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Abstract

We study how advancements in automation technology affect the division of aggregate income between capital and labor in the context of long-run growth. Our analysis focuses on the fundamental trade-off between the labor-displacing effect of automation and its positive productivity effect in an elementary task-based framework featuring a schedule of automation prices across tasks linked to the state of technology. We obtain general conditions for the automation technology and technical change driving automation to be labor-share displacing. We identify a unique task technology that reconciles the Kaldor facts with the presence of automation along the balanced growth path. We show that this technology aggregates to the Cobb–Douglas production function—thus providing novel task-based microfoundations for this workhorse functional form. We employ our theory to study the connection between the recent declines in the labor share and the unique nature of the current, IT-powered wave of automation.

Key words: Automation, labor share, Uzawa's theorem, Cobb-Douglas production function, capital-augmenting technological progress, balanced growth

JEL Classifications: D33, E25, O33, J23, J24, E24, O4.

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Automation has long been a cause of great concern to policy-makers, economists and workers due to its countervailing effects on labor income. On the one hand, automation displaces labor from tasks or activities, reducing labor income, but on the other hand, it is associated with productivity gains that reduce prices in the economy and increase real wages. This mechanism is at the heart of automation's effect on labor's share in aggregate income, which has recently been in the spotlight due to its steady decline in the data.[1] In fact, one hypothesis holds that it is modern incarnation of IT-powered automation that has eaten into the labor share, raising the follow-up question of what might differentiate this wave of automation from past waves that did not seem to have the same effect on the labor share.[2]

Economic theory offers only limited insights into how this trade-off works, leaving a number of key questions without general answers. For example, is the negative effect of displacement always accompanied by a positive productivity effect? What are the determinants of these effects in terms of technological fundamentals? Under what conditions, if any, is the net effect on the labor share positive? Can balanced growth with automation and constant factor shares be sustained to account for past growth experiences (i.e., the Kaldor facts)? Finally, is the current, IT-powered wave of automation different in ways that would suggest a suppressed productivity effect?

As for existing answers, neoclassical growth theory hints at a potential route of reconciling automation observations with past growth observations (the Kaldor facts) under the guise of capital-augmenting technical progress and the Cobb-Douglas production function, but by itself it is too aggregate to be helpful. The existing microfoundations for that aggregate production function, such as those given by Jones (2005) and Houthakker (1955), while more specific about the role of capital and technology in the economy, do no readily provide an operational notion of automation to fill this gap. A notable exception is the now seminal task-based theory

[1]Karabarbounis and Neiman (2013) measure this decline in multiple countries and conclude that the labor share has been falling globally since the 1980s. They attribute the decline to changing technology. Dao et al. (2017) provide updated evidence. While the decline in the aggregate labor share remains a controversial issue due to measurement challenges (Gutierrez and Piton, 2020; Koh et al., 2020), its decline in heavily automated and automating sectors is indisputable given their magnitude (e.g., US manufacturing).

[2]The constancy of the labor share is one of Kaldor's facts of growth (Kaldor, 1961). The monographs by Brynjolfsson and McAfee (2014), Ford (2009) and Frey (2020) highlight the growing concerns associated with modern automation and provide an anecdotal characterization of the core technologies that it involves. For example, evidence linking modern automation to labor displacement can be found in Acemoglu and Restrepo (2020), Autor and Salomons (2018), Hubmer (2020), Humlum (2019), and Graetz and Michaels (2018), to name a few.

of automation proposed by Acemoglu and Restrepo (2018) (AR18 hereafter), but their insights are also limited to a specific and stylized rebalancing mechanism tying productivity-enhancing innovations and structural changes to the displacement of existing and automated tasks by newly created and initially labor-intensive tasks.

The goal of the current paper is to fill this gap by providing a parsimonious task-based theory of automation that bridges these different strands of the literature and offers generalized answers to the questions listed above. To do so, while building on AR18's work, we fundamentally depart from their view of the task space as a dynamic object linked to innovation or structural changes associated with growth. Instead, the task space is static in our model and thus most appropriately interpreted as the *universe* of *all* feasible operations with matter that are potentially relevant for production. Firms use some of these tasks to produce the current set of goods, and in the background, tasks churn between being used, out of use, or reused.

The key enabling assumption is that the economy is sufficiently large and the time horizon sufficiently long so that, by the application of the law of large numbers, the properties of tasks in use become effectively divorced from any specific processes of structural change and innovation occurring throughout the economy.[3] This approach yields a general representation of technical change as an exogenous schedule of evolving productivity of capital across tasks, which, on the one hand, naturally anchors the discussion on the case of fully diffused progress that augments capital productivity across all tasks uniformly and, on the other hand, allows us to study a broader range of its possible "biased" incarnations. In particular, features such as the rents to R&D associated with automation innovation decrease in their importance to the division of income between labor and capital, since productivity of capital within tasks becomes a cumulative product of incremental innovations over their long lifetime.[4]

As in AR18, our specification of task technology assumes that there exists a sufficiently fine breakdown of production into a set of basic operations—called *tasks*—which in our case

[3]Our interpretation of the task space implies that even when the R&D process driving automation is directed toward some subset of tasks in the short run, the continual process of random structural changes occurring in the economy tends to reshuffle tasks and diffuse the impact of R&D on capital productivity across tasks, resulting in the effect of R&D on capital productivity being diffused across tasks in the long run.

[4]In models of R&D based on the quality ladders framework (Grossman and Helpman, 1991), rents accruing to R&D automation are determined by the marginal leap in productivity delivered by a new "automation recipe." As a result, in the long run, rents associated with the cumulative innovation that determines the overall productivity of capital up to that point within a task are small, and technical progress can be approximately seen as rent-free. AR18 track rents because new tasks emerge as the economy grows.

make capital and labor perfectly substitutable at a task-specific ratio. The existence of such a separation defines what *automation* and *automation capital* are, and not all capital needs to exhibit such a property.[5] After sorting tasks by that ratio—which yields *complexity* ranking of tasks—the usage of automation capital across tasks by a cost minimizing firm is determined via a cutoff rule. Automation is associated with any form of technical change that increases the productivity of capital across tasks and, by cost minimization, decreases the fraction of tasks assigned to labor. Our full model adds task-specific capital to explicitly link production of capital goods to completion of tasks. This endogenizes the price schedule of automating tasks.

The first substantive insight from our theory is that the assumed form of technical change driving automation is crucial for understanding how it affects the labor share in the economy. In particular, if technical progress is "diffused" in that it augments the productivity of capital broadly across tasks, automation need not be labor-share displacing. We derive conditions on technology for this to be the case, and the key to this result is the average productivity of labor over the range of already automated tasks. However, if technical progress is *sufficiently* biased toward marginal, first-to-be automated tasks, automation is *necessarily* associated with a decline in the labor share because the productivity effect is nil (a consequence of the envelope theorem). Interestingly, automation is *always* labor share displacing if it involves any structural changes that shift the mass of tasks towards the automated region. As we explain, such a form of technical change is equivalent to scattered jumps in the productivity of capital across the complexity spectrum that lowers the cost of automating a mass of previously nonmarginal tasks below the automation cutoff. This mechanism also applies in reverse, which is relevant in the context of the specific mechanism considered by AR18 by highlighting its generality in this respect.

To examine the long-run growth implications of our theory, we embed it into the neoclassical growth model. We impose balanced growth restrictions that feature automation and constant factor shares and seek to identify the underlying task technology. We show that these

[5]For example, consider a combination of a worker and a manual hammer in completing the task of nailing an object to a wall. A hammer is augmenting the capabilities of labor and it should not be thought as "automation capital" through the lens of our theory. However, in the same context, a pneumatic hammer can be thought of as "automation capital" because it autonomously completes a step in the production processes previously completed by a worker and a hammer.

conditions lead to a unique task technology that aggregates to the Cobb–Douglas production function—a result closely related to the known corollary to Uzawa's theorem.[6] In our full model featuring the production of task-specific capital, this technology boils down to the requirement that the power law governs the mass of tasks needed to produce machines that are specific to tasks of a given complexity rank. This result is reminiscent of Jones (2005) and Houthakker (1955) and it is appealing because the power law arises spontaneously in nature for well understood reasons, and thus, technology in our model has the potential of being nongeneric—as we discuss in Section 3.3.[7] The existence of balanced growth shows that there is no contradiction between automation and the Kaldor facts as long as the technical progress affecting the productivity of capital is diffused across tasks, but it also raises a follow-up question why the labor share may be declining due to current IT-powered automation—as hypothesized in the literature.

In this regard, our theory's answer is that IT may be the culprit because it represents a form of "complexity-biased" technical change. Specifically, we use our theory to propose a concrete model of how the emergence of IT-based automation adopted by profit-maximizing firms can endogenously lead to such an effect. The key feature of this model is that a capital-producing firm can use IT to "compress" the task load required to produce a machine (capital) at the expense of completing a fixed measure of some other tasks—with the degree of compression being optimally chosen by that firm. The idea is that this fixed set of tasks is associated with adding a computer chip and/or lines of computer code to obtain "smart" machines that optimize the use of hardware. To the extent that the technologies that drive the current wave of automation exhibit the characteristics of the proposed technology, they can be labor share-displacing, and the diffusion argument does not apply because of the specific nature of this technology. (Section 4 discusses motivating anecdotal evidence.)

To summarize our findings, it is helpful to invoke Leontief's analogy between humans and horses that AR18 reference in their work. Following up on that analogy, they ask: What differentiates humans from horses so that they will not share the fate of becoming redundant

[6]The microfoundation of the Cobb–Douglas production function is a new result. AR18 obtain a Cobb–Douglas production function under the assumption of unit elasticity between tasks (i.e., Cobb—Douglas aggregation of tasks) and a fixed automation margin.

[7]While this is not our focus, Section 3.3 discusses how this technology can endogenously arise as a result of an elementary process of innovation. See also the discussion of related literature at the end of this section.

in the course of modern automation? AR18's answer is that humans, unlike horses, create new and initially labor-intensive tasks that crowd out previously automated tasks—possibly associated with the replacement of obsolete goods or production techniques in the economy. In contrast, our theory does not take a stand on this issue, and gives a related but less specific answer: the fact that humans are fungible across a vast array of randomly churning tasks as the economy undergoes structural change can be enough of a defense line for labor to maintain its share. Put differently, it is not necessarily human creativity but also random task churning that can hold the line for labor by uniformly diffusing the effects of the productivity-enhancing innovation in automation technology, and this applies as long as average productivity of labor across the already automated tasks is not too low.

As discussed, our work is complementary (and most closely related) to the investigation opened up by AR18's work, and inspired by the existing microfoundations of the workhorse Cobb–Douglas production function given by Jones (2005) and Houthakker (1955).[8] Regarding the aforementioned microfoundations for the Cobb–Douglas production function, a notable distinguishing feature of our task-based microfoundation is that it does not require that aggregation occurs on an economy-wide level to obtain the Cobb–Douglas production function (approximately)—since the cost minimization in the use of capital and labor per task can be dispersed across heterogeneous firms. This addresses some of the criticisms of these microfoundations seen in the literature.[9] The shared feature is the Pareto distribution. We do not have a clear intuition for this connection other than that the Pareto distribution appears to deliver the right kind of curvature across structurally different models. As an example, the work on the search-based innovation along the lines of Kortum (1997) and its extension due to Ghiglino (2012) is particularly relevant in the context of providing further microfoundations for the Pareto distribution in our model.[10]

[8]Alongside other papers in this area, our work crucially builds on the task-based foundations of growth theory due to Zeira (1998). The recent work by Hubmer and Restrepo (2021) is also relevant and complementary in terms of its focus on the firm-level linkages between automation and declines in the firm-level labor share.

[9]The issue is that aggregation requires that firm optimization occurs on the economy-wide level, as discussed in Acemoglu (2009) (Section 15.8, p. 526). In particular, Acemoglu writes: "(...) existing evidence indicates that there are considerable differences in the production function across industries, and they cannot be well approximated by Cobb-Douglas production function. This suggests it would be interesting to combine the aggregation (...) with equilibrium interactions, which might delineate at what level the aggregation should take place and why (...)." We show how to overcome it in Online Appendix C and discuss it in Section 3.2.

[10]Under the interpretation of the notion of "idea" in this literature as a method to use capital to complete a task at some fixed level of productivity.

1 Baseline model of production

This section lays out our baseline theory of production in partial equilibrium and establishes its properties. We focus on decentralized setup to streamline the exposition.

1.1 Environment

The basic unit of production is a firm: an abstract optimizing unit representative of the economy as a whole. The firm takes prices as given and produces a homogeneous good sold in a competitive market for price $P > 0$. There are two factors of production: capital and labor. The user cost of capital is $r > 0$ and the wage rate is $w > 0$.

Technology

To produce one unit of output, the firm must complete a fixed measure of tasks indexed on the real line by $q \in \mathcal{Q} = \mathbb{R}_+ := [0, +\infty)$. A task is a basic operation that can be completed by employing either a unit of labor or $k(q)$ units of (automation) capital. The unit labor requirement is a normalization, and the underlying assumption is that the capital requirement and the labor requirement are i.i.d. across tasks—as we explicitly show in Online Appendix A.[11] There is no substitution between tasks; that is, completing a subset of tasks many times does not change the requirement to complete other tasks.[12] Tasks are sorted by capital requirement, implying that $k(q)$ is an increasing function and hence almost everywhere differentiable (a.e., hereafter). Throughout, we refer to q as task *complexity* rank. (It will become clear why we refer to it as a "rank.")

The measure of tasks that need to be completed to produce a unit of output is determined by a measure function $\mu : \mathscr{B}(\mathcal{Q}) \to \mathbb{R}_+ \cup \{+\infty\}$, where $\mathscr{B}(\mathcal{Q})$ denotes the Borel σ-algebra over $\mathcal{Q}$. We assume the measure μ is generated by a nonnegative Lebesgue measurable function g

[11]The setup considered here is equivalent to a setup featuring a variable labor requirement $l(q)$ under the assumption that $k(q)$ and $l(q)$ are independent across tasks. What permits this normalization is the fact that *all* tasks must be completed to produce a unit of output.

[12]The assumption that all tasks must be completed contrasts with related task models that allow for some degree of substitutability between tasks. For a fixed commodity that involves, say, tasks A and B, it is not clear what it physically means that completing task A twice is a substitute for completing task B. On the other hand, features such as a broader technology menu from which firms might be choosing, or differentiated goods, should be modeled explicitly in a microfounded model when such distinctions are critical for analysis. Our goal is to formulate the technology in a way that does not mix in preferences, hence this assumption.

referred to as density (not necessarily probability density); that is, for any Borel subset $\mathcal{S} \subseteq \mathcal{Q}$ of the complexity space, we have[13]

$$\mu(\mathcal{S}) = \int_{\mathcal{S}} g(q)\, dv, \tag{1}$$

where v is the Lebesgue measure of the real line. To simplify the analysis, we additionally assume that g has full support—which is largely without a loss in the given context—and that k is a strictly positive-valued function on at least part of the domain.[14] To ensure that production is feasible, we also assume that there exists a measurable partition of the complexity space so that total input usage is finite.

Assumption 1. *There exists a partition $\{\mathcal{Q}_k, \mathcal{Q}_l\}$ of $\mathcal{Q}$ with $\mathcal{Q}_k \cap \mathcal{Q}_l = \emptyset$ and $\mathcal{Q}_k \cup \mathcal{Q}_l = \mathcal{Q}$ such that $\int_{\mathcal{Q}_l} 1 d\mu < \infty$ and $\int_{\mathcal{Q}_k} k(q)\, d\mu < \infty$, k is positive-valued function on at least part of the domain, and g has full support.*

In summary, technology comprises a tuple $T := \{k, g\}$, or interchangably, $T = \{k, \mu\}$. If T obeys the above assumptions, we say that it is an *admissible task technology* and denote the set of such technologies by $\mathcal{T}$. If, in addition, g can be normalized to a probability density function (pdf), we say that technology T has a probabilistic representation.

1.2 Firm problem

Production technology exhibits constant returns to scale: to produce Y units of output the firm repeats the tasks needed to produce one unit of output Y times (units are sufficiently small to warrant $Y \in \mathbb{R}_+$). The profit maximizing firm chooses output level $Y > 0$ (scale) to maximize its flow profits given by $\Pi = PY - c(w, r) Y$, where PY is revenue, $c(w, r) Y$ is total cost, and $c(w, r)$ is both the marginal cost and the unit cost. Profit maximization under constant returns to scale is linear and hence ill-defined unless $P = c(w, r)$, in which case the

[13]The integral of a measurable function over a measure defines another measure—see, for example, Billingsley (1995) Theorem 16.9 (p. 212-213). By the Radon-Nikodym theorem—which provides conditions for the existence of the inverse of this mapping (obtaining g from a given μ)—the family of measures admitted by this formula includes all measures that are absolutely continuous with respect to the Lebesgue measure.

[14]The set on which $g(q) = 0$ can be eliminated from the domain with no impact on production (since inputs are zero). The space can be stretched to fill the real line. Irregular cases such as "fat" Cantor sets are of no economic relevance here.

firm is indifferent to the choice of output Y. Throughout, we assume that the output price P is such that profits are zero and Y clears the market.

The central element of the firm's optimization problem is thus the cost minimization problem that defines the unit/marginal cost $c(w,r)$. We need additional notation to define this cost. Let $\mathscr{P}$ be the collection of all partitions of the complexity space $\mathcal{Q}$ to measurable subsets $\mathcal{Q}_k$, $\mathcal{Q}_l$ such that $\mathcal{Q}_k \cap \mathcal{Q}_l = \emptyset$ and $\mathcal{Q}_k \cup \mathcal{Q}_l = \mathcal{Q}$. The unit cost then solves

$$c(w,r) := \min_{K,L,\{\mathcal{Q}_k,\mathcal{Q}_l\}\in\mathscr{P}} rK + wL \tag{2}$$

subject to

$$L = \int_{\mathcal{Q}_l} 1 d\mu = \mu(\mathcal{Q}_l) \text{ and } K = \int_{\mathcal{Q}_k} k(q)\, d\mu = \int_{\mathcal{Q}_k} k(q)\, g(q)\, dv. \tag{3}$$

The first constraint states that labor usage, L, is determined by the measure of the tasks assigned to labor, which is $\mu(\mathcal{Q}_l)$ due to the unit normalization of the labor requirement per task. The second constraint states that capital usage, K, is determined by the capital requirement function $k(q)$ integrated over the measure μ on the set $\mathcal{Q}_k$ of tasks assigned to capital, or equivalently, by the product $k(q)\, g(q)$ integrated over the Lebesgue measure v. We previously assumed that there exists a partition that makes production feasible, and so the cost minimization problem above is well defined. For later use, we denote the optimal factor intensity implied by the solution to (2) as $\frac{K}{Y}\left(\frac{w}{r}\right)$ and $\frac{L}{Y}\left(\frac{w}{r}\right)$, and refer to them throughout as *capital intensity* and *labor intensity*, respectively. (At this point, these can be either functions or correspondences.)

The lemma below establishes that the solution to the firm's cost minimization problem amounts to finding a cutoff value $q^* \geq 0$ that partitions the complexity space in such a way that all tasks below q^* are completed using capital and all tasks above q^* are completed using labor. The optimality of a cutoff rule is intuitive, but obtaining this result requires that the integrals underlying integrals are finite under a cutoff rule, which Assumption 1 ensures. (All omitted proofs from the text are in the appendix unless otherwise noted.)

Lemma 1. *Firm cost minimization in* (2) *involves a cutoff rule such that tasks on the interval* $[0, q^*]$ *are completed using capital and the remaining tasks are completed using labor, where* $q^* \in \mathbb{R}_+ \cup \{+\infty\}$ *is such that: i) for any* $0 < q^* < \infty$ *there exists* $\varepsilon > 0$ *such that for all*

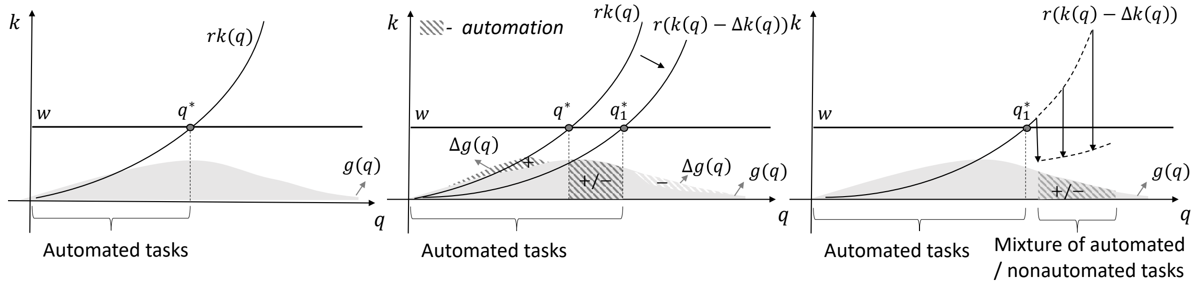

Figure 1: Cost minimization (a) and forms of technical change driving automation (b-c).

Notes: Panel a illustrates the cost–minimizing division of tasks into those assigned to capital and those assigned to labor. The task complexity rank is on the horizontal axis, and the cost of completing a task is on the vertical axis. The shaded area shows the density g of tasks that must be completed. The $rk(q)$ schedule is the cost of completing a task using capital and the w schedule is the cost of completing a task using labor. The cost–minimizing partition is given by cutoff q^*. Panel b illustrates the comparative statics associated with a downward shift in the capital requirement function k and a change in task density g. As shown, this leads to the *automation* of tasks in the striped region. Panel c provides an equivalent representation of the technical change associated with the change in density Δg depicted in panel b—after resorting tasks by $k(q)$. Here, $k(q)$ drops below w on a "scattered" set of positive mass of nonmarginal tasks over the indicated range.

$0 < \delta < \varepsilon$, $rk\left(q^* - \delta\right) \leq w$ *and* $rk\left(q^* + \delta\right) \geq w$, *or else* $q^* = 0$ *and* $rk\left(q^*\right) \geq w$, *or* $q^* = \infty$.

Except for countably many points of discontinuity and irregular cases such as k bounded from above, then, the cutoff q^* satisfies the cost indifference condition: $rk\left(q^*\right) = w$. The solution is unique as long as k is strictly increasing in the neighborhood of q^*. Otherwise, there can be a range of values, and market clearing conditions are needed to pin down the unique equilibrium value of q^*. We refer to this cutoff as the *automation cutoff*.

Figure 1a illustrates the firm's cost minimization problem. Task complexity q is on the horizontal axis and the factor cost of completing each task is on the vertical axis. The boundary of the shaded area depicts task density g. The cost of completing tasks using capital is given by the rk schedule and the cost of completing tasks using labor is given by the w schedule. As shown, to minimize costs, the firm uses capital to the left of the cutoff q^* where the cost of completing tasks using capital (rk schedule) intersects the cost of completing tasks using labor (w schedule), with the rest of the tasks being completed by labor. We refer to these tasks as *automated tasks*. We say that *automation* occurs whenever there is a decrease in the share of tasks assigned to labor.

As an example, Figure 1b depicts two possible forms of technical changes driving automation in our model: i) capital productivity-augmenting shift in k schedule by Δk and ii) complexity-reducing change in task density g by Δg. As shown, (i) increases the automation cutoff and leads to full automation of an existing mass of tasks corresponding to the striped area denoted by "$+/-$" in the figure, and (ii) removes a mass of tasks above the automation cutoff (striped area denoted by "$-$") and moves them below the automation cutoff (striped area denoted by "$+$"). Figure 1c shows that case (ii) can be equivalently interpreted as a "scattered jump" in the capital requirement below the cost of labor over some range of nonautomated tasks—which after resorting tasks in ascending order by capital requirement boils down to the depicted change in task density Δg in Figure 1b.

1.3 Aggregation properties of task technology

To characterize the connection between automation, technical change driving automation, and the labor share, it is convenient to introduce a notion of a production function. A production function of a firm is the maximum output Y that can be produced from fixed (aggregate) inputs K and L:

$$Y(T;K,L) := \sup\left\{\hat{Y}\in\mathbb{R}_+ : \exists_{q^*\in\mathcal{Q}} \text{ s.t. } K\geq \hat{Y}\int_0^{q^*} k(q)\,d\mu, L\geq \hat{Y}\int_{q^*}^{\infty} 1d\mu\right\}, \tag{4}$$

where $T=\{k,\mu\}\in\mathcal{T}$. The production function effectively defines the planning problem underlying production in our model.

Our first lemma shows that the monotonicity of capital requirement function k and the assumption that it is nonzero on at least part of the domain $\mathcal{Q}$ implies that determining $Y(T;K,L)$ for a given tuple K,L amounts to choosing a *factor utilization cutoff* q^* so that all inputs are *fully* utilized; that is, the technological constraints in (4) hold with *equality*. Note this cutoff is distinct from the automation cutoff implied by firm's cost minimization problem, albeit in equilibrium the two cutoffs must align.[15]

Lemma 2. *For any $K>0, L>0$ and technology $T=\{k,g\}\in\mathcal{T}$, there exists a unique factor utilization cutoff $q^*>0$ and $\hat{Y}>0$ such that $K=\hat{Y}\int_0^{q^*}k(q)\,d\mu$, $L=\hat{Y}\int_{q^*}^{\infty}d\mu$, and $Y(T;K,L)=\hat{Y}$, where $Y(T;K,L)$* is given by (4).

Our next result shows that the marginal products implied by the above production function

[15] On the firm level, note, the cost minimizing automation cutoff may not be unique but on the aggregate level market clearing in the input markets implies that among all cost minimizing cutoffs the one that prevails is the same as the factor usage cutoff associated with aggregate supply of inputs.

are well defined (almost everywhere) and we derive the formula for output elasticity with respect to each factor. In a competitive market environment output elasticities map onto factor shares, and so this formula is of interest.

Lemma 3. *Marginal products* $MPK := \frac{\partial Y(T;K,L)}{\partial K}$, $MPL := \frac{\partial Y(T;K,L)}{\partial L}$ *implied by* $Y(T;K,L)$ in (4) *are well defined (a.e.), with output elasticities given by*

$$\alpha := \frac{K}{Y}MPK = \left(1 + k\left(q^*\right)\frac{L}{K}\right)^{-1} \text{ and } \frac{L}{Y}MPL = 1 - \alpha \text{ (a.e.)}, \tag{5}$$

where q^* *is the factor utilization cutoff satisfying Lemma 2.*

Finally, our last technical result shows that, as far as aggregation goes, there is a degree of freedom in terms of the complexity space—which can be transformed by any nonnegative and invertible function with no impact on the aggregate production function. This implies that a single aggregate production function corresponds to a whole equivalence class of task technologies.

Lemma 4. *Suppose* $T_1 = \{k_1, \mu_1\} \in \mathcal{T}$ *and* $T_2 = \{k_2, \mu_2\} \in \mathcal{T}$ *are such that there exists a* μ_1, μ_2*-measurable (and invertible) map* $f : \mathcal{Q} \to \mathcal{Q}$ *so that* $k_1 \equiv k_2 \circ f^{-1}$ *(a.e.) and* $\mu_1 \equiv \mu_2 \circ f^{-1}$. *Then, the aggregate production function associated to each technology is identical; that is,* $Y(T_1; K, L) \equiv Y(T_2; K, L)$.

2 Effects of automation on labor share

We begin our analysis by first characterizing the link between technical change driving automation in our model and the labor share. We use these results to guide our discussion throughout the paper. As discussed in the introduction, technical progress should be thought of as the result of the history of capital productivity-enhancing innovations occurring at the task level and the process of diffusion of its effects as tasks randomly churn in and out of use across the economy. We outline a model of that in Section 3.3.

2.1 Comparative statics framework and definition

To represent capital-augmenting technical change as generally as possible, consider a infinitesimal perturbation of the task technology $T = \{k, g\} \in \mathcal{T}$ in the direction of some technology vector $\Delta T = \{\Delta k, \Delta g\}$, where $\Delta k(q)$ is a nonnegative and differentiable function and $\Delta g(q)$ is a differentiable function that shifts a mass of tasks away from labor and towards capital; that is, Δg is assumed to obey the restriction:

$$\int_0^{q^*} \Delta g(q)\, dv = -\int_{q^*}^{\infty} \Delta g(q)\, dv > 0, \tag{6}$$

where q^* is the initial equilibrium automation cutoff. Specifically, the perturbed technology is

$$T_\varepsilon := \{k_\varepsilon(q) := k(q) - \varepsilon \Delta k(q), g_\varepsilon(q) := g(q) + \varepsilon \Delta g(q)\}. \tag{7}$$

In the spirit of the variation calculus, we characterize the marginal impact of ε at $\varepsilon = 0$. We refer to such a marginal perturbation as *ΔT-biased capital-augmenting technical change* (or progress), and we also consider it coordinate-wise. Specifically, if $\Delta g \equiv 0$, we refer to this perturbation as *Δk-biased capital productivity-augmenting technical change*, and when $\Delta k \equiv 0$, we call it *Δg-biased complexity-reducing technical change* (*complexity-augmenting* when the sign in (6) is reversed).

While prices r, w are held constant, the price level P is assumed to adjust so that profits of the firm are zero at all times. The adjustment of the price level is key to redistributing the gains from automation between the factors of production and assessing the impact of automation and technical change on the labor share in a way that is consistent with the notion of equilibrium in our full model.

The effect of this perturbation in a "nonmarginal form" is illustrated in Figure 1b. As already discussed, the figure depicts a downward shift in the capital requirement by Δk and a shift in task density by Δg. Given our assumption that tasks are sorted by capital requirement, the latter form of technical change can be seen as equivalent to downward jumps in $k(q)$ below w across a scattered set on complexity domain—as shown in Figure 1c. After resorting tasks by $k(q)$, note, the technical change illustrated in that figure boils down to the change in density depicted in Figure 1b. As a result of this technical change, labor intensity $\frac{L}{Y}\left(\frac{w}{r}\right)$ decreases and capital intensity $\frac{K}{Y}\left(\frac{w}{r}\right)$ increases, and firm profits rise. Consequently, the equilibrium price P declines to restore the zero profit condition. The adjustment of the price level, note, has no feedback effect on the automation cutoff because that cutoff depends only on $\frac{w}{r}$.

Decomposition of effect of automation and technical change on labor share

The labor share in our setup corresponds to

$$LS_{\varepsilon} := \frac{w}{P_{\varepsilon}} \frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right), \tag{8}$$

where P_{ε} is the zero profit price level, $\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)$ is the labor requirement function that minimizes production cost and q_{ε}^{*} is the cost minimizing automation cutoff. With the exception of wage rate w—which is exogenous—all variables depend on technology, and hence ε in the subscript. Differentiating this expression with respect to ε shows that the labor share is affected via two distinct channels, which we label as *displacement effect* (DE) and the *productivity effect* (PE):

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0} = \underbrace{\frac{d\log \frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0}}_{\text{displacement effect DE}} + \underbrace{-\frac{d\log P_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}}_{\text{productivity effect PE}} . \tag{9}$$

Intuitively, *displacement effect* is associated with the displacement of labor by capital across tasks—represented by the striped area denoted by "–" in Figure 1b. The effect is in part attributed to the change in the automation cutoff and in part to the change in density g—as shown in that figure. Assuming that $k\left(q\right)$ is strictly increasing and differentiable at q^{*}, the cutoff satisfies the identity $k_{\varepsilon}\left(q_{\varepsilon}^{*}\right) \equiv \frac{w}{r}$, implying

$$\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0} = \frac{\Delta k\left(q^{*}\right)}{k'\left(q^{*}\right)} > 0, \tag{10}$$

where k' is shorthand for the derivative of k. The cutoff change is positive whenever $\Delta k\left(q^{*}\right) > 0$. The change in density, note, does not affect the cutoff—even though one can think of it as being driven by a decline in capital requirement, as depicted in Figure 1c.

The productivity effect is associated with a decline in the zero-profit price P_{ε}, as it raises the purchasing power of income in the economy and thus raises real wages. Differentiating the zero-profit condition $\left(w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right) + r\frac{K}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right) - P_{\varepsilon} \equiv 0\right)$ shows that, generally, the price impact is attributed to both *automation* of tasks and the *direct technical change effect*:

$$PE = \underbrace{-\frac{1}{P}\left(w\frac{d\frac{L}{Y}\left(q^{*}\right)}{dq^{*}} + r\frac{d\frac{K}{Y}\left(q^{*}\right)}{dq^{*}}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0} - \frac{1}{P}\int_{0}^{q^{*}}\frac{\Delta g\left(q\right)}{g\left(q\right)}k\left(q\right)d\mu}_{\text{automation}} \tag{11}$$

$$+ \underbrace{\frac{1}{P}\int_{0}^{q^{*}} r\Delta k\left(q\right)d\mu}_{\text{direct technical change effect}},$$

where, as noted, we use notation $P \equiv P_0$, $q^* \equiv q_0^*$, $\frac{L}{Y}_0 \equiv \frac{L}{Y}$, and so on and so forth. *Automation's* effect in (11) is in part driven by the change in automation cutoff (first two terms) and in part by the new automated tasks brought about by the change in task density (third term). *Direct technical change effect* is brought about by increased productivity of capital across the already automated tasks on the interval $(0, q^*)$. Since we restricted attention to the case of progress that increases productivity of capital, the productivity effect is nonnegative. The relative magnitude of these two effects is what determines the link between automation and the labor share.

2.2 Analysis and results

The proposition establishes the conditions on technical change and technology so that the net effect on the labor share is nonnegative. To understand this result, it is instructive to consider the two sources of variation separately. We discuss them one by one after the proposition to break down this result and narrow the key intuitions it conveys.

Proposition 1. $\Delta T = \{\Delta k, \Delta g\}$*-biased capital-augmenting technical progress: 1) changes the measure of nonautomated tasks completed by*

$$\frac{d\log S_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0} = -h\left(q^{*}\right)\left(\frac{\Delta k\left(q^{*}\right)}{k'\left(q^{*}\right)} + \frac{\int_{0}^{q^{*}}\frac{w}{r}\frac{\Delta g(q)}{g(q)}d\mu}{g\left(q^{*}\right)k\left(q^{*}\right)}\right) \text{ (a.e.)}, \tag{12}$$

and 2) changes the labor share by

$$\frac{d\log LS_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0} = \underbrace{\frac{d\log\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}{d\varepsilon}|_{\varepsilon=0}}_{\text{displacement effect DE}} + \underbrace{-\frac{d\log P_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}}_{\text{productivity effect PE}} \text{ (a.e.)} \tag{13}$$

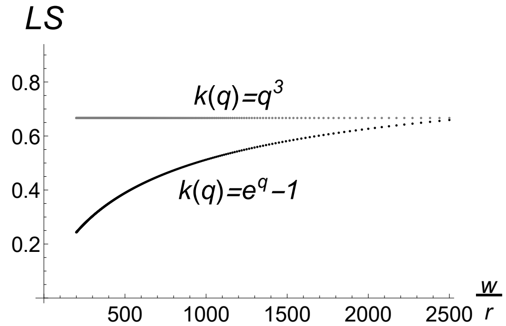

Figure 2: Labor share and the effect of a uniform capital productivity-augmenting technical progress.

Notes: The figure compares the trace of the labor share implied by the firm problem with respect to w/r in an exercise that assumes an exogenous decrease in r, fixed w, and zero profit P. (Distribution of q is Pareto pdf.)

where

$$DE = \frac{d \log S_\varepsilon\left(q_\varepsilon^*\right)}{d\varepsilon}|_{\varepsilon=0} = -h\left(q^*\right)\left(\frac{\Delta k\left(q^*\right)}{k'\left(q^*\right)} + \frac{\int_0^{q^*} \frac{w}{r}\frac{\Delta g(q)}{g(q)} d\mu}{g\left(q^*\right) k\left(q^*\right)}\right), \tag{14}$$

$$PE = \frac{1}{P}\int_0^{q^*} rk\left(q\right) d\mu + \frac{1}{P}\int_0^{q^*} \frac{\Delta g\left(q\right)}{g\left(q\right)}\left(w - rk\left(q\right)\right) d\mu, \tag{15}$$

$$= h\left(q^*\right) LS\left(\frac{\int_0^{q_0^*} \Delta k\left(q\right) d\mu}{k\left(q^*\right) g\left(q^*\right)} + \frac{\int_0^{q^*} \frac{\Delta g(q)}{g(q)}\left(\frac{w}{r} - k\left(q\right)\right) d\mu}{g\left(q^*\right) k\left(q^*\right)}\right)$$

and where q^* is the initial automation cutoff such that at that cutoff point $k\left(q\right)$ is strictly increasing and differentiable, $S\left(q^*\right) = \int_{q^*}^{\infty} d\mu$ is the survival function, and $h\left(q\right) := -\frac{dS(q)}{dq} = \frac{g(q)}{S(q)}$ is the hazard rate.

Effects of capital productivity-augmenting technical change

Regarding the capital productivity-augmenting component of technical change ($\Delta g \equiv 0$), the key implication of Proposition 1 is that, for the net effect to be nonnegative, technical progress must be "diffused" in that it must augment the productivity of capital across a broad range of already automated tasks. By the envelope theorem, the optimality of the automation cutoff q_0^* implies that the productivity effect is attributable to the direct effect of technical change in (11); that is, the effect of automation is *always* nil. Consequently, any *marginal* form of technical progress featuring $\Delta k\left(q^*\right) = k\left(q^*\right)$ and vanishing $\Delta k\left(q\right) \approx 0$ over the range of currently automated tasks on the interval $(0, q^*)$ *exclusively* entails the negative displacement effect—implying an unambiguously negative net effect. This is easily verified by plugging these values into (13) and noting that $PE = 0$ while $DE < 0$.

The second key implication of the proposition is that diffused technical progress by itself is not enough for the net effect to be positive ($PE + DE \geq 0$). The proposition provides a condition on technology so that this is the case.

To understand the intuition behind that condition, let us consider the case of *uniformly* capital productivity-augmenting technical progress; that is, let $\Delta k(q^*) = k(q^*)$. Replacing Δk with k in (13), implies that the productivity effect is

$$PE = \frac{1}{P} \int_0^{q^*} rk(q)\, d\mu = h(q^*)\, LS \frac{\int_0^{q^*} k(q)\, g(q)\, dv}{k(q^*)\, g(q^*)}. \tag{16}$$

The first equality from the left tells us that the productivity effect is associated with a reduction in the firm's spending on capital due to increased productivity of capital across the already automated tasks on the interval $(0, q^*)$. This is intuitive given our model features no substitution of one task for another task.[16] The second equality shows that the initial labor share and the average productivity of capital across the automated tasks relative to the marginal automated task is decisive about the magnitude of this effect. Therefore, a less convex $k(q)\, g(q)$ schedule is conducive to generating a larger productivity effect.

The strength of the displacement effect also turns out to depend on the convexity of the $k(q)$ schedule because this is what determines the change in the cutoff. The shape of density $g(q)$ is not relevant, and hence nonconvex $k(q)$ is most conducive to a positive net effect. In particular, in this case, we have

$$DE = \frac{d \log S_\varepsilon(q^*_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = -h(q^*)\, \frac{k(q^*)}{k'(q^*)}. \tag{17}$$

The first equality shows that displacement corresponds to the reduction in the total mass of tasks completed by labor—which is intuitive given the fact that the labor requirement per task is normalized to unity and we are considering the relative change in the labor share (change

[16] Had there been any substitution between tasks, the result would depend on the corresponding elasticity. However, for reasons discussed in footnote 12, we favor the formulation of technology without this feature.

in logs). As already mentioned, the second equality implies that the more convex the $k(q)$ schedule is, the smaller the change in the cutoff, and the smaller the displacement effect. (The hazard rate h is effectively unimportant because it scales both the productivity effect and the displacement effect.)

To show that the productivity effect can dominate, Figure 2 considers two distinct technologies: the first technology assumes $k(q) = q^3$ and the second technology assumes a less convex schedule given by $k(q) = e^q - 1$. In both cases, task density is given by the Pareto pdf: $g(q) \propto (q + .1)^{-2}$, and in both cases we solve for the zero profit equilibrium for a range of values for w/r for the given $k(q)$ schedules—which, note, is equivalent to a uniformly capital-augmenting progress considered here. As we can see, capital-augmenting progress corresponding to higher $\frac{w}{r}$ implies a flat path for the labor share in the first case and an increasing path in the second case.

Effects of complexity-reducing technical change

The complexity-reducing technical change does not affect the automation cutoff, but it nonetheless leads to positive displacement because a positive mass of "nonmarginal" tasks is reassigned from labor to capital—as shown in Figure 1b. In particular, the displacement effect is *always* negative and it pertains to the relative change in the mass of tasks assigned to labor:

$$DE = \frac{d \log S(q^*_\varepsilon)}{d\varepsilon}|_{\varepsilon=0} = -h(q^*) \frac{\int_0^{q^*} \frac{w}{r} \frac{\Delta g(q)}{g(q)} d\mu}{g(q^*) k(q^*)} < 0. \tag{18}$$

The productivity effect, in turn, is also *always* positive and depends on the net profit benefit from reassigning tasks, which comes from the difference between the relative cost of labor versus the cost of capital "$\frac{w}{r} - k(q)$" on the spectrum of the mass of reassigned tasks; that is,

$$PE = \frac{1}{P} \int_0^{q^*} \frac{\Delta g(q)}{g(q)} (w - rk(q)) \, d\mu = h(q^*) \, LS \left(\frac{\int_0^{q^*} \frac{\Delta g(q)}{g(q)} \left(\frac{w}{r} - k(q)\right) d\mu}{g(q^*) k(q^*)} \right). \tag{19}$$

The first equality shows that, as in the other case, the effect is driven by reduction in costs. The second equality translates this formula to a form that can be compared to the displacement effect above, from which we obtain that the net effect is *always* negative:

$$DE + PE = -\frac{h(q^*)}{g(q^*)\,k(q^*)} \left(\int_0^{q^*} \frac{\Delta g(q)}{g(q)} \left(\frac{w}{r}(1 - LS) + k(q)\,LS \right) d\mu \right) < 0. \tag{20}$$

The generality of this result may seem surprising but it is quite intuitive. Note that the assumed change in task density reduces production costs, which lowers the price level in the economy and increases the purchasing power of all income. In real terms, then, the benefit that accrues to labor is proportional to its initial share (LS). To see this from the above expression, set $k(q) \approx 0$ across the mass of reassigned tasks and note that even in that case the net effect is still negative and given by $-\frac{w}{r}(1 - LS)$. Intuitively, due to the displacement of labor from tasks, labor share declines by $\frac{w}{r}$ per displaced tasks, but the overall decline is only $\frac{w}{r}(1 - LS)$ because the decline in the price level recoups $\frac{w}{r}LS$. If the firm must pay for capital (i.e., when $k(q) > 0$), the negative net effect is larger because payments to capital reduce profits, with fraction $k(q)\,LS$ that would otherwise accrue to labor (measured in units of capital).

Discussion of results

The key lesson from the above results is that both the technology in place and the form of technical progress driving automation affect the labor share. In particular, observations of automation alone are insufficient to determine its effects on the labor share, which may be positive or negative.

As a general rule, Δk-biased capital productivity-augmenting technical change that is more "diffused" and improves the (relative) productivity of capital across a vast swath of tasks tends to boost the productivity effect. The net effect then depends on how productive capital is over the entire range of automated tasks vis-à-vis the marginal automated task, or equivalently, how productive labor is. As discussed in the introduction, the assumptions underlying the notion of tasks in our model favor such a diffused view of technical progress unless one can identify a specific reason to think otherwise (e.g., as we do in Section 4). A random churning of tasks that occurs in the economy will tend to diffuse the impact of even extremely marginal progress over time, and for this reason there is hope for labor to hold the line as long as it remains reasonably productive on the range of the automated tasks relative to the marginal task.

Structural changes that are complexity-reducing and push nonmarginal and nonautomated tasks towards the automated region are always labor-share displacing. A better empirical understanding of how structural changes and innovation affect task complexity is needed here.[17] To our way of thinking, it is not clear in which direction this effect goes as the economy grows. The next section shows that imposing balanced growth conditions in our framework is consistent with the absence of this channel in the past data.

Last but not least, in the medium run, positive rents to automation-enabling R&D may temporarily reduce the labor share in addition to the described above effects. Our analysis abstracts from such rents and focuses on the long-run effects of technical change.[18]

3 Balanced growth and automation

Can the observations of automation on a micro level be consistent with past growth experiences, namely the Kaldor facts (Kaldor, 1961)?[19] What can we learn about technical change and technology from these observations? Here we embed our model into a growth model and show that there is no contradiction between these observations as long as technical progress is uniformly capital productivity-augmenting, complexity-unbiased, and the underlying task technology takes a particular form that aggregates to the Cobb–Douglas production function.

3.1 Growth and general equilibrium

The overarching growth model is standard but we briefly describe the setup to lay out notation. Time is continuous, $t \in \mathbb{R}_+$ and $T_t \in \mathcal{T}$ denotes the exogenous technology path (under perfect foresight), which we summarize by the underlying sequence of production functions

[17] While more sophisticated *goods* or *services* may enter production as incomes grow, it does not imply that their production involves more complex *tasks*, because relation between goods and tasks is unclear. There are also anecdotes that suggest the opposite. For example, standardization in the process of mass production is complexity-reducing technical change. Digitization, by the nonrival nature of software, results in large productivity gains across selected tasks and can be thought of as Δg-biased complexity-reducing technical change (as shown in Figure 1c).

[18] See footnote 4. In the medium run, and hence in the context of the recent declines in the aggregate labor share that span just a few decades, rents to R&D associated with automation technology could be a factor. In fact, one of the hypotheses for the declining labor share in the aggregate data is that the rents associated with intangible capital have increased (Koh et al., 2020).

[19] For a review of the Kaldor facts, see Jones and Romer (2010).

$Y\left(T_t; K_t, \bar{L}\right)$ defined in (4).[20]

The economy is populated by a stand-in firm and a stand-in household. The household values consumption streams according to the present discounted value of the flow utility from consumption C_t given by $u\left(C_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma}$, $\sigma > 1$, where $0 < \rho < 1$ denotes the discount factor. The household inelastically supplies $\bar{L}$ units of labor and accumulates capital that it rents to firms. All markets are perfectly competitive and all prices are taken as given.

The *allocation* corresponds to any non-negative path of C_t, K_t, and Y_t, as well as factor utilization cutoff q_t^* associated with the definition of $Y\left(T_t; K_t, \bar{L}\right)$ uniquely pinned down by Lemma 3. The allocation obeys the economy-wide resource constraint

$$C_t + \dot{K}_t - \delta K_t = Y_t = Y\left(T_t; K_t, \bar{L}\right), \tag{21}$$

where $C_t \geq 0$ is consumption (in period t), $K_t \geq 0$ is capital used in production, $\dot{K}_t := \frac{dK_t}{dt}$ is the gross investment in capital, δK_t is the depreciation of capital, and $\bar{L}$ is labor used to production.

By welfare theorems, the *equilibrium allocation* solves the planning problem of maximizing lifetime utility $\int_0^\infty e^{-\rho t} u\left(C_t\right)$ subject to the resource constraint in (21) and given $K_0 > 0$. Supporting prices can be recovered from Lemma 3 and the path of $P_t > 0$ is undetermined and can be normalized to unity.

3.2 Conditions for balanced growth with automation

In a model with detailed microfoundations, the definition of a balanced growth path is more involved because it has to specify how various functional forms evolve over time. We use the standard approach of assuming "stable shape" conditions in the spirit of similar definitions used by, for example, Lucas and Moll (2014), Perla and Tonetti (2014), and Menzio and Martellini (2020). These assumptions are not without loss, but they are justified. The logic is that the underlying endogenous processes that determine these objects are stationary in the sense that they give rise to stable structural relationships within the economy.

Specifically, our definition requires that the growth rate of the capital requirement per task is independent of the task by assuming that $k\left(q\right)$ grows at a constant rate for all $q \in \mathcal{Q}$. We

[20]By a path we mean a function of time t.

impose a similar condition on the density function $g(q)$. Under this definition, note, growth can be driven only by technical progress that is uniformly factor-augmenting across tasks and complexity-unbiased. As we have argued, this view is consistent with the long-run perspective and the notion of task space in our model absent any correlating factor that directs capital-augmenting progress toward a particular subset of tasks or moves the mass of tasks within the already automated region. The formal definition is as follows:

Definition 1. A *balanced growth path with automation and constant factor shares* (BGP) comprises an allocation sequence: $Y_t = Y_0 e^{\gamma_Y t}$, $C_t = C_0 e^{\gamma_C t}$, $K_t = K_0 e^{\gamma_K t}$, $q_t^* = q_0^* e^{\gamma_{q^*} t}$ and a technology sequence $T_t = \{k_t(q) = k_0(q) e^{\gamma_k t}, g_t(q) = g_0(q) e^{\gamma_g t}\} \in \mathcal{T}$, such that $\gamma_Y > 0$,$\gamma_{q^*} > 0$ and $\alpha \equiv \frac{K_t}{Y_t} MPK_t$ is constant, where γ_k, γ_g, $Y_0 > 0$, $C_0 > 0$, $K_0 > 0$, $q_0^* > 0$ are scalars.

Our first result, summarized in Proposition 2, shows that BGP exists and requires that the initial task technology is of the form:

$$T_0 = \left\{k_0(q) = k_0 q^{\theta}, g_0(q) = g_0 q^{-\zeta-1}\right\}, \tag{22}$$

where, abusing notation, $k_0 > 0, g_0 > 0$ are scalars involved in the specification of similarly named functions $k_0(q)$, $g_0(q)$. We refer to this task technology as the *BGP task technology*.

The proof largely follows from Lemma 3 and the basic property that capital K and output Y grow at the same rate $\gamma := \gamma_K = \gamma_Y$ in the overarching growth model—which is also an intermediate step in the proof of the Uzawa steady-state growth theorem as found in, for example, Jones and Scrimgeour (2008). Specifically, by (5), for the factor shares to remain constant, $k_t(q_t^*) = k_0(q_0^* e^{\gamma_{q^*} t}) e^{\gamma_k t}$ must grow at the same rate γ, and so $k_0(q)$ must exhibit a constant elasticity with respect to q because q_t^* is also required to grow at a strictly positive rate $\gamma_{g^*} > 0$.[21] Accordingly, we must have $k_0(q) = k_0 q^{\theta}$ for some $\theta > 0$. Since the total labor supply is fixed at $\bar{L}$ and the resource feasibility requires $\bar{L} = Y_t \int_{q_t^*}^{\infty} g_t(q)\, dv$, it must be true that $\int_{q_t^*}^{\infty} g_t(q)\, dv$ declines at a constant rate γ to offset the constant growth of Y_t at rate γ. As we show in the proof, this yields $g_0(q) = g_0 q^{-\zeta-1}$, as stated.[22]

[21] Constant growth of $k(q^*)$ given constant growth of q^* implies $\frac{dk_0(q^*)}{dq^*}/k(q) = \theta$, for some constant $\theta > 0$, which solves to $k_0(q) = k_0 q^{\theta}$, for any constant $k_0 > 0$.

[22] $\theta > \zeta$ is required for the integrals to exist.

The fact that balanced growth restriction implies a unique functional form for g should not be surprising. The Pareto density function underlying BGP is the *only* function satisfying the so-called scale-free property; that is, Pareto density is the only density satisfying that, for any scalar $a > 0$, $g_0(aq) = \lambda(a) g_0(q)$ for some function $\lambda(a)$. This property is needed to ensure that on different parts of the complexity domain factor intensities exhibit consistent behavior so that growth can be balanced.[23]

Proposition 2. *If $\gamma_k < 0$ and $\gamma_g - \alpha\gamma_k > 0$, BGP exists and features: 1) $T_0 = \left\{k_0(q) = k_0 q^{\theta}, g_0(q) = g_0 q^{-\zeta-1}\right\}$, where $\zeta := \frac{\gamma_Y + \gamma_g}{\gamma_{q^*}}$, $\theta := \frac{\gamma_Y - \gamma_k}{\gamma_{q^*}}$; 2) $\gamma := \gamma_Y = \gamma_C = \gamma_g - \alpha\gamma_k$.*

The next result, summarized in Corollary 1 below, shows that the BGP technology aggregates to the constant returns to scale Cobb–Douglas (CD) production function—which is not surprising given what we already know from the Uzawa steady-state growth theorem. Specifically, recall that the Uzawa theorem implies that it must be possible to represent the technical progress driving growth as labor augmenting; that is, it must be possible to represent the production function in the neighborhood of the balanced growth path as $Y\left(T_0; K_t, a_t\bar{L}\right)$, for some constant growth path a_t referred to as labor-augmenting technical progress.[24] To obtain the CD production function from this result, more restrictions must be placed. For example, imposing the condition that the relative price of capital goods is declining at a constant rate along the balanced growth path suffices.[25] In our model, this condition is simply replaced by the requirement that there is a steady rate of automation.

Corollary 1. *The aggregate production function along the BGP is Cobb–Douglas; that is, output Y_t and marginal products MPK_t and MPL_t are consistent with those implied by*

$$Y_t(K, L) = A_t (Z_t K)^{\alpha} L^{1-\alpha} \tag{23}$$

where $\alpha = \frac{\zeta}{\theta}$ and $A_t > 0$, $Z_t > 0$ are scalars that grow at a constant rate.

[23] For the proof of this fact, see, for example, Newman (2004).

[24] We use the shorthand notation $\dot{x} := \frac{dx}{dt}$ throughout. For a detailed discussion and proof of the Uzawa theorem, see Jones and Scrimgeour (2004) and Jones and Scrimgeour (2008), or Acemoglu (2009) Theorem 2.6 (p. 60) and Theorem 2.7 (p. 63).

[25] We lack a good reference for this result and the outline of the proof can be found in Online Appendix H.

As Lemma 4 shows, the BGP task technology is a *unique* task technology that aggregates to the CD production function up to a monotone transformation of the complexity space $\mathcal{Q}$, which is a degree of freedom here by that lemma. The reason why a unique BGP task technology obtains is because we have imposed the condition that q^* grows at a constant rate.[26] Without this requirement, growth would not be balanced on the micro-level in a strict sense but it would remain balanced on the aggregate level. The definition of BGP could thus be extended to incorporate all cases that yield a Cobb–Douglas production function—and that lemma tells us what they are.

It is not difficult to see, for example, that the following monotone transformation $f(q) = \zeta^{-\frac{1}{\zeta}} q^{\frac{1}{\zeta}}$ applied to the BGP technology in (22) yields a technology that directly maps onto the parameters of the CD production function in (23):[27]

$$T^{CD} = \left\{ k(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{1-\alpha}{\alpha}, g(q) = A^{-1} q^{-2} \right\}. \tag{24}$$

We will refer to this technology as the *canonical CD task technology*. To give a better feel of the analytics underlying the above results, Example 1 explicitly derives the CD production function implied by the above technology and shows the equivalency between the above CD technology and the original BGP technology.

Example 1. Consider the canonical CD technology in (24). We first explicitly show it aggregates to CD production function. To that end, we use Lemma 2, calculate

$$\frac{L}{Y} = \int_{q^*}^{\infty} g(q)\, dq = A^{-1} q^{*-1}, \; \frac{K}{Y} = \int_{0}^{q^*} k(q)\, g(q)\, dq = (AZ)^{-1} q^{*\frac{1}{\alpha}-1}, \tag{25}$$

and eliminate q^* to obtain $Y = A(ZK)^{\alpha} L^{1-\alpha}$. Second, we note that from the proof of Corollary 1 that the BGP technology T_0 in Proposition 2 aggregates to $Y = \frac{1}{g_0} \left(\zeta^{\frac{\theta}{\zeta}-1} \frac{\theta-\zeta}{k_0} K \right)^{\frac{\zeta}{\theta}} L^{1-\frac{\zeta}{\theta}}$. Accordingly, both technologies give rise to the same production function for the matching parameters: $\alpha = \frac{\zeta}{\theta}$, $A = g_0^{-1}$ and $Z = \zeta^{\frac{\theta}{\zeta}-1} (\theta - \zeta) k_0^{-1}$. To see that CD technol-

[26]The restriction of this result to $\mathcal{T}$ can be generalized but we do not pursue such a generalization to simplify the exposition.

[27]Any task technology $T = \{k, g\}$ that aggregates to a CD production function can be represented in T^{CD} form after applying the transformation: $f(x) = S^{-1}\left(Ax^{-1}\right)$, where $S(q^*) = \int_{q^*}^{\infty} g dv$.

ogy can be obtained from BGP technology using the map $f(q) = \zeta^{-\frac{1}{\zeta}} q^{\frac{1}{\zeta}}$, note that the survival function associated with BGP technology is $\hat{S}(q) = g_0 \zeta^{-1} q^{-\zeta}$. Accordingly, we have $S(q) = \hat{S}\left(\zeta^{-\frac{1}{\zeta}} q^{\frac{1}{\zeta}}\right) = g_0 q^{-1} = A^{-1} q^{-1}$, $g(q) := -\frac{d}{dq} S(q) = A^{-1} q^{-2}$, and also $k(q) = k_0 \left(\zeta^{-\frac{1}{\zeta}} q^{\frac{1}{\zeta}}\right)^{\theta} = k_0 \zeta^{-\frac{\theta}{\zeta}} q^{\frac{\theta}{\zeta}} = Z^{-1} \frac{1-\alpha}{\alpha} q^{\frac{1}{\alpha}}$.

We conclude this section by stating a technical result implied by directly imposing the requirement that the task technology must aggregate to a Cobb-Douglas production function. This provides an alternative route of obtaining the above results without going through a definition of balanced growth. As we can see, k must be proportional to the hazard rate implied by g, and the range of k must cover the entire real line. This result also implies that measure μ is an infinite measure (i.e., $\mu(\mathcal{Q}) = \infty$). We comment on this property in the next section.

Proposition 3. *The aggregate production function is Cobb-Douglas with exponent $0 < \alpha < 1$ if task technology $T = \{k, g\} \in \mathcal{T}$ satisfies*

$$\alpha \frac{k'(q)}{k(q)} = h(q) := \frac{g(q)}{S(q)} \text{ (a.e.)}, \tag{26}$$

with $k(q) \to_{q \to 0} 0$, $k(q) \to_{q \to \infty} \infty$, $k(q) S(q) \to_{q \to 0} 0$. Accordingly, for all $\varepsilon > 0$, there exists a scalar $C_\varepsilon > 0$ such that for all $q \geq \varepsilon$ $k(q) = C_\varepsilon S(\varepsilon)^{\frac{1}{\alpha}} S(q)^{-\frac{1}{\alpha}}$. Furthermore, the implied measure is infinite, i.e., $\mu(\mathcal{Q}) = \infty$, or equivalently, $S(q) \to_{q \to 0} \infty$.

3.3 Properties of CD task technology and further extensions

CD task technology in (24) exhibits two key properties: 1) it has the potential for nongeneric microfoundations and 2) it can (approximately) account for firm/sectoral level heterogeneity of labor shares. We discuss these properties below and in the process sketch how to extend our model in various directions.

Nongeneric microfoundations

Capital requirement per task exhibits power law tails with index $\alpha + 1$—which is easy to see by evaluating the probability that capital requirement $\mathbf{k} = k(q) = q^{\frac{1}{\alpha}}$ associated with a randomly selected task is above some fixed value k conditionally on being above some base value $k_0 < k$:

$$Pr\left(\mathbf{k} \geq k | \mathbf{k} \geq k_0\right) = Pr\left(q \geq k^{\alpha} | q \geq k_0^{\alpha}\right) = \left(\frac{k}{k_0}\right)^{-\alpha}. \quad (27)$$

This property is appealing because power law arises spontaneously in nature, implying that the CD task technology gives hope of being nongeneric (Newman, 2004).

As shown in the literature, many elementary stochastic processes yield Pareto distribution (Gabaix, 2009; Newman, 2004). While it is beyond the scope of the current paper to provide fully fledged microfoundations for CD task technology, we highlight here our framework's potential in this direction by sketching a concrete mechanism that delivers such a result and, at the same time, operationalizes the notion of random task churning. We focus on the conditional distribution of capital requirement k in (27). (By Lemma 4, one can always transform the task space so that g is Pareto. The next section discusses why obtaining conditional (tail) Pareto pdf is sufficient.)

The setup is as follows. Suppose there is a countable number of feasible types of goods that arrive in the economy at a fixed Poisson rate $\mu > 0$ each, implying that their arrival time is an exponentially distributed random variable τ with parameter μ. Furthermore, suppose that the arrival of each good $i = \{1, 2, 3...\}$ brings a small mass of new tasks. (For now suppose these tasks are new, do not overlap across goods, and goods/tasks never come out of use. We return to this at the end.)

After a new task comes online, let the productivity of capital within this task be $k_0^{-1} > 0$, so that capital requirement is k_0. Suppose capital productivity grows at a fixed rate $\gamma > 0$ after entry; that is, after t periods a given task is online, capital requirement for this task is $k_0 \exp\left(-\gamma t\right)$. (We discuss at the end what happens when γ is an i.i.d. random variable.)

Deflating capital requirements by the average growth factor associated with capital productivity in the economy, $\exp\left(\gamma t\right)$, implies that the *deflated* capital requirements grow *before* they come online and they are constant thereafter. As a result, capital requirements across the *active* tasks are realizations of a random variable $k_0 \exp\left(\gamma \tau\right)$.

One of the fundamental properties that gives rise to Pareto distribution in many contexts is that exponential growth stopped after an exponentially distributed stopping time gives rise to a Pareto distribution (Newman, 2004).[28] In fact, a simple calculation shows that the probability

[28]For $\alpha = 1/3$, the unit of time must be large to avoid rapid introduction of goods into the economy.

$k_0 \exp(\gamma\tau)$ exceeds some fixed value k is

$$Pr\left(k_0 e^{\gamma\tau} \geq k\right) = Pr\left(\tau \geq \gamma^{-1}\log\frac{k}{k_0}\right) = e^{-\frac{\mu}{\gamma}\log\frac{k}{k_0}} = \left(\frac{k}{k_0}\right)^{-\frac{\mu}{\gamma}}. \tag{28}$$

Can γ be an i.i.d. random variable specific to a task, or can we have goods and tasks coming online and offline at some Poisson rate? The answer is affirmative. First, capital requirements for growth-deflated tasks that go offline and come back online at some fixed Poisson rate will exhibit a stationary *random growth process* implying cycles of growth and contraction that disperse deflated capital requirements. Second, i.i.d. γ will similarly induce a stationary *random growth process* on the task level (after deflating by average growth). *Random growth process* is known to give rise to Pareto distribution (Gabaix, 2009). Finally, accommodating overlapping tasks featuring higher productivity growth is also possible as long as the i.i.d. assumption of random growth can be maintained.[29]

Firm/sector heterogeneity of labor shares

As shown at the end of Proposition 3, the measure underlying CD task technology must be infinite ($\mu(\mathcal{Q}) = \infty$). What makes the measure infinite here is the exploding mass of tasks centered around $q = 0$. Since $k(q) \rightarrow_{q\rightarrow 0} 0$, production in the economy is still feasible, but capital is essential in production.[30] (Labor is also essential because $k(q) \rightarrow_{q\rightarrow\infty} \infty$.)

The fact that the measure of tasks is infinite is problematic because it implies that density g cannot be normalized to a pdf. A probabilistic representation of technology is useful to account for the firm- and sectoral-level heterogeneity in labor shares observed in the data while preserving the aggregation properties exhibited by the technology under our representative firm framework. In particular, had the measure been finite, we could equivalently recast our model economy as comprising a finite mass of heterogeneous firms that draw a finite number of tasks each from some common pdf underlying technology T, and yet the aggregate production function would be the same as in our representative firm model because all these firms would be drawing random tasks from the same pdf.[31] An extension along these lines is not

[29]For additional examples and the discussion of random growth, see Online Appendix G.

[30]Note that, for any $\varepsilon > 0$, we have $\int_{q_0}^{q_0+\varepsilon} q^{-2} \rightarrow_{q_0\rightarrow 0} +\infty$.

[31]As an example of a fully-fledged growth model with firm heterogeneity, consider the setup along the lines of Atkeson and Kehoe (2007). Let $\{T_t\}_{t=0}^{\infty}$ be the sequence of technology implying balanced growth in our

possible when the measure is infinite because it would require an infinite mass of firms, which is unreasonable. This shortcoming is related to the criticism of the existing microfoundations of the CD function, and it appears that it applies to our microfoundation as well.[32]

As we show in Online Appendix C, there is a straightforward way of addressing this issue. The idea is to relax the requirement that growth is *exactly* balanced—given the fact that the data is measured with an error and the Kaldor facts are statistical facts based on a finite sample of data. If so, an approximate balanced growth path would suffice to account for the Kaldor facts in a statistical sense, and this requirement is delivered in spades by the following domain-truncated version of the CD task technology:

$$\left\{k\left(q\right)=Z^{-1}\left(q+q_0\right)^{\frac{1}{\alpha}}\frac{1-\alpha}{\alpha},g\left(q\right)=A^{-1}\left(q+q_0\right)^{-2}\right\}, \tag{29}$$

where $q_0>0$ is the approximating parameter given $q_0=0$ yields (24). The underlying density of k is the Pareto pdf with index $\alpha+1$. An equivalent version of this technology is to truncate the domain to $[q_0,\infty)$ and use T^{CD} defined in (24).

Online Appendix C shows that this technology exhibits approximately the same balanced growth path for sufficiently low q_0 as the canonical CD technology and it further converges to the CD model's balanced growth path over time. Factor shares are approximately constant and they similarly converge to constant shares over time. Additionally, the above truncated version exhibits an appealing property implying that capital will not be used in production until a threshold productivity level is reached (in terms of Z or A). This property can provide task-based microfoundations for the model of industrial revolution by Hansen and Prescott (2002), or even models of poverty traps driven by learning-by-doing externality associated with using capital in production (Romer, 1986).

representative firm economy. Suppose that a subset of households called entrepreneurs has the technology to draw $N\in\{1,2,3...\}$ tasks at some fixed cost from the current $T_t=\{...,g_t\}$ technology according to pdf g_t and can establish firms. Each firm uses capital and labor to produce output by completing its own tasks in a cost minimizing way, and output is y_i^v for a given firm i, where $0<v<1$ is the span of control and y_i is the number of times all tasks are completed by firm i. Suppose that the firm distribution evolves, with old firms being retired as new firms draw technology from the improving technology frontier T_t. In such a setup, the aggregation properties exhibited by T_t would largely carry over to the heterogeneous firm setup after accounting for the effect of a limited span of control and positive profits in equilibrium.

[32]See footnote 9.

4 Effects of IT-powered automation on labor share

We next use our theory to show how the emergence of advanced IT adopted by profit-maximizing firms results in a complexity-biased technical change that lowers the labor share. The key feature of this model is that firms can use IT to "compress" the task load required to produce a machine (capital) specific to a given task at the expense of completing a fixed measure of some other tasks, resulting in an endogenous transformation of production technology across the economy.

The inspiration for this model came from the monograph by Brynjolfsson and McAfee (2014). Our reading of their work lead us to the interpretation that completing tasks generally requires physical power and cognitive input(s) to direct that power. Viewed from this perspective, while capital naturally provides mechanical power, automation requires a costly conversion of an oftentimes unstructured and cognition-intensive environment to a structured and cognition-free environment to fully remove labor from the equation. This is what brings complexity to automation and this is what makes it costly. The key idea captured by our model of IT is that such tasks can be automated using IT at a lower and more proportionate cost because IT equips capital with a form of "artificial cognition."

4.1 Extended model of production

The extended setup explicitly links capital to the task space and adds IT technology.

Task-specific capital

Define a machine of type q as a lasting embodiment of tasks, which, if completed once, can be used repeatedly to complete task q through the use of this machine until a Poisson event with arrival rate δ ends its useful lifetime. The technology to produce a machine of type q involves complexity space $\mathcal{Q}$ and a q-specific density $\tilde{g}_q(\hat{q})$ of tasks that, as in the baseline model, must be completed by either capital or labor. As a result, technology in the extended model is described by only the density function; that is, in capital goods producing sector(s), it is $\tilde{T}_q = \{\tilde{g}_q\}$, and in goods producing sector it is $T = \{g\}$. In particular, the schedule $k(q)$ becomes endogenous and it pertains to the (real) purchase price of a machine of type q.

The production of a task q-specific machine is assumed to happen in an instant of time and a sector of firms that produce machines is assumed to make zero profits. Accordingly, $k(q)$ satisfies the fixed point:

$$Pk(q) := \min_{\{\mathcal{Q}_k, \mathcal{Q}_l\} \in \mathscr{P}} \left\{ r \int_{\hat{q} \in \mathcal{Q}_k} k(\hat{q})\, \tilde{g}_q(\hat{q})\, dv + w \int_{\hat{q} \in \mathcal{Q}_l} \tilde{g}_q(\hat{q})\, dv \right\}. \tag{30}$$

As in the baseline model, the machine-producing firm seeks a measurable partition $\mathscr{P}$ of tasks that minimizes costs. By definition, tasks assigned to capital are in set $\mathcal{Q}_k$. The user cost of a machine is $rk(q)$ to maintain consistency with the baseline model, and in Online Appendix E we show that it is given by $r_t = P_t(1 + \rho - (1 - \delta)\gamma_{r,t})$, where $\gamma_{r,t}$ is the growth rate of $P_t k_t(q)$ at t. Tasks assigned to labor are in set $\mathcal{Q}_l$ and are treated analogously. (In terms of notation, note the distinction between the task that a machine is designed to complete, denoted by q, and the tasks that must be completed to produce this machine, denoted by $\hat{q}$.)

Next, we specialize this setup to focus attention on the balanced growth path technology, which we achieve by imposing the following assumption:

Assumption 2. $\tilde{g}_q(\hat{q}) = \lambda(q)\, g(\hat{q})$, *where* $g(\hat{q}) = A^{-1}\hat{q}^{-2}$ *and* $\lambda(q) = Z^{-1}q^{\frac{1}{\alpha}}$.

The above assumption implies that the density of tasks needed to produce a machine of type q involves some *base density* $g(\hat{q}) = A^{-1}\hat{q}^{-2}$—which we already know yields CD production function (see Example 1)—and a *task load* function $\lambda(q)$ that simply scales it. Intuitively, by that assumption, tasks that are more complex simply involve more tasks rather than differently distributed complexity of tasks.

The proposition below shows that this setup yields the capital requirement function associated with the CD task technology in (24). Accordingly, the production function is CD in the capital-producing sector(s) and hence also in the consumption sector as long as $g(q) = A^{-1}q^{-2}$ (as in Example 1).

Proposition 4. *The production function in the capital q-producing sector is Cobb–Douglas of the form:*

$$Y_q(K, L) = Zq^{-\frac{1}{\alpha}} A \left(Z \left(\frac{c(w, r)}{P} \right)^{-1} \frac{\alpha}{1-\alpha} K \right)^{\alpha} L^{1-\alpha}, \tag{31}$$

and the endogenous capital requirement function is

$$k(q) = \underbrace{Z^{-1} q^{\frac{1}{\alpha}}}_{=\lambda(q)} \frac{c(w,r)}{P}, \tag{32}$$

where $c(w,r)$ is the unit cost of production in the capital producing sector associated with the base technology $T = \{g\}$. If, in addition, $g(q) = A^{-1} q^{-2}$, the production function in the goods sector is also Cobb–Douglas and takes the form:

$$Y(K,L) = A\left(Z\left(\frac{c(w,r)}{P}\right)^{-1} \frac{\alpha}{1-\alpha} K\right)^{\alpha} L^{1-\alpha}. \tag{33}$$

IT-revolution in automation

We are now ready to lay out our task-based model of IT. The key idea here is that IT becomes available to the economy from a certain point in time onwards and was not available before (it is a breakthrough discovery akin to electricity or steam power). As mentioned, IT is a technology that allows capital-producing firms to "compress" the task load required to produce a machine (capital) at the expense of completing a fixed measure of some other tasks—with firms optimally choosing the scale of compression. This results in a transformation of production technology in the entire economy. The costs associated with operating IT technology are assumed to be borne each time a machine is produced. The formal definition of IT breakthrough is as follows:

Definition 2. A *breakthrough IT automation technology* comprises i) a task technology $T^{IT} = \{g^{IT}\}$ and ii) an associated strictly decreasing compression function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$, such that T^{IT} used $n \geq 0$ times "compresses" the task load in the production of machines of type $q \in \mathcal{Q}$ by factor $\kappa(n)$, implying transformed task density is $\tilde{g}_{q,n}(\hat{q}) = \kappa(n)\lambda(q) g(\hat{q})$. (Units are sufficiently small to justify the use of $n \in \mathbb{R}_+$.)

Assumption 3. $\kappa(n) = \kappa_0 \beta^{-1} n^{-\beta}$*, where* $0 < \beta < \alpha^{-1} - 1$ *and* $\kappa_0 > 0$ *are scalars.*

Assumption 3 specializes the functional forms to ensure that in the long run the arrival of IT technology is consistent with balanced growth. This specific functional form implies that a capital-producing firm can use IT to reduce the task load λ_q by $\beta \times 100$ percent at the expense of completing a fixed measure of tasks associated with a single application of T^{IT}—which from the firm's point of view is a fixed cost—and later will be denoted by b. The crucial assumption is the scalability of IT technology; that is, the fact that the firm can repeat the process n times to reduce the task load by $\beta \times 100$ percent n times.

In what follows, we examine the effects of the arrival of this technology in the specialized CD setup underlying Proposition 4. In particular, we assume the initial capital cost schedule is given by $k_0(q) = q^{\frac{1}{\alpha}}$.

4.2 Analysis and results

Based on Proposition 4, the breakthrough IT technology implies that the post-transformation capital price schedule is

$$k(q) = \min\left\{q^{\frac{1}{\alpha}}, \min_{n \geq 0} \kappa(n)\, q^{\frac{1}{\alpha}} + bn\right\}. \tag{34}$$

This follows from the fact that, initially, we had $k_0(q) = q^{\frac{1}{\alpha}}$, and the reduction of the task load scales down proportionally $\kappa(n)$—as noted under equation (32) in Proposition 4. As noted, the cost of completing tasks associated with IT is denoted by $b > 0$, which we do not need to specify explicitly because, from an atomless firm's point of view, this is just a constant. Of course, in equilibrium, b is linked to factor prices, resulting in a fixed point on the economy-wide level as the technology is applied.[33]

The inner minimization problem in (34) implies that the optimum occurs at $n^* = \left(\frac{\kappa_0}{b} q^{\frac{1}{\alpha}}\right)^{\frac{1}{1+\beta}}$. As we show in the appendix, the benefit from applying the breakthrough technology is strictly increasing in q, and hence the above problem implies a cutoff value $q_{\min} > 0$ such that the technology is applied only into the production of machines of type q above that cutoff. Using these results, the post-transformation schedule becomes

$$k(q) = \begin{cases} q^{\frac{1}{\alpha}} & q \leq q_{\min} \\ C q^{\frac{1}{\alpha}\frac{1}{1+\beta}} & q \geq q_{\min} \end{cases}, \tag{35}$$

[33]Our analysis below will show that solving for this fixed point would only reinforce our results because IT technology would be applied to a larger subset of the complexity space $\mathcal{Q}$.

where C is some scalar ensuring continuity; that is, C is such that $q_{\min}^{\frac{1}{\alpha}} = Cq_{\min}^{\frac{1}{\alpha}\frac{1}{1+\beta}}$.

It is clear from Example 1 that had $k(q) \propto q^{\frac{1}{\alpha}\frac{1}{1+\beta}}$ been applied globally to production of goods, we would have readily obtained a CD aggregate production function featuring a labor share given by $LS_1 = LS_0 - \beta\alpha$ instead of the initial $LS_0 = 1 - \alpha$. But the outer min operator implies that the breakthrough technology will never be adopted globally, resulting in a partial transformation of its upper portion (above $q_{\min}$). However, the impact of this feature—identical to that of the truncation of the domain introduced by $q_0 > 0$ in (29) and analyzed in Online Appendix C—turns out to vanish with the growth with automation. As a result, the decline in the labor share in this economy—while smaller initially—in the limit converges to $LS_1 = LS_0 - \beta\alpha$, where $LS_0 = 1 - \alpha$ is the initial labor share, and the economy converges to a new balanced growth path with a lower labor share. We summarize the results in the proposition below.

Proposition 5. *Suppose that the labor share is $LS_0 = 1 - \alpha$. The post-breakthrough labor share converges to $LS_1 = LS_0 - \alpha\beta$ as the economy further automates so that $q_{min}/q^* \to 0$.*

Discussion of results

Our model of IT is stylized and does not capture all the nuances that anecdotal evidence on the nature of automation may throw at us. However, as we see it, it is a step forward toward establishing concrete properties that IT technology must exhibit to be labor-share displacing in a concrete environment featuring a descriptively realistic notion of automation. As shown, a breakthrough IT automation technology lowers the labor share in our model because it exhibits three qualitative properties: 1) universal applicability, 2) task measure compression, and 3) scalability. The first property means that the breakthrough technology can be applied to most types of capital-producing tasks, implying a global impact across tasks. The second property means that IT effectively "compresses" the density of tasks in the production of capital goods at a fixed input of completing some other tasks. The third feature allows for its scalable application. All these properties are critical for technology to be labor-share displacing, but the exact functional forms are not.

Through the lens of our theory, then, the question of whether the modern IT-powered wave of automation is labor share displacing comes down to the question of whether the kind of enabling IT technologies that power the current wave of automation exhibit these properties. While answering this question conclusively is difficult, to the extent that most complex tasks involve sophisticated cognitive input(s), the above features are plausible when IT is used to provide "artificial cognition," which, for example, Brynjolfsson and McAfee (2014) consider to be one of the defining features of the current automation wave. In particular, Brynjolfsson and McAfee (2014) see automation as involving a combination of mechanical power and cognition to direct that power towards productive uses. They also see the previous phases of industrial revolution, which they refer to as the "first machine age," as largely using crude mechanical workarounds for the latter requirement. If we accept their view, then, the "first machine age" must have left us with a range of tasks that are prohibitively costly to automate without cognition, which immediately implies that the impact of the invention of "artificial cognition" must be complexity biased as in our model. As we see it, under that view, the question is more about the scope and the maturity of the existing technologies, as well as the exact nature of its impact.

As a caveat to our analysis, it is important to stress that the arrival of IT also brings offsetting structural changes that we do not model here. For example, IT has brought computer games that created labor-intensive jobs for a large number of highly trained programmers, which according to our earlier analysis would be a form of complexity-augmenting technical change. While the presence of such phenomena in the data is undeniable, their quantitative relevance in sustaining the labor share is less certain.

Lastly, the additional and novel aspect of the current wave of automation is digitization of human activity brought about by information and communication technology (ICT). Digitization enables an increased role of software in automating tasks (e.g., Uber services enabled by smartphones). Software can be particularly impactful because it is (largely) nonrival and for this reason the drop in the cost of performing a task by capital can be substantial, since on the digital domain nothing else is required and the price of hardware appears to be converging to a negligible level. The effect of digitization and software-based automation can thus have the effect of complexity-reducing technical change in the language of Section 2—which we have shown is *always* labor displacing. While digitization is an important aspect of technical change, our intention was to develop a more general framework that captures the impact of IT across a broader range of tasks, including those that require physical power to be completed.

5 Conclusions

We provided a general characterization of how technical progress driving automation and automation itself affect the division of income between capital and labor in the context of long-run growth. We found that forces that "diffuse" the effects of technical progress are conducive to constancy of the labor share as long as labor's productivity is not too low on the range of automated tasks. We argued that the diffused nature of technical progress is a natural consequence of random productivity-enhancing innovations and task churning. We have shown that a diffused and complexity-unbiased progress is consistent with automation observations and past growth experiences (the Kaldor facts), in the process obtaining a new microfoundation for the Cobb–Douglas production function. While we have argued that technical change of a more diffused nature can be favorable to labor, we also found that the modern wave of IT-powered automation involves a universal technology that, through the lens of our theory, can be complexity-biased and hence labor-share displacing.

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Appendix: Omitted proofs

Proofs of Lemma 1 and Lemma 2 The proofs are technical and not essential for what follows next. They can be found in Online Appendix B.

Proof of Lemma 3 The proof of the first, technical part of the lemma establishing differentiability is not essential for what follows next and it is in Online Appendix B. We focus here on the derivation of the marginal products. **Part II:** Consider fixed $K > 0$ and fixed $L > 0$. By Lemma 2, we know that positive inputs are associated with positive output $Y > 0$ and some positive cutoff $q^* > 0$. (Differentiability of these objects almost everywhere on the domain is established in Part I of the proof.) Consider an infinitesimal increment $dK > 0$ that adds to the capital stock. (We make it positive to ease the exposition but the reasoning applies to negative dK.) Given the corresponding change in the optimal cutoff $dq^* > 0$ to accommodate increment $dK > 0$ under Lemma 2, we can calculate $dY > 0$ from the labor input equation in that lemma (i.e., $L = Y \int_{q^*}^{\infty} 1 d\mu$), which gives

$$dL = (Y + dY) \int_{q^*+dq^*}^{\infty} 1 d\mu - Y \int_{q^*}^{\infty} 1 d\mu = 0. \tag{36}$$

By continuity property of Lebesgue integrals,[34] the equation can be solved. Accordingly, from the above equation, we calculate

$$Y = dY \frac{\int_{q^*+dq^*}^{\infty} 1 d\mu}{\int_{q^*}^{q^*+dq^*} 1 d\mu}. \tag{37}$$

By Lemma 2, we know

$$dK = \underbrace{(Y + dY) \int_{0}^{q^*+dq^*} k d\mu}_{K+dK} - \underbrace{Y \int_{0}^{q^*} k d\mu}_{K}. \tag{38}$$

[34]Lacking a textbook reference, we prove this property in the Online Appendix F.

Plugging in for Y from (37) above and dividing both sides by dY yields

$$\frac{dK}{dY} = \int\limits_{0}^{q^*+dq^*} k d\mu + \frac{\int_{q^*+dq^*}^{\infty} 1 d\mu}{\int_{q^*}^{q^*+dq^*} 1 d\mu} \int\limits_{q^*}^{q^*+dq^*} k d\mu. \tag{39}$$

(Recall that $k(q^*) > 0$, implying $dY > 0$; this is ensured by the fact that we started with $K > 0$ and $k(q)$ is an increasing function.) By the analog of the intermediate value theorem for Lebesgue integrals,[35] we can pick a real number $k(q^*) \leq \hat{k} \leq k(q^* + dq)$ such that $\int_{q^*}^{q^*+dq^*} k d\mu = \hat{k} \int_{q^*}^{q^*+dq^*} 1 d\mu$. Accordingly, the above equation simplifies to

$$\frac{dK}{dY} = \int\limits_{0}^{q^*+dq^*} k d\mu + \hat{k} \int\limits_{q^*+dq^*}^{\infty} 1 d\mu. \tag{40}$$

Again, by Lemma 2, and by definition of dK, dY, we know $\int_0^{q^*+dq^*} k(q)\, d\mu = \frac{K+dK}{Y+dY}$ and $\int_{q^*+dq^*}^{\infty} 1 d\mu = \frac{L+dL}{Y+dY}$, where from above we have $dL = 0$. Plugging in to the above equation, we obtain $\frac{dK}{dY} = \frac{K+dK}{Y+dY} + \hat{k} \frac{L}{Y+dY}$. Given dK, dY are infinitesimal, we can ignore them in the first two terms on the left-hand side and use $\frac{K}{Y}$ in place of $\frac{K+dK}{Y+dY}$. Multiplying both sides by $\frac{Y}{L}$, and using the fact that $k(q^*) \leq \hat{k} \leq k(q^* + dq)$ (a.e.)—which gives $\hat{k} \to k(q^*)$ as $dq^* \to 0$—we obtain $\frac{dK}{dY}\frac{Y}{K} = 1 + k(q^*) \frac{L}{K}$ (a.e.)—the inverse of output elasticity with respect to capital. The elasticity of output with respect to labor follows from Euler's law (i.e., $\frac{\partial Y}{\partial K} K/Y + \frac{\partial Y}{\partial L} L/Y = 1$, by constant returns to scale). Q.E.D.

Proof of Lemma 4 By Lemma 2, we need to show that, if $K, L, \hat{Y}$ satisfy $K = \hat{Y} \int_0^{q^*} k_2(q)\, d\mu_2$ and $L = \hat{Y} \int_0^{q^*} k_2(q)\, d\mu_2$, for some $q^* \in \mathcal{Q}$, we can find $q^{**} \in \mathcal{Q}$ such that $K = \hat{Y} \int_0^{q^{**}} k_1(q)\, d\mu_1$ and $L = \hat{Y} \int_0^{q^{**}} k_2(q)\, d\mu_1$. This follows trivially by the change-of-variables theorem,[36] which, for example, in the case of capital implies:

$$\int\limits_{[0,q^*]} k_2 d\mu_2 = \int\limits_{[0,q^*]} k_1 \circ f^{-1} d\mu_2 = \int\limits_{[0,f^{-1}(q^*)]} k_1 d\left(\mu_2 \circ f^{-1}\right) = \int\limits_{[0,f^{-1}(q^*)]} k_1 d\mu_1.$$

[35]See Wheeden and Zygmund (1977), Corollary 5.31, p. 75.
[36]See Bogachev (2007) Theorem 3.6.1 (p. 190).

(For Riemann integrals this result follows from the "u-substitution" method. The proof is more involved for Lebesgue integrals.)

Proof of Proposition 1 Given $\frac{L}{Y}\left(q_{\varepsilon}^{*},\varepsilon\right):=\int_{q_{\varepsilon}^{*}}^{\infty}\left(g_{0}\left(q\right)+\varepsilon\Delta g\left(q\right)\right)dq$, Lebesgue differentiation theorem yields

$$DE:=\frac{\frac{d\left(w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}}{w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}=h\left(q_{0}^{*}\right)\frac{\int_{q^{*}}^{\infty}\frac{\Delta g(q)}{g(q)}d\mu}{g\left(q^{*}\right)}-h\left(q^{*}\right)\frac{\Delta k\left(q^{*}\right)}{k'\left(q^{*}\right)},\tag{41}$$

where to obtain the last equality we used (10) and the definition of hazard rate: $h\left(q^{*}\right):=\frac{g(q^{*})}{\int_{q^{*}}^{\infty}g(q)dv}$. Using 1) $k\left(q^{*}\right)=\frac{w}{r}$, 2) $\int_{q^{*}}^{\infty}\frac{\Delta g(q)}{g(q)}d\mu\equiv\int_{q^{*}}^{\infty}\Delta g\left(q\right)dv=-\int_{0}^{q^{*}}\Delta g\left(q\right)dv$, and simplifying, we obtain the equations for DE as stated in the proposition. The price P_{ε} satisfies $P_{\varepsilon}-\left(w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)+r\frac{K}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)\equiv0$, which after differentiation gives

$$PE:=-\frac{dP_{\varepsilon}}{d\varepsilon}|_{\varepsilon=0}\frac{1}{P}=-\left(\frac{d\left(w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}|_{\varepsilon=0}+\frac{d\left(r\frac{K}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}|_{\varepsilon=0}\right)\frac{1}{P}.\tag{42}$$

Given $\frac{K}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right):=\int_{q_{\varepsilon}^{*}}^{\infty}\left(k\left(q\right)-\varepsilon\Delta k\left(q\right)\right)\left(1+\varepsilon\frac{\Delta g(q)}{g(q)}\right)d\mu$, we thus obtain

$$\begin{aligned}PE=&-(w\int_{q^{*}}^{\infty}\frac{\Delta g\left(q\right)}{g\left(q\right)}d\mu-wg\left(q^{*}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0}-r\int_{0}^{q^{*}}\Delta k\left(q\right)d\mu\\&+r\int_{0}^{q_{0}^{*}}\frac{\Delta g\left(q\right)}{g\left(q\right)}k\left(q\right)d\mu+rk\left(q^{*}\right)g\left(q^{*}\right)\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0})\frac{1}{P}.\end{aligned}\tag{43}$$

Since $q^{*}\equiv q_{0}^{*}$ satisfies $rk\left(q^{*}\right)\equiv w$, the terms involving "$\frac{dq_{\varepsilon}^{*}}{d\varepsilon}|_{\varepsilon=0}$" drop out. After basic manipulations, we obtain the first expression for PE in the proposition. Equivalently, note, we can write PE as

$$PE=-\underbrace{\frac{w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}{P_{0}}|_{\varepsilon=0}}_{LS}\left(\frac{\frac{d\left(w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}}{w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}+\frac{\frac{d\left(r\frac{K}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)\right)}{d\varepsilon}}{w\frac{L}{Y}_{\varepsilon}\left(q_{\varepsilon}^{*}\right)}|_{\varepsilon=0}\right)\tag{44}$$

to obtain

$$PE = LS\left(\frac{-\int_0^{q^*}\Delta k\left(q\right)d\mu+\left(\frac{w}{r}\int_{q^*}^{\infty}\frac{\Delta g(q)}{g(q)}d\mu+\int_0^{q^*}\frac{\Delta g(q)}{g(q)}k\left(q\right)d\mu\right)}{\int_{q^*}^{\infty}g\left(q\right)dv}\frac{r}{w}\right). \tag{45}$$

Similarly, using 1) and 2) above, and simplifying, we obtain the second equation for PE. It is now straightforward to derive $\frac{d\log S(q_\varepsilon^*)}{d\epsilon}|_{\varepsilon=0}$ by using $\frac{dq_\varepsilon^*}{d\varepsilon}|_{\varepsilon=0}$ in (10). We omit the details. (Note that the obtained expressions apply almost everywhere on $\mathcal{Q}$.) Q.E.D.

Proof of Proposition 2 The first part of the proof is in text. The proof that $\gamma := \gamma_K = \gamma_Y$ can be found in Jones and Scrimgeour (2008). We finish the proof by establishing the following omitted steps: 1) $g_0\left(q\right) = g_0 q^{-\zeta-1}$; 2) $\zeta = \frac{\gamma_Y+\gamma_g}{\gamma_{q^*}}$, $\theta = \frac{\gamma-\gamma_k}{\gamma_{q^*}}$, which guarantees $\theta < \zeta$, and also requires $\gamma - \gamma_k > 0$. (The statement of the proposition also includes 3) $\gamma = \gamma_g - \alpha\gamma_k$, which we omit from here because it follows from the next corollary.) To prove step 1), we note that

$$\frac{d\left(\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq\right)/dt}{\left(\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq\right)} = -\gamma, \tag{46}$$

since $\bar{L}$ is constant and, by Lemma 2, $\bar{L} = Y_t\int_{q^*(t)}^{\infty}g_0\left(q\right)e^{\gamma_g t}dq$. Differentiating this expression, dividing both sides by q^*, and using the fact that $\dot{q}^*/q^* = \gamma_{q^*}$, we obtain

$$\frac{g_0\left(q^*\right)}{\int_{q^*}^{\infty}g_0\left(q\right)dq} = \frac{\gamma+\gamma_g}{\gamma_{q^*}}\frac{1}{q^*}. \tag{47}$$

This equation must apply to all q^* (starting from initial q_0^*), since q^* is assumed to be growing at a strictly positive rate. Accordingly, it is an identity on that range. Define $f\left(q^*\right) := \int_{q^*}^{\infty}g_0\left(q\right)dq$ and note that f is a.e. differentiable with $f'\left(q^*\right) = -g_0\left(q\right)$ by Lebesgue's differentiation theorem.[37] Furthermore, note we can rewrite equation (47) as $\frac{f'(q^*)}{f(q^*)} \equiv -\frac{\gamma+\gamma_g}{\gamma_{q^*}}\frac{1}{q^*}$, which is an ODE and solves to $f\left(q^*\right) = Cq^{*-\frac{\gamma_Y+\gamma_g}{\gamma_{q^*}}}$ up to constant C, which implies $g_0\left(q\right) = -f'\left(q^*\right) = C\left(\frac{\gamma+\gamma_g}{\gamma_{q^*}}\right)q^{*-\frac{\gamma+\gamma_g}{\gamma_{q^*}}-1}$. Accordingly, $g_0\left(q\right) = g_0 q^{*-\zeta-1}$, where $\zeta := \frac{\gamma+\gamma_g}{\gamma_{q^*}}$, and $g_0 = C\frac{\gamma+\gamma_g}{\gamma_{q^*}}$ is a constant. As for step 2), we have already shown that $\zeta := \frac{\gamma+\gamma_g}{\gamma_{q^*}}$. The proof that $\theta := \frac{\gamma-\gamma_k}{\gamma_{q^*}}$ follows from the analogous reasoning applied to $K_t = Y_t\int_{q^*(t)}^{\infty}k_0\left(q\right)e^{\gamma_k t}g_0\left(q\right)e^{\gamma_g t}dq$, where we know from text that $k_0\left(q\right)$ is of the form

[37] Wheeden and Zygmund (1977) Theorem 7.2 (p. 100) and Theorem 7.11 (p. 107).

$k_0(q) = k_0 q^{\theta}$, where $\theta > 0$, or else $T \notin \mathcal{T}$. Accordingly, $\int_{q^*(t)}^{\infty} k_0(q) e^{\gamma_k t} g_0(q) e^{\gamma_g t} dq$ must be a constant, and θ can be obtained by solving

$$\frac{d\left(\int_{q^*(t)}^{\infty} k_0(q) e^{\gamma_k t} g_0(q) e^{\gamma_g t} dq\right)/dt}{\int_{q^*(t)}^{\infty} k_0(q) e^{\gamma_k t} g_0(q) e^{\gamma_g t} dq} = 0. \tag{48}$$

The approach to solve this equation is analogous. The solution is $k_0(q) g_0(q) = C\left(\frac{\gamma_g+\gamma_k}{\gamma_{q^*}}\right) q^{*\frac{\gamma_g+\gamma_k}{\gamma_{q^*}}-1}$. Using the formula for $g_0(q)$, and the fact that $k_0(q) = k_0 q^{\theta}$, we obtain $\theta = \frac{\gamma-\gamma_k}{\gamma_{g^*}}$. Q.E.D.

Proof of Corollary 1 Using Lemma 2, we have: $\frac{L}{Y} = g_0 \int_{q^*}^{\infty} q^{-\zeta-1} dq = g_0 \frac{1}{\zeta} q^{*-\zeta}$ and $\frac{K}{Y} = k_0 g_0 \int_0^{q^*} q^{\theta} q^{-\zeta-1} dq = \frac{k_0 g_0}{\theta-\zeta} q^{*\theta-\zeta}$. Eliminating q^*, yields $Y = \frac{1}{g_0} \zeta^{1-\frac{\zeta}{\theta}} \left(\frac{\theta-\zeta}{k_0}\right)^{\frac{\zeta}{\theta}} K^{\frac{\zeta}{\theta}} L^{1-\frac{\zeta}{\theta}}$, which is well defined since $\theta > \zeta > 0$, $g_0 > 0$, $k_0 > 0$.

Proof of Proposition 3 **Part I.** Necessity: If the relationship between Y, K and L is CD with an exponent α ($Y = AK^{\alpha} L^{1-\alpha}$, where $0 < \alpha < 1, A > 0$), the relationship between factor intensities is $\left(\frac{K}{Y}\right)^{\alpha} \left(\frac{L}{Y}\right)^{1-\alpha} \equiv \frac{1}{A}$, for some constant $A > 0$. Plugging in for factor intensities from Lemma 2, gives $\left(\int_0^{q^*} k(q) g(q) dv\right)^{\alpha} S(q^*)^{1-\alpha} \equiv \frac{1}{A}$, for any $q^* > 0$, where we must also have $S(q^*) < \infty$. By Lebesgue's differentiation theorem,[38] the derivative with respect to q^* is

$$\alpha \left(\int_0^{q^*} k(q) g(q) dv\right)^{\alpha-1} k(q^*) g(q^*) S(q^*)^{1-\alpha} \\ -(1-\alpha)\left(\int_0^{q^*} k(q) g(q) dv\right)^{\alpha} S(q^*)^{-\alpha} g(q^*) \equiv 0, \tag{49}$$

a.e., which simplifies to

$$\int_0^{q^*} k(q) g(q) dv \equiv \frac{\alpha}{1-\alpha} k(q^*) S(q^*) < \infty. \text{ (a.e.)} \tag{50}$$

[38]See Wheeden and Zygmund (1977) Theorem 7.2 (p. 100), Theorem 7.11 (p. 107), and the comment under the proof of Theorem 7.16 (p. 109). The derivative of the left-hand side is $\lim_{r\to 0} \left|\frac{\int_{[q_0,q^*+r]} k(q)g(q)dv - \int_{[q_0,q^*]} k(q)g(q)dv}{r}\right| = \lim_{r\to 0} \left|\frac{\int_{[q^*,q^*+r]} k(q)g(q)dv}{r}\right| = k(q) g(q)$ (a.e.), where the first equality follows from Theorem 5.24 and the last equality follows from Theorem 7.2, as referenced.

The original equation needs to hold up to an arbitrary constant $A > 0$, and so we do not lose sufficiency here by differentiating the original expression. We differentiate the above equation again with respect to q^*. The involved functions on the right-hand side, note, are monotone, implying they are differentiable (a.e.).[39] Differentiation gives

$$k\left(q\right)g\left(q\right)=\frac{\alpha}{1-\alpha}k'\left(q\right)S\left(q\right)-\frac{\alpha}{1-\alpha}k\left(q\right)g\left(q\right), \tag{51}$$

where we drop the asterisk over q because this distinction is no longer needed. Simplifying, we obtain $\alpha\frac{k'(q)}{k(q)} = h(q) := \frac{g(q)}{S(q)}$. **Part II.** Sufficiency: In addition to conditions identified above, we are seeking a particular solution with a constant C in such that (51) holds. We use $q^* \to 0$ to identify it. It is clear from (50) that $k(q^*) \to_{q^*\to 0} 0$ is the case, since integrability of k on $\mathcal{Q}$ implies $\int_0^{q^*} k(q^*)g(q^*) \to_{q^*\to 0} 0$. Accordingly, both the right-hand side in (51) ($\lim_{q^*\to 0}\frac{\alpha}{1-\alpha}k\left(q^*\right)S\left(q^*\right) = 0$) and the left-hand side ($\lim_{q^*\to 0}\int_0^{q^*} k\left(q\right)g\left(q\right) = 0$) must vanish in the limit.[40] Concluding, we need $k\left(q\right) \to_{q\to 0} 0$ and $\lim_{q^*\to 0} k\left(q^*\right)S\left(q^*\right)$. **Part III.** Properties: The unique solution to ODE in the lemma exists and is of the form $k\left(q\right) = c\left(\varepsilon\right)\exp\left(\alpha^{-1}H_\varepsilon\left(q\right)\right)$, where $H_\varepsilon\left(q\right) := \int_\varepsilon^q h\left(u\right)du$, and where $c\left(\varepsilon\right)$ is a constant implied by the solution of ODE that will need to take a particular value for a given $\varepsilon > 0$ (it is thus a function of the chosen $\varepsilon > 0$).[41] The relationship between the cumulative hazard function and the distribution G is given by

$$H_\varepsilon\left(q\right)=\int_\varepsilon^q\frac{g\left(u\right)}{S\left(u\right)}du=\int_\varepsilon^q\frac{1}{S\left(u\right)}\underbrace{\left(-\frac{d}{du}\left(S\left(u\right)\right)\right)}_{\equiv g(u)}du=\ln\left(\frac{S\left(\varepsilon\right)}{S\left(q\right)}\right). \tag{52}$$

Accordingly, for any given $\varepsilon > 0$, we have $k\left(q\right) = c\left(\varepsilon\right)S\left(\varepsilon\right)^{\frac{1}{\alpha}}S\left(q\right)^{-\frac{1}{\alpha}}$, for all $q \geq \varepsilon$. Since we cannot have $k \equiv 0$ by Assumption 1, the previous equation implies $k\left(q\right) \to_{q\to\infty} \infty$, since $S\left(q\right) \to_{q\to\infty} 0$.[42] **Part IV.** By contradiction, assume $\mu\left(\mathcal{Q}\right) < +\infty$, and note that, if so,

[39]Theorem 7.21 in Wheeden and Zygmund (1977).

[40]The last property follows from the mean value theorem and the existence of the integral.

[41]Since it is possible that $S\left(q\right) \to_{q\to 0} \infty$, we introduce a constant $\varepsilon > 0$ that bounds the initial condition away from q_0.

[42]$S\left(q\right)$ is a strictly decreasing function and bounded, which implies that this sequence must converge. The proof then follows the fact that the tail sum of any convergent series converges to zero, that is, if $\sum_{i=1}^{\infty} a_i$ converges, then $t_n = \sum_{i=n}^{\infty} a_i \to_{n\to\infty} 0$, which is a corollary from Cauchy's criterion of convergence for series. Specifically, define $a_i = \int_{q^*+i-1}^{q^*+i} 1d\mu$, note that $\sum_{i=1}^{\infty} a_i = \int_{q^*}^{\infty} 1\mu < \infty$ by Theorem 5.24 from Wheeden and

$S(q) \leq \mu(\mathcal{Q}) < +\infty$, for all $q \in \mathcal{Q}$. By the proof of this lemma (Part III above with $\varepsilon = 0$), we know that CD aggregation in this case requires: $k(q) = CS(q)^{-\frac{1}{\alpha}}$, for some constant $C > 0$. But this is a contradiction because Part II of this lemma requires $k(q) \to_{q\to 0} 0$, which is impossible to satisfy unless $k \equiv 0$, which contradicts Assumption 1. Concluding, $\mu(\mathcal{Q}) = \infty$, and by the continuity of measure, $S(q) \to_{q\to 0} \infty$.[43] Q.E.D.

Proof of Proposition 4 Note that the fixed point in (30) implies

$$Pk(q) := Z^{-1} q^{\frac{1}{\alpha}} \left(\int_0^{q^*} rk(q)\, g(q)\, dq + wS(q^*) \right), \tag{53}$$

where $S(q^*) = \int_{q^*}^{\infty} g(q)\, dq$ is the survival function associated with the base density g. Recall from the firm cost minimization that the cutoff q^* is $q^* \equiv q^*\left(\frac{w}{r}\right) = k^{-1}\left(\frac{w}{r}\right)$, which yields

$$S(q^*) \equiv S\left(q^*\left(\frac{w}{r}\right)\right) = S\left(k^{-1}\left(\frac{w}{r}\right)\right). \tag{54}$$

For the time being, assume that the fixed point $k(q)$ is such that its inverse is well defined. We will return to this. By (50) in proof of Proposition 3, the requirement that the production function in the capital q producing sector is Cobb-Douglas implies[44]

$$\int_0^{q^*} rk(q)\, g(q)\, dq \equiv \frac{\alpha}{1-\alpha} rk(q^*)\, S(q^*). \tag{55}$$

Substituting (54) and (55) into (53), we obtain $Pk(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{c(w,r)}{P}$, where $c(w,r) := S\left(k^{-1}\left(\frac{w}{r}\right)\right) \frac{w}{1-\alpha}$. We have now shown that the capital requirement function $k(q)$ is as stated up to the term $S\left(k^{-1}\left(\frac{w}{r}\right)\right) \frac{w}{1-\alpha}$, which is a constant from the firm's point of view. To see it is unit cost as stated, note that $L_q := S\left(q^*\left(\frac{w}{r}\right)\right)$ is the total labor input into the unit production of capital of type q by such a firm, and note that we can rewrite the previous equation as $c(w,r)(1-\alpha) = L_q w$. Guess and verify that the production function in the capital producing sectors is Cobb-Douglas with common share parameter α; then, by definition of α, the last

Zygmund (1977) and the properties of S, and apply the result for series.

[43] See Wheeden and Zygmund (1977) Theorem 10.1 (p. 163).

[44] To adopt this lemma to the production of capital, we associate output Y with the number of units of capital that are produced in the capital sector q and K, L with the total factor usage in the sector that produces capital of type q.

expression implies that $c(w,r)$ is the total unit cost of that firm because $L_q w$ is total labor cost. Following Example 1, it is straightforward to derive the production function. The guess is verified. Q.E.D.

Proof of Proposition 5 The first part of the proof is in text. The breakthrough technology is applied to capital producing task for all q such that $\frac{b}{\beta}\left(\frac{\kappa_0}{b}q^{\frac{1}{\alpha}}\right)^{\frac{1}{1+\beta}}\left(\left(\frac{\kappa_0}{b}\right)^{\frac{1}{1+\beta}}+1\right) \leq q^{\frac{1}{\alpha}}$. By monotonicity of this expression, it is clear there is $q_{\min} > 0$ such that the technology is applied for all $q \geq q_{\min}$ $(\beta > 0)$. Using (35) derived in text, we have: $\frac{K}{Y} \propto \left(\int_0^{q_{\min}} q^{\frac{1}{\alpha}-2}dq + C\int_{q_{\min}}^{q^*} q^{\frac{1}{\alpha}(1+\beta)-2}dq\right)$. Define the isoquant error as the difference between isoquants relative to the counterfactual $k(q) = Cq^{\frac{1}{\alpha}(1+\beta)}$ but applied globally (to all q); that is, let $\frac{K'}{Y'} = C\int_0^{q^*} q^{\frac{1}{\alpha}(1+\beta)-2}dq$, for the same constant C. After cumbersome manipulations, we obtain[45]

$$\varepsilon := \frac{|\frac{K}{Y}-\frac{K'}{Y'}|}{\frac{K'}{Y'}} = |\left(\frac{q_{\min}}{q^*}\right)^{1-\frac{1}{\alpha}} \frac{\left(C\left(\frac{q_{\min}}{q^*}\right)^{\frac{\beta}{\alpha}}(1-\alpha)-(q^*)^{-\frac{\beta}{\alpha}}(1-\alpha+\beta)(1+\beta)\right)}{(\alpha-1)}|.$$

Accordingly, $\varepsilon \to 0$ as $\frac{q_{\min}}{q^*} \to 0$, and if q^* is non-decreasing (economy automates). Consequently, the post-transformation isoquant converges uniformly to a CD isoquant and so does the labor share. The frontier isoquant at K, L supplied is what is relevant for the aggregate labor share. Q.E.D.

[45] Derivation of this formula is in the *Mathematica* notebook "Proposition_breakthrough.nb."

(Not intended for publication)
Online Appendix for "Understanding Growth Through Automation: The Neoclassical Perspective"*

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Abstract

This document contains the supplementary (online) appendix for the paper "Understanding Growth Through Automation: The Neoclassical Perspective."

Key words: Automation, labor share, Uzawa's theorem, Cobb-Douglas production function, capital-augmenting technological progress, balanced growth

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*The views expressed herein are solely those of the authors and do not necessarily reflect those of the International Monetary Fund, the Federal Reserve Bank of Philadelphia or the Federal Reserve System. Drozd (corresponding author): lukasz.drozd@phil.frb.org, Research Department, Federal Reserve Bank of Philadelphia, Ten Independence Mall, Philadelphia, PA 19106-1574; Taschereau-Dumouchel: Department of Economics, Cornell University, mt763@cornell.edu. Mendes Tavares: Research Department, International Monetary Fund, MMendesTavares@imf.org. The paper has been inspired by Guido Menzio's Society for Economic Dynamics 2021 plenary address. All errors are ours.

Appendix A: Normalization of labor requirement

Here we sketch the argument of how our assumption that the labor requirement is fixed across tasks can be thought of as a normalization in a more general setup with labor and capital requirements that differ by task but are mutually independent.

Suppose there is a separate capital and labor requirement function $k(q)$ and $l(q)$. Define $\hat{k}(q) := k(q)/l(q)$. Let q be ordered so that $\hat{k}(q)$ is increasing as assumed in text. Suppose these functions are measurable and the labor requirement is independent of (relative) capital requirement; that is, knowing $\hat{k}$ gives no information about l, implying for any q we have $\mathbb{E}\left\{l(q)\,|\hat{k}(q), q \in \mathcal{I}\right\} = \mathbb{E}\{l(q)\}$, where $\mathcal{I} = [a,b] \subset \mathcal{Q}$ is any bounded interval (we use expectation operator under a probability measure p induced on that interval; a $\sigma-$finite measure generates a conditional probability distribution on a bounded interval).

This implies that there exists a constant $C > 0$ such that for any bounded interval $\mathcal{I} = [a,b] \subset \mathcal{Q}$ we have

$$\int_{\mathcal{I}} l(q)\,d\mu = \int_{\mathcal{I}} C d\mu, \tag{1}$$

by assumption, since l is i.i.d. with respect to q (if this is not the case it would be possible to infer l from $\hat{k}$—that is, from q since $\hat{k}(q)$ is increasing—and we assume here that $\hat{k}(q)$ is strictly increasing on at least part of the domain).

Let us now normalize units of capital requirement and labor requirement by some positive constant $C > 0$; that is, abusing notation a bit, (re)define $k(q) := k(q)/C$ and $l(q) := l(q)/C$. Note that this only a change of units in which inputs are measured to ensure, by (1), that $\int_{\mathcal{I}} l(q)\,d\mu = \int_{\mathcal{I}} 1 d\mu$, as in the paper. On any bounded interval $\mathcal{I} = [a,b]$ we have

$$\int_{\mathcal{I}} k(q)\,d\mu = \int_{\mathcal{I}} \hat{k}(q)\,l(q)\,d\mu = \int_{\mathcal{I}} \hat{k}(q)\,d\mu \int_{\mathcal{I}} l(q)\,d\mu = \int_{\mathcal{I}} \hat{k}(q)\,d\mu,$$

where the first equality follows by definition of $\hat{k}$, the second equality follows from independence, and the last inequality follows from (1) by normalization. We have now obtained the result by showing that inputs are the same on the redefined and normalized space as on the original space.[1] We omit the details of extending this result to $\mathscr{B}(\mathcal{Q})$, which is standard but

[1] It must also be the case that the information we dropped is irrelevant for the firm, which we assume is the case. As a counterexample, suppose the firm—for whatever reason—chooses to do tasks with capital iff $l \geq 5$. In

cumbersome.

Appendix B: Omitted proofs of Lemmas 1, 2 and parts of 3

Proof of Lemma 1 **Part I:** We first show that technological constraints given by equation 4 in the paper are satisfied for finite inputs when a cutoff rule is used; in particular, we need to show that (i) the constant function $f(q) = 1$ is μ-integrable, or equivalently $g(q)$ is Lebesgue integrable on interval $[q^*, \infty)$, and that (ii) $k(q)$ is μ-integrable, or equivalently $k(q)\, g(q)$ is Lebesgue integrable on interval $[0, q^*]$, where q^* satisfies the requirement of the lemma. We prove it in two steps.

Step 1: Assume $q^* < \infty$ and $k(q^*) > 0$. To establish property (i) above, define $\mathcal{S} = [q^*, \infty)$. By contradiction, suppose $g(q)$ is not Lebesgue integrable on $\mathcal{S}$ (i.e., $\int_{\mathcal{S}} d\mu = \int_{\mathcal{S}} g(q)\, dv = +\infty$, since measurability is assumed and g is a non-negative function). By Assumption 1 in the paper, we know there is a measurable partition of $\mathcal{S}$ comprising two disjoint subsets $\mathcal{S}_l = \mathcal{Q}_l \cap \mathcal{S}, \mathcal{S}_k = \mathcal{Q}_l \cap \mathcal{S}$ such that $\int_{\mathcal{S}_l} d\mu < \infty$ and $\int_{\mathcal{S}_k} k(q)\, d\mu < \infty$, where $\{\mathcal{Q}_l, \mathcal{Q}_k\}$ is the partition implied by Assumption 1. Since $k(q)$ is an increasing function, we know

$$\infty > \int\limits_{\mathcal{S}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{S}_k} k(q^*)\, d\mu = k(q^*) \int\limits_{\mathcal{S}_k} d\mu,$$

which gives a contradiction by the following chain of evaluations:

$$\infty > \int\limits_{\mathcal{S}_l} 1 d\mu + \int\limits_{\mathcal{S}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{S}_l} 1 d\mu + k(q^*) \int\limits_{\mathcal{S}_k} d\mu = (1 + k(q^*)) \int\limits_{\mathcal{S}} 1 d\mu = +\infty. \tag{2}$$

To establish property (ii), we note that $0 \leq k(q) \leq \frac{w}{r} < \infty$ for all $q \leq q^*$, implying $\frac{r}{w} k(q) < 1$. This follows by the definition of cutoff q^* in the statement of lemma. Ac-

that case the ratio k/l would not be sufficient. This is not the case in our model because the firm only maximizes profits.

cordingly, $\int_0^{q^*} k(q)\, d\mu < \infty$ by the following chain of evaluations:

$$\infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \geq \int\limits_{\mathcal{Q}_l} \frac{r}{w} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu$$
$$= \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu \geq \left(\frac{r}{w} + 1\right) \int_0^{q^*} k(q)\, d\mu.$$

(For Step 2 below, note that the proof of property (i) actually does not depend on q^* in the statement of the lemma, and the proof of property (ii) does not depend on $k(q^*) > 0$.)

Step 2: This step covers degenerate cases: a) $k(q^*) = 0$ $(0 \leq q^* < \infty)$ or b) $q^* = +\infty$ (note: a and b is impossible by Assumption 1, since by that assumption $k(q)$ must be strictly positive for a sufficiently large q).

Case a: By definition of the cutoff in the statement of the lemma and the fact that $k(q)$ is an increasing function, we have $k(q) \geq \frac{w}{r} > 0$ for all $q > q^*$, and $k(q) = 0$ for all $q \leq q^*$ (the strictly inequality follows here from the hypothesis that $k(q^*) = 0$). Accordingly, we have established now property (i), since $\int_0^{q^*} k(q)\, d\mu = \int_0^{q^*} 0 d\mu = 0$. Recall that, as noted at the end of Step 1 above, the argument used in Step 1 above (proof of property ii) does not require $k(q^*) > 0$ as assumed in Step 1, and so property (ii) has already been proven there.

Case b: Note that $q^* = \infty$ implies $k(q) \leq \frac{w}{r}$ for all $q \in \mathcal{Q}$ by the cutoff rule stated in the lemma. Accordingly, by Assumption 1 in the paper, and the fact that $\frac{r}{w} k(q) \leq 1$ for all $q \in \mathcal{Q}$, property (i) follows from the evaluation:

$$\infty > \int\limits_{\mathcal{Q}_l} 1 d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu \geq \frac{r}{w} \int\limits_{\mathcal{Q}_l} k(q)\, d\mu + \int\limits_{\mathcal{Q}_k} k(q)\, d\mu = \left(\frac{r}{w} + 1\right) \int\limits_{\mathcal{Q}} k(q)\, d\mu.$$

To see that $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, we apply the argument used in Step 1 (proof of property i) to show that $\int_{q^{**}}^{\infty} 1\mu$ exists (is finite) for sufficiently large q^{**} such that $k(q^*) > 0$ (the existence of such a sufficiently large and finite q^{**} is ensured by the fact that $k(q)$ is strictly positive on at least part of the domain by Assumption 1 and, as noted, Step 1 (proof property i) did not actually rely on the assumption that q^* corresponds to the cutoff as defined in the statement of lemma). Since for any Lebesgue integral we have $\lim_{q^* \to \infty} \int_{q^*}^{\infty} 1\mu = 0$, we have now shown that both constraints in equation 4 in the paper are well defined when the cutoff rule is used.[2]

Part II: This part establishes that the proposed cutoff rule satisfies cost minimization. By contradiction, suppose that there exists a task partition $\mathcal{Q}_k = \mathcal{E}$ of a positive measure under μ that solves the minimization problem and that is different from that implied by the cutoff rule in the lemma (on a measurable set with a positive measure). If so, reassigning production of tasks in $\mathcal{A} = \mathcal{E} \cap \{q : q > q^*\}$ from capital to labor must reduce the cost because $rk(q) > w$ on that set by definition of the cutoff rule—since we are minimizing $rK + wL$—and analogously on set $\mathcal{A}^c$ on which we switch from using labor to capital. At least one of these sets must be of positive measure, contradicting cost minimization and establishing the result. Q.E.D.

Proof of Lemma 2 Consider the definition of the production function (equation 5 in the paper) with equality:

$$Y(K,L) := \sup \left\{ Y : \exists_{q^* \in \mathcal{Q}} \text{ s.t. } K = Y \int_0^{q^*} k(q)\, d\mu, L = Y \int_{q^*}^{\infty} 1 d\mu \right\}. \tag{3}$$

We split the proof to two steps: Step 1 shows the solution (Y,q^*) to the above equations exists. Step 2 shows the solution from Step 1 attains the supremum under the formulation stated in the paper (equation 5 in the paper).

Step 1: Note that the constraint in (3) implies that q^* satisfies

$$\frac{L}{K} = \frac{\int_{q^*}^{\infty} 1 d\mu}{\int_0^{q^*} k(q)\, d\mu}. \tag{4}$$

The integral in the numerator is finite whenever the integral in the denominator is nonzero. We have established this property in the proof of Lemma 1 (see Part I, Step 1). The key here is that when the denominator (or $K > 0$) is positive, then $k(q^*) > 0$, which in turn implies the existence (finiteness) of the integral in the numerator by the arguments used in the proof of Lemma 1 (see Part I, Step 1, proof of property i). Next, note the following basic properties of

[2] The proof follows the fact that the tail sum of any convergent series converges to zero, that is, if $\sum_{i=1}^{\infty} a_i$ converges, then $t_n = \sum_{i=n}^{\infty} a_i \to_{n\to\infty} 0$, which itself is a corollary from Cauchy's criterion of convergence for series. Specifically, define $a_i = \int_{q^*+i-1}^{q^*+i} 1 d\mu$, note that $\sum_{i=1}^{\infty} a_i = \int_{q^*}^{\infty} 1\mu < \infty$ by Theorem 5.24 from Wheeden and Zygmund (1977) and the hypothesis, and now apply the result for series.

the expression on the right-hand side of equation (4): i) the numerator can be made arbitrarily small as $q^* \to 0$, and since the numerator is increasing in q^*, the expression goes to ∞ as $q^* \to 0$; ii) the numerator goes to 0 when $q^* \to \infty$, and since the denominator is positive and increasing in q^*, the expression goes to 0 as $q^* \to \infty$ (the proof of this simple fact can be found in footnote 2); finally, iii) note that the expression is continuous with respect to q^* and strictly decreasing, and by all these properties it is bijective on $\mathbb{R}_+$.[3] Accordingly, there exists a unique $0 < q^* < +\infty$ that satisfies the two constraints (for any finite $L/K > 0$). Furthermore, the supremum is attained within this set. (Without a loss we can restrict attention to a compact domain of $\left(\hat{Y}, q^*\right)$ while maximizing a continuous function $f\left(\hat{Y}\right) = \hat{Y}$ on the set defined by (3). Accordingly, Weierstrass extreme value theorem ensures the existence of maximum.)

Step 2: We now turn to the question of whether this solution attains the supremum under the original definition of the production function given by equation 5 in the paper. For now assume $K > 0$. (We cover $K = 0$ at the very end.) Suppose, by the way of contradiction that there exists $\hat{Y}' > \hat{Y}$, $q^{*\prime} > 0$ such that $K > \hat{Y}' \int_0^{q^{*\prime}} k(q)\, g(q)\, dv, L \geq \hat{Y}' \int_{q^{*\prime}}^{\infty} g(v)\, dv$ (the case $K = \hat{Y}' \int_0^{q^{*\prime}} k(q)\, g(q)\, dv$, $L > \hat{Y}' \int_{q^{*\prime}}^{\infty} g(v)\, dv$ will follow by analogy and it is omitted). If so, the supremum of the original problem must exceed the one implied by 3, which, as we show next, leads to a contradiction. Note that the integrals exist at $q^{*\prime}$ by the hypothesis (the stated inequalities guarantee these integrals are finite). By the continuity of Lebesgue integrals (see footnote 3), we can pick $\Delta q^{*\prime} > 0$ such that $K > \hat{Y} \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$, which implies that there exists $\Delta \hat{Y} > 0$ such that $K = \left(\hat{Y}' + \Delta \hat{Y}'\right) \int_0^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dv$ (by continuity of the expression on the right). We must ensure that the integral in the last expression exists (is finite). Let $\bar{k} := \sup_{[q^{*\prime}, q^{*\prime}+\Delta q^{*\prime}] \subset \mathcal{Q}} k(q)$, which, note, must be a finite number. (If this was not the case, we would have had $k(q^{**}) = +\infty$ for any $q^{**} > q^{*\prime} + \Delta q^{*\prime}$—simply because $k(q)$ is increasing and it is defined everywhere on $\mathcal{Q}$.) The following chain of evaluations now shows that the integral in question exists as long as $\int_{q^{*\prime}}^{\infty} g(v)\, dv$ exists, which is the case by the hypothesis:

$$\infty > \bar{k} \int_{q^{*\prime}}^{\infty} g(q)\, dq > \bar{k} \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} g(q)\, dq > \int_{q^{*\prime}}^{q^{*\prime}+\Delta q^{*\prime}} k(q)\, g(q)\, dq.$$

[3] Lacking a textbook reference, we prove it in the Online Appendix E.

Returning to the main argument, the fact that g has full support implies $L > \hat{Y}' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} 1 g(v)\,dv$ by continuity of Lebesgue integrals.[4] But, if so, there exists $\hat{Y}'' = \hat{Y}' + \Delta\hat{Y}''$, for some $0 < \Delta\hat{Y}'' < \Delta\hat{Y}'$, such that $K \geq \hat{Y}'' \int_{0}^{q^{*\prime}+\Delta q^{*\prime}} k(q)\,g(q)\,dv$ and we maintain $L = \hat{Y}'' \int_{q^{*\prime}+\Delta q^{*\prime}}^{\infty} g(v)\,dv$, which is a contradiction of the fact that $\hat{Y}' > \hat{Y}$. $\hat{Y}' = \infty$ is not feasible because k is strictly positive on at least part of the domain (see Assumption 1 in the paper). The remaining case is easy to eliminate by instead considering "$-\Delta q^{*\prime}$" and we omit the details. If $K = 0$, note, there is not much to prove because $q^* = 0$. Q.E.D.

Proof of Lemma 3 (omitted parts from the paper) Part I shows existence and Part II derives the formulas and the proof is in the paper. **Part I:** The proof builds on the proof of Lemma 2. We have established in that lemma that the production function can be obtained from (3) and that a unique q^* exists that satisfies (4). By the second constraint then, we know that $Y(K, L), q^*$ satisfy

$$L = Y(K, L)\,\mu([q^*, \infty)), \tag{5}$$

which gives

$$q^*(Y, L) = S^{-1}\left(\frac{L}{Y}\right), \tag{6}$$

where $S(q) := \mu([q, \infty))$ is the survival function. The survival function under the assumptions made in the paper, by previous lemmas, is well-defined, positively-valued, continuous, strictly decreasing (because g has full support), and hence invertible and differentiable almost everywhere with a strictly negative derivative.[5] Accordingly, $S^{-1}\left(\frac{L}{Y}\right)$ exists and is differentiable a.e., since for functions of a single variable we have $[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$, which is well defined as long as f' is nonzero (which it is). This implies that the derivative of $q^*(Y, L)$ in (6) is well defined (a.e.). The production function $Y(K, L)$ can be recovered from capital usage equation of Lemma 2, which gives the identity:

$$f(Y(K, L), K) := Y(K, L) \int_{0}^{q^*(Y(K,L),L)} k(q)\,d\mu - K \equiv 0.$$

[4]As in footnote 3.

[5]See Theorem 7.21 (p. 111) in Wheeden and Zygmund (1977).

The implicit function theorem ensures that at the points of differentiability of $q^*(Y, L)$ the partial derivative $Y_K(K, L)$ is well defined as long as the partial derivative $f_Y(Y, K) := \frac{\partial f(Y,K)}{\partial Y}$ is non-vanishing (nonzero) and both f_Y, f_K are well defined—which is readily implied by the above functional form. The existence of the partial derivative with respect to L can be shown analogously and we omit it. **Part II:** In the appendix of the paper.

Appendix C: Growth properties of domain-truncated CD technology

Here we show how to obtain approximately balanced growth from the domain-truncated Cobb-Douglas task technology (TCD) of the form:

$$T_{q_0}^{TCD} = \left(\mathcal{Q} = [q_0, \infty), k(q) = Z^{-1} q^{\frac{1}{\alpha}} \frac{1-\alpha}{\alpha}, g(q) = A^{-1} q^{-2} \right), \tag{7}$$

where $q_0 > 0$. (For convenience, we modify domain $\mathcal{Q}$ instead of adding q_0 to effectively also shift the task domain.)

In this case the density function can be normalized by a constant to yield the standard Pareto probability density, implying that the implied measure μ is finite, and hence $T_{q_0}^{TCD}$ has probabilistic representation. We will show that this technology gives rise to approximately balanced growth and its predictions can be made statistically indistinguishable from the balanced growth path of the CD economy by picking sufficiently small q_0 given a finite sample of data.

To derive the aggregate production function implied by TCD technology, we follow the steps in Example 1 of the paper. If $q^* > q_0$, we obtain the following equation for the representative isoquant:

$$\frac{K}{Y}\left(\frac{L}{Y}\right) := (AZ)^{-1} \left(\left(A\frac{L}{Y} \right)^{1-\frac{1}{\alpha}} - q_0^{\frac{1}{\alpha}-1} \right). \tag{8}$$

As expected, $q_0 \to 0$ implies the production function is CD. However, the constraint $q^* \geq q_0$ may be binding, and in that case the above equation does not apply because Lemma 2 does not apply. Accordingly, we use the original definition of the production function and obtain $Y = ALq_0$, $K = 0$ when $\left(A\frac{L}{Y}\right)^{1-\frac{1}{\alpha}} \leq q_0^{\frac{1}{\alpha}-1}$, or equivalently $\frac{L}{Y} \geq (Aq_0)^{-1}$, which implies $q^* = q_0$ is binding. Figure 1 illustrates the obtained isoquant.

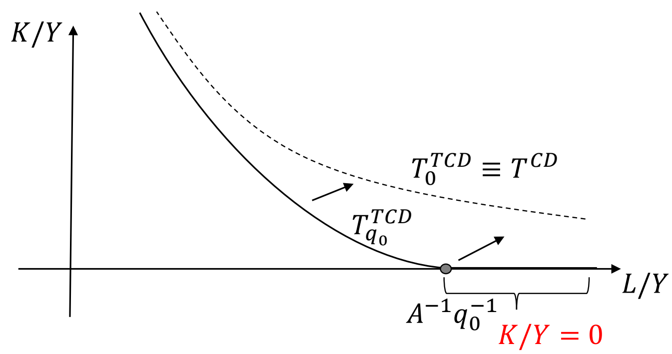

Figure 1: Representative isoquant of the production function implied by the $T_{q_0}^{TCD}$ technology.

Notes: The figure plots the representative isoquant of the production function implied by task technology $T_{q_0}^{TCD}$ (in text). The arrows indicate uniform convergence to CD isoquant as $q_0 \to 0$. On the flat portion, capital is not used in production and output is produced exclusively using labor.

The key property of this isoquant is that it *uniformly* converges to the CD isoquant ($q_0 = 0$) both when $q_0 \to 0$ and $AZ \to \infty$. This follows from the fact that we can bound the difference between the two isoquants by the expression:

$$\sup_{L/Y \geq 0} |\frac{K}{Y}\left(\frac{L}{Y}; q_0\right) - \frac{K}{Y}\left(\frac{L}{Y}; 0\right)| < \sup_{L/Y \geq 0} \frac{1}{\alpha^{-1} - 1} (AZ)^{-1} q_0^{\frac{1}{\alpha} - 1},$$

where $\frac{K}{Y}\left(\frac{L}{Y}; q_0\right)$ is the representative isoquant of the truncated CD technology ($q_0 > 0$) and $\frac{K}{Y}\left(\frac{L}{Y}; 0\right)$ is the representative isoquant of the CD technology. This bound follows from the fact that the "gap" between the two isoquants is decreasing with respect to $\frac{L}{Y}$ above $\frac{L}{Y} \geq (Aq_0)^{-1}$, as shown in the figure, and thus it is bounded by the "gap" at $\frac{L}{Y} = (Aq_0)^{-1}$, which itself narrows with $q_0 \to \infty$. This property does not imply that the production function converges uniformly, but it does imply that the production function converges uniformly on an arbitrarily bounded range of inputs. As we show next, after dividing each variable by the growth rate of technology, the model implies that the vector field on the phase space for normalized capital and consumption uniformly converges to that of the CD model.

Growth properties of domain-truncated CD task technology

Assume that Z_t and A_t grow at constant and strictly positive rates $\gamma_Z > 0, \gamma_A > 0$, respectively. Assume that q_0 is sufficiently small so that capital is used in equilibrium; that is, the economy stays on the increasing portion of the isoquant in Figure 1. We return to this at the end. To focus on how the growth path relates to the balanced growth path under CD technology, divide all variables except for labor by the balanced growth factor $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}$ of the CD model. For example, after this normalization, K_t becomes $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{K}_t$, C_t becomes $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}} \bar{C}_t$, and so on and so forth. Since A and Z both grow at strictly positive rates γ_A, γ_Z, $(A_t Z_t^{\alpha})^{\frac{1}{1-\alpha}}$ grows at rate γ can be easily calculated by differentiating this expression with respect to time. The normalized allocation solves the planning problem of the form:

$$\max_{(C_t, K_t)_t} \int_0^{\infty} e^{-(\rho+\gamma)t} u\left(\bar{C}_t\right) \tag{9}$$

subject to

$$\bar{C}_t + \dot{\bar{K}}_t + \gamma \bar{K}_t - \delta \bar{K}_t = \bar{Y}_t, \tag{10}$$

given $\bar{K}_0$, and $\bar{C}_t \geq 0$, $\bar{K}_t \geq 0$, where, by (8), output $\bar{Y}_t$ solves[6]

$$\bar{Y}_t = \bar{Y}_t\left(\bar{K}_t, \bar{L}; q_0\right) := \left(\frac{\bar{K}_t}{\bar{X}_t\left(\bar{Y}_t, \bar{L}; q_0\right)}\right)^{\alpha} \bar{L}^{1-\alpha}, \tag{11}$$

and where

$$\bar{X}_t\left(\bar{Y}_t, \bar{L}_t; q_0\right) := 1 - (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1} q_0^{\frac{1}{\alpha}-1}. \tag{12}$$

We refer to the above model as the TCD model while referring to the model in text as the CD model (which is the above but with $q_0 = 0$).

The fixed point that defines $\bar{Y}_t$ exists and is unique—as long as q_0 is not too high, which we assume. This follows from plugging (12) into (11) and noting the opposing monotonicity of the left- and right-hand side of the resulting equation with respect to $\bar{Y}$. Second, the above equation implies that the production function defined by (11) converges to the CD production function uniformly on any bounded domain, in particular for $\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0 > 0$ ($\bar{L}$ fixed).[7] The addition of an upper bound constraint $\bar{\bar{K}}$ is without a loss given that a sufficiently high

[6] See the Online Appendix B for an explicit derivation of the above formula.

[7] After plugging in from (11) to (12), it can be shown that for sufficiently low q_0 and $\bar{K} \geq K_0$, we can always find a unique X that solves the resulting equation. Plugging in that X to (11), we obtain unique value of output. In addition, X converges to 1 with both $q_0 \to 0$ and $A_t Z_t \to \infty$ for any $\bar{Y}_t > 0$ (uniformly after imposing a lower bound on $\bar{Y}_t$).

level of capital is unsustainable by the assumptions that consumption must be nonnegative and depreciation is a fraction of capital stock. The lower bound follows from our focus on a positive growth equilibrium. We return to this at the end of the section.

Equation (10) implies that the growth rate of capital $\gamma_{\bar{K},t}$ is

$$\gamma_{\bar{K},t}\left(\bar{K}_t, \bar{C}_t; q_0\right) := \frac{\dot{\bar{K}}_t}{\bar{K}_t} = \frac{\bar{Y}_t\left(\bar{K}_t, \bar{L}; q_0\right) + (\delta - \gamma)\bar{K}_t - \bar{C}_t}{\bar{K}_t}, \tag{13}$$

and hence, by the observations made, it converges uniformly to the growth rate at $q_0 = 0$ on the bounded domain $\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0$, both with respect to $q_0 \to 0$ and/or $t \to \infty$ (by which we mean $A_t Z_t \to \infty$). As a result, the growth rate of capital converges uniformly to the CD case, implying

$$\sup_{\bar{\bar{K}} \geq \bar{K} \geq \bar{K}_0, \bar{C} \geq 0} \left|\gamma_{\bar{K},t}\left(\bar{K}, \bar{C}; q_0\right) - \gamma_{\bar{K},t}\left(\bar{K}, \bar{C}; 0\right)\right| \to_{q_0 \to 0, t \to \infty} 0.$$

The Euler condition for the planning problem implies that the growth rate of consumption $\gamma_{\bar{C},t}$ is

$$\gamma_{\bar{C},t}\left(\bar{K}_t, \bar{C}_t; q_0\right) := \frac{\dot{\bar{C}}_t}{\bar{C}_t} = \frac{1}{\sigma}\left(MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) - \delta - \rho\right), \tag{14}$$

which, after basic manipulations detailed in the Online Appendix D below, can be linked to $MPK_t\left(\bar{K}_t, \bar{L}; 0\right)$ as follows

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{\alpha-1} MPK_t\left(\bar{K}_t, \bar{L}; 0\right)^{-1} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}, \tag{15}$$

where

$$MPK_t\left(\bar{K}_t, \bar{L}; 0\right) = \alpha\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)}\right)^{\frac{1}{\alpha}-1}.$$

Accordingly, we similarly obtain uniform convergence of consumption growth rate:

$$\sup_{\bar{K} \geq K_0} \left|\gamma_{\bar{C},t}\left(\bar{K}, \bar{C}; q_0\right) - \gamma_{\bar{C},t}\left(\bar{K}, \bar{C}; 0\right)\right| \to_{q_0 \to 0, A_t Z_t \to \infty} 0.$$

C and K are the two variables that define the phase space of the dynamic system that solves (9). As a result, as shown in Figure 2, the vector field for this system is a perturbed version of

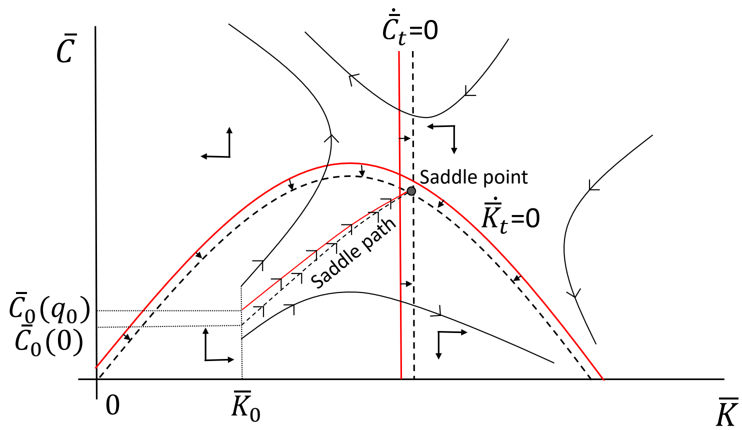

Figure 2: Phase diagram of the growth model with $q_0 > 0$ versus $q_0 = 0$ (dotted line).

Notes: The figure shows a phase diagram implied by the growth model featuring truncated Cobb-Douglas technology ($q_0 > 0$, solid lines) versus exact Cobb-Douglas technology featuring infinite measure ($q_0 = 0$, dotted lines). As shown in text, for $K > K_0$ all objects of the phase diagram exhibit uniform convergence to those associated with exact Cobb-Douglas technology, both with respect to $q_0 \to 0$ as well as time $t \to \infty$ (equivalently $A_t Z_t \to \infty$). Consequently, the optimal time path of capital and consumption along the saddle path is also similar as shown.

the one associated with the CD technology, with that perturbation uniformly vanishing with respect to both $q_0 \to 0$ and $A_t Z_t \to \infty$. Since qualitatively the phase diagram is standard, the solution that satisfies the usual transversality condition and nonnegativity conditions is the saddle path towards the intersection of the loci of points that imply stationary consumption and capital in the long-run. By the continuous dependence on the initial data theorem for differential equation, then, the time paths of each variable approach the CD case, and in the limit converge towards the common saddle point.

Let us now return to the omitted issue of capital being used along the growth path. When $K = 0$, note, the TCD technology implies that $MPK = \left(\frac{1}{\alpha} - 1\right)^{-1} AZq_0^{\frac{1}{\alpha}-1}$, which together with the Euler equation implies that capital will be accumulated as long as $MPK = \left(\frac{1}{\alpha} - 1\right)^{-1} AZq_0^{\frac{1}{\alpha}-1} > \rho + \delta + \sigma\gamma_A$, since consumption grows at rate γ_A when capital is not used nor accumulated ($K = 0$), and in that case consumption equals output, i.e., $C = Y = A\bar{L}q_0$. We can ensure this condition holds for q_0 sufficiently low given $A_0 Z_0$ as assumed, and because AZ grows at a strictly positive rate, we can be sure this condition will hold thereafter.

The global transitional dynamics implied by the TCD model is more complicated but it is appealing in its own right. In particular, this model can generate a stylized industrial revolution along the lines of Hansen and Prescott (2002) at low levels of capital and productivity Z. This happens when q_0 is not too low and AZ keeps on growing so that eventually capital starts being used in production (which gives rise to a stylized industrial revolution). The model can also generate a poverty trap when growth in Z comes from learning-by-doing externality associated with using capital as in Romer (1986).

Appendix D. Supplementary derivations for Online Appendix C

Production function for TCD task technology normalized by balanced growth factor: Plugging in $K_t = (A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{K}_t$, $Y_t= (A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{Y}_t$, $C_t = (A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{C}_t$ to the equation for TCD isoquant in text, we obtain

$$\frac{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{K}_t}{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{Y}_t} = (A_t Z_t)^{-1} \left(\left(A_t \frac{\bar{L}}{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{1-\frac{1}{\alpha}} - q_0^{\frac{1}{\alpha}-1} \right).$$

Simplifying terms and pulling out the first term in the last bracket, while raising both sides to the power α, we get

$$\left(A_t Z_t \frac{\bar{K}_t}{\bar{Y}_t} \right)^\alpha = \left(A_t \frac{\bar{L}}{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{\alpha-1} \left(1 - q_0^{\frac{1}{\alpha}-1} \left(\frac{A_t \bar{L}}{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}} \bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^\alpha,$$

which, given the fact that

$$\frac{A_t}{(A_t Z_t^\alpha)^{\frac{1}{1-\alpha}}} = \frac{A_t^{1-\frac{1}{1-\alpha}}}{Z_t^{\frac{\alpha}{1-\alpha}}} = \frac{A_t^{-\frac{\alpha}{1-\alpha}}}{Z_t^{\frac{\alpha}{1-\alpha}}} = (A_t Z_t)^{-\frac{\alpha}{1-\alpha}} = (A_t Z_t)^{-\frac{1}{\frac{1}{\alpha}-1}}$$

simplifies to

$$\left(\frac{\bar{K}_t}{\bar{Y}_t} \right)^\alpha = \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\alpha-1} \left(1 - q_0^{\frac{1}{\alpha}-1} (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^\alpha,$$

$$\left(\frac{\bar{K}_t}{\bar{Y}_t} \right)^\alpha \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{1-\alpha} = \left(1 - q_0^{\frac{1}{\alpha}-1} (A_t Z_t)^{-1} \left(\frac{\bar{L}}{\bar{Y}_t} \right)^{\frac{1}{\alpha}-1} \right)^\alpha,$$

and gives

$$\left(\bar{K}_t\right)^{\alpha}\left(\bar{L}\right)^{1-\alpha}=\bar{Y}_t\left(1-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}\right)^{\alpha}.$$

After some basic manipulation, the last expression yields the fixed point stated in text:

$$\bar{Y}_t\left(\bar{K}_t,\bar{L}_t;q_0\right)=\left(\frac{\bar{K}_t}{\bar{X}_t\left(Y_t,\bar{L};q_0\right)}\right)^{\alpha}\left(\bar{L}\right)^{1-\alpha}, \tag{16}$$

where

$$\bar{X}_t\left(Y_t,\bar{L};q_0\right)=1-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}.$$

Marginal product of capital MPK:

We use the last equation above and raise both sides to power $\frac{1}{\alpha}$ to obtain

$$\left(\bar{K}_t\right)^{1}\left(\bar{L}\right)^{\frac{1}{\alpha}-1}=\bar{Y}_t^{\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{\frac{1}{\alpha}-1}\bar{Y}_t^{\frac{1}{\alpha}}$$

hence obtain

$$\bar{K}_t=\bar{Y}_t^{\frac{1}{\alpha}}\bar{L}^{\frac{1}{\alpha}-1}-q_0^{\frac{1}{\alpha}-1}\left(A_tZ_t\right)^{-1}\bar{Y}_t,$$

$$\bar{K}_t=\left(A_tZ_t\right)^{-1}\bar{Y}_t\left(A_tZ_t\bar{Y}_t^{\frac{1}{\alpha}-1}\bar{L}^{\frac{1}{\alpha}-1}-q_0^{\frac{1}{\alpha}-1}\right)$$

and

$$A_tZ_t\bar{K}_t=\bar{Y}_t\left(A_tZ_t\left(\frac{\bar{L}}{\bar{Y}_t}\right)^{1-\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\right).$$

The above expression defines the production function $\bar{Y}\left(\bar{K}_t,\bar{L};q_0\right)$ implicitly via the expression:

$$A_tZ_t\bar{K}_t-\bar{Y}_t\left(A_tZ_t\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t,\bar{L};q_0\right)}\right)^{1-\frac{1}{\alpha}}-q_0^{\frac{1}{\alpha}-1}\right)\equiv 0$$

where $\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right)$ is given by (16). We use implicit function theorem and differentiate the above to calculate[8]

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\alpha^{-1}\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right)}\right)^{1-\frac{1}{\alpha}} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}.$$

For $q_0 = 0$, note, we obtain

$$MPK_t\left(\bar{K}_t, \bar{L}; 0\right) = \alpha\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)}\right)^{\frac{1}{\alpha}-1},$$

which is the expression for MPK for the CD production function given by $\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)$. Now, by (16), we know that

$$\bar{Y}\left(\bar{K}_t, \bar{L}; q_0\right) = \bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{-\alpha} \bar{Y}\left(\bar{K}_t, \bar{L}; 0\right).$$

Accordingly, we have

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{\alpha-1} \alpha^{-1}\left(\frac{\bar{L}}{\bar{Y}\left(\bar{K}_t, \bar{L}; 0\right)}\right)^{1-\frac{1}{\alpha}} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1}$$

and hence

$$MPK_t\left(\bar{K}_t, \bar{L}; q_0\right) = \left(\bar{X}_t\left(Y_t, \bar{L}; q_0\right)^{\alpha-1} MPK_t\left(\bar{K}_t, \bar{L}; 0\right)^{-1} - \left(A_t Z_t\right)^{-1} q_0^{\frac{1}{\alpha}-1}\right)^{-1},$$

which is the result stated in text.

Appendix E: User cost of capital in extended model

We derive the formula for the user cost of capital for our extended model, and it corresponds to the formula stated in text.

Let $R(q)$ be the user cost of a machine of type q, and let this be associated with some

[8] Derivation of the above expression is cumbersome and has been automated in the Mathematica notebook *MPK_TCD.nb*.

dividend earned for having this machine for one period of length dt and renting it out for the duration of that period. The key condition is that the profit from such an activity must be zero after accounting for the cost of acquisition, dividend, and resell value of the machine at $t + dt$.

The costs of buying a machine at t and holding it for one period of length dt comprises its nominal purchase price at time t, which is $P_t k_t(q)$, and the opportunity cost of funds $\rho P_t k_t(q)\, dt$ incurred over period of length dt (ρ is the interest rate). The resell price is $P_{t+dt} k_{t+dt}(q)$, but since with assumed Poisson probability $\delta\Delta$ the machine disintegrates, the expected residual value is $(1-\delta) P_{t+dt} k_{t+dt}(q)\, dt$. The zero profit condition is thus given by

$$\underbrace{R_t(q)}_{\text{user cost}} dt = \underbrace{(1+\rho) P_t k_t(q)\, dt}_{\text{acquisition cost}} - \underbrace{(1-\delta) P_{t+dt} k_{t+dt}(q)\, dt}_{\text{residual value after a period of use}}.$$

Assuming balanced growth, assume Pk grows at rate $\gamma_{k,t} > 1$ from one period to the next (from t to $t + dt$). This simplifies the above expression to $R_t(q) = (1 + \rho - (1-\delta)\gamma_{k,t}) P_t k_t(q)$. Given how we used r in the previous section, and assuming BGP, we obtain $r_t = (1 + \rho - (1-\delta)\gamma_k) P_t$.

Appendix F: Continuity of Lebesgue integrals

We lack a good reference for this result and outline the proof here for completeness. The claim is that the function

$$g(x) := \int_a^x f(q)\, dv$$

is a continuous function; that is,

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \int_a^x f(q)\, dv,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function, $a \in \mathbb{R}$, and "$\int$" pertains to a Lebesgue integral (v is the Lebesgue measure). *Proof:* Define an indicator function $1_S(q)$ that takes the value of 1 on the sub-scripted set S and rewrite the left-hand side as

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \lim_{x_n \to x_0} \int_{-\infty}^{+\infty} 1_{[a,x_n]}(q) f(q)\, dv.$$

Since $|1_{[a,x_n]}(q) f(q)| \leq |f(q)|$, we have $1_{[a,x_n]}(q) f(q) \to f(q)$ (a.e.) on $[a, x_0]$ (see from * below for formal argument). Given $|f(q)|$ is Lebesgue integrable by assumption, we can use dominated convergence theorem (DCT) and enter with the limit under the integral, which yields

$$\lim_{x_n \to x_0} \int_a^{x_n} f(q)\, dv = \int_{-\infty}^{+\infty} \lim_{x_n \to x_0} 1_{[a,x_n]}(q) f(q)\, dv.$$

*)Observe that $\lim_{x_n \to x_0} 1_{[a,x_n]}(q) = 1_{[a,x_0]}(q)$ (a.e.), since for any $q \leq x_0$, we have $1_{[a,x_n]}(q) = 1$, and for any $q > x_0$, there exists an N such that for all $n \geq N$ we have $1_{[a,x_n]}(q) = 0$. The set on which $1_{[a,x_n]}(q)$ and $1_{[a,x_0]}(q)$ disagree is of the form $[x_0, b_n]$, where $b_n \to x_0$, and hence its Lebesgue measure is zero in the limit, as claimed. Consequently, by DCT, we have

$$\int_{-\infty}^{+\infty} \lim_{x_n \to x_0} 1_{[a,x_n]}(q) f(q)\, dv = \int_{-\infty}^{+\infty} 1_{[a,x_0]}(q) f(q)\, dv,$$

which finishes the proof.

Appendix G: Feasibility of microfoundations for CD task technology

Here we discuss additional example of mechanical random processes that could give rise to Pareto distributed capital requirements. As in the paper, the discussion draws on Newman (2004) and especially Gabaix (2009).

Random growth Random growth model is one of the simplest mechanisms to obtain power law dynamics. To see how it could work within our framework, suppose that capital requirement on average declines at a rate $\gamma < 1$ per unit of time. That is, $\bar{k}_{t+dt} = \gamma \bar{k}_t$, where $\bar{k}_t$ is the mean value across all tasks (time being discrete $dt > 0$ or continuous $dt \to 0$). Furthermore, assume the distribution of the decline is uneven across individual tasks because innovations affect individual tasks differently, and as in the paper deflate each variable by growth factor γ^t. As is clear from the setup, some tasks may not decline at all in a given period—in which case the relative capital requirement deflated by average growth factor γ^t is rising—while other tasks may decline by more than the average and so their deflated requirement is falling. The important assumption here is that this process is i.i.d. across tasks. The discussion of random growth model in Gabaix (2009) now applies, including the discussion of the variations of this model that can generate a power law tail index below 1 when this reasoning is directly applied to capital requirement $k(q)$.

Endogenous technology verse function The fact that capital requirement is an inverse of productivity of capital can be used to obtain tail power law from the basic fact of taking an inverse of diffused observation (Sornette, 2002). Let $y = x^{\frac{1}{\zeta-1}}, \zeta - 1 > 0$ and suppose x is distributed according to some pdf $p_x(x)$ such that $p(x) \to C > 0$ as $x \to 0$. Then, the tail distribution of y follows a power law with exponent ζ. Of course, applying this result requires that the economy operates far into the tail of the distribution, or else it will not even approximately behave as our CD task technology.[9]

Yule process It is also possible to employ Yule's "speciation" process. As in the case of the example discussed in the paper, the key to this approach to endogenize Pareto distribution is the observation that a variable that grows exponentially and is stopped after an exponentially distributed time is Pareto distributed at the stopping time.[10] This extension is fairly involved and we omit it from. However, based on the insights from information theory, it is possible to obtain a bridge between our model and the combinatorial growth literature (Weitzman, 1998; Jones, 2021) and show that the resulting distribution that involves "speciation" of ideas is Pareto. Preliminary results are available upon a request.

Appendix H: Corollary to Uzawa's theorem

We lack a good reference for this result and outline it here for completeness. The appendix shows how an additional assumption of declining price of capital goods leads to Cobb–Douglas production function.

[9]The result follows from the change of variables formula.

[10]The key mathematical property is that an exponential of an exponentially distributed random variable is Pareto distributed, as the following calculation shows (X ~exponential, $Y = \exp(X)$):

$$Pr(Y \leq y) = Pr(\exp(X) \leq y) = Pr(X \leq \log(y)) = 1 - x^{-\lambda}.$$

By Theorem 2.6 and Theorem 2.7 in Acemoglu (2009), balanced growth path with positive and constant factor shares from $t \geq 0$ implies that one can find a sequence $\{a_t\}$ such that along that path $Y_t\left(K_t, \bar{L}\right) = Y_0\left(K_t, a_t\bar{L}\right)$ (Theorem 2.6) and $\frac{\partial Y_t\left(K_t,\bar{L}\right)}{\partial K_t} = \frac{\partial Y_0\left(K_t,a_t\bar{L}\right)}{\partial K_t}$, $\frac{\partial Y_t\left(K_t,\bar{L}\right)}{\partial \bar{L}} = \frac{\partial Y_0\left(K_t,a_t\bar{L}\right)}{\partial \bar{L}}$ (Theorem 2.7). Consider now the following definition: capital-augmenting progress occurs on the balanced growth path iff $k_t := K_t/a_t\bar{L}$ grows at a strictly positive rate. To see that this is a necessary and sufficient condition to imply that $Y_0\left(K_t, a_t\bar{L}\right)$ is CD, note we can express production along the balanced growth path as $f\left(k_t\right) := Y_0(k_t, 1)$, and that the constancy of the capital share implies $k_t f'\left(k_t\right)/f\left(k_t\right) = \alpha$ on that path, for some constant $0 < \alpha < 1$ and for all $t \geq 0$. Since k_t is growing and sweeps the entire domain (k_0, ∞), we obtain an ODE that solves to $f\left(k\right) = Ck^\alpha$ for some constant C, which yields the result: $Y_0\left(K, a\bar{L}\right) = CK^\alpha\left(a\bar{L}\right)^{1-\alpha}$. Concluding, CD production function obtains in any environment that restricts the balanced growth path to be such that k_t must grow over time, either by building it into the environment or requiring an equilibrium condition that implies that (e.g. a steadily declining price of capital goods).

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