

Abstract. *We analyze the optimal monetary policy in a pure endowment model with endogenously segmented asset markets. To obtain variations in asset prices similar to the ones that dominates US data, we study a model with stochastic risk aversion. The optimal monetary policy turns out to depend on these shocks in a simple way. When risk aversion is higher than average, so that price dividend ratios are lower than average, the optimal monetary policy is to inject less money than average. In this sense, optimal monetary policy is procyclical.*

Optimal Monetary Policy and Asset Prices in an Endogenously Segmented Market Model

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We study optimal monetary policy in a pure exchange model with endogenously segmented markets. We study how optimal monetary policy changes as risk aversion changes. The shocks to risk aversion produce changes in the market price of risk. We concentrate in these shocks because changes in the market price of risk arguably account for the larger part of the changes in prices of aggregate portfolios. We show that in this context optimal monetary policy is procyclical, i.e. when the market price of risk is high, so that asset prices are low, the optimal monetary injections should be large.

The model is very similar to the one in Alvarez, Atkeson and Kehoe (2001), where, in the spirit of the work by Baumol (1952) and Tobin (1956), agents must incur a fixed cost to transfer money between the asset market and the goods market. This fixed cost leads agents to trade assets and money only infrequently. In any given period only a fraction of agents are currently actively trading. Thus the asset market is segmented in the sense that, when the government injects money through an open market operation, only the currently active agents are on the other side of the transaction and only their marginal utilities determine asset prices rates.

When discussing asset pricing we focus on the interplay between monetary policy and asset market segmentation. We analyze how optimal monetary policy changes in the presence of preference shocks. The preference shocks we analyze are changes in risk aversion. We select this type of shocks based in the empirical literature such as Campbell and Shiller (1988), Campbell (1991), Campbell and Ammer (1993), Cochrane (1991), Cochrane (2001), that decomposes the sources of variability in the value of aggregate portfolios of equities. This literature decomposes the variability of asset prices in changes in expected discounted dividends, changes in interest rates and changes in the market price of risk. The findings in this literature is that the largest share of the variance of price dividend ratios and unexpected stock returns is explained by changes in expected required returns, and in particular, in the market price of risk.

The model is a standard cash-in-advance model except that agents must incur a fixed cost to transfer money between the goods market and the asset market. The household enters the beginning of each period with some cash in the goods market and then splits into a worker and a shopper. The worker sells the current endowment for cash and the shopper decides to either to buy goods with just the current real balances or incur the fixed cost to transfer cash either to or from the asset market and then buy goods. The household's endowment and thus the household's cash holdings are random and idiosyncratic.

The shopper follows a cutoff rule that define zones of activity and inactivity for trading cash and interest-bearing assets. Shoppers with high real balances incur the fixed cost and transfer cash to the asset market, while shoppers with low real balances incur the fixed cost and obtain cash from the asset market. Shoppers with intermediate real balances do not incur the fixed cost and simply spend their current real balances. Over time each household stochastically cycles through the zones of activity and inactivity as their idiosyncratic shocks

vary. If the fixed cost is zero all agents are active and the model reduces to the standard one in Lucas (1984).

Optimal monetary policy in this model depends on risk aversion. In this model agents trade, in a costly way, to insure idiosyncratic income shocks. Inflation, reduces the real income of all agents, so in the margin it compresses the income distribution and hence it reduces the need to use costly insurance. Of course if inflation is too large, all agents will have to trade: they will insure all the idiosyncratic risks, but will incur in lots of transaction costs. The optimal inflation rate balances these two forces. In this model inflation does not reduce total consumption, it just redistributes income across agents. Risk aversion has two opposite effects on the effect of inflation on welfare. On the one hand, the more risk averse agents are, the more agents trade, and hence the less they need of the reduction in cross sectional income dispersion that a small inflation produces. On the other hand, the more risk averse agents are, the more they value insurance. It turns out that, because the large impact that fixed cost have, the first effect dominates and hence the more risk averse agents are, the smaller the money injections are. Since high risk aversion leads to low prices of risky assets and leads to smaller money injections, the optimal monetary policy is procyclical.

In terms of the literature our paper is clearly related to the early work by Baumol (1952) and Tobin (1956). Jovanovic (1982), Romer (1986), and Chatterjee and Corbae (1992) developed general equilibrium versions of these models and used them to study how different constant inflation rates affect the steady state. There is also an extensive literature that dates back at least to Merton (1987) that considers asset market segmentation in models without money. See, for example, Hirshleifer (1988), Aiyagari and Gertler (1991), Cuny (1993), Allen and Gale (1994), Balasko, Cass and Shell (1995), Saito (1996) and Basak and Cuoco (1998) and the references cited therein. In contrast to all of these studies, however, this one examines optimal monetary policy with an emphasis on its relationship with asset prices.

1. The economy

The model is almost identical to the one used in Alvarez, Atkeson and Kehoe (2001).¹ We begin with a one country cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \dots$, a government, and a continuum of households of measure 1. Trade in this economy occurs in two separate locations: an asset market and a goods market. In the asset market, households trade cash and bonds which promise delivery of cash in the asset market in the next period, and the government introduces cash in the asset market via

¹There are two differences in the model in this paper: there are aggregate shocks and that the fixed cost of trading is specified in terms of utility, as opposed to goods. The first difference is to be able to analyze changes in asset prices. The second is to be able to produce analytical approximations to better understand optimal monetary policy.

open market operations. In the goods market, households use cash to buy goods subject to a cash-in-advance constraint and households sell their endowments for cash. Households incur a fixed cost of γ in utility for each transfer of cash between the asset market and the goods market. Except for this fixed cost the model is a standard cash-in-advance model as in Lucas (1984).

This economy has two four sources of uncertainty: idiosyncratic shocks to households' endowments, and three aggregate shocks: money growth, endowments and risk aversion. To simplify the exposition we abstract until the last section of the shocks in aggregate output and risk aversion since the strategy to solve for equilibrium allocations does not depend in on these shocks. The timing within each period $t \geq 1$ is illustrated in Figure 1. We emphasize the physical separation between markets by placing the asset market in the top half of the picture and the goods market in the bottom half. Households enter the period with the cash $P_{-1}y_{-1}$ they obtained from selling their endowments at $t - 1$, where P_{-1} is the price level and y_{-1} is their idiosyncratic random endowment at $t - 1$. The government conducts an open market operation in the asset market which determines the realization of money growth μ and the current price level P .

Each household then splits into a worker and a shopper. The worker sells the household endowment y for cash Py and rejoins the shopper at the end of the period. The shopper takes the household's cash $P_{-1}y_{-1}$ with real value $m = P_{-1}y_{-1}/P$ and shops for goods. The shopper can choose to pay the fixed cost γ to transfer cash Px with real value x to or from the asset market. This fixed cost γ is in terms of utility. If the shopper pays the fixed cost then the cash in advance constraint is $c = m + x$; otherwise this constraint is $c = m$.

Each household also enters the period with bonds that are claims to cash in the asset markets with payoffs contingent on both its idiosyncratic endowment y_{-1} and the rate of money growth μ in the current period. This cash can either be reinvested in the asset market or, if the fixed cost is paid, can be transferred to the goods market. In addition, if the fixed cost is paid, then cash from the goods market can be transferred to the asset market and used to buy new bonds. In Figure 1, letting B denote the current realization of the state-contingent bonds and $\int qB'$ the household's purchases of new bonds, the asset market constraint is $B = \int qB' + Px$ if the fixed cost is paid and $B = \int qB'$ otherwise. At the beginning of period $t + 1$, this household starts with cash Py in the goods markets and contingent bonds B' in the asset market.

In equilibrium some households choose to pay the fixed cost to transfer cash between the goods and asset markets while others do not. We refer to households that pay the fixed cost as *active* and refer to households who do not as *inactive*. Households with either sufficiently low real balances or sufficiently high real balances are active. The households with low real balances transfer cash from the asset market to the goods market while those with high real balances transfer cash in the opposite direction. Households with intermediate levels of real balances are in a *zone of inactivity* and simply consume their current real balances.

In Figure 1 and the body of the paper we assume that the shopper's cash-in-advance constraint binds and that in the asset markets households hold their assets in interest-bearing securities rather than cash. In Appendix A we provide sufficient conditions for this assumption to hold. In the rest of the section we flesh out this outline of the economy.

Each household's endowment y is independent and identically distributed across households and across time with distribution F which has density f . Let $Y = \int y f(y) dy$ be the constant aggregate endowment. Let $y^t = (y_0, \dots, y_t)$ denote a typical history of individual shocks up through period t and $f(y^t) = f(y_0)f(y_1) \dots f(y_t)$ the probability density over such histories. Let M_t denote the aggregate stock of money in period t , and let $\mu_t = M_t/M_{t-1}$ denote the growth rate of that money supply. Let $\mu^t = (\mu_1, \dots, \mu_t)$ denote the history of money growth shocks up through period t , and let $g(\mu^t)$ denote the probability density over such histories.

To make all households identical in period 0 we need to choose the initial conditions carefully. In period 0, households have \bar{B} units of government debt, which is a claim on \bar{B} dollars in the asset market in period 0. In this period there is no trade in goods and households simply trade bonds. In period 1 households also have y_0/μ_1 real balances in the goods market where y_0 also has distribution F and μ_1 is the money growth shock at the beginning of period 1.

The government issues one-period bonds with payoffs contingent on the aggregate state μ^t . In period t , given state μ^t , the government pays off outstanding bonds $B(\mu^t)$ in cash and issues claims to cash in the next asset market of the form $B(\mu^t, \mu_{t+1})$ at prices $q(\mu^t, \mu_{t+1})$. The government budget constraint in period $t \geq 1$, given state μ^t is

$$(1.1) \quad B(\mu^t) = M(\mu^t) - M(\mu^{t-1}) + \int_{\mu_{t+1}} q(\mu^t, \mu_{t+1}) B(\mu^t, \mu_{t+1}) d\mu_{t+1}.$$

In period 0, this constraint is $\bar{B} = \int_{\mu_1} q(\mu_1) B(\mu_1) d\mu_1$.

In the asset market in each period and state, households trade a complete set of one-period bonds that have payoffs next period which are contingent both on the aggregate event μ_{t+1} and their endowment realization y_t . An household in t with aggregate state μ^t and individual shock history y^{t-1} purchases $B(\mu^t, \mu_{t+1}, y^{t-1}, y_t)$ claims to cash that pay off in the next period contingent on the aggregate shock μ_{t+1} and the household's endowment shock y_t . We let $q(\mu^t, \mu_{t+1}, y_t)$ be the price of such a bond that pays one dollar in the asset market in period $t+1$ contingent on the relevant events. Because individual endowments are independent and identically distributed, we assume that these bond prices do not depend on the individual shock history y^{t-1} .

Instead of letting each household trade in all possible claims contingent on other households endowments, we suppose that each household trades only in claims contingent on the household's own endowment with the financial intermediary. This intermediary buys government bonds and trades in the household-specific contingent claims. This latter approach is much less cumbersome than the former and yields the same outcomes. Specifically, the

intermediary buys government bonds $B(\mu^{t+1})$ and sells household-specific claims of the form $B(\mu^{t+1}, y^t)$ to all the households to maximize profits for each aggregate state μ^{t+1}

$$\int_{y^t} q(\mu^{t+1}, y_t) B(\mu^{t+1}, y^{t-1}, y_t) f(y^t) dy^t - q(\mu^{t+1}) B(\mu^t, \mu_{t+1})$$

subject to the constraint $B(\mu^{t+1}) = \int_{y^t} B(\mu^{t+1}, y^t) f(y^t) dy^t$. From arbitrage it follows that $q(\mu^{t+1}, y_t) = q(\mu^{t+1}) f(y_t)$

Consider now the problem of an individual household. Let $P(\mu^t)$ denote the price level in the goods market in period t . In the goods market, in each period $t \geq 1$, households start with real balances $m(\mu^t, y^{t-1})$. They then choose transfers of real balances between the goods market and the asset market $x(\mu^t, y^{t-1})$, an indicator variable $z(\mu^t, y^{t-1})$ equal to zero if these transfers are zero and one if they are not and consumption $c(\mu^t, y^{t-1})$ subject to the cash-in-advance constraint

$$(1.2) \quad c(\mu^t, y^{t-1}) = m(\mu^t, y^{t-1}) + x(\mu^t, y^{t-1}) z(\mu^t, y^{t-1}),$$

where in (1.2) at $t = 1$, the term $m(\mu^t, y^{t-1})$ is given by y_0/μ_1 . New money balances in period $t + 1$ are given by $m(\mu^{t+1}, y^t) = P(\mu^t) y_t / P(\mu^{t+1})$.

In the asset market, each period households start with contingent claims $B(\mu^t, y^{t-1})$ to cash delivered in the asset market. They purchase new bonds and make cash transfers to or from the goods market subject to the sequence of budget constraints for $t \geq 1$

$$(1.3) \quad B(\mu^t, y^{t-1}) = \int_{\mu_{t+1}} \int_{y_t} q(\mu^t, \mu_{t+1}) B(\mu^t, \mu_{t+1}, y^{t-1}, y_t) f(y_t) d\mu_{t+1} dy_t + \\ + P(\mu^t) x(\mu^t, y^{t-1}) z(\mu^t, y^{t-1}).$$

In period $t = 0$, this asset market constraint is $\bar{B} = \int_{\mu_1} \int_{y_0} q(\mu_1) B(\mu_1, y_0) f(y_0) dy_0 d\mu_1$. Assume that both consumption and real bond holdings $B(\mu^t, y^{t-1})/P(\mu^t)$ are uniformly bounded by some large constants.

The problem of consumers is to maximize utility

$$(1.4) \quad \sum_{t=0}^{\infty} \beta^t \int \int [U(c(\mu^t, y^{t-1})) - \gamma z(\mu^t, y^{t-1})] g(\mu^t) f(y^{t-1}) d\mu^t dy^{t-1}$$

subject to the constraints (1.2) – (1.3).

The resource constraint is given by

$$(1.5) \quad \int c(\mu^t, y^{t-1}) f(y^{t-1}) dy^{t-1} = Y$$

for all t , μ^t , and the money market clearing condition is given by

$$(1.6) \quad M(\mu^t)/P(\mu^t) = \int (m(\mu^t, y^{t-1}) + x(\mu^t, y^{t-1}) z(\mu^t, y^{t-1})) f(y^{t-1}) dy^{t-1}$$

for all t , μ^t . An equilibrium is defined in the obvious fashion.

2. Characterizing equilibrium

Here we solve for the consumption and real balances of active and inactive households. We then characterize the link between the consumption of active households and asset prices.

Throughout we suppose that the cash-in-advance constraint always bind and the households hold only interest-bearing securities in the asset market. Under this supposition, a household's decision to pay the fixed cost to trade at date t affects only its current consumption and bond-holdings and does not directly affect the real balances it holds in the goods market at later dates.

Inactive households simply consume the real balances they currently hold in the goods market. More interesting is the consumption of active households. Since there is a complete set of state contingent bonds, once a household pays the fixed cost it equates its intertemporal marginal rate of substitution to that of other active households. Hence, since all households are identical ex-ante, all active households have a common consumption level $c_A(\mu^t)$ that only depends on the aggregate money shock μ^t and not on their idiosyncratic endowments.

We first construct the zones of activity and inactivity for an arbitrary consumption level c_A and then use the resource constraint to determine the equilibrium level. Define the function

$$(2.1) \quad h(m; c_A) = U(c_A) - U(m) - U'(c_A)[c_A - m] - \gamma.$$

Intuitively, this function measures the net gain to a household from switching from being an inactive household with consumption m to an active household with consumption c_A . The first two terms measure the direct utility gain within the current period from the switch while the third term measures the cost in utility terms of the required transfer of real balances from the asset market. Fixing c_A , it is optimal for a household with real balances m to trade and consume c_A if h is positive and not to trade and consume m if h is negative. Note that h is strictly convex in the argument m , it attains its minimum at $m = c_A$, and is negative at this minimum if $\gamma > 0$. Thus, h typically crosses zero twice.

Define the cutoffs $y_L(c_A, \mu), y_H(c_A, \mu)$ as the solutions to

$$(2.2) \quad h\left(\frac{y}{\mu}; c_A\right) = 0,$$

when both of these solutions exist. If (2.1) is negative for all $m < c_A$, set $y_L(c_A, \mu) = 0$ while if it is negative for all $m > c_A$, set $y_H(c_A, \mu) = \infty$. This cutoff rule is illustrated in Figure 2 (ignore, for now, the curve labelled quadratic approximation to h). Note that as the fixed cost γ goes to zero, $y_L(c_A, \mu)/\mu$ and $y_H(c_A, \mu)/\mu$ converge to c_A , so that all households become active.

The next Proposition develops an convenient analytical approximation for threshold functions $y_i(\mu, c_A)$ solving $h(y/\mu, c_A) = 0$ given c_A .

PROPOSITION 1. . Let $y_i(\mu, c_A)$ for $i = H, L$ be a solution of $h(y/\mu, c_A) = 0$, then we can write

$$y_i(\mu, c_A) / \mu = c_A \pm \Delta + o(\gamma), \text{ where } \Delta \equiv \sqrt{\frac{2\gamma}{-U''(c_A)}}$$

and where the approximation error $o(\gamma)$ is of order smaller than $\gamma^{1/2}$, i.e. $\lim_{\gamma \rightarrow 0} o(\gamma) / \gamma^{1/2} = 0$. If $U''' \geq 0$ everywhere, then

$$y_L(\mu, c_A) / \mu \geq c_A + \Delta \text{ and } y_H(\mu, c_A) / \mu \leq c_A + \Delta,$$

and if $U''' \leq 0$ everywhere, the inequalities are reversed. Thus if U is quadratic the approximation is exact, i.e. $o = 0$.

This proposition has an important corollary that shows that the range of inactivity widens a lot for small changes in the fixed cost when γ is small.

COROLLARY 2. Consider $y_L(\mu, c_A)$ and $y_H(\mu, c_A)$ as functions of γ . Recall that they are equal for $\gamma = 0$. Then we have

$$\frac{dy_H(\mu, c_A) / \mu}{d\gamma} \Big|_{\gamma=0} = +\infty \text{ and } \frac{dy_L(\mu, c_A) / \mu}{d\gamma} \Big|_{\gamma=0} = -\infty.$$

This corollary is similar to the results obtained by Abel and Eberly (1996 and 1998) in investment models with irreversibilities and fixed cost. Notice that we can also write the approximation as fraction of c_A , i.e. as

$$(2.3) \quad y(\mu, c_A) / \mu = c_A \left(1 \pm \sqrt{\frac{2\tilde{\gamma}}{\sigma}} \right)$$

where $\tilde{\gamma} \equiv \frac{\gamma/U'(c_A)}{c_A}$ and $\sigma \equiv \frac{-c_A U''(c_A)}{U'(c_A)}$

The terms on this approximation are easily interpreted. The term $\gamma/U'(c_A)$ is the fixed cost, converted into consumption units (recall γ is in utility units). Hence $\tilde{\gamma}$ is the fixed cost, relative to consumption of active agents. The term σ is the coefficient of relative risk aversion of active agents. The result are as expected: the width of the inaction region increases with the fixed cost and decreases with the relative risk aversion. Figure 2 illustrates how good this approximation is plotting h for a utility function U with a constant relative risk aversion coefficient σ and its quadratic approximation. The figure uses $\tilde{\gamma} = 0.005$, i.e. half of one percent, and $\sigma = 6$. Notice that the roots of h and its approximation are almost indistinguishable.

Given this form for the zones of activity and inactivity, we use the resource constraint to determine the equilibrium values of active households' consumption and corresponding cutoffs as follows. The cash-in-advance constraints together with constraints (1.5) and

(1.6) imply that the price level is $P(\mu^t) = M(\mu^t)/Y$, the inflation rate is $\pi_t = \mu_t$, real money holdings are $m(\mu^t, y^{t-1}) = y_{t-1}/\mu_t$ and the consumption of inactive households is $c(\mu^t, y^{t-1}) = y_{t-1}/\mu_t$. Substituting the inactive household's consumption into the resource constraint (1.5) and using the cutoff rules defined in (2.2) gives

$$(2.4) \quad c_A[F(y_L) + 1 - F(y_H)] + \frac{1}{\mu_t} \int_{y_L}^{y_H} yf(y)dy = Y$$

where we have suppressed explicit dependence of c_A , y_H , and y_L on μ^t . Clearly, these cutoff points and consumption levels of active households depend only on μ_t , while the consumption level of inactive households depends only on (μ_t, y_{t-1}) .

Fixing $\mu_t \geq 1$ and using (2.2) to solve for y_L and y_H as functions of c_A , it is clear that the left hand side of (2.4) is continuous and strictly monotonic in c_A . Thus any solution to these equations for the equilibrium values of active households' consumption and cutoffs is unique. These arguments give the following. (For details, see Appendix A.)

PROPOSITION 3. *The equilibrium consumption for households is given by*

$$c(\mu^t, y^{t-1}) = \begin{cases} y_{t-1}/\mu_t & \text{if } y_{t-1} \in (y_L(\mu_t), y_H(\mu_t)) \\ c_A(\mu_t) & \text{otherwise} \end{cases}$$

where the functions $y_L(\mu)$, $y_H(\mu)$, $c_A(\mu)$ are the solutions to (2.2) and (2.4).

In our analysis of asset prices, is useful to use the sequence of budget constraints (1.3) to substitute out for household's bond holdings and replace these constraints with a single date 0 constraint on households transfers of cash between the goods and asset markets. As we show in Appendix A, date zero dollar asset prices are determined by the first order condition for active households

$$(2.5) \quad \beta^t U'(c_A(\mu_t))g(\mu^t) = \lambda Q(\mu^t)P(\mu^t),$$

where λ is the multiplier on households' date zero budget constraint and $Q(\mu^t)$ is the price in dollars in the asset market at date 0 for a dollar delivered in the asset market at date t in state μ^t . Since all households are identical in period 0, the multipliers in the Lagrangian are the same for all of them.

2.1. Active agent's consumption and money injections

In this section we study how money injections affect the consumption of active households. We begin with a example with a discrete distribution of y and then move to the analysis of continuous distributions.

Consider first a simple example in which y takes on three values $y_0 < y_1 < y_2$, with probabilities f_0, f_1, f_2 , respectively. We conjecture an equilibrium in which, when money growth is $\bar{\mu}$, households with the central value of the endowment y_1 choose not to trade

and those with low and high endowments y_0 and y_2 choose to trade. Under this conjecture, for money growth shocks μ close to $\bar{\mu}$, from the resource constraint, active households each consume an equal share of the active households' aggregate endowment plus the inflation tax levied on inactive households minus the fixed cost, or

$$(2.6) \quad c_A(\mu) = \frac{y_0 f_0 + y_2 f_2}{f_0 + f_2} + \left(1 - \frac{1}{\mu}\right) \frac{y_1 f_1}{f_0 + f_2}.$$

The corresponding cutoffs $y_L(c_A(\mu), \mu)$, $y_H(c_A(\mu), \mu)$ are found from (2.2). This conjecture is valid as long as $y_0 < y_L(c_A(\bar{\mu}), \bar{\mu}) < y_1 < y_H(c_A(\bar{\mu}), \bar{\mu}) < y_2$.

Clearly, an increase in the money growth rate μ raises the inflation tax levied on each inactive household's real balances. In equilibrium, these inflation tax revenues must go to active households. In this example, the number of active households does not vary with the money injection, so the consumption of each active household increases. Specifically,

$$(2.7) \quad \frac{dc_A}{d\mu} = \frac{1}{\mu^2} \frac{y_1 f_1}{f_0 + f_2},$$

which is the ratio of the total consumption of inactive households to that of active households.

Consider next the case in which y has a continuous density. Differentiating (2.2)–(2.4) gives

$$(2.8) \quad \begin{aligned} & \frac{dc_A}{d\mu} \{ [F(y_L) + 1 - F(y_H)] \\ & \quad \mu f(y_L) \left(c_A - \frac{y_L}{\mu} \right) \eta_L - \mu f(y_H) \left(c_A - \frac{y_H}{\mu} \right) \eta_H \} \\ & = \frac{1}{\mu} \int_{y_L}^{y_H} \frac{y}{\mu} f(y) dy + \left(\frac{y_L}{\mu} - c_A \right) f(y_L) \frac{y_L}{\mu} + \left(c_A - \frac{y_H}{\mu} \right) f(y_H) \frac{y_H}{\mu} \end{aligned}$$

where $\eta_i = U''(c_A)(c_A - y_i/\mu)/[U'(c_A) - U'(y_i/\mu)]$. From (2.2) it follows that $y_L/\mu < c_A < y_H/\mu$. Thus, η_H and η_L are positive and so is the term in braces on the left side of (2.8). On the right side of (2.8), the first term is positive and the last two are negative, so without further restrictions, the sign of the right side of (2.8) is ambiguous. The first term measures the effect of the inflation tax on the consumption of inactive households, holding fixed the zone of inactivity. The last two terms measure the change in the consumption of inactive households that results from the change in the zone of inactivity. The fraction $f(y_L)$ of households at the lower edge of the zone with real balances y_L/μ become active and the fraction $f(y_H)$ of households at the upper edge of the zone with real balances y_H/μ become inactive households. As long the number of households at these edges is not too large the consumption of active households increase.

To understand the effects of the growth rate of money μ on active agents consumption c_A , we use the analytical approximation for the thresholds $y_i(\mu, c_A)$ and solve the remaining

part of the model, i.e. we replace the approximation for $y_i(\mu, c_A)$ of Proposition 1 to obtain the following equation for active agent's consumption $c_A(\mu)$:

$$(2.9) \quad \bar{y} = c_A(\mu) [F((c_A(\mu) - \Delta)\mu) + 1 - F((c_A(\mu) + \Delta)\mu)] \\ + \int_{(c_A(\mu) - \Delta)\mu}^{(c_A(\mu) + \Delta)\mu} \frac{y}{\mu} dF(y)$$

We first consider the case where y has symmetric distribution, with a p.d.f. that is increasing at the left of its mean value, and decreasing at the left. Specifically we consider a distribution of y with c.d.f. F with support $[y_1, y_2]$ that satisfies:

$$F(\bar{y} - x) = 1 - F(\bar{y} + x)$$

for all $\bar{y} - x$ in the support. Notice that symmetry implies that $F(\bar{y}) = 1/2$ and whenever F is differentiable, its p.d.f. f , satisfies $f(\bar{y} + x) = f(\bar{y} - x)$. It should not be surprising that with the symmetry in the distribution F and in the thresholds implied by a quadratic utility, active agents consume the average income \bar{y} if there is no inflation.

PROPOSITION 4. . *If y is symmetric around \bar{y} then $c_A(\mu) = \bar{y}$ at $\mu = 1$.*

The next proposition analyzes the effect of monetary expansions on $c_A(\mu)$. In the symmetric case, at least for small inflation rates, c_A is increasing in μ . As explained above there are two effects μ on c_A . One effect is that as μ increases, the real income of the non-active agents is reduced, and hence c_A must increase. The other effect is that an increase in μ increases the number of non-active agents. This effect tends to decrease the consumption of active agents. In the symmetric case, with a p.d.f. f with a maximum in \bar{y} the first effect dominates. This effect dominates because, with the assumed shape of f , there are relatively few agents close edges of the inaction region, relative to the agents at the center of this region. That is, in terms of the discussion in the paragraph just below the expression for $dc_A/d\mu$ (2.8), $f(y_L)$ and $f(y_H)$ are sufficiently "small". In the next proposition we assume that F is symmetric and convex for y below \bar{y} . This means that its p.d.f. f will be increasing below \bar{y} and decreasing above \bar{y} .

PROPOSITION 5. . *Let U quadratic, y be symmetric.. Furthermore assume that F is convex for $y \leq \bar{y}$, Assume that $y_H(\mu) \leq y_2$. Then $c_A(\mu)$ is weakly increasing in μ at $\mu = 1$. If F is strictly convex in $(y_L(1), \bar{y})$ then $dc_A(\mu)/d\mu > 0$ at $\mu = 1$.*

The next example uses a uniform distribution, and hence it violates the assumption from the previous proposition that F is convex below \bar{y} . In this case $c_A(\mu)$ is constant, at least for a range of values around $\mu = 1$ so that the thresholds $y_H(\mu)$ and $y_L(\mu)$ are in the support. In this example the density f is constant, so, in terms of the discussion above (2.8), there are many agents shifting between active and non-active.

Uniform Example. Take y to be uniform in $[y_1, y_2]$, and denote its density $f = 1/(y_2 - y_1)$. Then, $c_A(\mu) = \bar{y}$ as long as $y_H(\mu) \leq y_2$.

We now introduce a generalization of the uniform example. In this generalization there is a mass point at $\bar{y}(1 - \varepsilon)$. This example allows to set parameters so that y has a c.d.f. F which is symmetric and strictly convex for $y \leq \bar{y}$. In this case Proposition 5 that ensures that $dc_A/d\mu > 0$. This example allows also to set parameters so that F is not symmetric, a case that will be of interest to analyze optimal monetary policy and to compare with other distributions that we solve numerically, such as log-normal, where mean and median do not coincide.

Main Example. Assume that y is distributed as

$$(2.10)y = \begin{cases} \bar{y}(1 - \varepsilon) & \text{with probability } \pi \\ \text{uniform } [y_1, y_2] & \text{with probability } 1 - \pi \end{cases}$$

where $(y_1 + y_2)/2 = \bar{y}$.

In this example, at least for μ close to 1 and ε small, a fraction π of agents are always non-active since they will have a real income close to \bar{y} .

PROPOSITION 6. . Assume utility is quadratic and that the endowment is as in (2.10). As long as $y_L(\mu) < \bar{y}(1 - \varepsilon) < y_H(\mu)$, consumption of active agents is increasing in μ and given by:

$$c_A(\mu) = \frac{\bar{y}}{1 - \pi} \left(1 - \frac{\pi(1 - \varepsilon)}{\mu} \right).$$

To get some intuition for this proposition set $\varepsilon = 1$, so the distribution is symmetric and F is convex for y below \bar{y} . In this case, as suggested by our previous propositions, $c_A(\mu) = \bar{y}$ and $dc_A/d\mu > 0$ at $\mu = 1$. For all μ in the range specified in the proposition, c_A is increasing in μ . In particular, $dc_A/d\mu = \bar{y}\pi(1 - \varepsilon)/(1 - \pi)\mu^2 > 0$, which is, itself, increasing in π . The intuition for the last result is clear: as π increases, a larger fraction of income is controlled by non-active agents, hence the inflationary taxes levied on them and redistributed to the active agents are higher.

3. Optimal Monetary Policy

In this section we analyze the optimal monetary policy. SUMMARIZE RESULTS.

To analyze the optimal monetary policy we continue with the assumption that the cash in advance constraint binds. In this case equilibrium allocations, given μ , are optimal. To see this, define $L(\mu)$ as the expected utility of agents starting the period with a cross section distribution on nominal balances described by the c.d.f. F and inflation rate μ . To verify that the equilibrium is constraint optimal, assume that, for a given μ , the planner can choose $c_A(\mu)$, $y_L(\mu)$ and $y_H(\mu)$, so that, implicitly, it has the same trading technology

as the agents. We show that the resulting constraint optimal allocation is the same as the equilibrium allocation. Let

$$(3.1) \quad L(\mu) \equiv \max_{c_A, y_L, y_H} \int_{y_L}^{y_H} U(y/\mu) f(y) dy + [F(y_L) + 1 - F(y_H)] (U(c_A) - \gamma)$$

subject to

$$\bar{y} = [F(y_L) + 1 - F(y_H)] c_A + \int_{y_L}^{y_H} \frac{y}{\mu} f(y) dy.$$

We have:

PROPOSITION 7. . *The planner's choice of c_A , y_H and y_L solving (3.1) satisfy*

$$h(y_i(\mu) / \mu, c_A(\mu)) = 0$$

for $i = L, H$, and hence they coincide with the market allocations.

This result should not be surprising, given that the planner has access to the same technology as the agents. The proof proceeds by showing that the first order conditions for y_i for $i = H, L$ and c_A imply that $h(y_i/\mu, c_A) = 0$. Given that market equilibrium is characterized by a static problem and that, for each μ , the allocations are constraint efficient, we use the envelope theorem to derive the effect μ on the expected utility $L(\mu)$. The only effect of monetary policy in this model is to change the pre-trade distribution of real endowments. A straightforward application of the envelope theorem gives:

$$(3.2) \quad \partial L(\mu) / \partial \mu = \frac{1}{\mu^2} \left(\int_{y_L(\mu)}^{y_H(\mu)} [U'(c_A(\mu)) - U'(y/\mu)] y dF(y) \right)$$

This equation has the following interpretation. As μ increases, the pre-trade distribution of real endowments decreases stochastically. This reduces the consumption of each non-active agent in an amount proportional to his nominal income. On the other hand, this income goes to active agents, who have a common consumption $c_A(\mu)$. Thus, for each level of the nominal income y , we compare $U'(c_A(\mu))$ with $U'(y/\mu)$; this difference if multiplied by y because the transfer is proportional to the nominal income. Finally we integrate these differences using the distribution of pre-trade income F for the non-active agents.

We let μ^* denote the optimal inflation rate, i.e. the value of μ that makes $dL(\mu) / d\mu = 0$. We want to characterize μ^* as a function of the givens of the model, i.e. the utility function U , the fixed cost γ and the distribution of the endowments F . It should be clear from our previous analysis of $c_A(\mu)$ that the distribution of endowments y , conditional on agents being non-active, is crucial to understand μ^* . We make this point using a simple example with a stable distribution F and a quadratic utility U . For this example we assume that μ is a range so that the inaction region is contained in $[y_1, y_2]$, the support of y , i.e.e., $y_1 < y_L(\mu) < y_H(\mu) < y_2$. In both cases, straightforward algebra gives that (3.2) becomes

$$\partial L(\mu) / \partial \mu = \frac{-U''(c_A)}{\mu^2} \int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} [y/\mu - c_A(\mu)] y f(y) dy$$

Example: effect of F in optimal μ^ .* Assume that $F(y) = \delta (y^{1-\alpha} - 1) / (1 - \alpha) - \kappa$ with support in $[y_1, y_2]$ for some constants $\delta > 0$, κ and $\alpha > 0$, so that $f(y) = \delta y^{-\alpha}$. Then for μ such that $y_1 < y_L(\mu) < y_H(\mu) < y_2$:

$$\frac{dL(\mu)}{d\mu} = \begin{cases} > 0 & \text{if } \alpha \in (0, 1] \\ = 0 & \text{if } \alpha = 1 \\ < 0 & \text{if } \alpha \in (1, 2] \end{cases}$$

(see the appendix for the details).

We now interpret and explain this example. If $dL/d\mu > 0$ for all μ in this range, then it must be that the optimal inflation μ^* is even higher than the inflation rate that makes the inaction region to stretch up to the upper bond of the domain of y , i.e., $y_H(\mu) = y_2$. Conversely, if $dL/d\mu < 0$ in this range then it must be that the optimal inflation μ^* is even lower than the inflation rate that makes $y_L(\mu) = y_1$. The intuition for these results is relatively simple. Recall that inflation reduces every non-active agents consumption by an amount proportional to his nominal income, and increases the common level of all active agents consumption. The impact in the expected utility depends on the net effect. Setting α to different values, changes the relative fractions (densities) of non-active agents with different nominal incomes. As α increases, the fraction of non-active agents with lower nominal incomes -and hence high marginal utility- increases. Since non-active agents with high nominal income have low marginal utility, as α increases inflation increases welfare by less; as $\alpha < 1$, inflation decreases welfare.

The distributions used in the previous example, except for the case of $\alpha = 1$ which gives the uniform distribution, are not symmetric. If we consider symmetric distributions, with a c.d.f. F convex for $y \leq \bar{y}$, the optimal inflation rate is positive, i.e. $\mu^* > 1$. We have,

PROPOSITION 8. *Assume that U is quadratic and y has support $[y_1, y_2]$ and c.d.f. F that is symmetric, differentiable, convex for all $y \leq \bar{y}$, where \bar{y} is its mean. Then the optimal inflation μ^* is positive, i.e. $\mu^* > 1$. Furthermore assume that the range of inaction for $\mu = 1$ is in the interior of the support, so that $y_2 > \bar{y} + \Delta$. Then the optimal inflation is bounded, i.e. $\mu^* < \bar{\mu} \equiv y_2 / (\bar{y} - \Delta)$.*

First we give the intuition for the lower bound of $\mu^* > 1$. Around the no inflation case, i.e. $\mu = 1$, a small increase in μ makes the distribution of pre-trade real endowments, y/μ more concentrated. Then, relative to the $\mu = 1$ case, this more concentrated distribution allows to save in transaction costs. The intuition of why the μ^* is bounded is very simple. Suppose that $\mu = \bar{\mu}$, a value large enough so that all agents are traders, hence the pre-trade real distribution of endowments is so low that $y_2/\bar{\mu} = \bar{y} - \Delta$, i.e. the agent with the highest endowment is indifferent between trading or not. Hence, with this high μ it is optimal for all agents to be active and consume $c_A = \bar{y}$. This allocation is dominated for the one with $\mu = 1$, that has the same consumption of active agents, but where non-active agents are

better off. To see this, consider the case of agents that, with $\mu = 1$, have endowments close to \bar{y} . Then it must be the case that $\mu^* < \bar{\mu}$.

The next proposition analyzes $(\mu/U'(c_A))(dL/d\mu)$ which measures the effect on ex-ante utility of a one percent increase in inflation, in units of consumption goods. Using the approximation (2.3), we obtain

$$\frac{\mu}{U'(c_A)} \frac{d^2L}{d\mu d\sigma} \Big|_{\mu=1} = \sigma \int_{\bar{y}-\bar{y}\sqrt{2\bar{\gamma}/\sigma}}^{\bar{y}+\bar{y}\sqrt{2\bar{\gamma}/\sigma}} [y - \bar{y}] \frac{y}{\bar{y}} dF(y)$$

This expression is obtained from the quadratic and symmetric case, by replacing Δ and using that $c_A(1) = \bar{y}$. For positive γ we have shown that the optimal inflation is positive, i.e. $\mu^* > 1$, so we know that $(\mu/U'(c_A))(dL/d\mu) \geq 0$ at $\mu = 1$. We are interested in analyzing the effect of risk aversion of the welfare effects of inflation. Inspection of this expression shows that risk aversion, σ , has two different effects. On the one hand, the more risk averse agents are, the more important is in welfare terms the decrease in dispersion in the pre-trader real income achieved by inflation. On the other hand, the more risk averse agents are, there is more trade in equilibrium, i.e. higher risk aversion implies that the inaction region is smaller, and hence there is less benefits of inflation. We show that, for small γ the second effects dominates, and hence $(\mu/U'(c_A))(d^2L/d\mu d\sigma) < 0$, i.e. for higher risk aversion, inflation increases welfare by less. That the second effect dominates the impact on $dL/d\mu$ should not be surprising in view that we have shown that a small fixed cost can have large effects on the width of the inaction region.

PROPOSITION 9. . Consider the case where F is symmetric, convex for $y \leq \bar{y}$ and differentiable at its mean value \bar{y} . Then

$$\frac{\mu}{U'(c_A)} \frac{d^2L}{d\mu d\sigma} \Big|_{\mu=1} = -f(\bar{y}) \sqrt{4/\sigma^2} + o(\gamma)$$

where $g(\gamma) = o(\gamma)$ means that $\lim_{\gamma \rightarrow 0} g(\gamma)/\gamma = 0$.

In the next proposition we return to the example described in (2.10) where utility is quadratic, and the distribution F is described by the parameters π, ε , and \bar{f} .

PROPOSITION 10. . Let denote $\mu^*(\Delta, \bar{y}, \bar{f}, \pi, \varepsilon)$ the optimal monetary policy, i.e the value of μ that solves

$$(3.3) \quad 0 = \frac{dL}{d\mu} = \frac{\pi}{\mu^2} (U'(c_A(\mu)) - U'(\bar{y}/\mu)) \bar{y} + \frac{1 - \pi}{\mu^2} \int_{(c_A(\mu) - \Delta)\mu}^{(c_A(\mu) + \Delta)\mu} [U'(c_A(\mu)) - U'(y/\mu)] y \bar{f} dy$$

Then for π large enough there is an interior solution. The optimal inflation $\mu^*(\Delta, \bar{y}, \bar{f}, \pi, \varepsilon)$ is increasing in Δ , increasing in \bar{f} , decreasing in π and decreasing in \bar{y} . If $\varepsilon = 0$, $\mu^* > 1$. Moreover, if $\varepsilon > 0$ and π and $\Delta \bar{f}$ are large enough, $\mu^* < 1$.

To understand this proposition let's first consider the case where $\varepsilon = 0$, so that the distribution is symmetric. This case differs from the uniform, in that there is a mass point at the mean value \bar{y} . Expression (3.3) displays the two forces that lead to the optimal inflation μ^* . As shown in Proposition 8 the optimal inflation rate is positive in this case since $dL/d\mu > 0$ for $\mu = 1$. Indeed for $\mu = 1$, as argue above $c_A(\mu) = \bar{y}$ and hence (3.3) has only the second term,

$$0 = \frac{dL}{d\mu} = (1 - \pi) \int_{\bar{y}-\Delta}^{\bar{y}+\Delta} [U'(\bar{y}) - U'(y)] y \bar{f} dy > 0$$

This expression is positive since it gives higher weights, i.e. higher $y\bar{f}$, to the higher differences of the marginal utilities $U'(\bar{y}) - U'(y)$. The larger weights reflect, as explained above, that redistributions induced by inflation are proportional to nominal income. Since these weights are larger for large y , this expression is positive. This expression is increasing in Δ since, for larger inaction region, the difference for the weights in the boundaries of the inaction region are larger. Now we use expression (3.3) to explain the two forces that determine μ^* . As shown in Proposition 6, c_A increases with μ . This leads to a decrease in $dL/d\mu$, since the redistribution in favor of the active agents has a smaller marginal benefit, i.e. $U'(c_A(\mu))$ is smaller. The second force is that increases in μ , reduce the consumption of the inactive agents with nominal income \bar{y} , of which there is a mass point π , to \bar{y}/μ . This increases their marginal utility, and hence decreases the benefits of distributing resources away from them. We now argue that μ^* is increasing in Δ . Recall that at $\mu = 1$ we argue that $dL/d\mu > 0$, and furthermore that it is increasing in Δ . We also explain why $dL/d\mu$ is decreasing in μ . The forces that make $dL/d\mu$ increasing in Δ also apply for $\mu > 1$. Hence for larger Δ , the optimal inflation μ^* is larger. Recall that, using the approximation (2.3) that Δ is increasing in the normalized fixed cost $\tilde{\gamma}$ and decreasing in risk aversion σ . Hence the optimal inflation rate μ^* is decreasing in σ . Finally, for $\varepsilon > 0$, then $c_A(\mu)$ is smaller than \bar{y} even for $\mu = 1$. This reduces the optimal inflation rate μ^* , and even allows the possibility of the inflation rate to be negative, i.e. $\mu^* < 1$.

We end this section with a numerical example, where we verify that the optimal inflation rate μ^* is increasing in the relative risk aversion coefficient σ and decreasing in the fixed cost $\tilde{\gamma}$. The example has lognormal y , with σ_y denoting the standard deviation of $\log(y)$, and with the mean of log normalized so that $Ey = \bar{y} = 1$. Appendix C gives the details of the computations. The fixed cost $\tilde{\gamma}$ is expressed as in (2.3), in goods, by dividing it by the marginal utility of the average endowment, and relative to the average endowment $\bar{y} = 1$. Preferences are given by a utility function U with constant relative risk aversion σ . We use this example to illustrate that our approximations work very well, even though the distribution of y is not symmetric - the median of y is smaller than its mean - and that U is not quadratic - the utility function has $U''' > 0$ everywhere. We set $\tilde{\gamma} = 0.005$, $\sigma_y = 0.075$ and produce two sets of figures, one for $\sigma = 1$ and one for $\sigma = 6$. The plot containing

$c_A(\mu)$, $y_L(\mu)/\mu$ and $y_H(\mu)/\mu$ for each σ shows that these three functions are parallel to each other, and that the difference between $c_A(\mu)$ and $y_L(\mu)/\mu$ and the difference between $y_H(\mu)/\mu$ and $c_A(\mu)$ are approximately the same for each μ , as the approximation in Proposition 1 states. The magnitude of $c_A(\mu) - y_L(\mu)/\mu$ is essentially equal to the approximation $\Delta = \bar{y} (2 \tilde{\gamma} / \sigma)^{1/2}$ of Proposition 1. Consider the case of $\sigma = 1$, then the approximation gives $\Delta = (1/100)^{1/2} = 0.1$, and for $\sigma = 2$, we have $\Delta = (1/600)^{1/2} \cong 0.04$, which are just the magnitudes of the differences in the figures. The plots labelled utility display expected utility $L(\mu)$ for different values of μ . From these two plots we can see that the optimal inflation rate μ^* is negative and decreasing in σ . For instance for $\sigma = 1$, is approximately -0.75% a year and for $\sigma = 6$ it is approximately -1.25% a year. The plot labelled fraction of traders, displays $F(y_L(\mu)) + 1 - F(y_H(\mu))$ for different values of μ . As expected, comparing these plots with the ones plotting expected utility, the value that maximizes utility roughly coincides with the one that minimized the fraction of traders, and hence the transaction costs. The fact that $\mu^* < 1$ is because the distribution of y is not symmetric, as analyzed in our previous examples.

4. Accounting for the variation in aggregate stocks prices and returns

We briefly review the literature that decomposes the variations on aggregate stock returns. This serves as motivation for our choice of shocks.

Campbell (1991) shows that variation in unexpected excess returns of a stock can be decomposed into variation in three components: unexpected future dividends, unexpected future excess returns, and unexpected future interest rates. He estimates that the variations in unexpected excess returns of broad stock indexes is accounted for variation in future dividends and future expected returns, in approximately the same magnitude. He finds that variation in unexpected future interest rates is not important in accounting for unexpected stock excess returns.

Campbell and Shiller (1988) and Cochrane (1991) shows that variation in dividend price ratio of a stock can be decomposed into two components: the forecast of dividend price ratios to future returns and the forecast of dividend price ratios to future returns. He estimates that almost all the variation in price dividend ratios of aggregate stock portfolios is accounted for the forecast of dividend price ratios on future returns. Furthermore, Cochrane (2001) reviews the estimates in the literature, and conclude that dividend price ratios forecast future excess returns, not just returns.

Both Campbell's and Cochrane's decompositions are obtained using the log-linear approximation for the identity defining returns introduced by Campbell and Shiller (1988). This approximation writes the log return as the growth rate of dividends plus the current dividend yield minus the discounted future dividend yield. Denoting stock prices by S_t , and dividends by D_t , and returns by R_{t+1} we have that $R_{t+1} = (S_{t+1} + D_{t+1})/S_t$. A log-linear

approximation to this equation gives

$$r_{t+1} = (d_{t+1} - d_t) + (d_t - s_t) + k - \rho(d_{t+1} - s_{t+1})$$

where lowercase letters denotes logs of the corresponding uppercase letters, ρ is a constant given by $(S/D) / (1 + (S/D))$ and k is constant of no interest.

We now obtain Campbell decomposition for innovations in excess returns. Defining the excess returns as $e_{t+1} \equiv r_{t+1} - i_{t+1}$, where i_{t+1} is the real interest rate, multiplying $e_{t+j} = r_{t+j} - i_{t+1}$ by ρ^j , using the Campbell Shiller approximation, adding up the terms of different $t + j$, and taking limits

$$e_{t+1} = \frac{k}{1 - \rho} + \sum_{j=0}^{\infty} \rho^j (d_{t+1+j} - d_{t+j}) - \sum_{j=1}^{\infty} \rho^j e_{t+1+j} - \sum_{j=0}^{\infty} \rho^j i_{t+1+j}.$$

Taking time t conditional expectation E_t in both sides, and subtracting from e_{t+1} , we obtain Campbell's decomposition. He estimates a monthly VAR for excess returns, interest rates, and other variables useful to predict returns such as detrended interest rates and dividend price ratios. Using the estimated VAR, Campbell computes the innovations and decomposes the variance of unexpected returns on aggregate portfolios $var(e_{t+1} - E_t e_{t+1})$ by computing the variance and covariance of each of the three infinite sums of the left hand side. Campbell (1991) Table 4, finds that the variance of the innovations on future discounted sum of dividends growth rates and the variance of the innovations on the future discounted sum of excess returns are about 1/3 of the variance of innovations on excess returns, and negatively correlated. He also finds that the variance of the innovations in the discounted sum of future interest rates is very small. Campbell and Ammer (1993) finds that for the post war period most of the variance of unexpected stock returns in accounted by changes in the sum of discounted future excess returns.

Cochrane's decomposition is also obtained by using a log-linear approximation introduced by Campbell and Shiller (1988). Using this approximation the log of dividend price ratios can be written as the discounted sum of future dividend growth rates minus the discounted sum of future returns. In particular Cochrane's decomposition is taking by using Campbell and Shiller approximation, solving for $s_t = p_t$ and interating it forward, so that

$$(4.1) \quad s_t - d_t = \frac{\rho k}{1 - \rho} + \sum_{j=0}^{\infty} \rho^j (d_{t+1+j} - d_{t+j}) - \sum_{j=0}^{\infty} \rho^j r_{t+1+j}.$$

Multiplying both sides by the demeaned log dividend price ratios $[s_t - d_t - E(s_t - d_t)]$, and taking expectations,

$$var(s_t - p_t) = cov\left(s_t - d_t, \sum_{j=0}^{\infty} \rho^j (d_{t+1+j} - d_{t+j})\right) - cov\left(s_t - d_t, \sum_{j=0}^{\infty} \rho^j r_{t+1+j}\right).$$

Cochrane (1991) estimates each covariance approximating the infinite sum by finite sums, say of 15 terms for annual data. Cochrane estimates the first covariance to be around

zero for real data. Cochrane (2001) Tables 20.1 and 20.2, reviews the literature on predictability of excess returns, where dividend price ratios are useful to predict future excess returns, especially for multiperiod excess returns, but not useful to predict future dividend growth rates -either one or multiperiod ones- Alternatively, by adding and subtracting $\sum_{j=0}^{\infty} \rho^j i_{t+1+j}$ on the right hand side of $s_t - d_t$, using the definition of excess returns e_t , we can write

$$\begin{aligned} \text{var}(s_t - d_t) &= \text{cov}\left(s_t - d_t, \sum_{j=0}^{\infty} \rho^j (d_{t+1+j} - d_{t+j})\right) - \text{cov}\left(s_t - d_t, \sum_{j=0}^{\infty} \rho^j e_{t+1+j}\right) \\ &\quad - \text{cov}\left(s_t - d_t, \sum_{j=0}^{\infty} \rho^j i_{t+1+j}\right). \end{aligned}$$

Dividing both sides of this equation of $\text{var}(s_t - d_t)$ we obtain that 1 equals the sum of OLS coefficient of price dividends on discounted sum of future growth rate of dividends, dividend price ratios and interest rates. Regressions of long term excess returns, i.e. $\sum_{j=0}^J e_{t+1+j}$ on dividend price ratios $d_t - s_t$ produces coefficient with values close to 1 for $J = 5$ or 7 (see Tables 20.2 and 21.2 in Cochrane 2001, and Table 5 in Campbell and Cochrane 1998).

5. Asset prices and Optimal Monetary Policy

Now we introduce two aggregate shocks in the model: shocks to aggregate output \bar{y} and shocks to risk aversion σ . Using our previous results for the optimal inflation rate, we show that monetary policy is procyclical, in the sense that money injection, and inflation, are higher when equity prices are higher.

We let time t preferences be described by a utility function $U(c_t, \sigma_t)$ with constant relative risk aversion σ_t which we assume to be random. We also let the aggregate income \bar{y}_t be random, and in particular we assume that the growth rate of aggregate consumption $w_{t+1} \equiv \bar{y}_{t+1}/\bar{y}_t$ is iid. We introduce shocks to \bar{y}_t to be able to talk about securities that carry a risk premium. We introduce shocks to risk aversion σ_t so that the market price of risk is random, and then price-dividend ratios of risky securities changes. We assume that shocks to σ_t are persistent and that the growth rate of aggregate consumption is iid because it will imply that expected future dividends growth is constant and that interest rates do not move much. These choices are based to reproduce stylized facts of the data reviewed in the previous section.

The equilibrium for the model with the aggregate shocks is of the same form as for the case analyzed above: for each period we compute c_A , y_L and y_L for each distribution with average income \bar{y}_t risk aversion σ_t and money growth μ_t . We let F denote the distribution of y/\bar{y}_t i.e. of the idiosyncratic income, relative to the aggregate income. To avoid that the fixed cost becomes negligible as outputs grows, we assume that the fixed cost γ_t is varies with the aggregate output \bar{y}_t , in particular we assume that the fixed cost γ_t , which is measured in utility terms, is given by $\gamma_t = \tilde{\gamma} \times \bar{y}_t^{1-\sigma_t}$ for some constant $\tilde{\gamma} > 0$. In this

case the fixed cost is constant in terms of goods, if we evaluate the utility at the aggregate endowment.

Recall that the monetary model has a binding cash in advance, and so velocity is constant and equal to one. It follows that the gross inflation rate between period t and period $t + 1$, denoted by π_{t+1} is given by $\pi_{t+1} \equiv P_{t+1}/P_t = \mu_{t+1}/w_{t+1}$. Given that we assume that the utility function is a power of consumption, and that we normalize the fixed cost accordingly, it is immediate to verify that active agents consumption and the thresholds are homogenous of degree one in \bar{y}_t , i.e.

$$\begin{aligned} c_A(\pi_t, \sigma_t, \bar{y}_t) &= c_A(\pi_t, 1, \sigma_t) \bar{y}_t \\ y_L(\pi_t, \sigma_t, y_t) &= y_L(\pi_t, \sigma_t, 1) \bar{y}_t \\ y_L(\pi_t, \sigma_t, y_t) &= y_L(\pi_t, \sigma_t, 1) \bar{y}_t \end{aligned}$$

for all σ_t , π_t and \bar{y}_t . Thus, normalizing $\bar{y}_t = 1$, the equilibrium allocation are solutions to $h = 0$, i.e.

$$(5.1) \quad U(c_A(\pi, \sigma), \sigma) - U(y_i(\pi, \sigma)/\pi, \sigma) - U'(c_A(\pi, \sigma), \sigma)(c_A - y_i(\pi, \sigma)/\pi) = \tilde{\gamma}$$

for $i = H, L$ and the resource constraint

$$(5.2) \quad c_A(\pi, \sigma) [F(y_L(\pi, \sigma)) + 1 - F(y_H(\pi, \sigma))] + \int_{y_L(\pi, \sigma)}^{y_H(\pi, \sigma)} \frac{y}{\pi} dF(y) = 1$$

where inflation is given by $\pi = \mu/w$. Notice that the growth rate of output w does not enter in (5.1) and (5.2). It only influences inflation.

Now we turn to the prices of risky assets. Because of the assumptions that market are complete and that agents are ex-ante identical, asset prices are computed using the marginal utility of active agents, i.e.

$$U'(c_A, \sigma) = \left[\bar{y} c_A \left(\frac{\mu}{w}, \sigma \right) \right]^{-\sigma}.$$

For instance, if we denote by S_t the time t price of a claim to the aggregate output stream $\{\bar{y}_{t+s}\}$, then

$$U' \left(c_A \left(\frac{\mu_t}{w_t}, \sigma_t \right), \sigma_t \right) \bar{y}_t S_t = \sum_{s=1}^{\infty} \beta^s E_t \left\{ U' \left(c_A \left(\frac{\mu_{t+s}}{w_{t+s}}, \sigma_{t+s} \right), \sigma_{t+s} \right) \bar{y}_{t+s} \right\}$$

Then empirical evidence reviewed in the previous section finds that shocks to expected returns account for most of the variation of equity prices and are very persistent. To accomplish that we assume that the shocks to risk aversion σ_t are stationary but very persistent. To understand how these shocks deliver that behavior of asset prices, assume that inflation and σ are constant, so that $c_A(\pi, \sigma)$ is constant. We argue that if σ increases once and for all, price dividend ratios decreases. To see this, consider the case where σ is constant, $c_A(\pi, \sigma)$ is constant and $\log(w)$ is normally distributed with mean m and

standard deviation σ_w . In this case it is well known that the expected (log) return of equity over the (log) risk free rate is equal to $\sigma \sigma_w^2$. It is clear that if all the future expected excess return of equity increases, i.e. if there is a once and for all increase in risk aversion σ , then the current price dividend ratio must decrease. This case is interesting because the changes in the excess expected return of equity for the case where risk aversion σ changes once and for all approximates the changes in the excess expected returns of equity when the shocks to σ are very persistent. Hence, price dividend ratios decrease if risk aversion σ increases in a persistent way.

We now discuss how optimal monetary policy co-moves with asset prices. Denote the optimal monetary injection by $\hat{\mu}(w, \sigma)$. From our previous discussion it is clear that the optimal time t inflation rate for an economy with aggregate output shocks and risk aversion σ_t is the same as the one for an economy with no aggregate output shocks and risk aversion σ_t . Thus, $\hat{\mu}(w, \sigma) = \mu^*(\sigma)/w$ where $\mu^*(\sigma)$ denotes the optimal inflation rate, and money growth rate, for an economy with constant aggregate output, as analyzed in the previous sections. Hence, the optimal inflation rate in $\pi(\sigma) = \hat{\mu}(w, \sigma)/w = \mu^*(\sigma)$, is decreasing in the risk aversion σ , since in the previous sections we show that $\mu^*(\sigma)$ is decreasing in risk aversion σ . In the previous section we show that $\mu^*(\sigma)$ was decreasing in σ , so if we combine this feature with the fact discussed in the paragraph above that price dividend ratios decrease with σ , we conclude that the optimal monetary policy is procyclical.

6. Conclusion

We have developed a model in the spirit of Baumol (1952) and Tobin (1956) that captures the idea that when a government injects money through an open market operation only a fraction of the households in the economy are on the other side of the transaction and hence money injections have distributional effects in addition to their standard Fisherian effects. We have deliberately kept the model simple to allow an analytical solution. In this model optimal monetary policy is a substitute for costly private insurance, and hence it depend on the needs for insurance as well as the amount of insurance that private markets provide. We show that, as agents become more risk averse, private markets provides much more insurance, and hence monetary policy is less important in this regard. As a consequence, times where agents are very risk averse are times where the price of risky securities is low and where it is optimal to inject less money. In this sense, the optimal monetary policy is procyclical.

7. References

- Abel, Andrew B. and Janice Eberly (1996), "Optimal Investment with Costly Reversibility" *The Review of Economic Studies*, Vol. 63, No. 4. pp. 581-593.
- Abel, Andrew B. and Janice Eberly (1998), "The Mix and Scale of Factors with Irreversibility and Fixed Costs of Investment", *Carnegie-Rochester Conference Series on Public Policy* 48, pp. 101-135, with Andrew B. Abel.
- Allen, Franklin and Douglas Gale. (1994) "Limited Market Participation and Volatility of Asset Prices," *American Economic Review*, 84, 933-955.
- Aiyagari, S. R. and M. Gertler. (1991) "Asset Returns with Transaction Costs and Uninsured Individual Risk," *Journal of Monetary Economics*, 27: 311-31.
- Alvarez, Fernando and Andrew Atkeson. (1997) "Money and Exchange Rates in the Grossman-Weiss-Rotemberg model," *Journal of Monetary Economics* 40, (no. 3 December): pp. 619-40.
- Alvarez, Fernando, Andrew Atkeson and Patrick Kehoe, "Money, Interest Rates, Exchange Rates with Endogenously Segmented Markets", forthcoming in the *Journal of Political Economy*.
- Balasko, Y., D. Cass, and K. Shell. (1995) "Market Participation and Sunspot Equilibria," *Review of Economic Studies*, 62: 491-512.
- Barr, David, G. and Campbell, John Y. (1997) "Inflation, Real Interest Rates, and the Bond Market: A Study of UK Nominal and Index-Linked Government Bond Prices," *Journal of Monetary Economics*, 39(3), 361-83.
- Basak, Suleyman and Cuoco, Domenico. (1998) "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies*, 11(2): 309-41.
- Baumol, William J. (1952) "The Transactions Demand for Cash: An Inventory Theoretic Approach," *Quarterly Journal of Economics*, 66 (November): 545-56.
- Campbell, John Y. and Cochrane, John (1998), "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", *Journal of Political Economy*, 2, 107.
- Campbell, John Y. , Robert J. Shiller, (1986), "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors" ,*Review of Financial Studies*, Vol. 1, No. 3. (Autumn, 1988), pp. 195-228.

- Campbell, John Y. . “A Variance Decomposition of Stock Returns”, *Economic Journal*, 101, 157-179.
- Campbell, John Y., and Ammer, John. (1993), “What Moves the Stock and Bond Markets? A Variance Decomposition for Long-Term Asset Returns,” *Journal of Finance*, 48(1), March 1993, pages 3-37.
- Campbell, John Y., Andrew W. Lo, and A. Craig MacKinlay. (1997) *The Econometrics of Financial Markets*. Princeton, N. J., Princeton University Press.
- Chatterjee, Satyajit, and Dean Corbae. (1992) “Endogenous Market Participation and the General Equilibrium Value of Money”, *Journal of Political Economy*, 100 (no. 3, June), 615–46.
- Cochrane, John. (1991) “Explaining the Variance of Price Dividend Ratios”, *Review of Financial Studies*, 5, 243-280.
- Cuny, C. J. (1993) “The Role of Liquidity in Futures Markets Innovation,” *Review of Financial Studies*, 6: 57-78.
- Dotsey, Michael and Peter Ireland. (1995) “Liquidity Effects and Transactions Technologies,” *Journal of Money Credit and Banking*, 27(4): 1440-57.
- Grossman, Sanford, and Laurence Weiss. (1983) “A Transactions-Based Model of the Monetary Transmission Mechanism,” *American Economic*.
- Jovanovic, Boyan. (1982) “Inflation and Welfare in the Steady State”, *Journal of Political Economy* 90 (no. 3, June): 561-577.
- Hirshleifer, D. (1988) “Residual Risk, Trading Costs, and Commodity Futures Risk Premia,” *Review of Financial Studies*, 1: 173-193.
- Lucas, Robert E., Jr. 1982. Interest rates and currency prices in a two country world. *Journal of Monetary Economics*, vol. 10, 335-360.
- Lucas, Robert E. Jr. (1984) “Money in a Theory of Finance,” *Carnegie Rochester Conference Series on Public Policy*, 21(0), 9-45.
- Lucas, Robert E. Jr. (1990) “Liquidity and Interest Rates”, *Journal of Economic Theory* 50 (no. 2, April): 237-264.
- Merton, R. C. (1987) “A Simple Model of Capital Market Equilibrium with Incomplete Information,” *Journal of Finance*, 42: 483-510.

- Romer, David. (1986) "A Simple General Equilibrium Version of the Baumol-Tobin Model," *Quarterly Journal of Economics* 101 (no. 4, November): 663–85.
- Rotemberg, Julio J. (1984) "A Monetary Equilibrium Model with Transactions Costs", *Journal of Political Economy* 92 (no. 1, February): 40-58.
- Rotemberg, Julio, J. 1985. Money and the terms of trade. *Journal of International Economics*, vol 19, 141-160.
- Saito, M. (1996) "Limited Participation and Asset Pricing," working paper, University of British Columbia.
- Tobin, James. (1956) "The Interest-Elasticity of the Transactions Demand for Cash," *Review of Economics and Statistics* 38 (no. 3, August): 241–47.

Appendix A

In this appendix we provide sufficient conditions to ensure that households never hold over cash in either the goods or asset market. This is almost exactly the same as in Alvarez, Atkeson and Kehoe (2001). To allow for the possibility that the household may hold cash, we modify the household's constraints as follows. In the goods markets, we denote unspent real balances that the shopper might carry over from goods shopping by $a(\mu^t, y^{t-1})$ and rewrite the constraints (1.2) and (??)

$$(7.1) \quad a(\mu^t, y^{t-1}) = m(\mu^t, y^{t-1}) + x(\mu^t, y^{t-1})z(\mu^t, y^{t-1}) - c(\mu^t, y^{t-1}),$$

$$(7.2) \quad m(\mu^{t+1}, y^t) = \frac{P(\mu^t)}{P(\mu^{t+1})}[y_t + a(\mu^t, y^{t-1})],$$

and add the cash-in-advance constraint

$$(7.3) \quad a(\mu^t, y^{t-1}) \geq 0.$$

In the asset market, we replace the constraints (1.3) with the sequence of budget constraints for $t \geq 1$

$$(7.4) \quad B(\mu^t, y^{t-1}) = \int_{\mu_{t+1}} \int_{y_t} q(\mu^t, \mu_{t+1}) B(\mu^t, \mu_{t+1}, y^{t-1}, y_t) f(y_t) d\mu_{t+1} dy_t + \\ N(\mu^t, y^{t-1}) - N(\mu^{t-1}, y^{t-2}) + P(\mu^t)x(\mu^t, y^{t-1})z(\mu^t, y^{t-1}),$$

where $N(\mu^{t-1}, y^{t-2})$ is cash held over from the previous asset market, $N(\mu^t, y^{t-1})$ is cash help over into the next asset market, and with $N(\mu^t, y^{t-1}) \geq 0$ and $N(\mu^{t-1}, y^{t-2}) = N_0$ in period $t = 1$. In period $t = 0$, this asset market constraint is $\bar{B} = \int_{\mu_1} \int_{y_0} q(\mu_1) B(\mu_1, y_0) f(y_0) dy_0 d\mu_1 + N_0$. Otherwise, the household's problem is unchanged.

We develop our sufficient conditions in several steps. We first characterize the household's optimal choice of c and x given prices and arbitrary rules for m, a , and z , and summarize these results in Lemma 1. We then characterize the household's trading rule z given an arbitrary rule for m, a and the optimal rules for c and x and summarize these results in Lemma 2. These lemmas complete the proof of proposition 1 in the text. In Lemma 3, we provide sufficient conditions on the money growth process and endowments process to ensure that a and N are always zero.

First use the sequence of budget constraints (7.4) to substitute out for agent's bond holdings and replace these constraints with a single date 0 constraint on agents transfers of cash between the goods and asset markets. Any bounded allocation and bondholdings that satisfies (7.4) also satisfies a date 0 budget constraint

$$(7.5) \quad \sum_{t=0}^{\infty} \int Q(\mu^t) \int_{y^{t-1}} \left\{ P(\mu^t)x(\mu^t, y^{t-1})z(\mu^t, y^{t-1}) + \right.$$

$$N(\mu^t, y^{t-1}) - N(\mu^t, y^{t-2}) \} f(y^{t-1}) dy^{t-1} d\mu^t \leq \bar{B}.$$

Thus, the household's problem can be restated as follows. Choose real money holdings m and a , trading rule z , consumption and transfers c and x and cash in the asset market N , subject to constraints (7.1)-(7.3) and (7.5).

Consider first a household's optimal choice of consumption $c(\mu^t, y^{t-1})$ and transfers of dollar real balances $x(\mu^t, y^{t-1})$ given prices $Q(\mu^t), P(\mu^t)$, arbitrary feasible choices of real money holdings $m(\mu^t, y^{t-1})$ and $a(\mu^t, y^{t-1})$ and a trading rule $z(\mu^t, y^{t-1})$. These choices maximize the Lagrangian corresponding to the household's problem. Let $\nu(\mu^t, y^{t-1})$ be the multiplier on (7.1), and λ be the multiplier on (7.5). The first order condition corresponding to c is $\beta^t U'(c(\mu^t, y^{t-1}))g(\mu^t)f(y^{t-1}) = \nu(\mu^t, y^{t-1})$. The first order condition corresponding to x is $\lambda Q(\mu^t)P(\mu^t)z(\mu^t, y^{t-1})f(y^{t-1}) = \nu(\mu^t, y^{t-1})z(\mu^t, y^{t-1})$. For those states such that $z(\mu^t, y^{t-1}) = 1$, these two first order conditions imply

$$(7.6) \quad \beta^t U'(c(\mu^t, y^{t-1}))g(\mu^t) = \lambda Q(\mu^t)P(\mu^t).$$

Since all households are identical at date 0, the multipliers in the Lagrangian are the same for all households. We summarize this discussion as follows

Lemma 1. All households who choose to pay the fixed cost for a given aggregate state μ^t have identical consumption $c(\mu^t, y^{t-1}) = c_A(\mu^t)$, for some function c_A . Households who choose not to pay the fixed cost have consumption $c(\mu^t, y^{t-1}) = m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1})$.

Next consider a household's optimal choice of whether to pay the fixed cost to trade given prices $Q(\mu^t), P(\mu^t)$ and arbitrary feasible choices of real money holdings in the goods market $m(\mu^t, y^{t-1}), a(\mu^t, y^{t-1})$. From Lemma 1, we have the form of the optimal consumption and transfer rules corresponding to the choices of $z = 1$ and $z = 0$. Substituting these rules into (1.4) and (7.5) gives the problem of choosing $c_A(\mu^t)$ and $z(\mu^t, y^{t-1})$ to maximize

$$(7.7) \quad \sum_{t=1}^{\infty} \beta^t \int \int [U(c_A(\mu^t)) - \gamma] z(\mu^t, y^{t-1})g(\mu^t)f(y^{t-1})d\mu^t dy^{t-1} + \\ \sum_{t=1}^{\infty} \beta^t \int \int U(m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1}))(1 - z(\mu^t, y^{t-1}))g(\mu^t)f(y^{t-1})d\mu^t dy^{t-1}$$

subject to the constraint

$$(7.8) \quad \bar{B} \geq \sum_{t=1}^{\infty} \int \int Q(\mu^t) [N(\mu^t, y^{t-1}) - N(\mu^{t-1}, y^{t-2})] f(y^{t-1})d\mu^t dy^{t-1} + \\ \sum_{t=1}^{\infty} \int \int Q(\mu^t)P(\mu^t) [c_A(\mu^t) - (m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1}))] z(\mu^t, y^{t-1})f(y^{t-1})d\mu^t dy^{t-1}.$$

Let η denote the Lagrange multiplier on (7.8) and consider the following variational argument. Consider a state (μ^t, y^{t-1}) . The increment to the Lagrangian of setting $z(\mu^t, y^{t-1}) = 1$ in this state is

$$(7.9) \quad \beta^t [U(c_A(\mu^t)) - \gamma] g(\mu^t)f(y^{t-1}) -$$

$$\eta Q(\mu^t)P(\mu^t) \left[c_A(\mu^t) - (m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1})) \right] f(y^{t-1})$$

which is simply the direct utility gain $U(c_A(\mu^t))$ minus the cost of the required transfers. The increment to the Lagrangian of setting $z(\mu^t, y^{t-1}) = 0$ in this state is

$$(7.10) \beta^t U \left((m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1})) \right) g(\mu^t) f(y^{t-1})$$

which is simply the direct utility gain since there are no transfers. The first order condition with respect to c_A is

$$\beta^t U'(c_A(\mu^t)) g(\mu^t) = \eta Q(\mu^t) P(\mu^t).$$

Subtracting (7.10) from (7.9) and using (7.6) gives the cutoff rules defined by (2.2). More formally, we have

Lemma 2: Given active households' consumption $c_A(\mu^t)$, a household chooses $z(\mu^t, y^{t-1}) = 0$ if $m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1}) \in \left(\frac{y_L(c_A(\mu^t), \mu_t)}{\mu_t}, \frac{y_H(c_A(\mu^t), \mu_t)}{\mu_t} \right)$ and they choose $z(\mu^t, y^{t-1}) = 1$ otherwise.

These lemmas complete the proof of proposition 1. To complete our asset pricing formulas we need to compute the equilibrium value of the multiplier λ . Given the equilibrium values of consumption computed in proposition 1 we have that λ solves

$$(7.11) \sum_{t=1}^{\infty} \int \beta^t U'(c_A(\mu_t)) \int_{y_L(\mu_t)}^{y_H(\mu_t)} \frac{M(\mu^t)}{Y} \left[c_T(\mu_t) - \frac{y}{\mu_t} \right] f(y) dy g(\mu^t) d\mu^t = \frac{\bar{B}}{\lambda}.$$

Households will not want to store cash in the asset market if nominal interest rates are positive. Thus, to ensure that $N = 0$, it is sufficient to check that nominal interest rates are always positive. We now turn to the problem of developing conditions sufficient to ensure that households never want to store cash in the goods market. Assume that households have CRRA utility of the form $U(c) = c^{1-\sigma}/(1-\sigma)$. Let $Q(\mu^t)$ and $P(\mu^t)$ be the prices constructed above when a and N are assumed equal to zero. Consider first the consumption of a household who deviates from the strategy of never holding cash from one period to the next in the goods market. From Lemmas 1 and 2, we have that, holding fixed a plan $\{a_t(\mu^t, y^{t-1})\}$ for holding cash in the goods market, this deviant household's consumption choices are similar to those of a household who does not hold cash in the goods market. In particular, in those states of nature in which the deviant chooses to pay to the fixed cost to trade, from Lemma 1, his consumption satisfies the first order condition $\beta^t U'(c_A^d(\mu^t)) g(\mu^t) = \eta^d Q(\mu^t) P(\mu^t)$ where η^d is the Lagrange multiplier on this household's date zero budget constraint. Thus, in those states in which the deviant household pays the fixed cost to trade, it equates its marginal rate of substitution to that of other active households who do not deviate. Given the assumption of CRRA utility, this implies that $c_A^d(\mu^t) = \theta c_A(\mu_t)$ for all μ^t for some fixed factor of proportionality θ . In those states of nature in which the deviant household does not choose to pay the fixed cost, its consumption is

$c^d(\mu^t, y^{t-1}) = m^d(\mu^t, y^{t-1}) - a^d(\mu^t, y^{t-1})$, and its decision whether to pay the fixed cost is determined by the cutoffs $y_L(\theta c_T(\mu_t), \mu_t)$ and $y_H(\theta c_T(\mu_t), \mu_t)$ described in Lemma 2. Using the fact that $m^d(\mu^t, y^{t-1}) = (y_{t-1} + a^d(\mu^{t-1}, y^{t-2}))/\mu_t$ and, in the event that the deviant household pays the fixed cost, $x^d(\mu^t, y^{t-1}) = \theta c_T(\mu_t) - (m_t^d(\mu^t, y^{t-1}) - a^d(\mu^t, y^{t-1}))$, the factor of proportionality θ (and the implied Lagrange multiplier η^d) corresponding to any fixed plan $\{a_t(\mu^t, y^{t-1})\}$ for holding cash in the goods market must be set so that the deviant household's date zero budget constraint holds with equality. The relevant budget constraint is written

$$\begin{aligned} \bar{B} &= \sum_{t=1}^{\infty} \int \int Q(\mu^t) P(\mu^t) \left[\theta c_T(\mu_t) - (m(\mu^t, y^{t-1}) - a(\mu^t, y^{t-1})) \right] \\ &\quad \times z(\mu^t, y^{t-1}) f(y^{t-1}) d\mu^t dy^{t-1} \end{aligned}$$

where $z(\mu^t, y_{t-1}) = 1$ if $(y_{t-1} + a^d(\mu^{t-1}, y^{t-2}))/\mu_t - a^d(\mu^t, y^{t-1}) \in [y_L(\theta c_T(\mu_t), \mu_t)/\mu_t, y_H(\theta c_T(\mu_t), \mu_t)/\mu_t]$ and $z(\mu^t, y_{t-1}) = 0$ otherwise.

Next observe that, since the cutoffs $y_L(\theta c_T(\mu_t), \mu_t)$ and $y_H(\theta c_T(\mu_t), \mu_t)$ are monotonically increasing in θ for all values of μ_t , no deviant household would choose a plan $\{a_t(\mu^t, y^{t-1})\}$ for holding cash in the goods market such that the implied factor of proportionality θ was so small such that $y_H(\theta c_T(\mu_t), \mu_t) \leq y_L(c_T(\mu_t), \mu_t)$ for all possible realizations of μ_t . To see this, observe that the consumption of such a deviant household would lie below the consumption we have constructed for a household that never holds cash in the goods market in every possible state of nature μ^t, y^{t-1} , and thus the utility of such a deviant household would have to be lower than that of a household that never held cash in the goods market.

Lemma 3. Let $\bar{\theta}$ to be the supremum over the set of θ which satisfy (??). Then, it is optimal for a household to never hold over cash in the goods market if, for all $a \geq 0, \mu_t$ and $\theta \geq \bar{\theta}$

$$\begin{aligned} &U' \left(\frac{y_H(\theta c_A(\mu_t), \mu_t)}{\mu_t} \right) \\ &> \beta \int_{\mu_{t+1}} \int_{y_L(\theta c_A(\mu_{t+1}), \mu_{t+1}) - a}^{y_H(\theta c_A(\mu_{t+1}), \mu_{t+1}) - a} U' \left(\frac{y_t + a}{\mu_{t+1}} \right) \frac{f(y_t)}{\mu_{t+1}} g(\mu_{t+1} | \mu^t) dy_t d\mu_{t+1} + \\ &\beta \int_{\mu_{t+1}} \frac{U'(\theta c_A(\mu_{t+1}))}{\mu_{t+1}} \\ &\quad \times [F(y_L(\theta c_A(\mu_{t+1}), \mu_{t+1}) - a) + 1 - F(y_H(\theta c_A(\mu_{t+1}), \mu_{t+1}) - a)] g(\mu_{t+1} | \mu^t) d\mu_{t+1}. \end{aligned}$$

Proof. Given any plan $\{a_t(\mu^t, y^{t-1})\}$ for holding cash in the goods market and associated value of θ , the highest consumption that a deviant household could have at date t is $y_H(\theta c_T(\mu_t), \mu_t)/\mu_t$ and thus the smallest marginal utility of consumption it could have at that date is $U'(y_H(\theta c_T(\mu_t), \mu_t)/\mu_t)$. The terms on the right-hand side are the expected value of the product of the marginal utility of consumption and the return to holding currency

in the goods market ($1/\mu_{t+1}$) at date $t + 1$. Thus, the condition in the lemma ensures that such a household always prefers to consume its real balances at t rather than carry them over into period $t + 1$ at rate of return $1/\mu_{t+1}$. Therefore, this condition implies that there is no plan for holding cash in the goods market that gives higher utility than the plan of never holding cash in the goods market.

Appendix B

In this appendix we collect the proof of the new results.

Proof of Proposition (1). First we show that the approximation error is of order smaller than $\gamma^{1/2}$. We denote by $y_i(\gamma)$ the solution of $h(y(\gamma), c_A) = 0$. By using a third order Taylor expansion of $U(y/\mu)$ around c_A

$$\gamma = -\frac{1}{6}U'''(\tilde{c}_A(\gamma)) [y_L(\gamma) - c_A]^3 - \frac{1}{2}U''(c_A) [y_L(\gamma) - c_A]^2$$

where $\tilde{c}_A(\gamma)$ is some value in $[y_L(\gamma), c_A]$. Defining $d(\gamma) = y(\gamma)/\mu - \tilde{y}(\gamma)/\mu$, and using $\tilde{y}(\gamma)/\mu = c_A - \Delta$

$$\gamma = -\frac{1}{6}U'''(\tilde{c}_A(\gamma)) [d(\gamma) - \Delta]^3 - \frac{1}{2}U''(c_A) [d(\gamma) - \Delta]^2$$

rearranging

$$1 = \left(-\frac{1}{6}U'''(\tilde{c}_A(\gamma)) [d(\gamma) - \Delta] - \frac{1}{2}U''(c_A) \right) \left[\frac{d(\gamma) - \Delta}{\gamma^{1/2}} \right]^2$$

and taking limits on both sides

$$1 = \lim_{\gamma \rightarrow 0} \left(-\frac{1}{6}U'''(\tilde{c}_A(\gamma)) [d(\gamma) - \Delta] - \frac{1}{2}U''(c_A) \right) \left[\frac{d(\gamma) - \Delta}{\gamma^{1/2}} \right]^2$$

using that $\lim_{\gamma \rightarrow 0} \tilde{y}(\gamma)/\mu = \lim_{\gamma \rightarrow 0} y(\gamma)/\mu = c_A$, then $\lim_{\gamma \rightarrow 0} d(\gamma) = 0$, using its definition $\lim_{\gamma \rightarrow 0} \Delta(\gamma) = 0$, by the mean value theorem, $\lim_{\gamma \rightarrow 0} \tilde{c}_A(\gamma) = c_A$, and by replacing $\Delta = (-2\gamma/U''(c_A))^{1/2}$,

$$1 = -\frac{1}{2}U''(c_A) \lim_{\gamma \rightarrow 0} \left[\frac{d(\gamma)}{\gamma^{1/2}} - \left(\frac{2}{-U''(c_A)} \right)^{1/2} \right]^2 = \lim_{\gamma \rightarrow 0} \left[\frac{d(\gamma)}{\gamma^{1/2}} \left(\frac{-U''(c_A)}{2} \right)^{1/2} - 1 \right]^2$$

hence there are two values for the limit, either $\lim_{\gamma \rightarrow 0} d(\gamma)/\gamma^{1/2} = 0$, in which case the result is established, or

$$(7.12) \lim_{\gamma \rightarrow 0} \frac{d(\gamma)}{\gamma^{1/2}} = 2 \left(\frac{2}{-U''(c_A)} \right)^{1/2}.$$

We now show that the second case leads to a contradiction with $y_L(\gamma) < c_A$ for all γ . By definition of $d(\gamma)$,

$$\frac{y(\gamma) - c_A}{\gamma^{1/2}} = - \left(\frac{2}{-U''(c_A)} \right)^{1/2} + d(\gamma)$$

and taking limits using (7.12)

$$\lim_{\gamma \rightarrow 0} \frac{y(\gamma) - c_A}{\gamma^{1/2}} = - \left(\frac{2}{-U''(c_A)} \right)^{1/2} + \lim_{\gamma \rightarrow 0} d(\gamma) = \left(\frac{2}{-U''(c_A)} \right)^{1/2} > 0.$$

Thus $\lim_{\gamma \rightarrow 0} d(\gamma) / \gamma^{1/2} = 0$. A similar argument holds for y_H .

Finally we show that if $U''' \geq 0$, then $y_L/\mu \geq c_A - \Delta$ and $y_L/\mu \leq c_A + \Delta$. Consider a third order Taylor expansion of h , as a function of y/μ , around c_A :

$$h(y/\mu, c_A) = -\frac{1}{6}U'''(\tilde{c}_A(\gamma)) [y/\mu - c_A]^3 - \frac{1}{2}U''(c_A) [y/\mu - c_A]^2 - \gamma$$

Define $\tilde{h}(y/\mu, c_A) = (1/2)U'''(c_A) [y/\mu - c_A]^2 - \gamma$. By definition $y_i(\mu, c_A)$ solves $h = 0$ and $(c_A \pm \Delta)$ solve $\tilde{h} = 0$. For $y/\mu \geq c_A$, $h \leq \tilde{h}$ when $U''' \geq 0$, and since h is increasing in y/μ in that range, $y_H/\mu \geq c_A - \Delta$. For $y/\mu \leq c_A$, $h \geq \tilde{h}$ when $U''' \geq 0$, and since h is decreasing in y/μ in that range, $y_L/\mu \leq c_A + \Delta$. The reverse argument applies when $U''' \leq 0$. *QED*.

Proof of Proposition 4. To see this notice that $c_A(\mu)$ must solve

$$\begin{aligned} & [\bar{y} - c_A] [F((c_A - \Delta)\mu) + 1 - F((c_A + \Delta)\mu)] \\ &= \phi(\mu, (c_A - \Delta)\mu, (c_A + \Delta)\mu) \end{aligned}$$

where

$$\phi(\mu, d_1, d_2) \equiv \int_{d_1}^{d_2} \left(\frac{y}{\mu} - \bar{y} \right) dF(y).$$

For $c_A = \bar{y}$, $\mu = 1$, symmetry implies that

$$\phi(1, \bar{y} - \Delta, \bar{y} + \Delta) = \int_{\bar{y} - \Delta}^{\bar{y} + \Delta} (y - \bar{y}) dF(y) = 0$$

hence if $[F((c_A - \Delta)\mu) + 1 - F((c_A + \Delta)\mu)] < 1$ then $c_A = \bar{y}$. Otherwise every agent is active, and hence $c_A = \bar{y}$. *QED*.

Proof of Proposition 5. Differentiating both sides of

$$c_A [F((c_A - \Delta)\mu) + 1 - F((c_A + \Delta)\mu)] + \int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} \frac{y}{\mu} f dy = \bar{y}$$

with respect to μ ,

$$\begin{aligned} 0 &= c'_A [F((c_A - \Delta)\mu) + 1 - F((c_A + \Delta)\mu)] \\ &+ c_A [f(y_L)(c_A - \Delta) - f(y_H)(c_A + \Delta)] \\ &- f(y_L)(c_A - \Delta)^2 + f(y_H)(c_A + \Delta)^2 - \frac{1}{\mu^2} \int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} y dF(y) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 &= c'_A [F((c_A - \Delta)\mu) + 1 - F((c_A + \Delta)\mu)] \\ &- \frac{1}{\mu^2} \int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} y dF(y) + f(y_L)(c_A - \Delta)\Delta + f(y_H)(c_A + \Delta)\Delta \end{aligned}$$

Hence $dc_A(\mu)/d\mu > 0$ iff

$$-\frac{1}{\mu^2} \int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} y dF(y) + f(y_L)(c_A - \Delta)\Delta + f(y_H)(c_A + \Delta)\Delta < 0.$$

By the previous proposition at $\mu = 1$, $c_A(\mu) = \bar{y}$ and $f(y_H) = f(y_L) = \bar{f}$. Using these assumptions, when $\mu = 1$

$$\begin{aligned} & -\frac{1}{\mu^2} \int_{(c_A-\Delta)\mu}^{(c_A+\Delta)\mu} y dF(y) + f(y_L)(c_A - \Delta)\Delta + f(y_H)(c_A + \Delta)\Delta \\ = & -\int_{(\bar{y}-\Delta)\mu}^{(\bar{y}+\Delta)\mu} y dF(y) + 2\bar{f}\Delta c_A \end{aligned}$$

By hypothesis, $f(y_H) = \bar{f}$, hence

$$\int_{(c_A-\Delta)\mu}^{(c_A+\Delta)\mu} y dF(y) \geq \int_{(c_A-\Delta)\mu}^{(c_A+\Delta)\mu} y \bar{f} dy = \bar{f} \left[\frac{(c_A + \Delta)^2 - (c_A - \Delta)^2}{2} \right] = \bar{f} 2c_A \Delta$$

with strict inequality if F is strictly convex in $(y_L(1), \bar{y})$. *QED*.

Proof of Proposition 6. Active agents consumption c_A satisfies

$$\begin{aligned} \bar{y} = & \{ c_A(1 - \pi) \bar{f}((c_A - \Delta)\mu - y_1) + 1 - \pi - (1 - \pi) \bar{f}((c_A + \Delta)\mu - y_1) \\ & 1 - \pi - (1 - \pi) \bar{f}((c_A + \Delta)\mu - y_1) \} \\ & + \frac{1 - \pi}{\mu} \int_{(c_A-\Delta)\mu}^{(c_A+\Delta)\mu} y \bar{f} dy + \frac{\pi}{\mu} \bar{y}(1 - \varepsilon) \end{aligned}$$

where $\bar{f} = 1/(y_2 - y_1)$. Direct computations gives

$$\begin{aligned} & \bar{y} - \frac{\pi(1 - \varepsilon)}{\mu} \bar{y} \\ = & c_A \left[- (1 - \pi) \bar{f} \Delta \mu + 1 - \pi - (1 - \pi) \bar{f} \Delta \mu \right] \\ & + \frac{(1 - \pi) \mu^2}{\mu} \bar{f} \left[\frac{(c_A + \Delta)^2 - (c_A - \Delta)^2}{2} \right] \\ = & c_A(1 - \pi) - c_A(1 - \pi) \mu \bar{f} \Delta 2 + (1 - \pi) \mu \bar{f} 2c_A \Delta = (1 - \pi) c_A \end{aligned}$$

which gives the desired result. *QED*.

Proof of Proposition 7. Consider the Lagrangian

$$\begin{aligned} L(\mu) = & \max_{c_A, y_L, y_H} \int_{y_L}^{y_H} U(y/\mu) dF(y) + [F(y_L) + 1 - F(y_H)] [U(c_A) - \gamma] + \\ & + \lambda \left[\bar{y} - \int_{y_L}^{y_H} \frac{y}{\mu} dF(y) - [F(y_L) + 1 - F(y_H)] c_A \right] \end{aligned}$$

The first order condition with respect to c_A is

$$(7.13) [F(y_L) + 1 - F(y_H)] U'(c_A) - \lambda [F(y_L) + 1 - F(y_H)] = 0,$$

the first order condition with respect to y^H is

$$(7.14) U(y_H/\mu) f(y_H) - f(y_H) U(c_A) + \lambda f(y_H) \left[c_A - \frac{y_H}{\mu} \right] - f(y_H) \gamma = 0$$

and the and first order condition for y_L :

$$(7.15) -U(y_L/\mu) f(y_L) + f(y_L) U(c_A) + \lambda f(y_L) \left[\frac{y_L}{\mu} - c_A \right] + f(y_H) \gamma = 0$$

Substituting λ from (7.13) into (7.14) and (7.15) we obtain

$$\begin{aligned} U(c_A) - U(y_H/\mu) - U'(c_A) \left[c_A - \frac{y_H}{\mu} \right] &= \gamma \\ U(c_A) - U(y_L/\mu) - U'(c_A) \left[c_A - \frac{y_L}{\mu} \right] &= \gamma \end{aligned}$$

which are the solutions to $h(\frac{y}{\mu}; c_A) = 0$. *QED*

Proof of Example: effect of F in optimal μ^* . Simple algebra shows that

$$\frac{dL}{d\mu} = \frac{-U''(c_A) \delta}{\mu^\alpha} \int_{(c_A-\Delta)}^{(c_A+\Delta)} [y - c_A] y^{1-\alpha} dy,$$

so we only need to sign $\int_{(c_A-\Delta)}^{(c_A+\Delta)} [y - c_A] y^{1-\alpha} dy$. To so so notice that

$$\frac{1}{2\Delta} \int_{(c_A-\Delta)}^{(c_A+\Delta)} ([y - c_A] y^{1-\alpha}) dy = E[\eta(y, c_A; \alpha)]$$

where the expectation is taking with respect to y being uniform in $[c_A - \Delta, c_A + \Delta]$, so it has expected value c_A and where the function η is $\eta(y, c_A; \alpha) = [y - c_A] y^{1-\alpha}$. The function η satisfies

$$0 = \eta(c_A, c_A; \alpha) = \eta(E[y], c_A; \alpha).$$

Direct computation of the second derivative of η shows that η is linear in y if $\alpha = 1$, convex if $\alpha \in (0, 1]$ and concave if $\alpha \in (1, 2]$. *QED*

Proof of Proposition 8. The proof follows from the following two lemmas.

Lemma 1. Assume that U is quadratic and that F is symmetric, convex for all $y \leq \bar{y}$, where \bar{y} is its mean, and differentiable at \bar{y} . Then $dL/d\mu > 0$ when evaluated at $\mu = 1$.

Proof. By definition

$$\begin{aligned} \frac{dL}{d\mu} &= \frac{1}{\mu^2} \left(\int_{(c_A(\mu)-\Delta)\mu}^{(c_A(\mu)+\Delta)\mu} [U'(c_A(\mu)) - U'(y/\mu)] y f(y) dy \right) \\ &= \frac{-U''(\bar{y})}{\mu^2} \left(\int_{\bar{y}-\Delta}^{\bar{y}+\Delta} [y - \bar{y}] y f(y) dy \right) \end{aligned}$$

using $c_A(\mu) = \bar{y}$, and $U'(y/\mu) = U'(\bar{y}) + U''(\bar{y})(y/\mu - \bar{y})$. Notice that, by symmetry,

$$\int_{\bar{y}-\Delta}^{\bar{y}+\Delta} [y - \bar{y}] f(y) dy = 0.$$

Hence,

$$\int_{\bar{y}-\Delta}^{\bar{y}+\Delta} [y - \bar{y}] y f(y) dy > 0$$

since the density $yf(y)$ puts more mass to higher values of y . *QED*.

Lemma 2. Assume that U is quadratic and that F is symmetric, convex for all $y \leq \bar{y}$, where \bar{y} is its mean, the support of y is $[y_1, y_2]$ and F is differentiable at $y = y_2$. Then $dL/d\mu < 0$ when evaluated at $\mu = \bar{\mu} \equiv y_2/(\bar{y} - \Delta)$.

Proof. First notice that at $\mu = \bar{\mu}$ inflation is so high that every agent is a trader: $c_A(\bar{\mu}) = \bar{y}$ and $y_L(\mu, c_A) = y_2$. This can be easily verified since

$$\int_{(c-\Delta)\bar{\mu}}^{(c+\Delta)\bar{\mu}} yf(y) dy = \int_{y_2}^{y_2} yf(y) dy = 0 \text{ and } F(y_L) = F(y_2) = 1$$

and hence the resource constraint is satisfied at $c_A(\bar{\mu}) = \bar{y}$. It is optimal for all agents to be traders since $h(y_2/\bar{\mu}, c_A) = h(\bar{y} - c_A, c_A) = h(\bar{y} - \Delta, \bar{y}) = 0$, and thus $h(y/\mu, \bar{y}) \geq 0$ for any $y \leq y_2$. Now we consider $\mu = \bar{\mu} - \varepsilon$ for a small positive ε and evaluate $dL/d\mu$,

$$\begin{aligned} \frac{dL}{d\mu} &= -\frac{U''(c_A(\mu))}{\mu^2} \int_{(c_A(\mu)-\Delta)\mu}^{(c_A(\mu)+\Delta)\mu} [y/\mu - c_A(\mu)] y dF(y) \\ &= -\frac{U''(c_A(\mu))}{\mu^2} \int_{(c_A(\mu)-\Delta)\mu}^{y_2} [y/\mu - c_A(\mu)] y dF(y) \\ &\leq -\frac{U''(c_A(\mu))}{\mu^2} \left[\frac{y_2}{\mu} - c_A(\mu) \right] \int_{(c_A(\mu)-\Delta)\mu}^{y_2} y dF(y) < 0 \end{aligned}$$

where the first equality holds for ε small enough, the first inequality uses continuity of $c_A(\mu)$ so that for small ε , $c_A(\mu)$ is approximately \bar{y} , and the third inequality uses that $\bar{y} < y_2$. *QED*.

Proof of Proposition 9. Direct computations gives

$$\frac{\mu}{U'(c_A)} \frac{dL}{d\mu} \Big|_{\mu=1} = \sigma \int_{\bar{y}-\bar{y}\sqrt{2\gamma/\sigma}}^{\bar{y}+\bar{y}\sqrt{2\gamma/\sigma}} [y - \bar{y}] \frac{y}{\bar{y}} dF(y) > 0$$

We now show that $\lim_{\gamma \rightarrow 0} w(\gamma)/\gamma = 0$ where $w(\gamma) = \int_{\bar{y}-\bar{y}\sqrt{2\gamma/\sigma}}^{\bar{y}+\bar{y}\sqrt{2\gamma/\sigma}} [y - \bar{y}] \frac{y}{\bar{y}} dF(y)$. Making a first order approximation,

$$w'(\gamma) = f\left(\bar{y}\left(1 + \sqrt{2\gamma/\sigma}\right)\right) \sqrt{4/\sigma^2}$$

and hence $w(\gamma) = w(0) + w'(0)\gamma + o(\gamma)$ for $w'(\gamma) = f(\bar{y})\sqrt{4/\sigma^2}$ and $w(0) = 0$. Direct computations give

$$\frac{\mu}{U'(c_A)} \frac{d^2L}{d\mu d\sigma} \Big|_{\mu=1} = \int_{\bar{y}-\bar{y}\sqrt{2\gamma/\sigma}}^{\bar{y}+\bar{y}\sqrt{2\gamma/\sigma}} [y - \bar{y}] \frac{y}{\bar{y}} dF(y) - f\left(\bar{y}\left(1 + \sqrt{2\gamma/\sigma}\right)\right) 2\bar{y}^2 (2\gamma/\sigma)$$

For the other term

$$\lim_{\gamma \rightarrow 0} f\left(\bar{y}\left(1 + \sqrt{2\gamma/\sigma}\right)\right) 2\Delta\bar{y} (2\gamma/\sigma) / \gamma = 4f(\bar{y})/\sigma.$$

QED.

Proof of Proposition 10. The first term of $dL/d\mu$ can be simplified to

$$\begin{aligned}
& \frac{\pi}{\mu^2} (U'(c_A) - U'(\bar{y}/\mu)) \bar{y} \\
&= \frac{\pi}{\mu^2} (U'(c_A) - U'(c_A) - U''(c_A) (\bar{y}/\mu - c_A)) \bar{y} \\
&= \frac{-U''(c_A) \pi}{\mu^2} (\bar{y}/\mu - c_A) \bar{y} = \frac{-U''(c_A) \pi}{\mu^2} \bar{y}^2 \left(\frac{1}{\mu} - \frac{\mu - \pi(1 - \varepsilon)}{(1 - \pi)\mu} \right) \\
&= \frac{-U''(c_A) \pi}{\mu^2} \bar{y}^2 \left(\frac{(1 - \pi)\mu - \mu + \pi(1 - \varepsilon)}{(1 - \pi)\mu} \right) \\
&= \frac{-U''(c_A) \pi}{\mu^2} \bar{y}^2 \left(\frac{\pi - \pi\mu - \pi\varepsilon}{(1 - \pi)\mu} \right) = \frac{-U''(c_A) \pi^2}{\mu^2} \bar{y}^2 \left(\frac{1 - \mu - \varepsilon}{(1 - \pi)\mu} \right)
\end{aligned}$$

The second term of $dL/d\mu$ is

$$\begin{aligned}
& \frac{1 - \pi}{\mu^2} \left(\int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} [U'(c_A) - U'(y/\mu)] y \bar{f} dy \right) \\
&= \frac{1 - \pi}{\mu^2} \left(\int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} [U'(c_A) - U'(c_A) - U''(c_A) (y/\mu - c_A)] y f(y) dy \right) \\
&= \frac{-U''(c_A)}{\mu^2} (1 - \pi) \left(\int_{(c_A - \Delta)\mu}^{(c_A + \Delta)\mu} (y/\mu - c_A) y f(y) dy \right) \\
&= \frac{-U''(c_A)}{\mu^2} (1 - \pi) \bar{f} \times \\
& \quad \left(\frac{1}{\mu} \frac{((c_A + \Delta)\mu)^3 - ((c_A - \Delta)\mu)^3}{3} - c_A \frac{((c_A + \Delta)\mu)^2 - ((c_A - \Delta)\mu)^2}{2} \right) \\
&= -U''(c_A) (1 - \pi) \bar{f} \left(\frac{(c_A + \Delta)^3 - (c_A - \Delta)^3}{3} - c_A \frac{(c_A + \Delta)^2 - (c_A - \Delta)^2}{2} \right) \\
&= -U''(c_A) (1 - \pi) \bar{f} \times \\
& \quad \left\{ \frac{c_A [(c_A + \Delta)^2 - (c_A - \Delta)^2] + \Delta [(c_A + \Delta)^2 + (c_A - \Delta)^2]}{3} \right. \\
& \quad \left. - \frac{c_A [(c_A + \Delta)^2 - (c_A - \Delta)^2]}{2} \right\} \\
&= -U''(c_A) (1 - \pi) \bar{f} \left(\frac{2}{3} \Delta (c_A^2 + \Delta^2) - c_A [(c_A + \Delta)^2 - (c_A - \Delta)^2] \left(\frac{1}{2} - \frac{1}{3} \right) \right) \\
&= -U''(c_A) (1 - \pi) \bar{f} \left(\frac{2}{3} \Delta (c_A^2 + \Delta^2) - c_A [2c_A \Delta] \frac{1}{6} \right) \\
&= -U''(c_A) (1 - \pi) \bar{f} \left(\frac{2}{3} \Delta (c_A^2 + \Delta^2) - c_A^2 \Delta \frac{1}{3} \right) = -U''(c_A) \frac{(1 - \pi) \bar{f} \Delta}{3} [c_A^2 + 2\Delta^2]
\end{aligned}$$

combining both

$$\begin{aligned}
& \frac{-U''(c_A) \pi^2}{\mu^2} \bar{y}^2 \left(\frac{1 - \mu - \varepsilon}{(1 - \pi)\mu} \right) \\
\frac{dL}{d\mu} &= -U''(c_A) \left(\frac{\pi^2 \bar{y}^2}{\mu^2} \left(\frac{1 - \mu - \varepsilon}{(1 - \pi)\mu} \right) + \frac{(1 - \pi)}{3} \bar{f} \Delta [c_A^2 + 2\Delta^2] \right)
\end{aligned}$$

then $dL/d\mu = 0$ implies

$$\begin{aligned} \pi^2 \bar{y}^2 \left(\frac{\mu - (1 - \varepsilon)}{1 - \pi} \right) &= \mu^3 \left(\frac{(1 - \pi)}{3} \bar{f} \Delta \left[\left(\frac{\bar{y}}{1 - \pi} \right)^2 \left(\frac{\mu - \pi(1 - \varepsilon)}{\mu} \right)^2 + 2\Delta^2 \right] \right) \\ &= \frac{(1 - \pi)}{3} \bar{f} \Delta \left[\left(\frac{\bar{y}}{1 - \pi} \right)^2 \mu^3 \left(\frac{\mu - \pi(1 - \varepsilon)}{\mu} \right)^2 + \mu^3 2\Delta^2 \right] \\ &= \frac{(1 - \pi)}{3} \bar{f} \Delta \left[\left(\frac{\bar{y}}{1 - \pi} \right)^2 \mu (\mu - \pi(1 - \varepsilon))^2 + \mu^3 2\Delta^2 \right] \end{aligned}$$

or

$$(7.16) (\mu - (1 - \varepsilon)) = \frac{\bar{f} \Delta}{3\pi^2} \left[\mu (\mu - \pi(1 - \varepsilon))^2 + \frac{(1 - \pi)^2}{\bar{y}^2} \mu^3 2\Delta^2 \right]$$

The left hand side of (7.16) linear, increasing in μ and equals zero at $\mu = 1 - \varepsilon$. The right hand side equals zero at $\mu = 0$, and it is increasing for $\mu \geq \pi(1 - \varepsilon)$ and convex in μ . Hence this equation has, at most, two positive solutions. For this equation to have a positive solution Δ , \bar{f} , $(1 - \pi)$ have to be small enough. In particular if $\pi = 0$ it has no positive solutions, and $dL/d\mu > 0$ for $\mu > 0$. Let's denote the smallest positive solution by $\mu^*(\pi, \bar{f}, \Delta, \bar{y}, \varepsilon)$. The smaller solution, μ^* , is the one that describes optimal policy. This can be verified by analysing both sides of the equation for $dL/d\mu$, which gives that $dL/d\mu > 0$ for $\mu < \mu^*$ and $dL/d\mu > 0$ for $\mu > \mu^*$. Thus higher positive solution of (7.16) is a local minimum. From the analysis of the right hand side of (7.16) it easy to verify that μ^* is increasing in Δ , increasing in \bar{f} , and decreasing in π . It is easy to see that as $\pi \rightarrow 1$, the solution goes to

$$\frac{3}{\bar{f} \Delta} = [\mu^* (\mu^* - (1 - \varepsilon))]$$

in which case μ^* can be smaller than one if $\bar{f} \Delta$ are large enough. *QED*.

Appendix C

Consider an example in which y is log-normal with $\log y$ having a normal distribution with mean zero and variance σ_y^2 . In this case, the resource constraint is

$$(c_A + \gamma) [N(\log y_L; 0, \sigma_y) + 1 - N(\log y_H; 0, \sigma_y)] + \frac{1}{\bar{\mu}} \frac{1}{\sigma_y \sqrt{2\pi}} \int_{\log y_L}^{\log y_H} \exp(w) \exp\left(-\frac{1}{2} \left(\frac{w}{\sigma_y}\right)^2\right) dw = \exp\left(\frac{\sigma_y^2}{2}\right).$$

where $N(\log y_L; 0, \sigma_y)$ is the cdf of a normal mean zero standard deviation σ_y evaluated at $\log y_L$. We can compute the integral in the resource constraint as follows

$$\begin{aligned} & \int_{y_L}^{y_H} U\left(\frac{y}{\mu}\right) dF(y) \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \frac{1}{\sigma_y \sqrt{2\pi}} \int_{\log y_L}^{\log y_H} \frac{\exp(w(1-\sigma))}{1-\sigma} \exp\left(-\frac{1}{2} \left(\frac{w}{\sigma_y}\right)^2\right) dw \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \frac{1}{(1-\sigma)\sigma_y \sqrt{2\pi}} \int_{\log y_L}^{\log y_H} \exp\left(w(1-\sigma) - \frac{1}{2} \left(\frac{w}{\sigma_y}\right)^2\right) dw \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \frac{1}{(1-\sigma)\sigma_y \sqrt{2\pi}} \int_{\log y_L}^{\log y_H} \exp\left(\frac{1}{2} \frac{\sigma_y^4 (1-\sigma)^2}{\sigma_y^2}\right) \exp\left(-\frac{1}{2} \frac{w^2 - 2w(1-\sigma)\sigma_y^2 + [\sigma_y^2(1-\sigma)]^2}{\sigma_y^2}\right) dw \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \frac{1}{(1-\sigma)\sigma_y \sqrt{2\pi}} \exp\left(\frac{\sigma_y^2(1-\sigma)^2}{2}\right) \int_{\log y_L}^{\log y_H} \exp\left(-\frac{1}{2} \frac{(w - \sigma_y^2(1-\sigma))^2}{\sigma_y^2}\right) dw \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \left(\frac{1}{1-\sigma}\right) \exp\left(\frac{\sigma_y^2(1-\sigma)^2}{2}\right) [N(\log y_H, \sigma_y^2(1-\sigma), \sigma_y) - N(\log y_L, \sigma_y^2(1-\sigma), \sigma_y)] \\ & \int_{y_L}^{y_H} U\left(\frac{y}{\mu}\right) dF(y) \\ &= \left(\frac{1}{\mu}\right)^{1-\sigma} \left(\frac{1}{1-\sigma}\right) \exp\left(\frac{\sigma_y^2(1-\sigma)^2}{2}\right) [N(\log y_H, \sigma_y^2(1-\sigma), \sigma_y) - N(\log y_L, \sigma_y^2(1-\sigma), \sigma_y)] \end{aligned}$$

Notice that if we set $\sigma = 1$, then we get

$$\int_{y_L}^{y_H} U\left(\frac{y}{\mu}\right) dF(y) = \int_{y_L}^{y_H} \frac{y}{\mu} dF(y)$$

so we can use this expression for the resource constraint

$$\begin{aligned} & \frac{1}{\bar{\mu}} \frac{1}{\sigma_y \sqrt{2\pi}} \int_{\log y_L}^{\log y_H} \exp(w) \exp\left(-\frac{1}{2} \left(\frac{w}{\sigma_y}\right)^2\right) dw \\ &= \left(\frac{1}{\mu}\right) \exp\left(\frac{\sigma_y^2}{2}\right) [N(\log y_H, \sigma_y^2, \sigma_y) - N(\log y_L, \sigma_y^2, \sigma_y)] \end{aligned}$$

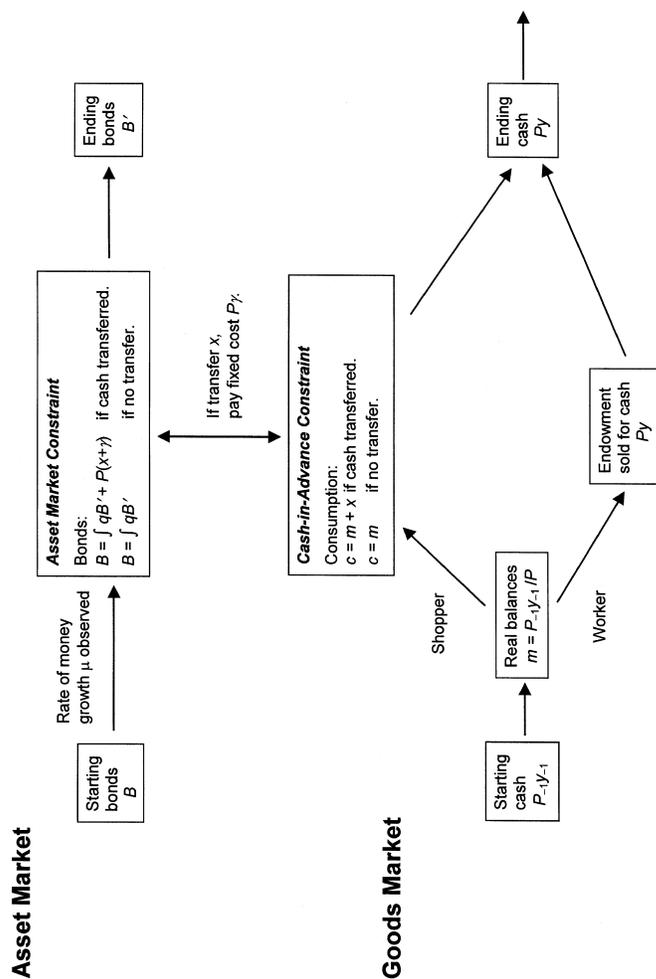
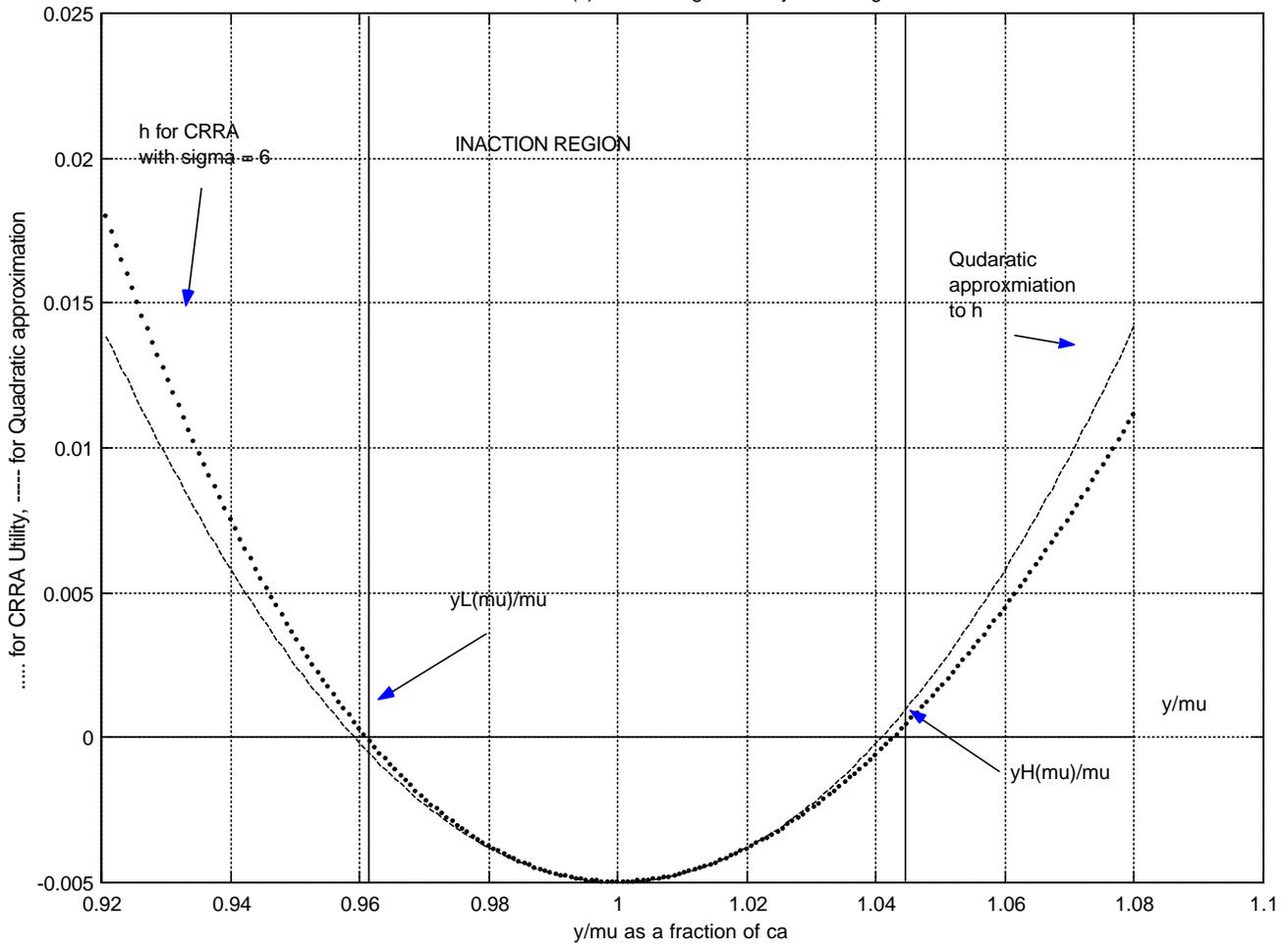
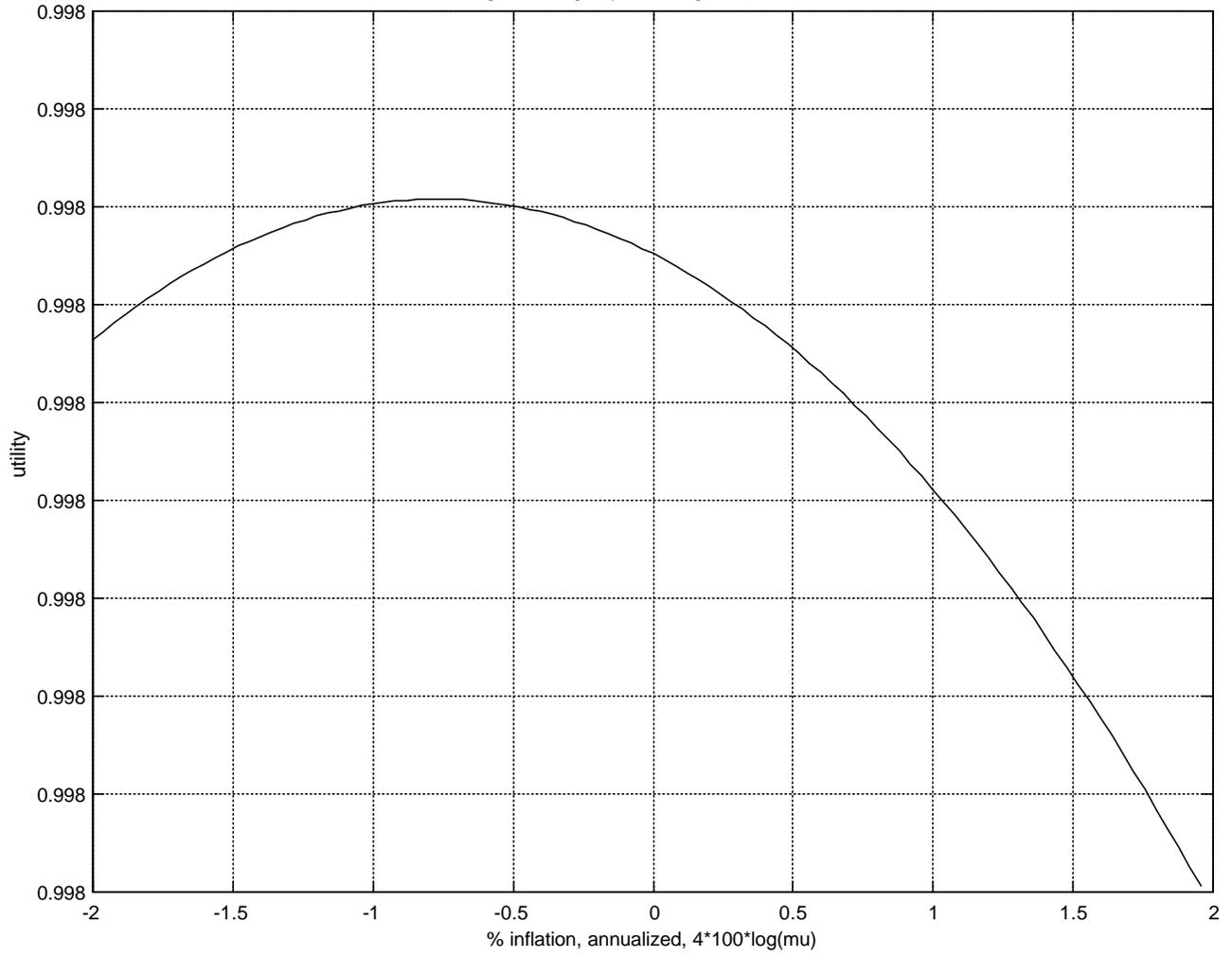


FIG. 1.—Timing in the two markets

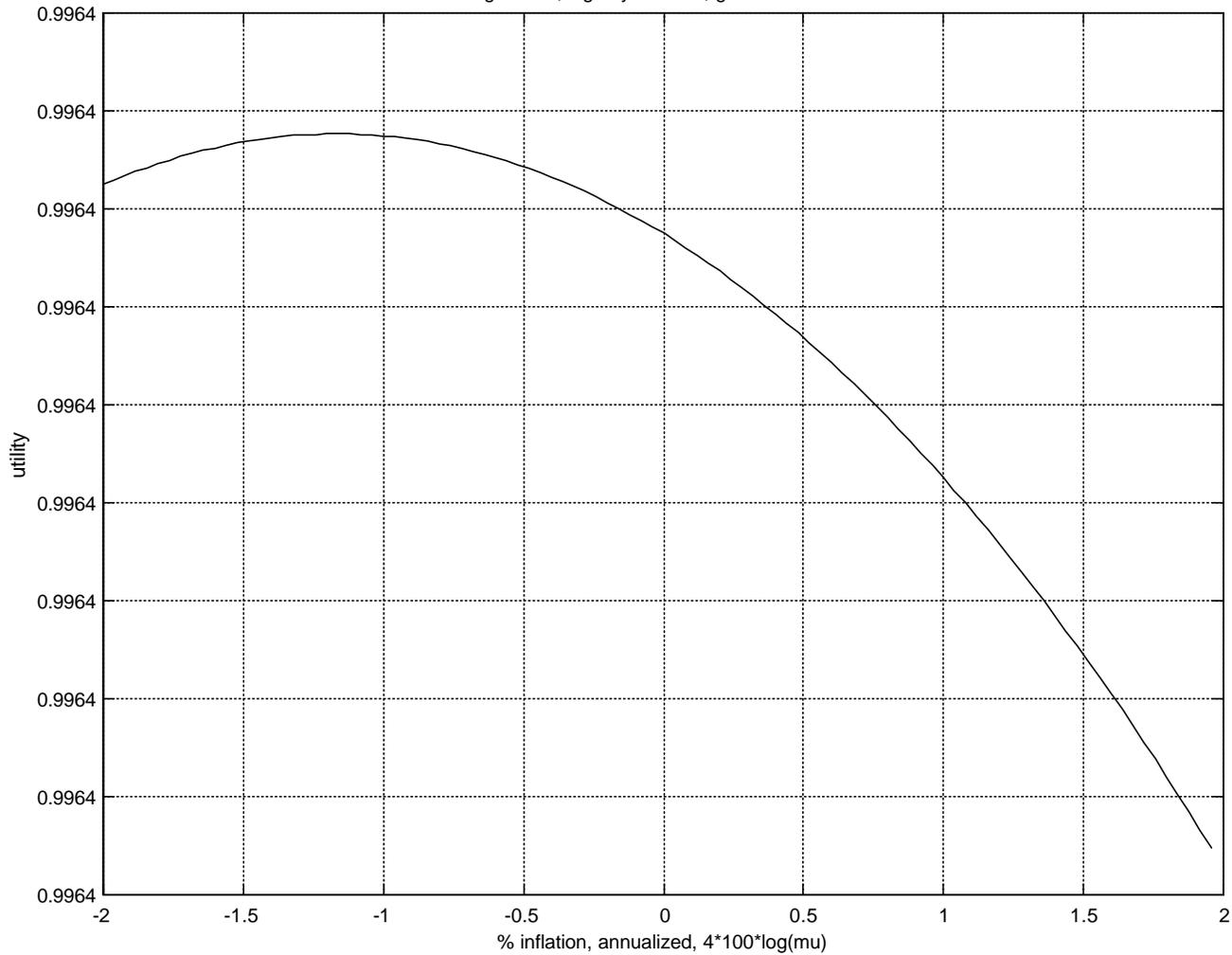
h function(s): Net Change in Utility of trading



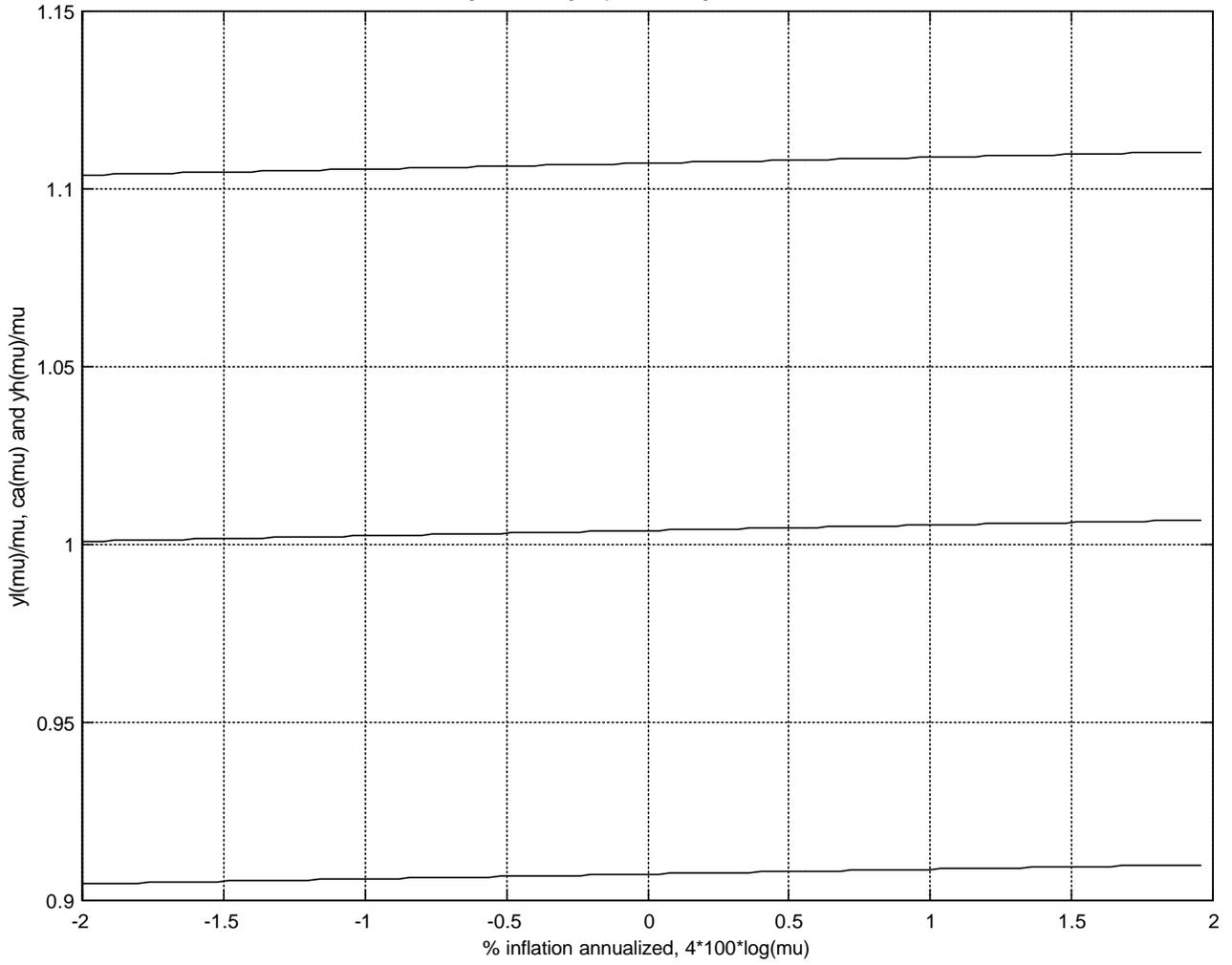
sigma=1, sigmay=0.075, gamma = 0.005



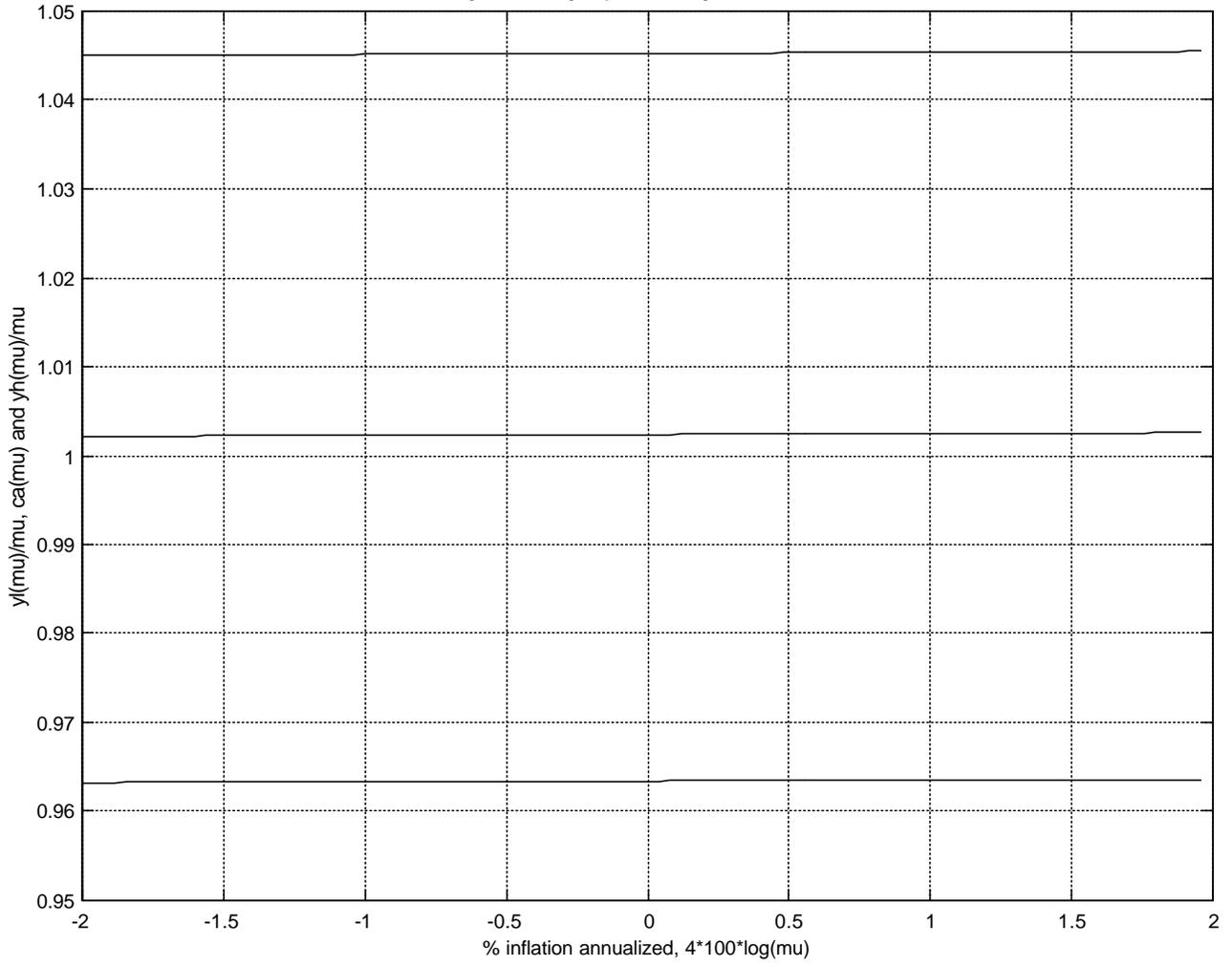
sigma = 6, sigmay = 0.075, gamma = 0.005



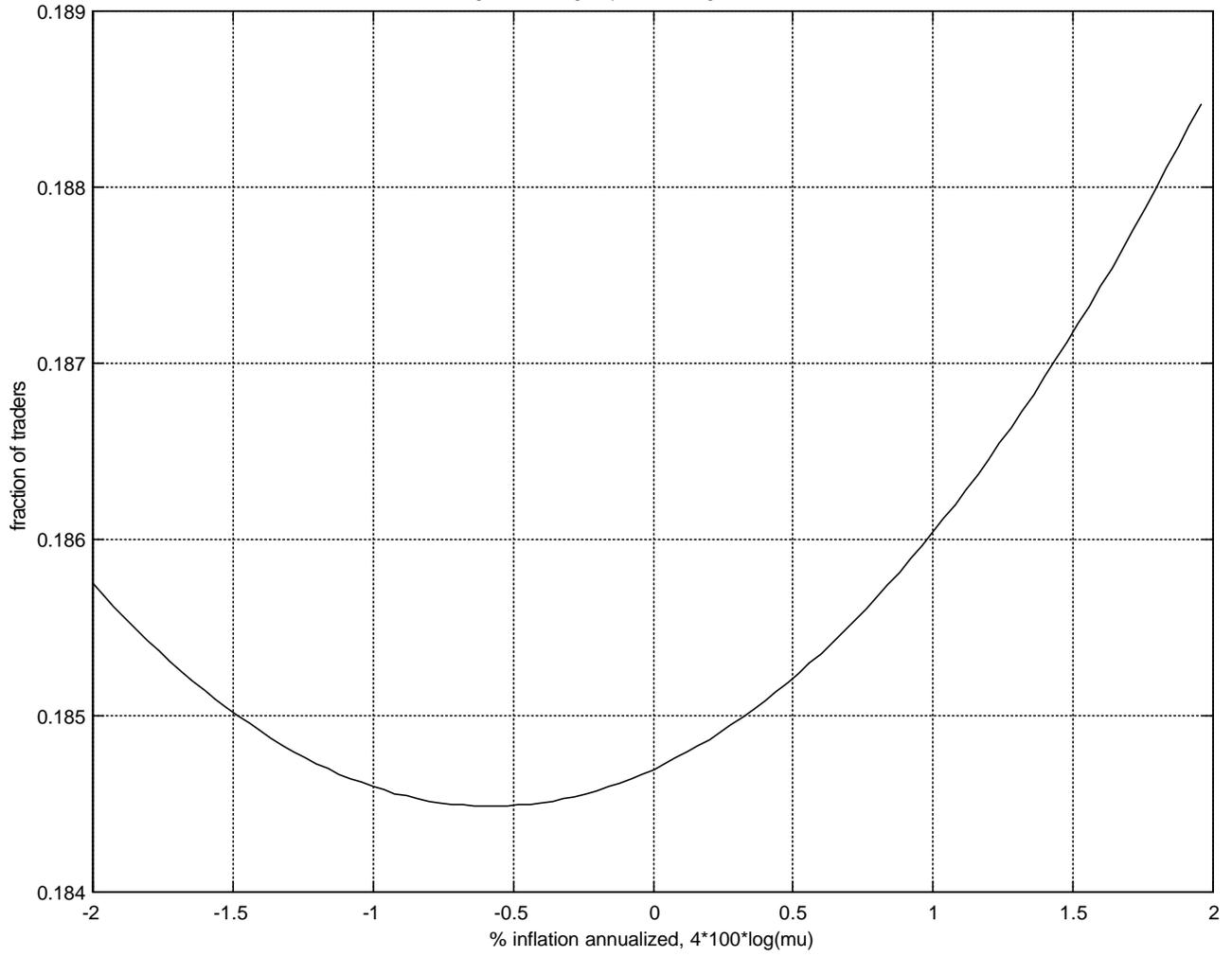
sigma = 1, sigmay = 0.075, gamma = 0.005



sigma = 6, sigmay = 0.075, gamma = 0.005



sigma = 1, sigmay = 0.075, gamma = 0.005



sigma = 6, sigmay = 0.075, gamma = 0.005

