned}$$

This is an increasing function. Moreover, when  $R = \hat{R}$  we obtain:

$$\hat{\tau}^C \equiv \frac{e_2}{\beta} \left[ u'(c_2(\bar{\theta}, \underline{\theta}) - v(n_1(\underline{\theta}))) - \beta^3 \int \int u'(c_2(\theta, \hat{\theta}) - v(n_1(\hat{\theta}))) dF(\hat{\theta}) dF(\theta) \right].$$

Therefore, if  $\tau \leq \hat{\tau}^C$ , we have an equilibrium with  $R \leq \hat{R}$ .

## Proof of Lemma 1

We prove the second statement first. Define the function

$$w(\theta) \equiv R \theta \frac{p_1(\theta)}{p_2},$$

then equation (20) can be rewritten as a functional equation in  $w(\cdot)$ , that is

$$f(w(\hat{\theta}), \hat{\theta}) = \beta w(\hat{\theta}) \int u'(e_2 - w(\hat{\theta})n(\hat{\theta}) + w(\theta)n(\theta) - v(n(\theta))) dF(\theta) - \hat{\theta} \quad \forall \hat{\theta}$$

where  $n(\hat{\theta}) = v'^{-1}(w(\hat{\theta}))$  and hence  $dn(\hat{\theta})/dw(\hat{\theta}) = 1/v''(v'^{-1}(w(\hat{\theta})))$ . Using the implicit function theorem we obtain

$$\frac{dw(\hat{\theta})}{d\hat{\theta}} = \left[ \frac{\partial f(w(\hat{\theta}), \hat{\theta})}{\partial w(\hat{\theta})} \right]^{-1}$$

where

$$\begin{aligned} \frac{\partial f(w(\hat{\theta}), \hat{\theta})}{\partial w(\hat{\theta})} &= \frac{\hat{\theta}}{w(\hat{\theta})} - \left[ n(\hat{\theta}) + \frac{w(\hat{\theta})}{v''(n(\hat{\theta}))} \right]. \\ &\beta w(\hat{\theta}) \int u''(e_2 - w(\hat{\theta})n(\hat{\theta}) + w(\theta)n(\theta) - v(n(\theta))) dF(\theta). \end{aligned}$$

Given that  $u$  is concave, it follows that

$$\frac{dw(\hat{\theta})}{d\hat{\theta}} > 0$$

completing the proof.

To prove the first part of the lemma notice that

$$\beta R \frac{p_1(\hat{\theta})}{p_2} \int u'(e_2 - w(\hat{\theta})n(\hat{\theta}) + w(\theta)n(\theta) - v(n(\theta)))dF(\theta) = 1$$

and observe that  $w(\hat{\theta})n(\hat{\theta})$  is increasing in  $\hat{\theta}$ .

### Proof of Proposition 4

From market clearing we know that

$$c_1^U(\theta; \zeta) = \theta n^U(\theta; \zeta),$$

where  $n^U(\theta; \zeta) = n^{FB}(\theta; \zeta)$ , as we have shown in section 3.1. From equation (3)-(5), it follows that

$$c_1^U(\theta; \zeta) = \theta v'^{-1}(\theta/\mu(\zeta)), \quad (34)$$

where  $\mu(\zeta)$  solves the implicit function

$$u'^{-1}\left(\frac{\mu(\zeta)}{\beta}\right) + \int v\left(v'^{-1}\left(\frac{\theta}{\mu(\zeta)}\right)\right) dF(\theta; \zeta) = e_2.$$

Notice that the right-hand side of this expression is decreasing in  $\mu(\zeta)$ . Moreover, it is increasing in  $\zeta$  because the integrand  $v(v'^{-1}(\theta/\mu(\zeta)))$  is increasing in  $\theta$  and, by first-order stochastic dominance, the integral is increasing in  $\zeta$ . Applying the implicit function theorem, it follows immediately that  $\mu'(\zeta) > 0$ . Hence, differentiating expression (34) we obtain

$$\frac{\partial c_1(\theta; \zeta)}{\partial \zeta} = -\left(\frac{\theta}{\mu(\zeta)}\right)^2 \frac{\mu'(\zeta)}{v''(\theta/\mu(\zeta))} < 0$$

which concludes the proof.

### Proof of Proposition 5

Introduce the parameter  $\zeta$  in the functional equation (30)

$$\hat{\theta} - \beta W(x(\hat{\theta})) \int u'(I(x(\theta)) - x(\hat{\theta})) dF(\theta; \zeta) = 0. \quad (35)$$

Denote the associated mapping with  $T(x; \zeta)$ .

Consider two values  $\zeta_I$  and  $\zeta_{II}$ , with  $\zeta_I > \zeta_{II}$ . Let  $x_I(\cdot)$  be the solution to (35) for  $\zeta_I$ . First, define

$$x^0 = T(x_I; \zeta_{II})$$

and notice that  $x^0 \geq x_I$ . To prove this statement notice that  $u'(I(x_I(\theta)) - x_I(\hat{\theta}))$  is a decreasing function of  $\theta$  (by Lemma 1) and, thus,  $\int u'(I(x_I(\theta)) - x_I(\hat{\theta})) dF(\theta; \zeta)$  is decreasing in  $\zeta$ . Suppose that  $x^0(\hat{\theta}) < x_I(\hat{\theta})$  for some  $\hat{\theta}$ , then,

$$\begin{aligned} \hat{\theta} &= \beta W(x_I(\hat{\theta})) \int u'(I(x_I(\theta)) - x_I(\hat{\theta})) dF(\theta; \zeta_I) \geq \\ &\geq \beta W(x_I(\hat{\theta})) \int u'(I(x_I(\theta)) - x_I(\hat{\theta})) dF(\theta; \zeta_{II}) > \\ &> \beta W(x_I(\hat{\theta})) \int u'(I(x_I(\theta)) - x^0(\hat{\theta})) dF(\theta; \zeta_{II}) = \hat{\theta}, \end{aligned}$$

gives a contradiction.

Next, define the sequence of functions  $(x^0, x^1, \dots)$  using the recursion

$$x^{j+1} = T(x^j; \zeta_{II}).$$

Since  $x^0 \geq x_I$  and  $T$  is a monotone operator (Lemma 3), it follows that this sequence is monotone, with  $x^{j+1} \geq x^j$ . Moreover,  $T$  is a contraction, so this sequence has a limit point, which coincides with the fixed point  $x_{II}$  which characterizes the equilibrium for  $\zeta_{II}$ . Therefore,  $x_{II} \geq x_I$ .

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