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Benjamin Lester
Research Department, Federal Reserve Bank of Philadelphia

Ali Shourideh
Carnegie Mellon University

Venky Venkateswaran
NYU – Stern School of Business

Ariel Zetlin-Jones
Carnegie Mellon University

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Benjamin Lester
Federal Reserve Bank of Philadelphia

Ali Shourideh
Carnegie Mellon University

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Abstract

We incorporate a search-theoretic model of imperfect competition into a standard model of asymmetric information with unrestricted contracts. We characterize the unique equilibrium, and use our characterization to explore the interaction between adverse selection, screening, and imperfect competition. We show that the relationship between an agent's type, the quantity he trades, and the price he pays is jointly determined by the severity of adverse selection and the concentration of market power. Therefore, quantifying the effects of adverse selection requires controlling for market structure. We also show that increasing competition and reducing informational asymmetries can decrease welfare.

Keywords: Adverse Selection, Imperfect Competition, Screening, Transparency, Search Theory
JEL Codes: D41, D42, D43, D82, D83, D86, L13

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1 Introduction

Many important markets suffer from adverse selection, including the markets for insurance, credit, and certain financial securities. There is mounting evidence that many of these markets also feature some degree of imperfect competition.[1] And yet, perhaps surprisingly, the effect of imperfect competition on prices, allocations, and welfare in markets with adverse selection remains an open question.

Answering this question is important for several reasons. For one, many empirical studies attempt to quantify the effects of adverse selection in the markets mentioned above.[2] A natural question is to what extent these estimates—and the conclusions that follow—are sensitive to the assumptions imposed on the market structure. There has also been a recent push by policymakers to make these markets more competitive and less opaque.[3] Again, a crucial, but underexplored question is whether these attempts to promote competition and reduce information asymmetries are necessarily welfare-improving.

Unfortunately, the ability to answer these questions has been constrained by a shortage of appropriate theoretical frameworks.[4] A key challenge is to incorporate nonlinear pricing schedules—which are routinely used to screen different types of agents—into a model with asymmetric information and imperfect competition. This paper delivers such a model: we develop a novel, tractable framework of adverse selection, screening, and imperfect competition.

The key innovation is to introduce a search-theoretic model of imperfect competition (a la Burdett and Judd, 1983) into an otherwise standard model with asymmetric information and nonlinear contracts. Within this environment, we provide a full analytical characterization of the unique equilibrium, and then use this characterization to study both the positive and normative issues highlighted above.

First, we show how the structure of equilibrium contracts—and hence the relationship between an agent's type, the quantity that he trades, and the corresponding price—is *jointly* determined by the severity of the adverse selection problem and the degree of imperfect competition. In particular, we show that equilibrium offers separate different types of agents when competition is relatively intense or adverse selection is relatively severe, while they typically pool different types of agents in markets where

[1] For evidence of market power in insurance markets, see Brown and Goolsbee (2002), Dafny (2010), and Cabral et al. (2014). For evidence of market power in various credit markets, see, e.g., Ausubel (1991), Calem and Mester (1995), and Crawford et al. (2015). In over-the-counter financial markets, a variety of data suggest that dealers extract significant rents; indeed, this finding is hard-wired into workhorse models of this market, such as Duffie et al. (2005).

[2] See the seminal paper by Chiappori and Salanie (2000), and Einav et al. (2010a) for a comprehensive survey.

[3] Increasing competition and transparency in health insurance markets is a cornerstone of the Affordable Care Act, while the Dodd-Frank legislation addresses similar issues in over-the-counter financial markets. In credit markets, on the other hand, legislation has recently focused on *restricting* how much information lenders can demand or use from borrowers.

[4] As Chiappori et al. (2006) put it, "there is a crying need for [a model] devoted to the interaction between imperfect competition and adverse selection."

principals have sufficient market power and adverse selection is sufficiently mild. Second, we explore how total trading volume—which, in our environment, corresponds to the utilitarian welfare measure—responds to changes in the degree of competition and the severity of adverse selection. We show that increasing competition or reducing informational asymmetries is only welfare-improving in markets in which *both* market power is sufficiently concentrated and adverse selection is sufficiently severe.

Before expanding on these results, it is helpful to lay out the basic ingredients of the model. The agents, whom we call "sellers," are endowed with a perfectly divisible good of either low or high quality, which is private information. The principals, whom we call "buyers," offer menus containing price-quantity combinations to potentially screen high- and low-quality sellers.[5] Sellers can accept at most one contract, i.e., contracts are exclusive. To this otherwise canonical model of trade under asymmetric information, we introduce imperfect competition by endowing the buyers with some degree of market power. The crucial assumption is that each seller receives a stochastic number of offers, with a positive probability of receiving only one. Hence, when a buyer formulates an offer, he understands that it will be compared with an alternative offer with some probability, which we denote by π, and it will be the seller's only option with probability $1-\pi$. This formulation allows us to capture the perfectly competitive case by setting $\pi = 1$, the monopsony case by setting $\pi = 0$, and everything in between.

For the general case of imperfect competition, with $\pi \in (0,1)$, the equilibrium involves buyers mixing over menus according to a nondegenerate distribution function.[6] Since each menu is comprised of two price-quantity pairs (one for each type), the main equilibrium object is a probability distribution over four-dimensional offers. An important contribution of our paper is developing a methodology that allows for a complete, yet tractable, characterization of this complicated equilibrium object.

We begin by showing that any menu can be summarized by the indirect utilities it offers to sellers of each type. Next, we establish an important property: in any equilibrium, all menus that are offered by buyers are ranked in exactly the same way by both low- and high-quality sellers. This property, which we call "strictly rank-preserving," implies that all equilibrium menus can be ranked along a single dimension. The equilibrium, then, can be described by a distribution function over a unidimensional variable—say, the indirect utility offered to low-quality sellers—along with a strictly monotonic function

[5] The use of the labels "buyers" and "sellers" is merely for concreteness and corresponds most clearly with an asset market interpretation. These monikers can simply be switched in the context of an insurance market, so that the "buyers" of insurance are the agents with private information and the "sellers" of insurance are the principals.

[6] Mixing is to be expected for at least two reasons. First, this is a robust feature of nearly all models in which buyers are both monopsonists and Bertrand competitors with some probability, even without adverse selection or nonlinear contracts. Second, even in perfectly competitive markets, it is well known that pure strategy equilibria may not exist in an environment with both adverse selection and nonlinear contracts.

mapping this variable to the indirect utility offered to the high-quality seller. We show how to solve for these two functions, obtaining a full analytical characterization of all equilibrium objects of interest, and then establish that the equilibrium is unique. Interestingly, our approach not only avoids the well-known problems with existence of equilibria in models of adverse selection and screening, but also requires no assumptions on off-path beliefs to get uniqueness. We then use this characterization to explore the implications of imperfect competition in markets suffering from adverse selection.

First, we show that the structure of menus offered in equilibrium depends on both the degree of competition, captured by π, and the severity of the adverse selection problem, which is succinctly summarized by a single statistic that is largest (i.e., adverse selection is most severe) when: (*i*) the fraction of low-quality sellers is large; (*ii*) the potential surplus from trading with high-quality sellers is small; and (*iii*) the information cost of separating the two types of sellers, as captured by the difference in their reservation values, is large. Given these summary statistics, we show that separating menus are more prevalent when competition is relatively strong or when adverse selection is relatively severe, while pooling menus are more prevalent when competition is relatively weak and adverse selection is relatively mild. Interestingly, holding constant the severity of adverse selection, the equilibrium may involve all pooling menus, all separating menus, or a mixture of the two, depending on the degree of competition. This finding suggests that attempts to infer the severity of adverse selection from the distribution of contracts that are traded should take into account the extent to which the market is competitive.

Next, we examine our model's implications for welfare, defined as the objective of a utilitarian social planner. In our context, this objective maps one-for-one to the expected quantity of high-quality goods traded. A key finding is that competition can worsen the distortions related to asymmetric information and, therefore, can be detrimental to welfare. When adverse selection is mild, these negative effects are particularly stark: welfare is actually (weakly) maximized under monopsony, or $\pi = 0$.

When adverse selection is severe, however, welfare is inverse U-shaped in π, i.e., an *interior* level of competition maximizes welfare. To understand why, note that an increase in competition induces buyers to allocate more of the surplus to sellers (of both types) in an attempt to retain market share. All else equal, increasing the utility offered to low-quality sellers is good for welfare: by relaxing the low-quality seller's incentive compatibility constraint, the buyer is able to exchange a larger quantity with high-quality sellers. However, ceteris paribus, increasing the utility offered to high-quality sellers is bad for welfare: it tightens the incentive constraint and forces buyers to trade less with high-quality sellers. Hence, the net effect of an increase in competition depends on whether the share of the surplus offered to high-quality sellers rises faster or slower than that offered to low-quality sellers.

When competition is low, buyers earn a disproportionate fraction of their profits from low-quality sellers. Therefore, when buyers have lots of market power, an increase in competition leads to a faster increase in the utility offered to low-quality sellers, since buyers care relatively more about retaining these sellers. As a result, the quantity traded with high-quality sellers and welfare rise with competition. When competition is sufficiently high, profits come disproportionately from high-quality sellers. In this case, increasing competition induces a faster increase in the utility offered to high-quality sellers and, therefore, a decrease in expected trade and welfare. These results suggest that promoting competition—or policies that have similar effects, such as price supports or minimum quantity restrictions—can have adverse effects on welfare in markets that are sufficiently competitive and face severe adverse selection.

Next, we study the welfare effects of providing buyers with more information—specifically, a noisy signal—about the seller's type. As in the case of increasing competition, the welfare effects of this perturbation depend on the severity of the two main frictions in the model: imperfect competition and adverse selection. When adverse selection is relatively mild or competition relatively strong, reducing informational asymmetries can actually be detrimental to welfare. The opposite is true when adverse selection and trading frictions are relatively severe. In sum, these normative results highlight how the interaction between these two frictions can have surprising implications for changes in policy (or technological innovations), underscoring the need for a theoretical framework such as ours.

Our baseline model, which we describe in Section 2 and analyze in Sections 3–5, was designed to be as simple as possible in order to focus on the novel interactions between adverse selection and imperfect competition. In Sections 6 and 7, we analyze several relevant extensions and variants of our model. In Section 6, we endogenize the level of competition by letting buyers choose the intensity with which they "advertise" their offers. This allows us to study how the severity of adverse selection can influence the market structure, and the ensuing welfare implications. In Section 7, we consider a more general market setting with an arbitrary meeting technology, where sellers can meet any number of buyers (including zero). We show how to derive the equilibrium in this setting, using the techniques from our benchmark model, and confirm that our main welfare results hold for certain popular meeting technologies.[7]

[7] In the Appendix, we explore a number of additional extensions: we relax the assumption of linear utility to analyze the canonical model of insurance under private information; we allow the degree of competition to differ across sellers of different quality; we show how to incorporate additional dimensions of heterogeneity, including horizontal and vertical differentiation; and we consider the case of $N > 2$ types or qualities.

Literature Review. Our paper contributes to the extensive body of literature on adverse selection and, specifically, the role of contracts as screening devices. Most of this literature has either assumed a monopolistic market structure (a la Stiglitz, 1977) or perfect competition (a la Rothschild and Stiglitz, 1976).[8] The main novelty of our analysis is to synthesize a standard model of adverse selection and screening with the search-theoretic model of imperfect competition developed by Burdett and Judd (1983). While this model of imperfect competition has been used extensively in both theoretical and empirical work, to the best of our knowledge none of these papers address adverse selection and screening.[9]

A recent paper by Garrett et al. (2014) exploits the Burdett and Judd (1983) model in an environment with screening contracts and asymmetric information, but the asymmetric information is over the agents' *private values*. This key difference implies that the role of screening—and how it interacts with imperfect competition—is ultimately very different in our paper and theirs.[10]

More closely related to our work is the literature that studies adverse selection and nonlinear contracts in an environment with *competitive search*, such as Guerrieri et al. (2010).[11] As in our paper, Guerrieri et al. (2010) present an explicit model of bilateral trade without placing any restrictions on contracts, beyond those arising from the primitive frictions. There are, however, several important differences. First, we study how perturbations to the search technology affect market power, and the interaction between the resulting distortions and the underlying adverse selection problem, while Guerrieri et al. (2010) and others focus on the role of search frictions in providing incentives (through the probability of trade) and not on market power per se. Second, depending on parameters, our equilibrium menus can be pooling, separating, or a combination of both; the equilibrium in Guerrieri et al. (2010) always features separating equilibria. In this sense, our approach has the potential to speak to a richer set of observed outcomes. Finally, we obtain a unique equilibrium without additional assumptions or refinements, whereas uniqueness in Guerrieri et al. (2010) relies on a restriction on off-equilibrium beliefs.

An alternative approach to modeling imperfect competition is through product differentiation, as in Villas-Boas and Schmidt-Mohr (1999), Veiga and Weyl (2016), Mahoney and Weyl (2014), Townsend and Zhorin (2014), and Bénabou and Tirole (2016).[12] These papers vary competition by perturbing the

[8]For recent contributions to this literature that assume perfectly competitive markets, see, e.g., Bisin and Gottardi (2006), Chari et al. (2014), and Azevedo and Gottlieb (2017).

[9]Carrillo-Tudela and Kaas (2015) analyze a related labor market setting with adverse selection using an on-the-job search model, but their focus is quite different from ours.

[10]With private values, screening is useful only for rent extraction. Increasing competition reduces these rents, and hence the incentive to screen, causing welfare to rise. With common values, increasing competition strengthens incentives to separate, causing welfare to (eventually) decline.

[11]Also see Kim (2012), Guerrieri and Shimer (2014), and Chang (2017), among others.

[12]Also see Fang and Wu (2016), who propose a slightly different model of imperfect competition.

importance of an orthogonal attribute of each contract, which is interpreted as "distance" in a Hotelling interpretation or "taste" in a random utility, discrete choice framework. We take a different approach to modeling (and varying) competition that allows us to hold preferences—and thus the potential social surplus—constant. We also reach different conclusions about the desirability of competition. For example, Bénabou and Tirole (2016) highlight a tradeoff from increasing competition when agents allocate effort between multiple, imperfectly observable or contractible tasks. However, without multitasking, they find that competition improves welfare, even in the presence of adverse selection. This is also the case in Mahoney and Weyl (2014), who restrict attention to single-price contracts. Veiga and Weyl (2016) also restrict attention to a single contract, but with endogenous "quality," and find that welfare is maximized under monopoly. In comparison to these papers, we find that competition can be beneficial or harmful. Though a number of differences (e.g., multidimensional heterogeneity, the contract space, the equilibrium concept) preclude a direct comparison, we interpret the results in these papers as providing a distinct but complementary insight about the interaction between competition and adverse selection.

2 Model

The Environment. We consider an economy with two buyers and a unit measure of sellers. Each seller is endowed with a single unit of a perfectly divisible good. Buyers have no capacity constraints, i.e., they can trade with many sellers. A fraction $\mu_l \in (0, 1)$ of sellers possess a low (l) quality good, while the remaining fraction $\mu_h = 1 - \mu_l$ possess a high (h) quality good. Buyers and sellers derive utility v_i and c_i, respectively, from consuming each unit of a quality $i \in \{l, h\}$ good, with $v_l < v_h$ and $c_l < c_h$. We assume that there are gains from trading both high- and low-quality goods, i.e., that $v_i > c_i$ for $i \in \{l, h\}$.

There are two types of frictions in the market. First, there is *asymmetric information*: sellers observe the quality of the good they possess while buyers do not, though the probability μ_i that a randomly selected good is quality $i \in \{l, h\}$ is common knowledge. In order to generate the standard "lemons problem," we focus on the case in which $v_l < c_h$.

The second type of friction is a *search* friction: as we describe in detail below, the buyers in our model will make offers, but the sellers will not necessarily sample (or have access to) all offers. In particular, we assume that a fraction $1 - p$ of sellers will be matched with—and hence receive an offer from—a single buyer, which we assume is equally likely to be either buyer. The remaining fraction of sellers, p, will be matched with both buyers. A seller can only trade with a buyer if they are matched. Throughout the paper, we refer to sellers who are matched with one buyer as "captive," since they only have one option for trade, and we refer to those who are matched with two buyers as "noncaptive."

Given these search frictions, a buyer understands that, conditional on being matched with a particular seller, this seller will be captive with probability $1 - \pi$ and noncaptive with probability π, where

$$\pi = \frac{p}{\frac{1}{2}(1-p) + p} = \frac{2p}{1+p}. \tag{1}$$

This formalization of search frictions is helpful for deriving and explaining our key results in the simplest possible manner. For one, it allows us to vary the degree of competition with a single parameter, π, nesting monoposony and perfect competition as special cases.[13] Second, since the current formulation ensures that all sellers are matched with at least one buyer, a change in π varies the degree of competition without changing the potential gains from trade or "coverage" in the market; this is particularly helpful in isolating the effects of competition on welfare. However, it is important to stress that our equilibrium characterization and the ensuing results extend to markets with an arbitrary number of buyers and more general meeting technologies; see Section 7.

Offers, Payoffs, and Definition of Equilibrium. We model the interaction between a seller and the buyer(s) that she meets as a game in which the buyer(s) choose a mechanism and the seller chooses a message to send to each buyer she meets. A buyer's mechanism is a function that maps the seller's message into an offer, which specifies a quantity of numeraire to be exchanged for a certain fraction of the seller's good.[14] The seller's message space can be arbitrarily large: it could include the quality of her good, whether or not she is in contact with the other buyer, the details of the other buyer's mechanism, and any other (even not payoff-relevant) information. Importantly, we assume that mechanisms are exclusive, in the sense that a seller can choose to accept the offer generated by only one buyer's mechanism, even when two offers are available.

In Appendix C, we apply insights from the delegation principle (Peters, 2001; Martimort and Stole, 2002) to show that, in our environment, it is sufficient to restrict attention to menu games where buyers offer a menu of two contracts.[15] In particular, letting x denote the quantity of good to be exchanged for t units of numeraire, a buyer's offer can be summarized by the menu $\{(x_l, t_l), (x_h, t_h)\} \in \left([0,1] \times \mathbb{R}_+\right)^2$,

[13] Given the relationship in (1), it turns out that varying p or π is equivalent for all of our results below. We choose π because it simplifies some of the equations.

[14] The mechanisms we consider are assumed to be deterministic, but otherwise unrestricted. Stochastic mechanisms present considerable technical challenges and raise other conceptual issues that are, in our view, tangential to our key results.

[15] To be more precise, we show that the (distribution of) equilibrium allocations in any game where buyers offer the general mechanisms described above coincide with those in another game in which buyers only offer a menu of two contracts.

where (x_i, t_i) is the contract intended for a seller of type $i \in \{l, h\}$. A seller who owns a quality i good and accepts a contract (x, t) receives a payoff $t + (1-x)c_i$, while a buyer who acquires a quality i good at terms (x, t) receives a payoff $-t + xv_i$. Meanwhile, a seller with a quality i good who does not trade receives a payoff c_i, while a buyer who does not trade receives zero payoff.

Let $\mathbf{z}_i = (x_i, t_i)$ denote the contract that is intended for a seller of type $i \in \{l, h\}$, and let $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$. A buyer's strategy, then, is a distribution across menus, $\Phi \in \Delta\big(([0,1] \times \mathbb{R}_+)^2\big)$. A seller's strategy is much simpler: given the available menus, a seller chooses the menu with the contract that maximizes her payoffs or mixes between menus if she is indifferent. Conditional on a menu, the seller chooses the contract that maximizes her payoffs. In what follows, we will take the seller's optimal behavior as given.

A symmetric equilibrium is thus a distribution $\Phi^\star(\mathbf{z})$ such that:

1. *Incentive compatibility*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$t_i + c_i(1 - x_i) \quad \geqslant \quad t_{-i} + c_i(1 - x_{-i}) \quad \text{for } i \in \{l, h\}. \tag{2}$$

2. *Buyer's optimize*: for almost all $\mathbf{z} = \{(x_l, t_l), (x_h, t_h)\}$ in the support of $\Phi^\star(\mathbf{z})$,

$$\mathbf{z} \in \arg\max_{\mathbf{z}} \sum_{i \in \{l,h\}} \mu_i (v_i x_i - t_i) \left[1 - \pi + \pi \int_{\mathbf{z}'} \chi_i(\mathbf{z}, \mathbf{z}') \Phi^\star(d\mathbf{z}')\right], \tag{3}$$

where

$$\chi_i(\mathbf{z}, \mathbf{z}') = \left\{ \begin{array}{c} 0 \\ \frac{1}{2} \\ 1 \end{array} \right\} \quad \text{if} \quad t_i + c_i(1 - x_i) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} t_i' + c_i(1 - x_i'). \tag{4}$$

The function χ_i reflects the seller's optimal choice. We assume that a seller who is indifferent between menus randomizes with equal probability. Within a given menu, we assume that sellers do not randomize; for any incentive compatible contract, sellers choose the contract intended for their type, as in most of the mechanism design literature (see, e.g., Myerson (1985), Dasgupta et al. (1979)).

3 Properties of Equilibria

Characterizing the equilibrium described above requires solving for a distribution over four-dimensional menus. In this section, we first show that each menu can be summarized by the indirect utilities that it delivers to each type of seller, so that equilibrium strategies can be defined by a joint distribution over two-dimensional objects. Then, we establish that the marginal distributions of offers intended for each type of seller have fully connected support and no mass points. Finally, we establish that, in equilibrium, the two contracts offered by any buyer are ranked in exactly the same way by both low- and high-type sellers: a low-type seller prefers buyer 1's offer to buyer 2's offer if and only if a high-type seller does as well. This property of equilibria, which we call "strictly rank-preserving," simplifies the characterization even more, as the marginal distribution of offers for high-quality sellers can be expressed as a strictly monotonic transformation of the marginal distribution of offers for low-quality sellers.

3.1 Utility Representation

As a first step, we establish that any menu can be summarized by two numbers, (u_l, u_h), where $u_i = t_i + c_i(1 - x_i)$ denotes the utility received by a type $i \in \{l, h\}$ seller from accepting a contract $\mathbf{z}_i$.

Lemma 1. *In any equilibrium, for almost all* $\mathbf{z} \in \text{supp}(\Phi^\star)$*, it must be that* $x_l = 1$ *and* $t_l = t_h + c_l(1 - x_h)$*.*

In words, Lemma 1 states that all equilibrium menus require that low-quality sellers trade their entire endowment, and that their incentive compatibility constraint always binds. This is reminiscent of the "no-distortion-at-the-top" result in the taxation literature, or that of full insurance for the high-risk agents in Rothschild and Stiglitz (1976).

Corollary 1. *In equilibrium, any menu of contracts* $\{(x_l, t_l), (x_h, t_h)\} \in \left([0,1] \times \mathbb{R}_+\right)^2$ *can be summarized by a pair* (u_l, u_h) *with* $x_l = 1$*,* $t_l = u_l$*,*

$$x_h = 1 - \frac{u_h - u_l}{c_h - c_l}, \text{ and} \tag{5}$$

$$t_h = \frac{u_l c_h - u_h c_l}{c_h - c_l}. \tag{6}$$

Notice that, since $0 \leqslant x_h \leqslant 1$, feasibility requires that the pair (u_l, u_h) satisfies

$$c_h - c_l \geqslant u_h - u_l \geqslant 0. \tag{7}$$

In what follows, we will often refer to the requirement $u_h \geqslant u_l$ as a "monotonicity constraint." Note that, when this constraint binds, Corollary 1 implies that $x_h = 1$ and $t_h = t_l$.

3.2 Recasting the Buyer's Problem and Equilibrium

Lemma 1 and Corollary 1 allow us to recast the problem of a buyer as choosing a menu of indirect utilities, (u_l, u_h), taking as given the distribution of indirect utilities offered by the other buyer. For any menu (u_l, u_h), a buyer must infer the probability that the menu will be accepted by a type $i \in \{l, h\}$ seller. In order to calculate these probabilities, let us define the marginal distributions

$$F_i(u_i) = \int_{\mathbf{z}_i'} \mathbf{1}\left[t_i' + c_i\left(1 - x_i'\right) \leqslant u_i\right] \Phi\left(d\mathbf{z}_i'\right)$$

for $i \in \{l, h\}$. In words, $F_l(u_l)$ and $F_h(u_h)$ are the probability distributions of indirect utilities arising from each buyer's mixed strategy. When these distributions are continuous and have no mass points, the probability that a contract intended for a type i seller is accepted is simply $1 - \pi + \pi F_i(u_i)$, i.e., the probability that the seller is captive plus the probability that he is noncaptive but receives another offer less than u_i. However, if $F_i(\cdot)$ has a mass point at u_i, then the fraction of noncaptive sellers of type i attracted to a contract with value u_i is given by $\tilde{F}_i(u_i) = \frac{1}{2}F_i^-(u_i) + \frac{1}{2}F_i(u_i)$, where $F_i^-(u_i) = \lim_{u \nearrow u_i} F_i(u)$ is the left limit of F_i at u_i. Given $\tilde{F}_i(\cdot)$, a buyer solves

$$\max_{u_l \geqslant c_l,\ u_h \geqslant c_h} \mu_l\left(1 - \pi + \pi\tilde{F}_l(u_l)\right)\Pi_l(u_l, u_h) + \mu_h\left(1 - \pi + \pi\tilde{F}_h(u_h)\right)\Pi_h(u_l, u_h) \tag{8}$$

subject to (7), with

$$\Pi_l(u_l, u_h) \equiv v_l x_l - t_l = v_l - u_l \tag{9}$$

$$\Pi_h(u_l, u_h) \equiv v_h x_h - t_h = v_h - u_h \frac{v_h - c_l}{c_h - c_l} + u_l \frac{v_h - c_h}{c_h - c_l}. \tag{10}$$

In words, $\Pi_i(u_l, u_h)$ is the buyer's payoff conditional on the offer u_i being accepted by a type i seller. We refer to the objective in (8) as $\Pi(u_l, u_h)$.

Before proceeding, note that $\Pi_h(u_l, u_h)$ is increasing in u_l: by offering more utility to low-quality sellers, the buyer relaxes the incentive constraint and can earn more profits when he trades with high-quality sellers. As a result, one can easily show that the profit function $\Pi(u_l, u_h)$ is (at least) weakly supermodular. This property will be important in several of the results we establish below.

Using the optimization problem described above, we can redefine the equilibrium in terms of the distributions of indirect utilities. In particular, for each u_l, let

$$U_h(u_l) = \{u'_h \in \arg\max \Pi(u_l, u'_h) \mid u'_h \geqslant c_h \cap c_h - c_l \geqslant u'_h - u_l \geqslant 0\}.$$

The equilibrium can then be described by the marginal distributions $\{F_i(u_i)\}_{i\in\{l,h\}}$ together with the requirement that a joint distribution function must exist. In other words, a probability measure Φ over the set of feasible pairs (u_l, u_h) must exist such that, for each $u_l > u'_l$ and $u_h > u'_h$

$$\begin{aligned}
1 &= \Phi(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l)\}, \hat{u}_l \in [c_l, v_h]) \\
F_l^-(u_l) - F_l(u'_l) &= \Phi(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_l \in (u'_l, u_l)\}), \qquad (11) \\
F_h^-(u_h) - F_h(u'_h) &= \Phi(\{(\hat{u}_l, \hat{u}_h); \hat{u}_h \in U_h(\hat{u}_l), \hat{u}_h \in (u'_h, u_h)\}). \qquad (12)
\end{aligned}$$

Note that this definition imposes two requirements. The first is that buyers behave optimally: for each u_l, the joint probability measure puts a positive weight only on $u_h \in U_h(u_l)$. The second is aggregate consistency, i.e., that F_l and F_h are marginal distributions associated with a joint measure of menus.

3.3 Basic Properties of Equilibrium Distributions

In this section, we establish that, in equilibrium, the distributions $F_l(u_l)$ and $F_h(u_h)$ are continuous and have connected support, i.e., there are neither mass points nor gaps in either distribution.

Proposition 1. *The marginal distributions F_l and F_h have connected support. They are also continuous, with the possible exception of a mass point in F_l at v_l.*

As in Burdett and Judd (1983), the proof of Proposition 1 rules out gaps and mass points in the distribution by constructing profitable deviations. A complication that arises in our model, which does not arise in Burdett and Judd (1983), is that payoffs are interdependent, e.g., a change in the utility offered to low-quality sellers changes the contract—and hence the profits—that a buyer receives from high-quality sellers. We prove the properties of F_l and F_h described in Proposition 1 sequentially: we first show that F_h is continuous and strictly increasing, and then apply an inductive argument to prove that F_l has connected support and is continuous, with a possible exception at the lower bound of the support. An important step in the induction argument, which we later use more generally, is to show that the objective function $\Pi(u_l, u_h)$ is strictly supermodular. We state this here as a lemma.

Lemma 2. *The profit function is strictly supermodular, i.e.,*

$$\Pi\left(u_{l1}, u_{h1}\right) + \Pi\left(u_{l2}, u_{h2}\right) \geqslant \Pi\left(u_{l2}, u_{h1}\right) + \Pi\left(u_{l1}, u_{h2}\right), \; \forall u_{i1} \geqslant u_{i2}, \; i \in \{l, h\}$$

with strict inequality when $u_{i1} > u_{i2}, \; i \in \{l, h\}$.

As noted above, the supermodularity of the buyer's profit function reflects a basic complementarity between the indirect utilities offered to low- and high-quality sellers. An important implication is that the correspondence $U_h\left(u_l\right)$ is weakly increasing. We use this property to construct deviations to rule out gaps and mass points in the distribution F_l almost everywhere in its support; later, in Section 4, we show that these mass points only occur in a knife-edge case. Hence, generically, the marginal distribution F_l has connected support and no mass points in its support.

3.4 Strict rank-preserving

In this section, we establish that every equilibrium has the property that the menus being offered are *strictly rank-preserving*—that is, low- and high-quality sellers share the same ranking over the set of menus offered in equilibrium—with the possible exception of the knife-edge case discussed above. We prove this result by showing that the mapping between a buyer's optimal offer to low- and high-quality sellers, $U_h(u_l)$, is a well defined, strictly increasing function. We start with the following definition.

Definition 1. *For any subset* $\mathcal{U}_l$ *of* $\operatorname{Supp}\left(F_l\right)$*, an equilibrium is* ***strictly rank-preserving over*** $\mathcal{U}_l$ *if the correspondence* $U_h\left(u_l\right)$ *is a strictly increasing function of* u_l *for all* $u_l \in \mathcal{U}_l$*. An equilibrium is* ***strictly rank-preserving*** *if it is strictly rank-preserving over* $\operatorname{Supp}\left(F_l\right)$*.*

Equivalently, an equilibrium is strictly rank-preserving if, for any $\left(u_l, u_h\right)$ and $\left(u_l', u_h'\right)$ in the equilibrium support, $u_l > u_l'$ if and only if $u_h > u_h'$. Given this terminology, we now establish a key result.

Theorem 1. *All equilibria are strictly rank-preserving over the set* $\operatorname{Supp}\left(F_l\right) \setminus \{v_l\}$*.*

Theorem 1 follows from the facts established above. In particular, the strict supermodularity of $\Pi(u_l, u_h)$ implies that $U_h\left(u_l\right)$ is a weakly increasing correspondence. However, since $F_l\left(\cdot\right)$ and $F_h\left(\cdot\right)$ are strictly increasing and continuous, we show that $U_h(u_l)$ can neither be multi-valued nor have flats. Intuitively, if there exists a $u_l > \min \operatorname{Supp}(F_l) \equiv \underline{u}_l$ and $u_h' > u_h$ such that $u_h, u_h' \in U_h(u_l)$, then the supermodularity of $\Pi(u_l, u_h)$ implies that $[u_h, u_h'] \subset U_h(u_l)$. Since $F_h(\cdot)$ has connected support, if U_h were a correspondence for some u_l, then this would imply that $F_l(\cdot)$ must have a mass point at u_l, which contradicts Proposition 1. Similarly, if there exists u_h and $u_l' > u_l$ offered in equilibrium such that $U_h(u_l') = U_h(u_l) = u_h$, then F_h would feature a mass point, in contradiction with Proposition 1. Hence, $U_h(u_l)$ must be a strictly increasing *function* for all $u_l > \underline{u}_l$.

If $F_l(\cdot)$ is continuous everywhere, then every menu offered in equilibrium is accepted by the same fraction of low- and high-quality noncaptive sellers; we state this formally in the following Corollary.

Corollary 2. *If* F_l *and* F_h *are continuous, then* $F_h(U_h(u_l)) = F_l(u_l)$.

Taken together, Theorem 1 and Corollary 2 simplify the construction of an equilibrium. Specifically, when an equilibrium exists in which the marginal distributions are continuous, then the equilibrium can be described compactly by the marginal distribution $F_l(u_l)$ and the function $U_h(u_l)$.

4 Construction of Equilibrium

In this section, we use the properties established above to construct equilibria. Then, we show that the equilibrium we construct is unique. In this sense, we characterize the entire set of equilibrium outcomes.

4.1 Special Cases: Monopsony and Perfect Competition

To fix ideas, we first characterize equilibria in the well-known special cases of $\pi = 0$ and $\pi = 1$, i.e., when sellers face a monopsonist and when they face two buyers in Bertrand competition, respectively. As we will see, several features of the equilibrium in these two extreme cases guide our construction of equilibria for the general case of $\pi \in (0, 1)$.

When $\pi = 0$, so that each seller meets with at most one buyer, the buyers solve

$$\max_{(u_l, u_h)} \qquad \mu_l(\nu_l - u_l) + \mu_h \left[\nu_h - u_h \frac{\nu_h - c_l}{c_h - c_l} + u_l \frac{\nu_h - c_h}{c_h - c_l} \right],$$

subject to the monotonicity and feasibility constraints in (7). The solution to this problem, summarized in Lemma 3 below, is standard and, hence, we omit the proof.

Lemma 3. *Suppose* $\pi = 0$*, and let*

$$\phi_l \equiv 1 - \frac{\mu_h}{\mu_l} \left(\frac{\nu_h - c_h}{c_h - c_l} \right). \tag{13}$$

If $\phi_l > 0$*, then the unique equilibrium has* $u_l = c_l$ *with* $x_l = 1$ *and* $u_h = c_h$ *with* $x_h = 0$*; if* $\phi_l < 0$*, then* $u_l = u_h = c_h$ *with* $x_l = x_h = 1$*; and if* $\phi_l = 0$*, then* $u_l \in [c_l, c_h]$ *with* $x_l = 1$ *and* $u_h = c_h$ *with* $x_h \in [0, 1]$.

The parameter ϕ_l is a summary statistic for the adverse selection problem: it represents the net marginal cost (to the buyer) of delivering an additional unit of utility to a low-quality seller. It is strictly less than 1 because the direct cost of an additional unit of transfer to a low-quality seller is partially offset by the indirect benefit of relaxing this seller's incentive constraint, which allows the buyer to trade more with a high-quality seller. This indirect benefit is captured by the second term on the right-hand side: when this term is large, ϕ_l is small, the cost of trading with high-quality sellers is low, and adverse selection is *mild*. Conversely, when this term is small, ϕ_l is large, it is costly to trade with high-quality sellers, and therefore adverse selection is relatively *severe*. According to this measure, adverse selection is thus severe when the relative fraction of high-quality sellers, μ_h/μ_l, is small; the gains from trading with high-quality sellers, $\nu_h - c_h$, are relatively small; and/or the information rents associated with separating high- and low-quality sellers, $c_h - c_l$, are large.

When $\phi_l > 0$, the net cost to a buyer of increasing u_l is positive, so she sets u_l as low as possible, i.e., $u_l = c_l$. This implies that the high-quality seller is entirely shut out, i.e., $x_h = 0$. Otherwise, when $\phi_l < 0$, increasing u_l yields a net benefit to the buyer. As a result, a buyer raises u_l until the monotonicity constraint in (7) binds, i.e., she pools high- and low-quality sellers, offering $u_h = u_l = c_h$.

Before proceeding to the perfectly competitive case, we highlight two features of the equilibrium under monopsony. First, buyers offer *separating menus* ($u_h > u_l$) when ϕ_l is positive and *pooling menus* ($u_h = u_l$) when ϕ_l is negative. Second, they make non-negative payoffs on both types when $\phi_l > 0$, but lose money on low-quality sellers when $\phi_l < 0$. In other words, the equilibrium features *cross-subsidization* when ϕ_l is negative, but not when ϕ_l is positive.

When competition is perfect, i.e., when $\pi = 1$, our setup becomes the same as that in Rosenthal and Weiss (1984), and similar to that of Rothschild and Stiglitz (1976). In this case, when $\phi_l \geqslant 0$, the unique equilibrium is in pure strategies, with buyers offering the standard "least-cost separating" contract; type l sellers earn $u_l = \nu_l$ and type h sellers trade a fraction of their endowment at a unit price of ν_h, such that the incentive constraint of the low-quality seller binds. However, when $\phi_l < 0$, there is no pure strategy equilibrium.[16] In this case, an equilibrium in mixed strategies emerges, as in Rosenthal and Weiss (1984) and Dasgupta and Maskin (1986).[17] Each buyer mixes over menus, all of which involve negative profits from low-quality sellers, offset exactly by positive profits from high-quality sellers, leading to zero profits. The marginal distribution $F_l(\cdot)$ is such that profitable deviations

[16] All buyers offering the least-cost separating contract cannot be an equilibrium, as a pooling offer is profitable. All buyers offering a pooling contract cannot be an equilibrium either, since it is vulnerable to a cream-skimming deviation, wherein a competing buyer draws away high-quality sellers by offering a contract with $x < 1$ but at a higher price.

[17] Luz (2017) shows that the equilibrium is unique.

are ruled out. The following lemma summarizes these results.

Lemma 4. *When $\pi = 1$, the unique equilibrium is: (i) if $\phi_l \geqslant 0$, then $u_l = v_l$, $x_l = 1$, $u_h = \frac{v_h(c_h - c_l) + v_l(v_h - c_h)}{v_h - c_l}$, and $x_h = \frac{v_l - c_l}{v_h - c_l}$; (ii) if $\phi_l < 0$, then the symmetric equilibrium is described by the distribution*

$$F_l(u_l) = \left(\frac{u_l - v_l}{\mu_h(v_h - v_l)}\right)^{-\phi_l}, \tag{14}$$

with $\text{Supp}(F_l) = [v_l, \bar{v}]$ and $F_h(u_h) = F_l(U_h(u_l))$, where $\bar{v} = \mu_h v_h + \mu_l v_l$ and $U_h(u_l)$ satisfies

$$\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l \Pi_l(u_l, U_h(u_l)) = 0. \tag{15}$$

As with $\pi = 0$, equilibrium when $\pi = 1$ features no cross-subsidization when $\phi_l \geqslant 0$ and cross-subsidization when $\phi_l < 0$. However, unlike the case with $\pi = 0$, equilibrium with $\pi = 1$ features separating contracts for all values of ϕ_l. These properties guide our construction of equilibria in the next section, when we study the general case of $\pi \in (0, 1)$.

4.2 General Case: Imperfect Competition

We now describe how to construct equilibria when $\pi \in (0, 1)$. Recall that an equilibrium is summarized by a distribution $F_l(u_l)$ and a strictly increasing function $U_h(u_l)$. A key determinant of the structure of equilibrium menus is whether the monotonicity constraint in (7) is binding. When it is slack, the local optimality (or first-order) condition for u_l, along with the strict rank-preserving condition that relates $F_h(U_h(u_l)) = F_l(u_l)$ together characterize the equilibrium distribution $F_l(u_l)$. The function $U_h(u_l)$ then follows from the requirement that all menus $(u_l, U_h(u_l))$ must yield the buyer equal profits. When the monotonicity constraint is binding, the policy function is, by definition, $U_h(u_l) = u_l$.[18]

Our analysis of $\pi = 0$ and $\pi = 1$ points to the importance of ϕ_l. Recall that when $\phi_l > 0$, the monotonicity constraint was always slack, i.e., $U_h(u_l) > u_l$ for all $u_l \in \text{Supp}(F_l)$. When $\phi_l < 0$, on the other hand, the monotonicity constraint was binding only when $\pi = 0$ and slack at $\pi = 1$. Guided by these results, we discuss our construction separately for the $\phi_l > 0$ and the $\phi_l < 0$ cases.[19]

[18]Of course, $u_h = U_h(u_l)$ must be locally optimal as well, but this condition is implied by the joint requirements on u_l and $U_h(u_l)$ described above.

[19]The equilibrium when $\phi_l = 0$ has a slightly different structure and, for the sake of brevity, we relegate analysis of this knife-edge case to Appendix D.

Case 1: $\phi_l > 0$. Given the analysis of $\pi = 0$ and $\pi = 1$, we conjecture that the monotonicity constraint is slack for any $\pi \in (0,1)$. Proposition 2 confirms that this is indeed the case.

Proposition 2. *For any $\pi \in (0,1)$ and $\phi_l > 0$, there exists an equilibrium in which* (i) *F_l satisfies the differential equation*

$$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)}(v_l - u_l) = \phi_l, \tag{16}$$

with the boundary condition $F_l(c_l) = 0$; and (ii) *$U_h(u_l) > u_l$ and satisfies the equal profit condition*

$$(1-\pi+\pi F_l(u_l))\left[\mu_h \Pi_h(u_l, U_h(u_l)) + \mu_l (v_l - u_l)\right] = \mu_l(1-\pi)(v_l - c_l). \tag{17}$$

Equation (16) is derived by taking the first-order condition of (8) with respect to u_l—holding u_h fixed—and then imposing the strict rank-preserving property.[20] This necessary condition is familiar from basic production theory. The left-hand side is the marginal benefit to the buyer of increasing u_l, i.e., the product of the semi-elasticity of demand and the profit per trade. The right-hand side, ϕ_l, is the marginal cost of increasing the utility of the low-quality seller, taking into account the fact that increasing u_l relaxes the incentive constraint.[21] Note that, even though (16) ensures that local deviations by a buyer from an equilibrium menu are not profitable, completing the proof requires ensuring that there are no profitable global deviations as well; we establish that this is true in Appendix A.2.1.

The boundary condition requires that the lowest utility offered to the low-quality seller is c_l. From (17), and the fact that $F_l(c_l) = 0$, we find $U_h(c_l) = c_h$, so that the worst menu offered in equilibrium coincides with the monopsony outcome. Intuitively, if the worst menu offers more utility to low-quality sellers than c_l, the buyer could profit by decreasing u_l and u_h; the gains associated with trading at better terms with the low types would exceed the losses associated from trading less quantity with high types, precisely because $\phi_l > 0$. Given that $\underline{u}_l = c_l$, if the worst equilibrium menu offers more utility to high-quality sellers than c_h, then a buyer offering this menu could profit by decreasing u_h; his payoff from trading with high types would increase without changing the payoffs from trading with low types.

The final equilibrium object, $U_h(u_l)$, is characterized by the equal profit condition: the left side of (17) defines the buyer's payoff from the menu $(u_l, U_h(u_l))$, while the right side is the profit earned from the worst contract offered in equilibrium. Figure 1 plots the two equilibrium functions in this region.

[20] As we discuss in the proof of Proposition 2, this first-order condition requires three assumptions: that $u_h > u_l$ for all menus; that there is no mass point at the lower bound of the support of $F_l(u_l)$; and that the implied quantity traded by the high-quality seller is interior in all trades, i.e., $0 < x_h = (u_h - u_l)/(c_h - c_l) < 1$, except possibly at the boundary of the support of F_l. All of these assumptions are confirmed in equilibrium.

[21] It is straightforward to derive a closed-form solution for $F_l(u_l)$ from (16); see equation (1) in the Appendix.

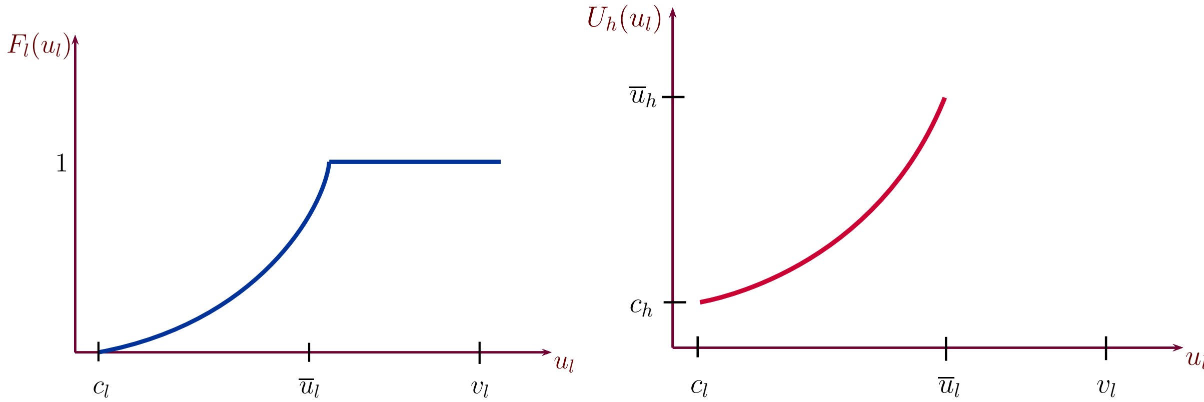

Figure 1: Equilibrium for $\pi \in (0,1)$, $\phi_l > 0$. The left panel plots the CDF $F_l(u_l)$ and the right panel plots the mapping $U_h(u_l)$.

Notice from (16) that, since $\phi_l > 0$, our equilibrium has $v_l > u_l$ for all menus in equilibrium, so that buyers earn strictly positive profits from trading with low-quality sellers. It is straightforward to show that buyers also earn strictly positive profits from trading with high-quality sellers. Hence, in this region, the equilibrium features no cross-subsidization, as was the case for $\pi = 0$ and $\pi = 1$. Finally, it is also worth noting that the equilibrium distribution of offers converges to the limiting cases as π converges to both 0 and 1; in the former case, the distribution converges to a mass point at the monopsony outcome, while in the latter case, the distribution converges to a mass point at the least-cost separating outcome.

Case 2: $\phi_l < 0$**.** In this region of the parameter space, the equilibrium features a pooling menu when $\pi = 0$ and a distribution of separating menus when $\pi = 1$. This leads us to conjecture that the equilibrium for $\pi \in (0,1)$ can feature pooling, separating, or a mixture of the two, depending on the value of π. The following lemma formalizes this conjecture and shows the existence of a threshold utility for the offer to low-quality sellers, $\hat{u}_l$, such that all offers with $u_l < \hat{u}_l$ are pooling menus, while all offers with $u_l > \hat{u}_l$ are separating menus.[22] Depending on whether this threshold lies at the lower bound, the upper bound, or in the interior of the support of $F_l(u_l)$, there are three possible cases, respectively: all equilibrium offers are separating menus, all are pooling menus, or there is a mixture with some pooling menus (offering relatively low utility to the seller) and some separating menus (offering higher utility). Later, in Proposition 4, we provide conditions on ϕ_l and π under which each case obtains.

Proposition 3. *For any $\pi \in (0,1)$ and $\phi_l < 0$, there exists an equilibrium in which:* (i) *there exists a threshold*

[22] At this point, it may seem arbitrary to conjecture that pooling occurs at the bottom of the distribution and separation at the top. As we will discuss later in the text, the reason this is ultimately true is that the cream-skimming deviation—which makes the pooling offer suboptimal—becomes more attractive as the indirect utility being offered increases.

$\hat{u}_l$ *such that, for any* u_l *in the interior of* $\text{Supp}(F_l)$*:*

1. *if* $u_l \leqslant \hat{u}_l$, $U_h(u_l) = u_l$ *and* F_l *satisfies*

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} (\mu_h v_h + \mu_l v_l - u_l) = 1 , \tag{18}$$

2. *if* $u_l > \hat{u}_l$, $U_h(u_l) > u_l$ *and* F_l *satisfies (16).*

(ii) $U_h(\underline{u}_l) = c_h$ *and* $U_h(\overline{u}_l) = \overline{u}_l$.

To understand the first set of (necessary) conditions in Proposition 3, consider the region where the buyers offer pooling menus. Here, buyers trade off profit per trade against the probability of trade, with no interaction between offers and incentive constraints. As a result, the equilibrium in this pooling region behaves as in the canonical Burdett and Judd (1983) single-quality model, with the buyer's payoff equal to the average value $\mu_h v_h + (1 - \mu_h) v_l$. This yields (18). In the region where buyers offer separating menus, $F_l(u_l)$ is characterized by the local optimality condition (16), exactly as in the $\phi_l > 0$ case. Recall from our discussion that this differential equation accounts explicitly for the effect of an offer u_l on the seller's incentive constraint. In this region, $U_h(u_l)$ is determined by the equal profit condition.

The second part of the result describes boundary conditions for the worst and best menus offered in equilibrium. The first condition requires that the worst menu yields utility c_h to high-quality sellers. To see why, suppose the worst menu is a pooling menu with $U_h(\underline{u}_l) = \underline{u}_l > c_h$. Then, lowering both u_h and u_l leads to strictly higher profits. If the worst menu is separating with $U_h(\underline{u}_l) > c_h$, then a downward deviation in only u_h is feasible and strictly increases profits. The second condition requires that the best menu offered in equilibrium is a pooling menu. Intuitively, if the best menu offered in equilibrium were a separating menu, then $x_h < 1$. This cannot be optimal when $\phi_l < 0$: the buyer can trade more with the high-quality seller by increasing the utility offered to low-quality sellers. Since this is already the best menu in equilibrium, this deviation has no impact on the number of sellers the buyer attracts but yields strictly higher profits.

Given these properties, we now establish two critical values—$\phi_1(\pi)$ and $\phi_2(\pi)$, with $\phi_2(\pi) < \phi_1(\pi) < 0$—that determine which of the three cases described above emerge in equilibrium. When $\phi_l < \phi_2(\pi)$, the threshold $\hat{u}_l = \overline{u}_l$ and there is an *all pooling* equilibrium. When $\phi_l > \phi_1(\pi)$, the monotonicity constraint is slack almost everywhere, so that $\hat{u}_l = \underline{u}_l$, and the equilibrium features *all separating* menus. Finally, if ϕ_l lies between these two critical values, we have a *mixed* equilibrium, with an intermediate threshold $\hat{u}_l \in (\underline{u}_l, \overline{u}_l)$. Figure 2 illustrates $U_h(u_l)$ for all three possibilities.

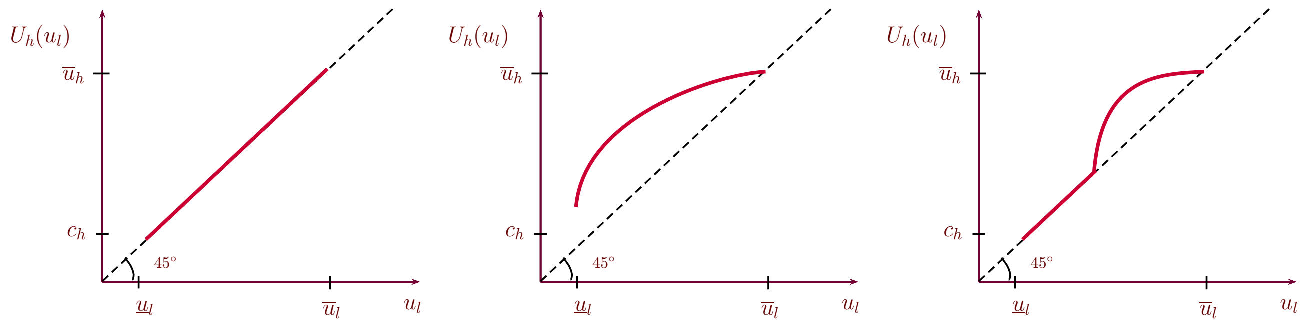

Figure 2: The mapping $U_h(u_l)$ for all pooling, all separating, and mixed equilibria when $\phi_l < 0$

Proposition 4. *For any $\pi \in (0, 1)$, there exist cutoffs $\phi_2(\pi) < \phi_1(\pi) < 0$ such that an all pooling equilibrium exists for all $\phi_l \leqslant \phi_2(\pi)$, a mixed equilibrium exists for all $\phi_l \in (\phi_2(\pi), \phi_1(\pi))$, and an all separating equilibrium exists for all $\phi_l \in (\phi_1(\pi), 0)$.*

Intuitively, for a pooling menu (u_l, u_l) to be offered in equilibrium, the cream-skimming deviation $(u_l - \varepsilon, u_l)$ for some $\varepsilon > 0$ cannot yield strictly higher profits. To see how incentives to cream-skim vary with ϕ_l and π, notice that there are two sources of higher profits from the menu $(u_l - \varepsilon, u_l)$, relative to the candidate pooling menu. First, it decreases the loss conditional on trading with a low-quality seller. Second, it reduces the probability of trading with a noncaptive low-quality seller; since the buyer loses money on these sellers, this reduction in trading probability raises profits. The cost of cream-skimming is that the buyer earns lower profits on high-quality sellers. Therefore, incentives to cream-skim are weak—and thus pooling is easier to sustain—when high-quality sellers are relatively abundant (ϕ_l very negative) and/or there are relatively few noncaptive sellers (π is small).

The higher the level of utility being offered in a pooling menu, the more vulnerable it is to cream-skimming. Hence, if such a deviation is profitable at the lowest candidate value, c_h, then pooling cannot be sustained at all: this is the condition that determines the cutoff $\phi_1(\pi)$. Similarly, the cutoff $\phi_2(\pi)$ defines the boundary at which cream-skimming is not profitable even at the best pooling menu, $\overline{u_l}$. We derive these thresholds formally and provide a full equilibrium characterization in Appendix A.2.2.

Notice that, in all three cases, $u_l > v_l$ (since $u_l \geqslant c_h > v_l$) so that buyers always suffer losses when trading with low-quality sellers. Hence, as in the extreme cases of $\pi = 0$ and $\pi = 1$, there is cross-subsidization in every equilibrium when $\phi_l < 0$. Finally, as in the case of $\phi_l > 0$, the equilibrium distribution converges to the limiting cases as π converges to both 0 and 1.

Figure 3 summarizes the various types of equilibria and the regions in which each one obtains. The x- and y-axes represent the intensity of competition and severity of adverse selection, respectively. Recall that the latter is summarized by ϕ_l, which is a function of μ_h, the fraction of high-quality goods, as well as the valuations v_h, c_h, c_l. For concreteness, we use μ_h to vary ϕ_l on the y-axis—a higher fraction of high-quality goods implies a lower ϕ_l and, therefore, milder adverse selection.[23]

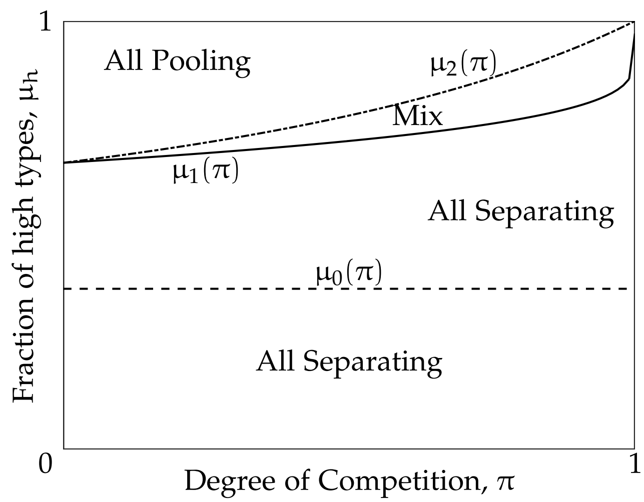

Figure 3: Equilibrium regions

4.3 Uniqueness

In Theorem 2, below, we establish that the equilibria constructed above are unique. For intuition, we sketch the arguments here for $\phi_l \neq 0$.[24] First, we show that equilibria do not feature a mass point, even at v_l. Next, when $\phi_l > 0$, we prove that equilibria do not feature pooling menus on a positive measure subset of F_l; since equilibria have no mass points and must be separating almost everywhere, the equilibrium we construct in Proposition 2 describes the unique equilibrium. Finally, when $\phi_l < 0$, we demonstrate uniqueness in steps. First, we show that any equilibrium features pooling at the upper bound of the support of F_l. Second, we prove that any equilibrium features at most one interval of pooling menus followed by at most one interval of separating menus. Third, we prove that the equilibria characterized in Proposition 4 are mutually exclusive, so that equilibria without mass points are unique. Since no equilibrium features mass points when $\phi_l < 0$, these results establish the uniqueness of the equilibrium characterized in Proposition 4.

Theorem 2. *For any $\pi \in (0, 1)$ and $\phi_l \in \mathbb{R}$, there exists an equilibrium and it is unique.*

[23]The boundaries are redefined accordingly: $\mu_h \leqslant \mu_0$ if and only if $\phi_l \geqslant 0$ and $\mu_h \leqslant \mu_j(\pi)$ if and only if $\phi_l \geqslant \phi_j(\pi)$, $j \in \{1, 2\}$.

[24]In Appendix D, we also prove uniqueness for the knife-edge case of $\phi_l = 0$.

Note that we obtain a unique equilibrium without any refinements or restrictions on off-path behavior. This is because buyers' payoffs are well defined for any offer. In particular, since buyers are not capacity-constrained, the fraction of type $i \in \{l, h\}$ sellers who accept any offer (u_l, u_h) is uniquely determined by the (exogenous) meeting technology and the (endogenous) distribution of offers.[25]

4.4 Discussion

The equilibrium characterized above has a number of testable implications. To start, we highlight three robust predictions about the properties of equilibrium menus. First, the strict rank-preserving property suggests a positive correlation between the contracts that buyers offer to different types of sellers: those buyers who make attractive offers to low-quality sellers will also make attractive offers to high-quality sellers. Hence, in equilibrium, buyers do not specialize in trading with a particular type of seller, but rather trade with equal frequency across all types. Second, whether buyers pool different types of sellers or separate them (using a menu of options) depends crucially on the severity of the two frictions.[26] Separation is more likely when adverse selection is relatively severe—so that the information costs of trading with high-quality sellers are large relative to the benefits—and competition is relatively strong—so that the payoffs from cream-skimming are relatively high.[27] Pooling is more likely when competition among buyers is weak and adverse selection is mild. Third, the theory predicts that menus that are less attractive, from the perspective of sellers, are more likely to be pooling. In other words, those who are posting offers with relatively unattractive terms should be offering fewer options and should account for a smaller share of observed transactions.

Our analysis also has implications about dispersion. In the region with separating menus, the model predicts dispersion *within* and *across* types. This is true both for quantities traded (coverage in an insurance context or loan size in a credit market context) as well as prices (premia or interest rates, respectively). The extent of dispersion—both the support and the standard deviation of the quantity/price distributions—is determined by the interaction of competition (measured by π) and adverse selection (measured by ϕ_l). This joint dependence calls into question the practice of identifying imperfect competition or asymmetric information in isolation using cross-sectional dispersion. For example, a common

[25] Refinements are often necessary in models with capacity-constrained buyers, when two types of sellers would like to accept an off-path offer, and the probability that each type is able to execute the trade is not pinned down.

[26] This result stands in stark contrast to, e.g., Guerrieri et al. (2010). In that model, and many like it, the quantity traded with high-quality sellers is *independent* of the distribution of types in the market; trade with high-quality sellers is distorted even if the fraction of low-quality goods in the market is arbitrarily small.

[27] Consistent with our findings, Decarolis and Guglielmo (2016) find evidence of greater cream-skimming by health insurance providers when the market is more competitive.

empirical strategy to identify adverse selection is to test the correlation between the quantity an agent trades and her type, as measured by ex-post outcomes.[28] In our equilibrium, depending on market structure, the correlation between the seller's quality and the quantity she sells can be either negative or zero. As a result, using the relationship between quantity and type without accounting for the imperfect nature of competition is likely to yield misleading conclusions. A similar concern applies to the strategy of identifying search frictions from price dispersion.[29] In markets where adverse selection is a concern, the magnitude of cross-sectional variation in terms of trade is also a function of selection-related parameters. Obtaining an accurate assessment of trading frictions in such settings thus requires controlling for the underlying distribution of types.

5 Increasing Competition and Reducing Information Asymmetries

Many markets in which adverse selection is a first-order concern are experiencing dramatic changes. Some of these changes are regulatory in nature; for example, as we describe in greater detail below, there are several recent policy initiatives to make health insurance markets and over-the-counter markets for financial securities more competitive and transparent. Other changes derive from technological improvements; for example, advances in credit scoring reduce information asymmetries in loan markets.

In this section, we use the framework developed above to examine the likely effects of these types of changes on economic activity. Our metric for economic activity is the utilitarian welfare function, which measures the expected gains from trade that are realized in equilibrium. We show that increasing competition or reducing information asymmetries can worsen the distortions from adverse selection—thereby *decreasing* the expected gains from trade—when markets are relatively competitive. As a result, initiatives to make these markets more competitive or transparent are only welfare-improving when both frictions are relatively severe, i.e., when buyers have a lot of market power (i.e., when π is low) and the adverse selection problem is relatively severe (i.e., when ϕ_l is high).

While these comparative statics are certainly informative, one may be concerned that they reflect an inefficiency in the particular game we postulate between buyers and sellers. At the end of this section, we compare equilibrium outcomes to a constrained efficient benchmark. We argue that, in the region of the parameter space where $\phi_l > 0$, equilibrium gains from trade coincide with those in the constrained

[28] This technique for identifying adverse selection has been applied to a number of markets, following the seminal paper by Chiappori and Salanie (2000); recent examples include Ivashina (2009), Einav et al. (2010b), and Crawford et al. (2015).

[29] Using price dispersion to help identify search frictions is standard in the industrial organization literature; see, e.g., Gavazza (2016).

efficient benchmark. This suggests that the comparative statics results that we described above are not a consequence of the particular game we have modeled, but rather a more fundamental feature of markets with adverse selection and imperfect competition.

5.1 Utilitarian Welfare

As noted above, our metric for economic activity will be the objective of a utilitarian planner, defined as the expected gains from trade realized between buyers and sellers, or

$$\begin{aligned} W(\pi, \mu_h) &= (1-\mu_h)(v_l - c_l) + \mu_h \left\{ \frac{2-2\pi}{2-\pi} \int \left[x_h(u_l)(v_h - c_h) \right] dF_l(u_l) \right. \\ & \left. + \frac{\pi}{2-\pi} \int \mu_h \left[x_h(u_l)(v_h - c_h) \right] d\left(F_l(u_l)^2 \right) \right\}, \end{aligned} \tag{19}$$

where, in a slight abuse of notation, we let

$$x_h(u_l) = 1 - \frac{U_h(u_l) - u_l}{c_h - c_l}. \tag{20}$$

The first term in (19) represents the gains from trade generated by low-quality goods; since all sellers receive at least one offer and $x_l = 1$ in every trade, all low-quality goods are transferred to the buyer. The second term captures the expected gains from trade between buyers and captive high-quality sellers. In particular, from equation (1), we can write the measure of captive sellers as $1 - p = \frac{2-2\pi}{2-\pi}$. A randomly selected captive high-quality seller transfers $x_h(u_l)$ to the buyer and consumes the remaining $1 - x_h(u_l)$ herself, where u_l is drawn from $F_l(u_l)$. Finally, the last term in (19) captures the expected gains from trade between buyers and noncaptive high-quality sellers. A measure $p = \frac{\pi}{2-\pi}$ of sellers are noncaptive and, since noncaptive sellers choose the maximum indirect utility among the two offers they receive, they trade an amount $x_h(u_l)$, where u_l is drawn from $F_l(u_l)^2$.

5.2 Increasing Competition

We first study the effects of increasing competition, which has been a common policy response to address perceived failures in markets for insurance, credit, and certain types of financial securities.[30] We do so

[30] For example, a recent report by the Congressional Budget Office (2014) argues for "fostering greater competition" in health insurance plans by developing "policies that would increase the average number of sponsors per region," which would then "increase the likelihood that beneficiaries would select low-cost plans." Similarly, the U.S. Treasury (2010) argued that the Consumer Financial Protection Bureau "will make consumer financial markets more transparent – and that's good for everyone: The agency will give Americans [...] the tools they need to comparison shop for the best prices and the best loans, which will [...] increase competition and innovations that benefit borrowers." A similar rationale underlies the Core Principles

by examining the relationship between welfare and competition, as captured by π. In Proposition 5, we establish that welfare is maximized at $\pi = 0$ when the adverse selection problem is relatively mild. However, when the adverse selection problem is severe, we show that W is hump-shaped in π; i.e., there is an *interior* level of competition that maximizes welfare in this region of the parameter space.

Proposition 5. *If $\phi_l \leqslant 0$, welfare is maximized at $\pi = 0$. Otherwise, it is maximized at a $\pi \in (0, 1)$.*

The first result in Proposition 5 is straightforward. Since a monopsonist offers a pooling contract in this region of the parameter space, all gains from trade are realized. Competition only serves to increase incentives to cream-skim. When these incentives are sufficiently strong, equilibrium menus offer high-quality sellers a higher price but a lower quantity to trade in order to ensure that such a deviation is not profitable, causing a decline in welfare.

The second result is less obvious. To see the intuition, first note that, as π increases, buyers allocate more surplus to sellers: $F_l(u_l)$ shifts to the right (in the sense of first-order stochastic dominance) and $\overline{u}_l$ increases. Second, and crucially, $x_h(u_l)$ is hump-shaped in u_l: it increases near the monopsony offer c_l, and it decreases when u_l is sufficiently close to the competitive offer, v_l. When π is close to zero, $\overline{u}_l$ is relatively small, and the distribution of offers is clustered near the monopsony contract; a small increase in π causes a rightward shift in the density of offers to values of u_l associated with *higher* values of x_h, increasing the gains from trade realized between buyers and high-quality sellers. When π is close to 1, $\overline{u}_l$ is close to v_l, and the distribution of offers is clustered near the competitive contract; a small increase in π causes a shift toward values of u_l associated with *lower* values of x_h.

Therefore, understanding why welfare is hump-shaped in π ultimately requires understanding why $x_h(u_l)$ is hump-shaped in u_l. Note that, ceteris paribus, an increase in u_l relaxes the type l seller's incentive compatibility constraint, allowing buyers to raise x_h. In contrast, ceteris paribus, an increase in u_h tightens the type l seller's incentive compatibility constraint, requiring buyers to lower x_h. Thus, as offers to both types increase, the net effect on x_h depends on which one rises faster—formally, whether $U_h'(u_l)$ is greater or less than 1. Figure 4 illustrates this relationship between the quantity traded with high types, x_h, and the rate at which u_h and u_l increase within the set of equilibrium menus being offered. The figure reveals that u_l rises faster than u_h for smaller values of u_l, so that x_h is increasing in this region. However, as u_l nears v_l, u_h rises faster, and thus x_h is decreasing in this region.

To explain the hump-shape of welfare, then, we need to understand why $U_h'(u_l) < 1$ for low levels of

and Other Requirements for Swap Execution Facilities (Commodity Futures Trading Commission (2013)), issued under the Dodd-Frank Wall Street Reform and Consumer Protection Act, which requires that a swap facility sends a buyer's request for price quotes to a minimum number of sellers before a trade can be executed.

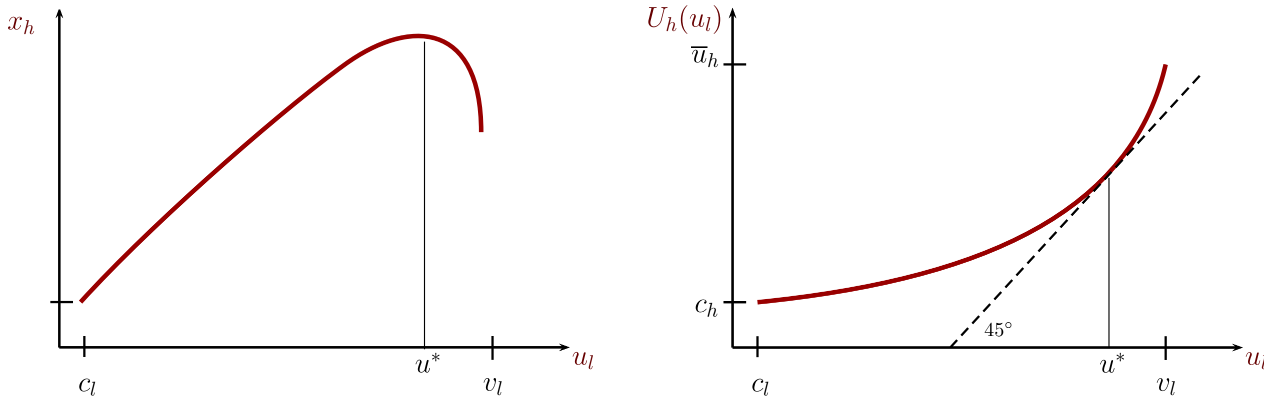

Figure 4: Trade (x_h) and Utility (U_h) of high-quality seller as functions of u_l when $\pi > 0$.

u_l and $U_h'(u_l) > 1$ for high levels of u_l. While this slope is a complicated equilibrium object, determined by the interaction of an individual buyer's optimal strategy and the equilibrium distribution of offers, the basic intuition can be understood through two opposing forces. First, it is cheaper for buyers to provide utility to the low type (relative to the high type) because doing so has the additional benefit of relaxing the incentive constraints; we call this the "incentive effect," and this force tends to reduce the slope, $U_h'(u_l)$. Second, as u_l rises, buyers have more incentive to attract type h sellers, relative to type l sellers; formally, one can show that $\Pi_h(u_l, U_h(u_l))/\Pi_l(u_l, u_h)$ is increasing in u_l. This effect, which we call the "composition effect," leads them to increase u_h at faster rates at higher u_l.

To illustrate these two forces more clearly, consider the following optimality condition that any equilibrium menu $(u_l, U_h(u_l))$ must satisfy:[31]

$$U_h'(u_l) = \underbrace{\frac{\phi_l}{\phi_h}}_{\text{incentive effect}} \underbrace{\frac{\Pi_h(u_l, U_h(u_l))}{\Pi_l(u_l)}}_{\text{composition effect}}, \tag{21}$$

where $\phi_h = (v_h - c_l)/(c_h - c_l)$ is the marginal cost of providing an additional unit of utility to type h sellers—i.e., $\phi_h = \frac{d\Pi_h}{du_h}$—and for notational convenience $\Pi_l(u_l) \equiv \Pi_l(u_l, u_h)$. The first term, the incentive effect, is the ratio of the marginal costs of providing utility to the two types of sellers. Since this term is strictly less than 1, all else equal, the incentive effect leads to more aggressive competition for the low type and, therefore, to u_h rising more slowly than u_l.

The second term, the ratio of profits, can be larger or smaller than 1, depending on u_l. When u_l

[31]This equation combines the optimality condition (16) for u_l, the corresponding optimality condition for u_h, $\frac{\pi f_h}{1-\pi+\pi F_h}\Pi_h = \phi_h$, and the strict rank-preserving property $F_l(u_l) = F_h(U_h(u_l))$, which implies $f_l = f_h U_h'(u_l)$.

is close to the monopsony outcome, $\Pi_h \approx 0$, so the composition effect is also less than 1, and we have $U_h'(u_l) < 1$. However, as u_l approaches the upper bound ν_l, this second term overwhelms the incentive effect, resulting in $U_h'(u_l) > 1$. In fact, one can show that $\lim_{u_l \to \nu_l} \frac{\Pi_h}{\Pi_l} = \lim_{u_l \to \nu_l} \ U_h' = \infty$.[32] This is why $x_h'(u_l) < 0$ close to the Bertrand outcome.

5.3 Reducing Information Asymmetries

We now study the welfare consequences of reducing informational asymmetries. This exercise sheds light on the implications of certain policy initiatives, as well as the effects of various technological innovations. For example, an important debate in insurance, credit, and financial markets centers around information that the informed party (the seller in our context) is required to disclose and the extent to which such information can be used by the uninformed party (the buyers in our model) to discriminate.[33] Moreover, technological developments in these markets also have the potential to decrease informational asymmetries, as advanced record-keeping and more sophisticated scoring systems (e.g., credit scores) provide buyers with more and/or better information about sellers' intrinsic types.

To study such changes, we introduce a noisy *public* signal $s \in \{0, 1\}$ about the quality of each seller.[34] The signal is informative, so that $\Pr(s = 1|h) = \Pr(s = 0|l) > 0.5$. Since the signal is public, the buyers may condition their offers on it, i.e., offer separate menus for sellers with $s = 0$ and $s = 1$. Thus, the economy has two subgroups, $j \in \{0, 1\}$, with the fraction of high-quality sellers in subgroup j given by

$$\mu_{hj} = \frac{\mu_h \ \Pr(s = j \mid h\ )}{\mu_h \ \Pr(s = j \mid h\ ) + \mu_l \ \ (1 - \Pr(s = j \mid l\ ))} \ .$$

Note that the average across subgroups is equal to the unconditional fraction of high types, i.e., $\mathbb{E}[\mu_{hj}] = \mu_h$. The equilibrium outcome for each subgroup can be constructed using the procedure in Section 4 with the appropriate μ_{hj}. Welfare is then given by the average welfare across subgroups, i.e., $\mathbb{E}[W(\pi, \mu_{hj})]$.

[32] Indeed, one can show that, if this limit were finite, the equality in (21) cannot hold: the right-hand side must be strictly less than the left-hand side. Intuitively, if the ratio of profits is finite in the limit, buyers have incentive to offer a lower u_h. The only way to discourage such deviations is to make high types more profitable—in the limit, infinitely so. See the working paper version, Lester et al. (2015b), for a detailed discussion.

[33] In insurance markets, these questions typically concern an individual's health factors, both observable (e.g., age or gender) and unobservable (e.g., pre-existing conditions) to the insurance provider. In credit markets, similar questions arise with respect to observable characteristics that can legally be used in determining a borrower's creditworthiness, as well as the amount of information about a borrower's credit history that should be available to lenders (e.g., how long a delinquency stays on an individual's credit history). In financial markets, the relevant issue is not only whether a seller discloses relevant information about an asset to a buyer, but also whether the payoff structure of the asset is sufficiently *transparent* for sellers to distinguish good from bad assets. For example, in order to support "sustainable securitisation markets," the Basel Committee on Banking Supervision and the International Organization of Securities Commissions established a joint task force to identify criteria for "simple, transparent, and comparable" securitized assets. See http://www.bis.org/bcbs/publ/d304.pdf.

[34] The restriction to a binary signal is only for simplicity. It is easy to introduce richer information structures.

When buyers do not observe a signal (or, equivalently, are not permitted to condition their offers on it), welfare is simply $W(\pi, \mathbb{E}[\mu_{hj}])$. Hence, whether the signal increases or decreases welfare, respectively, depends on whether $W(\pi, \mu_h)$ is convex or concave in μ_h in the relevant region.

Before proceeding, two comments are in order. First, our focus is on the effect of a *small* increase in the information available to buyers; that is, we are interested in signals that induce a local mean-preserving spread around μ_h. Very informative signals always improve welfare—for example, if buyers receive a perfect signal about sellers' types, then all gains from trade are realized—but this is not a very interesting or realistic experiment. Second, we focus on the region with $\mu_h < \mu_0$, so that $\phi_l > 0$, which is more tractable and shows interesting interactions between competition and additional information.[35] Moreover, in this region, W is *linear* in μ_h when $\pi = 0$ or $\pi = 1$. Hence, imposing monopsony or perfect competition would lead us to the conclusion that additional information has no effect on welfare.

Proposition 6 shows that W has a strictly convex region when π is sufficiently low, implying that more information is beneficial when markets are close to (but not at) the monopsony benchmark. Alternatively, when markets are relatively (but not perfectly) competitive, W has a strictly concave region, implying that more information actually reduces welfare.

Proposition 6. *There exist $\underline{\pi}, \overline{\pi} \in (0,1)$ such that:* (i) *for all $\pi \in (0, \underline{\pi})$, there exists $0 < \underline{\mu}_h < \overline{\mu}_h < \mu_0$ such that W is strictly convex on the interval $[\underline{\mu}_h, \overline{\mu}_h]$; and* (ii) *for all $\pi \in (\overline{\pi}, 1)$, there exists $0 < \underline{\mu}'_h < \overline{\mu}'_h < \mu_0$ such that W is strictly concave on the interval $[\underline{\mu}'_h, \overline{\mu}'_h]$.*

To see the intuition behind Proposition 6, recall from the previous subsection that trade with the high-quality seller (and thus welfare) is governed by the interaction of the incentive effect and the relative profit (or composition) effect. The consequences of more information can be understood in terms of these two forces, too. In particular, a lower ϕ_l drives down the first term in (21), which encourages more competition for low-quality sellers and, hence, boosts trade and welfare. Now, from (13), we see that ϕ_l is a concave function of μ_h. Since the additional signal induces a mean-preserving spread of μ_h, it results in a lower ϕ_l on average, which, ceteris paribus, increases trade. This mechanism makes more information desirable. The effect from relative profits goes in the opposite direction. In equilibrium, milder adverse selection raises profits from high types relative to low types, which increases U'_h and hence decreases trade. Close to monopsony, since the incentive effect dominates, more information raises welfare. The opposite happens when π is close to 1, and the effect on relative profits dominates.

[35] Numerical simulations suggest that additional information always reduces welfare when $\mu_h > \mu_0$.

5.4 Constrained Efficiency

The analysis above establishes that, when $\phi_l > 0$, increasing competition and reducing information asymmetries can have nonmonotonic effects on the equilibrium volume of trade, and hence on the (utilitarian) welfare measure. However, one might worry that this nonmonotonicity is an artifact of the particular game we study, and not a robust feature of markets with asymmetric information. To address this concern, in Appendix B, we adopt a mechanism design approach to derive a constrained efficient benchmark. A direct mechanism prescribes transfers of goods and numeraire for each buyer and seller based on their reported types, which include the quality of their good and the other agents with whom they are matched. An allocation is constrained efficient if it is implementable by a direct mechanism and maximizes a Pareto-weighted sum of utilities, subject to feasibility (only matched agents can trade), individual rationality (each agent receives a payoff at least as high as in equilibrium), incentive compatibility (types are reported truthfully), and exclusivity (sellers can trade with at most one buyer).

A key result is that the expected volume of trade in this benchmark coincides with that in our equilibrium when $\phi_l > 0$. This implies that a benevolent planner cannot propose a trading mechanism that would strictly increase trading volume or welfare, relative to our equilibrium allocation, given $\phi_l > 0$ and $\pi \in [0, 1]$. In contrast, when $\phi_l \in [\phi_2, 0]$—where ϕ_2 is defined in Proposition 4—then the constrained efficient allocation implies full pooling ($x_h = 1$) in all trades while the expected trading volume of high-quality goods in equilibrium is strictly less than 1. In this region, a benevolent planner could make everyone better off by inducing more cross-subsidization from high- to low-quality sellers. The source of this inefficiency is similar to that which arises in many models with adverse selection and competition (see, e.g., Rothschild and Stiglitz, 1976; Guerrieri et al., 2010): buyers' incentives to cream-skim high-quality sellers limit equilibrium cross-subsidization.[36]

Finally, another natural question is whether the welfare-maximizing allocation is attained in an environment where the level of competition is determined endogenously by the choices of market participants. In the next section, we take up this question by extending our analysis to study an environment where the market structure—summarized by π—is endogenous.[37]

[36] Hence, as in many models of adverse selection (i.e., hidden information about the common value of the asset), policies that incentivize pooling can potentially improve welfare when the equilibrium is constrained inefficient. In the context of our model, these policies would include restrictions on the type of contracts that buyers offer (e.g., ruling out separating contracts) or mandates that sellers must trade a minimum quantity. Note, however, that if sellers had hidden information about their private valuation, one may obtain constrained efficiency in an environment with directed search, implying no role for policy interventions (see, e.g., Albrecht et al. (2014)).

[37] A related exercise is to consider interventions that mimic the effects of increasing or decreasing competition. In Lester et al. (2015b), we study what happens when the government enters a market suffering from adverse selection as a "large buyer," as it has in, e.g., the markets for student loans, health insurance, or certain financial assets. We show that, by offering to buy any

6 Endogenous Market Structure

In this section, we allow buyers to choose how intensely they advertise their offers to sellers. This exercise has two benefits: the *degree of competition* will be endogenous; and the measure of sellers who are contacted by at least one buyer, or *coverage*, will also be endogenous. This allows us to study which features of the environment determine the market structure and the corresponding welfare implications.

To this end, suppose that, in addition to choosing a menu of contracts to offer, each buyer $k \in \{1, 2\}$ also chooses the effort or intensity with which his offer will be advertised to sellers. Exerting effort is costly, but increases the likelihood that each seller observes the offer. Formally, we assume that buyer k can choose the probability $\hat{\pi}^k$ that each seller observes his offer by incurring a cost $C(\hat{\pi}^k)$, which is a continuously differentiable, strictly increasing, and strictly convex function with $C(0) = C'(0) = 0$ and $C'(1) = \infty$.[38] Note that $\hat{\pi}^k$ represents a slightly different object than π in our benchmark model, since it affects both competition and coverage. However, what is crucial is that—just like π in our earlier analysis—$\hat{\pi}^{-k}$ is the conditional probability that a seller who buyer k meets has a second offer. Hence, in a symmetric Nash equilibrium, $\hat{\pi}^k = \hat{\pi}^{-k} \equiv \hat{\pi}$ remains the key determinant of the level of competition.

Taking as given the other buyer's advertising intensity, $\hat{\pi}^{-k}$, and the distribution of offers that he makes to sellers of type $i \in \{l, h\}$, $F_i^{-k}(u_i^{-k})$, buyer k chooses a tuple $(\hat{\pi}^k, u_l^k, u_h^k)$ to maximize

$$\sum_{i \in \{l,h\}} \mu_i \left[\hat{\pi}^k \left(1 - \hat{\pi}^{-k}\right) + \hat{\pi}^k \hat{\pi}^{-k} F_i^{-k}\left(u_i^k\right)\right] \Pi_i\left(u_l^k, u_h^k\right), \tag{22}$$

subject to the participation and incentive constraints described in the benchmark model, with $\Pi_i(\cdot, \cdot)$ defined in (9)–(10). Factoring out $\hat{\pi}^k$ from (22), one can see that the choice of $\hat{\pi}^k$ and (u_l^k, u_h^k) are separable. Hence, given $\hat{\pi}^{-k}$, the first-order conditions on u_l^k and u_h^k are exactly as they were before (replacing π with $\hat{\pi}^{-k}$), while the first-order condition determining the optimal choice of $\hat{\pi}^k$ is

$$C'\left(\hat{\pi}^k\right) = \sum_{i \in \{l,h\}} \mu_i \left[1 - \hat{\pi}^{-k} + \hat{\pi}^{-k} F_i^{-k}\left(u_i^k\right)\right] \Pi_i\left(u_l^k, u_h^k\right). \tag{23}$$

In a symmetric equilibrium, where $\hat{\pi}^1 = \hat{\pi}^2 \equiv \hat{\pi}$, equation (23) implies that the marginal cost of increasing $\hat{\pi}$ is equal to the equilibrium profits characterized in Propositions 2 and 3. Since these profits are

quantity at a fixed price, the government can increase sellers' outside option and promote more competition, which recreates the effects of increasing π. Such an intervention can increase welfare only when both market power and the distortions arising from adverse selection are severe. Otherwise, we show that such programs can be detrimental to welfare even if, in principle, the intervention makes non-negative profits.

[38]Note that this implies a fraction $\left(1 - \hat{\pi}^1\right)\left(1 - \hat{\pi}^2\right)$ of sellers receive zero offers. This is the sense in which coverage is endogenous in the current setup, whereas we fixed this fraction (to zero) in our benchmark model.

decreasing in $\hat{\pi}$, the next result follows almost immediately.

Proposition 7. *For any $\phi_l < 1$, there exists a unique symmetric equilibrium, with $\hat{\pi}^\star \in (0,1)$ and $\{F_i^\star(u_i)\}_{i\in\{l,h\}}$ as described in Propositions 2 and 3.*

In Lemma 5, below, we offer comparative statics with respect to the fraction of high-quality sellers, μ_h.[39] Recall that there exists a μ_0 such that $\phi_l \geqslant 0$ if and only if $\mu_h \leqslant \mu_0$. We show that the equilibrium $\hat{\pi}^\star$ is U-shaped in μ_h, achieving a minimum at $\mu_h = \mu_0$.

Lemma 5. *The equilibrium advertising intensity $\hat{\pi}^\star$ is decreasing in μ_h if $\mu_h < \mu_0$ and increasing otherwise.*

To understand the intuition, consider first the case of "severe adverse selection," i.e., when $\mu_h < \mu_0$ or, equivalently, when $\phi_l > 0$. In this region, once information rents are taken into account, the buyer's payoff from trading with low-quality sellers is larger than the payoff from trading with high-quality sellers (even if $v_h - c_h > v_l - c_l$). Thus, from the buyer's perspective, an increase in μ_h in this region actually worsens the pool of potential sellers and, as a result, buyers optimally choose a lower $\hat{\pi}$. The opposite is true when $\mu_h > \mu_0$, where we say adverse selection is "mild." In this region, after adjusting for information rents, it is relatively more profitable to trade with high-quality sellers, and thus buyers optimally choose larger values of $\hat{\pi}$ as the fraction of high-quality sellers increases.

Lemma 5 has implications for the relationship between the composition of high- and low-quality sellers in a market and the (endogenous) level of competition that prevails. In particular, this result suggests that competition for customers should be strongest in markets with less uncertainty about sellers' types (i.e., extreme values of μ_h), and weakest in markets with more uncertainty about sellers' types (i.e., intermediate values of μ_h).[40]

The model with endogenous $\hat{\pi}$ also allows us to connect some of our welfare results to more concrete implications for policy. To see this, suppose $C(\hat{\pi}^k) = Ac(\hat{\pi}^k)$ for some positive constant $A > 0$, and consider the effect of taxing buyers' advertising intensities according to a proportional tax, $\tau\hat{\pi}$. For simplicity, suppose all tax proceeds are then simply rebated to the agents. In Lemma 6, below, we establish that welfare is increasing in τ in some regions of the parameter space. That is, a policy making it *more costly* for buyers to contact sellers can *improve* welfare.

Lemma 6. *Suppose $\phi_l > 0$. There exists an $\widetilde{A} > 0$ such that welfare is increasing in τ for all $A < \widetilde{A}$.*

[39] Comparative statics for other parameters may be similarly derived by considering the effect of perturbing each parameter on equilibrium profits.

[40] One can also show that there exist cost functions that are consistent with *any* $\hat{\pi}^\star$, so that endogenizing competition does not rule out certain types of equilibrium.

The result in Lemma 6 follows closely from the fact that welfare is hump-shaped in $\hat{\pi}$, even taking into account that an increase in $\hat{\pi}$ increases coverage. As a result, when A is sufficiently small, $\hat{\pi}^\star$ is large and a decrease in $\hat{\pi}^\star$—brought about by an increase in τ—causes welfare to rise.

7 Large Markets and Meeting Technologies

We now show how our analysis and results extend to an environment with an arbitrarily large number of buyers and sellers and a more general *meeting technology*. Suppose there is a measure b of buyers and a measure s of sellers. As in our benchmark model, buyers send offers and sellers receive them. The meeting technology dictates the number of offers each buyer sends and where these offers end up.

Formally, let η denote the (expected) number of offers that each buyer sends; let $\lambda = \frac{\eta b}{s}$ denote the ratio of offers to sellers; and let P_n denote the probability that each seller receives $n \in \{0\} \cup \mathbb{N}$ offers. A meeting technology, then, can be summarized by a pair (λ, P_n).[41] For a buyer, a meeting technology implies that an offer he sends is received by a seller with $n-1$ other offers with probability Q_n, where $nP_n = \lambda Q_n$ for all $n \in \mathbb{N}$. Following the convention in the literature, we let $Q_0 = 1 - \sum_{n=1}^{\infty} Q_n$.

7.1 Characterizing Equilibrium

As in our benchmark model, we restrict attention to symmetric equilibria, where $\{F_i(u_i)\}_{i\in\{l,h\}}$ summarizes the distribution of menus being offered by buyers. Taking this distribution as given, an individual buyer makes an offer (u_l, u_h) that solves

$$\max_{u_l,u_h} \sum_{i\in\{l,h\}} \mu_i \left[\sum_{n=1}^{\infty} Q_n F_i^{n-1}(u_i) \right] \Pi_i(u_l, u_h), \tag{24}$$

where, again, $\Pi_i(u_l, u_h)$ is defined in (9)–(10). Importantly, the objective in (24) can be rewritten as

$$[1 - Q_0] \sum_{i\in\{l,h\}} \mu_i \left[1 - \tilde{\pi} + \tilde{\pi} G_i(u_i)\right] \Pi_i(u_l, u_h), \tag{25}$$

[41] This formulation of a meeting technology is slightly more general than what is commonly used in the existing literature (e.g., Eeckhout and Kircher, 2010), in that we allow the "queue length" λ to depend on the meeting technology.

where $\tilde{\pi} = 1 - Q_1/(1 - Q_0)$ is the probability that an offer is received by a seller that has at least one other offer, conditional on being received by a seller, and

$$G_i(u_i) = \frac{1}{\tilde{\pi}} \sum_{n=2}^{\infty} \frac{Q_n}{1 - Q_0} F_i^{n-1}(u_i) \tag{26}$$

is the probability that the seller accepts the offer u_i, given that they own a good of quality $i \in \{l, h\}$.

Notice immediately that (25) has the same form as our objective function in the two-buyer case, replacing π with $\tilde{\pi}$ and $F_i(u_i)$ with $G_i(u_i)$. As a result, our characterization of equilibrium in Propositions 2 and 3 is preserved, and the distribution $G_i(u_i)$ is uniquely defined in all regions of the parameter space. Moreover, from (26), it is easy to show that $G_i(u_i)$ uniquely determines the distribution of offers made by buyers, $F_i(u_i)$. Using these results, one can easily determine the type of contracts that are offered in equilibrium[42]; the distribution of offers that are made to each type of seller, $F_i(u_i)$, which is the solution to (26); and the prices and quantities that are ultimately traded in equilibrium.

7.2 Competition, Coverage, and Equilibrium Gains from Trade

In our benchmark model, we studied the effects of changing the probability that a seller received two offers, π. We now explore similar comparative statics within the context of a general meeting technology. In particular, we let P_n and λ (and hence Q_n) depend on a parameter α. This formulation is intentionally general: a change in α could correspond to a change in the measure of buyers, a change in the expected number of offers per buyer, or a change in the technology that matches offers to sellers.

As in Section 6, we focus on the case where $\phi_l > 0$ and define the utilitarian welfare measure

$$W(\alpha) = \sum_{n=1}^{\infty} P_n(\alpha) \left[\mu_h (v_h - c_h) \int x_h(u_l)\, d\left(F_l^n(u_l)\right) + \mu_l (v_l - c_l) \right] + \sum_{i=l,h} \mu_i c_i.$$

As in our benchmark model, when $\phi_l > 0$, the distribution $G_l(u_l)$ solves the differential equation

$$\frac{\tilde{\pi} g_l(u_l)}{1 - \tilde{\pi} + \tilde{\pi} G_l(u_l)} = \frac{\phi_l}{v_l - u_l} \tag{27}$$

with support $[c_l, \overline{u}_l(\alpha)]$ such that $G_l(c_l) = 0$ and $G_l(\overline{u}_l(\alpha)) = 1$.

[42]This is done by comparing ϕ_l, which is unchanged, to ϕ_1 and ϕ_2, which are updated by replacing π with $\tilde{\pi}$.

Solving (27) and imposing equal profits implies that the mapping $U_h(u_l)$ must satisfy

$$\left(\frac{v_l - c_l}{v_l - u_l}\right)^{\phi_l} \left[\mu_l \left(v_l - u_l\right) + \mu_h \Pi_h \left(u_l, U_h \left(u_l\right)\right)\right] = \mu_l \left(v_l - c_l\right),$$

exactly as in the case of two buyers. An immediate, and important, implication is that $x_h(u_l)$ is hump-shaped in u_l and independent of α. Hence, a change in α only affects the distribution of offers that are made, summarized by F_l, and the distribution of offers that sellers receive, summarized by P_n.

As a result, the effects of a change in α can be decomposed as follows:

$$\begin{aligned} W'\left(\alpha\right) = & \sum_{n=1}^{\infty} \frac{\partial P_n\left(\alpha\right)}{\partial \alpha} \left[\mu_h \left(v_h - c_h\right) \int x_h \left(u_l\right) d \left(F_l^n \left(u_l; \alpha\right)\right) + \mu_l \left(v_l - c_l\right)\right] + \\ & \sum_{n=1}^{\infty} P_n\left(\alpha\right) \left[\mu_h \left(v_h - c_h\right) \frac{\partial}{\partial \alpha} \int_{c_l}^{\overline{u}_l(\alpha)} x_h\left(u_l\right) d\left(F_l^n\left(u_l; \alpha\right)\right)\right], \end{aligned}$$

where, for the purpose of clarity, we've made the dependence of F_l on α explicit. The first term in the equation above was absent in our benchmark model, but captures a standard effect in models with frictions: the effect of a change in α on the set of sellers who are able to trade, or what we call the *coverage* effect. The second term captures the effect that we focused on in our benchmark model: the effect of a change in α on the distribution of offers, or what we call the *competition* effect.

For example, suppose increasing α leads to a first-order stochastic dominant (FOSD) shift in the number of offers that sellers receive. In this case, the coverage effect would be positive, since fewer sellers receive zero offers. However, the competition effect could be negative, since an increase in α leads to a FOSD shift in the distribution of offers F_l. As in our benchmark model, when α is sufficiently large, this shift puts more weight on the downward-sloping region of $x_h\left(u_l\right)$, thus reducing welfare.

Which of these two effects dominates typically depends on the details of the meeting technology. Consider, for example, the Poisson meeting technology with $\lambda(\alpha) = \alpha$ and $P_n\left(\alpha\right) = \frac{e^{-\alpha}\alpha^n}{n!}$. This is perhaps the most popular meeting technology in the literature (see, e.g., Butters (1977) and Hall (1977) for early examples, and Burdett et al. (2001) for a more recent example), and an increase in α clearly leads to a FOSD shift in the distribution of offers that sellers receive. We show in the Appendix that when $\phi_l > 0$, there exists an α^* such that welfare is decreasing in α for all finite $\alpha > \alpha^*$. Therefore, as in our benchmark model, some frictions can increase welfare *even after* accounting for the coverage effect.

The same is not true, however, for all meeting technologies. For example, consider the Geometric meeting technology with $\lambda(\alpha) = \alpha/(1-\alpha)$ and $P_n(\alpha) = \alpha^n(1-\alpha)$, which was studied recently by, e.g., Lester et al. (2015a). Under this meeting technology, when $\phi_l > 0$, we show in the Appendix that the coverage effect always dominates the competition effect, so that welfare is increasing in α. Intuitively, the coverage effect is relatively strong because the fraction of sellers who fail to receive an offer, P_0, falls slowly in α as $\alpha \to 1$, whereas the competition effect vanishes more quickly.[43]

8 Conclusion

In their survey of the literature on insurance markets, Einav et al. (2010a) note that, despite substantial progress in understanding the effects of adverse selection,

> "There has been much less progress on empirical models of insurance market competition, or on empirical models of insurance contracting that incorporate realistic market frictions. One challenge is to develop an appropriate conceptual framework. Even in stylized models of insurance markets with asymmetric information, characterizing competitive equilibrium can be challenging, and the challenge is compounded if one wants to allow for realistic consumer heterogeneity and market imperfections."

In this paper, we overcome this challenge and develop a tractable, unified framework to study adverse selection, screening, and imperfect competition. We provide a full analytical characterization of the unique equilibrium, and use it to study both positive and normative issues.

Our framework can be exploited and extended to address a variety of important issues. On the applied side, our equilibrium provides a new structural framework that can be used to jointly identify the extent of adverse selection and imperfect competition in various markets, and to study how the interaction of these two frictions affects the distribution of contracts, prices, and quantities that are traded. On the theoretical side, one natural extension is to study the analog of our model with *nonexclusive contracts*; although this would complicate the analysis considerably, it would also make our framework suitable to analyze certain markets where exclusivity is hard to enforce. We leave these exercises for future work.

[43]Note, however, that one can augment the Geometric meeting technology to ensure full coverage by setting $\lambda(\alpha) = \alpha/(1-\alpha)$ and $P_n(\alpha) = \alpha^{n-1}(1-\alpha)$ for $n \in \mathbb{N}$, with $P_0 = 0$. This specification removes the positive effects of increased coverage on welfare, leaving only the negative competition effect as $\alpha \to 1$.

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Screening and Adverse Selection in Frictional Markets*

Online Appendix

Benjamin Lester
Federal Reserve Bank of Philadelphia

Ali Shourideh
Carnegie Mellon University

Venky Venkateswaran
NYU – Stern School of Business

Ariel Zetlin-Jones
Carnegie Mellon University

September 20, 2017

Appendices

*The views expressed here are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System.

A Omitted Proofs

This section contains proofs of the results presented in the main text.

A.1 Proofs from Section 3

A.1.1 Proof of Lemma 1

Proof. Both results are similar to existing results (see, for example, Dasgupta and Maskin (1986)), and thus we keep the exposition brief. To establish that $x_l = 1$ in all equilibrium menus, suppose by way of contradiction that some equilibrium menu $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ has $x_l < 1$ and $t_l \in \mathbb{R}_+$, yielding a low-quality seller utility u_l. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}'_l, \mathbf{z}_h)$ with $x'_l = x_l + \epsilon$ for $\epsilon \in (0, 1 - x_l]$ and $t'_l = t_l + \epsilon c_l$. Note that $u'_l = u_l$, so that $\mathbf{z}_l$ and $\mathbf{z}'_l$ are accepted with the same probability, but

$$x_l v_l - t_l < x_l v_l - t_l + \epsilon(v_l - c_l) = x'_l v_l - t'_l,$$

so that $\mathbf{z}'_l$ earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, no equilibrium menu features $x_l < 1$.

To establish that a low-quality seller's incentive compatibility constraint binds in all equilibrium menus, suppose by way of contradiction that some equilibrium menu $\mathbf{z} = (\mathbf{z}_l, \mathbf{z}_h)$ has $t_l > t_h + c_l(1 - x_h)$. Now, consider a deviation $\mathbf{z}' = (\mathbf{z}_l, \mathbf{z}'_h)$ with $x'_h = x_h + \epsilon$ and $t'_h = t_h + \epsilon c_h$ for $\epsilon \in \left(0, \frac{t_l - t_h - c_l(1 - x_h)}{c_h - c_l}\right]$, which is a nonempty interval by assumption. The upper bound on ϵ ensures that the incentive compatibility constraint on type l sellers is not violated. In addition, note that $u'_h = u_h$, so that $\mathbf{z}_h$ and $\mathbf{z}'_h$ are accepted with the same probability, but

$$x_h v_h - t_h < x_h v_h - t_h + \epsilon(v_h - c_h) = x'_h v_h - t'_h,$$

so that $\mathbf{z}'_h$ earns the buyer a higher payoff when it is accepted, implying existence of a profitable deviation. Therefore, in all equilibrium menus, the type l seller's incentive constraint binds. ∎

A.1.2 Proof of Proposition 1 and Lemma 2

We prove the proposition through the following sequence of lemmas.

Lemma 1. $F_h(\cdot)$ *has no flats.*

Proof. Suppose by way of contradiction that $F_h(\cdot)$ is flat in an interval (u_{h1}, u_{h2}). In other words, there exists $(u_{l2}, u_{h2}) \in \text{Supp}(F_l) \times \text{Supp}(F_h)$ such that, for some $\bar{\varepsilon} > 0$, the distribution F_h satisfies $F_h(u_{h2}) = F_h(u_{h2} - \varepsilon)$ for all $\epsilon \in [0, \bar{\varepsilon}]$. We prove that there must exist a profitable deviation. The particular deviation we construct depends on whether $u_{l2} < u_{h2}$ or $u_{l2} = u_{h2}$ and whether F_l is flat on an interval containing u_{l2} or not. We consider each relevant case in turn:

1. Suppose that $u_{l2} < u_{h2}$. In this case, a deviation to $(u_{l2}, u_{h2} - \varepsilon')$ with $\varepsilon' < \varepsilon$ is feasible and must be profitable because such a deviation increases profits earned from trading with h types but does not change the fraction of h types attracted.

2. Suppose that $u_{l2} = u_{h2}$ and F_l is flat below u_{l2}. In this case, a deviation of the form $(u_{l2} - \varepsilon', u_{h2} - \varepsilon')$ for a small but positive ε' is profitable since it increases profits per trade (from both l and h type sellers) but does not change the fraction of either type attracted.

3. Suppose $u_{l2} = u_{h2}$ and F_l is not flat below u_{l2}. Such a situation is depicted in Figure 1. Point A represents the contract (u_{l2}, u_{h2}). Since F_h is flat by assumption, the area between the two **red** dashed lines must not contain any equilibrium menu. Since F_l is not flat below u_{l2} by assumption

and there are no menus in the area between the red dashed lines, an equilibrium contract must exist in the region where point D is located; recall, since $u_h \geqslant u_l$, point D cannot lie below the lower red dashed line. Let point D represent such an equilibrium menu. In addition, let B represent a menu with the same offer to the low type as D but offers u_{h2} to the high type. Similarly, let C represent a menu with the same offer to the low type as A and the same offer to the high type as D.

For any distributions, F_l and F_h, the profit function, $\Pi(u_l, u_h)$, is weakly supermodular so that

$$\Pi_A + \Pi_D \leqslant \Pi_C + \Pi_B.$$

Since both D and A are offered in equilibrium, we must have that $\Pi_A = \Pi_D \geqslant \Pi_C, \Pi_B$. This implies that $\Pi_A = \Pi_B$. Additionally, since F_h is flat between B and E (and these menus offer the same u_l), it must be that $\Pi_E > \Pi_B$. Therefore, this is a profitable deviation.

■

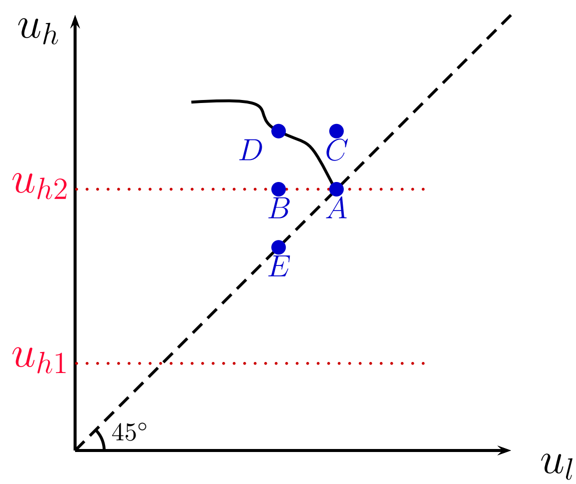


Figure 1: A graphical illustration of why F_h cannot be flat.

Lemma 2. $F_l(\cdot)$ *has no flats.*

Proof. Suppose by way of contradiction that F_l is flat in an interval (u_{l1}, u_{l2}). Without loss of generality, we can complete the measure Φ to include menus with the first element given by u_{l1} and u_{l2}. Since the profit function is weakly supermodular, then the policy correspondence must be weakly increasing. Now consider the policy correspondences $U_h(u_{l1})$ and $U_h(u_{l2})$. Note that $Cl(U_h(u_{l1}))$ and $Cl(U_h(u_{l2}))$ cannot be disjoint—if they were, then there would be a flat in the support of F_h, which contradicts Lemma 1. Let $\hat{u}_h$ be a common value in the two sets. We present a depiction of such a situation in Figure 2 below.

Holding $\hat{u}_h$ fixed, the profit function must be linear over the set (u_{l1}, u_{l2}), since $F_l(\cdot)$ is flat by assumption. Therefore, all the menus on the line AB must also deliver profits equal to equilibrium profits. However, since profits earned from trading with h types are increasing in u_l, the marginal benefit of a change in u_h is changing along the line AB. As a result, it is possible to construct an upward or downward deviation along AB that increases profits, implying the existence of a profitable deviation. ■

Lemma 3. Φ *has no mass point.*

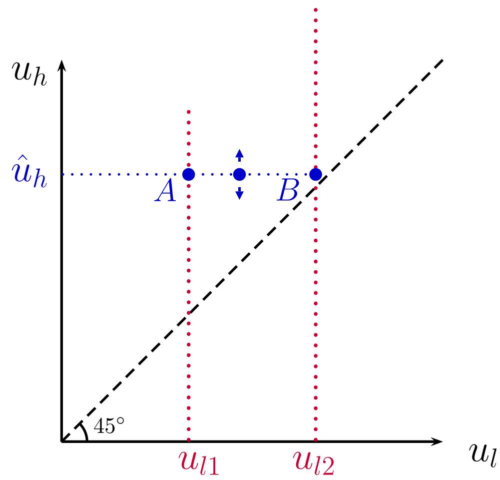


Figure 2: A graphical illustration of why F_l cannot be flat.

Proof. Suppose by way of contradiction that Φ has a mass point at the menu (u_l, u_h). Let m denote the mass at this menu. Since for any such menu, a deviation of the form $(u_l + \varepsilon_1, u_h + \varepsilon_2)$ for small $\varepsilon_1, \varepsilon_2$ (one of which is positive or negative) must be feasible, profits earned from the mass of sellers attracted to such deviation must be zero:

$$\mu_l \pi \frac{m}{2} \Pi_l(u_l) + \mu_h \pi \frac{m}{2} \Pi_h (u_l, u_h) = 0.$$

If the menu (u_l, u_h) is interior to the constraint set—that is, if $c_h - c_l > u_h - u_l > 0$—then a simple deviation along u_l or u_h will be feasible and profitable. However, it is possible that (u_l, u_h) is on the boundary of the set and, as a result, not all deviations are feasible. There are two relevant possibilities:

1. Suppose that the menu with mass, (u_l, u_h), satisfies $u_h = u_l + c_h - c_l$. In such a case, the menu features no trade with the high type. Therefore, it must be that $\Pi_h \leqslant 0$. Since equilibrium profits are strictly positive, it must be that $\Pi_l > 0$. Hence, an infinitesimal increase in u_l, which is feasible, increases profits.

2. Suppose that the menu with mass, (u_l, u_h), satisfies $u_h = u_l$. Then (u_l, u_h) is a pooling menu. Therefore, the profits from the high type must be positive. As a result, the buyer offering this contract could increase profits with an infinitesimal increase in u_h (which would attract a mass of high types) while holding u_l constant.

■

Lemma 4. $F_h(\cdot)$ *does not have a mass point.*

Proof. Suppose by way of contradiction that F_h has a mass point. From Lemma 3, we know that this mass point could not have been created from a mass point in Φ. Therefore, if F_h has a mass point at $\hat{u}_h$, it must be that a positive measure set of the form $\{(u_l, \hat{u}_h)\}$ exists. Figure 3 depicts this possibility.

Note that at one of the points on the line, profits from the h type, $\Pi_h(u_l, \hat{u}_h)$, must be nonzero since Π_h is strictly increasing in u_l. Therefore, a small deviation upward or downward increases profits; this implies the existence of a profitable deviation and yields the necessary contradiction. ■

To show that F_l has no mass points, we make use of the strict supermodularity of the profit function, which only relies on the continuity of F_h. We therefore provide a proof of the strict supermodularity of the profit function here.

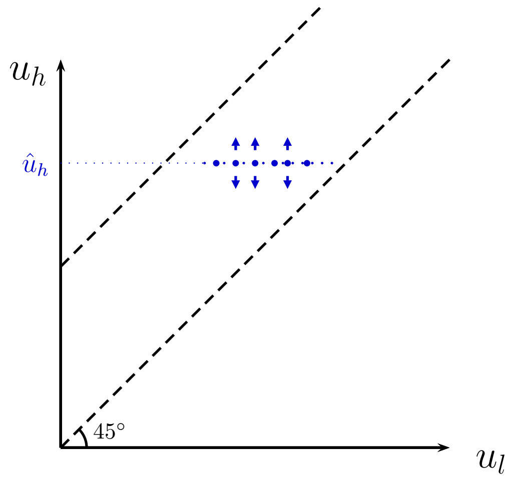


Figure 3: A graphical illustration of why F_h cannot have a mass point.

Proof of Lemma 2. Suppose $u_{l2} > u_{l1}$ and $u_{h2} > u_{h1}$. Then, letting $\xi_1 \equiv \frac{v_h - c_h}{c_h - c_l} > 0$ and $\xi_2 \equiv \frac{v_h - c_l}{c_h - c_l} > 0$,

$$\begin{aligned} & \Pi(u_{l1}, u_{h2}) - \Pi(u_{l1}, u_{h1}) \\ = \; & \mu_h \{[1-\pi+\pi F_h(u_{h2})]\Pi_h(u_{l1}, u_{h2}) - [1-\pi+\pi F_h(u_{h1})]\Pi_h(u_{l1}, u_{h1})\} \\ = \; & \mu_h \{[1-\pi+\pi F_h(u_{h2})]\,[v_h + \xi_1 u_{l1} - \xi_2 u_{h2}] - [1-\pi+\pi F_h(u_{h1})]\,[v_h + \xi_1 u_{l1} - \xi_2 u_{h1}]\} \\ < \; & \mu_h \{[1-\pi+\pi F_h(u_{h2})]\,[v_h + \xi_1 u_{l2} - \xi_2 u_{h2}] - [1-\pi+\pi F_h(u_{h1})]\,[v_h + \xi_1 u_{l2} - \xi_2 u_{h1}]\} \\ = \; & \Pi(u_{l2}, u_{h2}) - \Pi(u_{l2}, u_{h1}), \end{aligned}$$

where the inequality follows from the fact that F_h is strictly increasing, and hence

$$\pi\xi_1(u_{l2} - u_{l1})[F_h(u_{h2}) - F_h(u_{h1})] > 0.$$

■

Lemma 5. F_l *is continuous except possibly at* v_l.

Proof. Suppose by way of contradiction that F_l is not continuous and thus has a mass point at some $\hat{u}_l$. Again, by Lemma 3, it must be that a positive measure set of the form $S = \{(\hat{u}_l, u_h)\}$ exists. It is immediate that $\Pi_l(\hat{u}_l) = 0$; otherwise, it would be profitable to increase or decrease u_l by ε if $\Pi_l(\hat{u}_l) > 0$ or $\Pi_l(\hat{u}_l) < 0$, respectively. If $\Pi_l(\hat{u}_l) = 0$, then it must be $\hat{u}_l = v_l$. ■

A.2 Proofs from Section 4

A.2.1 Proof of Proposition 2

Proof. We first show that the equilibrium allocations constructed in (16) and (17) are indeed separating and interior. Our construction ensures that local deviations are not profitable. Below we prove that the global deviations are not profitable as well.

Verifying Allocations Are Separating and Interior. Note that the solution to the differential equation in (17), together with boundary condition $F_l(c_l) = 0$, must satisfy

$$1 - \pi + \pi F_l(u_l) = (1-\pi)(v_l - c_l)^{\phi_l}(v_l - u_l)^{-\phi_l}. \tag{1}$$

Therefore, from (17), $U_h(u_l)$ must satisfy

$$U_h(u_l) = \frac{1}{\mu_h \frac{v_h - c_l}{c_h - c_l}} \left[\mu_h v_h + \mu_l v_l - \mu_l \phi_l u_l - \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l} \right].$$

For the allocation to be separating, we must verify that

$$u_l + c_h - c_l \geqslant U_h(u_l) > u_l, \forall u_l \in \mathrm{Supp}(F_l), \tag{2}$$

where

$$\mathrm{Supp}(F_l) = \left[c_l, v_l - (1-\pi)^{\frac{1}{\phi_l}} (v_l - c_l) \right].$$

The second inequality in (2), $U_h(u_l) > u_l$, is satisfied if and only if

$$\mu_h v_h + \mu_l v_l > \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l} + u_l \tag{3}$$

for all $u_l \in \mathrm{Supp}(F_l)$. Let $H(u_l)$ denote the right-hand side of (3). We argue that $H(\cdot)$ is strictly concave and attains its maximum at a value $u_l^* \in [c_l, v_l]$ with $H(u_l^*) < \mu_h v_h + \mu_l v_l$, implying that (3) is satisfied for all $u_l \in \mathrm{Supp}(F_l)$. To see this, note that

$$\begin{aligned} H'(u_l) &= -\phi_l \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l - 1} + 1 && (4) \\ H''(u_l) &= \phi_l(\phi_l - 1) \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l - 2} < 0, && (5) \end{aligned}$$

where the inequality in (5) is implied by the fact that $0 < \phi_l < 1$. Also, since $\phi_l < 1$, $H'(v_l) = -\infty$ and $H'(c_l) = 1 - \phi_l \mu_l > 0$, the maximum of $H(u_l)$ is attained on the interior of $[c_l, v_l]$.

The function $H(u_l)$ is maximized at u_l^* given by

$$u_l^* = v_l - (\phi_l \mu_l)^{\frac{1}{1-\phi_l}} (v_l - c_l)$$

with

$$H(u_l^*) = v_l + (v_l - c_l) \mu_l^{\frac{1}{1-\phi_l}} \phi_l^{\frac{\phi_l}{1-\phi_l}} [1 - \phi_l].$$

Since $c_h \geqslant v_l$ and $\phi_l < 1$, it is immediate that

$$(\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} < 1 \leqslant \frac{(c_h - c_l)(v_h - v_l)}{(v_l - c_l)(v_h - c_h)},$$

which implies

$$(v_l - c_l) \mu_l (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} < \mu_h (v_h - v_l).$$

Hence,

$$(v_l - c_l) \mu_l (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} (1 - \phi_l) < \mu_h (v_h - v_l)$$

and

$$\max_{u_l \in [c_l, v_l]} H(u_l) = H(u_l^*) = v_l + (v_l - c_l) \mu_l (\phi_l \mu_l)^{\frac{\phi_l}{1-\phi_l}} (1 - \phi_l) < \mu_h (v_h - v_l) + v_l$$

as needed.

We now establish that the first inequality in (2) is true, which requires showing that

$$\frac{\mu_h v_h + \mu_l v_l - \mu_l \phi_l u_l - \mu_l (v_l - c_l)^{1-\phi_l} (v_l - u_l)^{\phi_l}}{\mu_h \frac{v_h - c_l}{c_h - c_l}} \leqslant u_l + c_h - c_l,$$

or, equivalently,

$$\mu_h c_l + \mu_l v_l \leqslant u_l + \mu_l \left(v_l - c_l\right)^{1-\phi_l} \left(v_l - u_l\right)^{\phi_l}, \forall u_l \in \mathrm{Supp}\left(F_l\right) \subset \left[c_l, v_l\right]. \tag{6}$$

Since, the right side of (6) is a concave function, it takes its minimum values at the extremes of the interval $[v_l, c_l]$. These values are given by v_l and $\mu_l v_l + \mu_h c_l$, both of which are at least as large as the left side of (6). Hence, (6) must be satisfied for all $u_l \in [v_l, c_l]$, as needed.

Global Deviations. Note that our conditions (16) and (17) imply that local deviations with respect to u_h and u_l are not profitable. It, thus, remains to show that, for all $\left(u_l', u_h'\right)$, $\Pi\left(u_l', u_h'\right) \leqslant \mu_l (1-\pi)(v_l - c_l)$. We consider two types of deviations:

1. Consider first deviation menus with $u_h' > \max \mathrm{Supp}\left(F_h\right) = \bar{u}_h$. Such deviations attract all type h sellers, so that $1 - \pi + \pi F_h\left(u_h'\right) = 1$. If $u_l' > \max \mathrm{Supp}\left(F_l\right) = \bar{u}_l$, then the profits from this menu are given by
$$\mu_l\left(v_l - u_l'\right) + \mu_h \Pi_h\left(u_l', u_h'\right).$$
Since $\phi_l > 0$, this function is decreasing in u_l' and u_h', and therefore
$$\mu_l\left(v_l - u_l'\right) + \mu_h \Pi_h\left(u_l', u_h'\right) < \mu_l\left(v_l - \bar{u}_l\right) + \mu_h \Pi_h\left(\bar{u}_l, \bar{u}_h\right) = \mu_l (1-\pi)(v_l - c_l).$$
When $u_l' \leqslant \bar{u}_l$, the partial derivative of $\Pi\left(u_l', u_h'\right)$ with respect to u_l' is
$$\begin{aligned} &-\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right) + \mu_l \pi f_l\left(u_l'\right)\left(v_l - u_l'\right) + \mu_h \frac{v_h - c_h}{c_h - c_l} \geqslant \\ &\qquad -\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right) + \mu_l \pi f_l\left(u_l'\right)\left(v_l - u_l'\right) + \mu_h\left(1-\pi+\pi F_l\left(u_l'\right)\right)\frac{v_h - c_h}{c_h - c_l} = 0. \end{aligned}$$
Thus, for a given value of u_h', we must have
$$\Pi\left(u_l', u_h'\right) \leqslant \Pi\left(\bar{u}_l, u_h'\right) < \Pi\left(\bar{u}_l, \bar{u}_h\right) = \mu_l (1-\pi)(v_l - c_l),$$
where the last inequality follows from the fact that Π_h is decreasing in u_h'. Thus, such global deviations are unprofitable.

2. Next consider deviations with $u_h' \in [c_h, \bar{u}_h]$. In this case, there must exist $\tilde{u}_l$ such that $u_h' = U_h\left(\tilde{u}_l\right)$ and thus $F_h\left(u_h'\right) = F_l\left(\tilde{u}_l\right)$. We can thus write the profits obtained from the deviation menu (u_l', u_h') as
$$\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(v_l - u_l'\right) + \mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\Pi_h\left(u_l', u_h'\right). \tag{7}$$
We show that the function defined by (7) is strictly concave in u_l' for values of $u_l' \in \mathrm{Supp}\left(F_l\right)$ and decreasing for values of $u_l' > \bar{u}_l$ so that this function is maximized at the value of u_l', which equates its partial derivative with zero. By (16), this partial derivative is zero when evaluated at $u_l' = \tilde{u}_l$, which completes the proof.

   Note that for $u_l' \in \mathrm{Supp}\left(F_l\right)$, since Π_h is linear in u_l', the second derivative of (7) with respect to u_l' is given by
$$\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(v_l - u_l'\right).$$

Using the form of the distribution F_l given by (1), we may rewrite this second derivative as

$$\begin{aligned}\frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)\left(\nu_l-u_l'\right) &= \frac{\partial^2}{\partial\left(u_l'\right)^2}\mu_l\left(1-\pi\right)\left(\nu_l-c_l\right)^{\phi_l}\left(\nu_l-u_l'\right)^{1-\phi_l} \\ &= \left(\phi_l-1\right)\phi_l\mu_l\left(1-\pi\right)\left(\nu_l-c_l\right)^{\phi_l}\left(\nu_l-u_l'\right)^{-1-\phi_l}<0\end{aligned}$$

so that (7) is strictly concave in u_l' for values of $u_l' \in \operatorname{Supp}\left(F_l\right)$. For values $u_l' > \tilde{u}_l$, $1-\pi+\pi F_l\left(u_l'\right)=1$, and thus (7) satisfies

$$\mu_l\left(\nu_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\Pi_h\left(u_l',u_h'\right).$$

The derivative of this function with respect to u_l' is given by

$$-\mu_l+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\frac{\nu_h-c_h}{c_h-c_l}<-\mu_l+\mu_h\frac{\nu_h-c_h}{c_h-c_l}=-\mu_l\phi_l<0.$$

Therefore, (7) is maximized at a value of u_l', which equates the partial derivative of (7) with zero. This value must satisfy

$$-\mu_l\left(1-\pi+\pi F_l\left(u_l'\right)\right)+\mu_l\pi f_l\left(u_l'\right)\left(\nu_l-u_l'\right)+\mu_h\left(1-\pi+\pi F_l\left(\tilde{u}_l\right)\right)\frac{\nu_h-c_h}{c_h-c_l}=0.$$

Note that since (7) is strictly concave, at most one u_l' exists that satisfies the above. The differential equation (16) implies that $u_l'=\tilde{u}_l$ is a solution to the above equation. This implies that (7) must be maximized at $u_l'=\tilde{u}_l$.

A.2.2 Proofs of Propositions 3 and 4

We prove these propositions together. To begin, let ϕ_1 be the value of ϕ_l that satisfies

$$c_h \geqslant \nu_l+\frac{\pi(1-\mu_l)\left(\nu_h-\nu_l\right)}{\left(1-\pi\right)\left[\left(1-\pi\right)^{\frac{1-\phi_l}{\phi_l}}-1\right]} \tag{8}$$

with equality. Similarly, let ϕ_2 be the value of ϕ_l that satisfies

$$1-\pi \geqslant \frac{\mu_h\nu_h+\mu_l\nu_l-\nu_l}{(1-\phi_l)(\mu_h\nu_h+\mu_l\nu_l-c_h)} \tag{9}$$

with equality. We first argue that (8) represents a lower bound on ϕ_l and (9) represents an upper bound on ϕ_l, which lies below the lower bound defined by (8). In other words, the inequalities (8) and (9) partition the set $(-\infty,0]$. We then prove that the equilibrium described in Proposition 4 exists—that is, in each case, no profitable local or global deviations exist when buyers use the equilibrium strategies defined jointly by Propositions 3 and 4.

Lemma 6. *(8) is satisfied if and only if $\phi_1 \leqslant \phi_l < 0$ and (9) is satisfied if and only if $\phi_l \leqslant \phi_2$. Moreover, $\phi_2 < \phi_1 < 0$.*

Proof. First, note that equation (8), which implicitly determines the threshold ϕ_1, can be rewritten as

$$\left(1-\pi\right)^{\frac{1-\phi_l}{\phi_l}} \geqslant \frac{\pi}{1-\pi}\frac{\nu_h-\nu_l}{c_h-\nu_l}\mu_h+1, \tag{10}$$

or, after taking logs and substituting for ϕ_l, can be rewritten as

$$\frac{\mu_h (\nu_h - c_h)}{c_h - c_l - \mu_h (\nu_h - c_l)} \log(1-\pi) - \log(\mu_h \pi (\nu_h - \nu_l) + (1-\pi)(c_h - \nu_l)) - \log[(1-\pi)(c_h - \nu_l)] \geqslant 0. \tag{11}$$

We show that the left side of (11) is a decreasing function of μ_h, that (11) is strictly satisfied when μ_h is such that $\phi_l = 0$, and that (11) is weakly violated when $\mu_h = 1$. Hence, there is a unique threshold μ_1 (and implied threshold ϕ_1) such that for all $\mu_h \leqslant \mu_1$ such that $\phi_l < 0$, the separating condition (8) is satisfied. Differentiating the left side of (11) with respect to μ_h, we obtain

$$\log(1-\pi) \frac{(\nu_h - c_h)(c_h - c_l)}{[c_h - c_l - \mu_h (\nu_h - c_l)]^2} - \frac{\pi (\nu_h - \nu_l)}{\mu_h \pi (\nu_h - \nu_l) + (1-\pi)(c_h - \nu_l)},$$

which is negative for all $\pi \leqslant 1$. Next, as $\phi_l \to 0$ from below, it is immediate that (10) is satisfied since the left-hand side tends to infinity. As $\mu_h \to 1$, the term $(1-\phi_l)/\phi_l \to -1$ and so (10) tends to the requirement that

$$1 \geqslant \pi \frac{\nu_h - \nu_l}{c_h - \nu_l} + (1-\pi),$$

which is violated since $c_h < \nu_h$.

Next, consider equation (9), which implicitly determines the threshold ϕ_2. Substituting for ϕ_l, one can show the inequality (9) is equivalent to

$$\mu_h (\nu_h - \nu_l) \left[1 + (1-\pi)\frac{\nu_h - c_h}{c_h - c_l}\right] \geqslant \nu_h - \nu_l + (c_h - \nu_l)(1-\pi)\frac{\nu_h - c_h}{c_h - c_l}. \tag{12}$$

Clearly, (12) represents a lower bound on μ_h, or, equivalently, an upper bound on ϕ_l. Note that this equation is necessarily satisfied at $\mu_h = 1$. It is immediate that when μ_h is such that $\phi_l = 0$, equation (9) is violated since $c_h > \nu_l$.

We now establish that $\phi_2 < \phi_1$ by proving that $\phi_l \leqslant \phi_2$ implies $\phi_l < \phi_1$. Suppose $\phi_l \leqslant \phi_2$ and let $\bar{\nu} = \mu_h \nu_h + \mu_l \nu_l$, so that we can write (9) as

$$1 - \pi \geqslant \frac{\bar{\nu} - \nu_l}{(1-\phi_l)(\bar{\nu} - c_h)}. \tag{13}$$

Below, we will use the fact that (13) implies

$$1 - \phi_l \geqslant \frac{\bar{\nu} - \nu_l}{(\bar{\nu} - c_h)(1-\pi)} > \frac{\bar{\nu} - \nu_l}{\bar{\nu} - c_h} \quad \Rightarrow \quad -\phi_l > \frac{c_h - \nu_l}{\bar{\nu} - c_h}.$$

To prove that (8) is violated when $\phi_l \leqslant \phi_2$, note that (8) can be rearranged as

$$(1-\pi)\left[(1-\pi)^{\frac{1-\phi_l}{\phi_l}} - 1\right](c_h - \nu_l) - \pi \mu_h (\nu_h - \nu_l) \geqslant 0$$

which can be simplified to

$$(1-\pi)(\bar{\nu} - c_h) + (1-\pi)^{\frac{1}{\phi_l}} (c_h - \nu_l) \geqslant \bar{\nu} - \nu_l. \tag{14}$$

We will show that (14) is violated if (13) holds. Towards this end, define a function

$$H(\pi) = (1-\pi)(\bar{\nu} - c_h) + (1-\pi)^{\frac{1}{\phi_l}} (c_h - \nu_l)$$

so that we must show $H(\pi) < \bar{\nu} - \nu_l$. We argue that $H(\cdot)$ is a strictly convex function that is decreasing at $\pi = 0$ and that, if π satisfies (13), then $H(\pi) < H(0) = \bar{v} - v_l$, which completes the proof.

First, note that $H(\cdot)$ is strictly convex, since $\phi_l < 0$, and

$$\begin{aligned} H'(\pi) &= -(\bar{v} - c_h) - \frac{1}{\phi_l}(1-\pi)^{\frac{1}{\phi_l}-1}(c_h - v_l), \\ H''(\pi) &= \frac{1}{\phi_l}\left(\frac{1}{\phi_l} - 1\right)(1-\pi)^{\frac{1}{\phi_l}-2}(c_h - v_l) > 0. \end{aligned}$$

Next, observe that $H(0) = \bar{v} - v_l$, $H'(0) \leqslant 0$ when $-\phi_l \geqslant (c_h - v_l)/(\bar{v} - c_h)$ and $\lim_{\pi \to 1} H(\pi) = \infty$. Thus, there is a unique value $\pi^s > 0$ such that for all $\pi < \pi^s$, $H(\pi) \leqslant \bar{v} - v_l$.

Next, let $\hat{\pi}$ denote the value of π such that (13) is satisfied with equality. We will prove that $H(\hat{\pi}) < \bar{v} - v_l$, so that $H(\pi) < \bar{v} - v_l$ for all $\pi \leqslant \hat{\pi}$.

Using the expression for $H(\pi)$, we have

$$H(\hat{\pi}) = \frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)}(\bar{v} - c_h) + \left(\frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)}\right)^{\frac{1}{\phi_l}}(c_h - v_l). \tag{15}$$

Straightforward algebra can be applied to (15) to show that $H(\hat{\pi}) < \bar{v} - v_l$ if and only if

$$\left(\frac{c_h - v_l}{\bar{v} - c_h}\right)^{\phi_l}\left(\frac{\bar{v} - v_l}{\bar{v} - c_h}\right)^{1-\phi_l} > (-\phi_l)^{\phi_l}(1-\phi_l)^{1-\phi_l}. \tag{16}$$

Since $(\bar{v} - v_l)/(\bar{v} - c_h) = 1 + (c_h - v_l)/(\bar{v} - c_h)$, if we let $B(x) = x^{\phi_l}(1+x)^{1-\phi_l}$, then (16) can be written as the requirement that

$$B\left(\frac{c_h - v_l}{\bar{v} - c_h}\right) > B(-\phi_l).$$

It is straightforward to show that $B'(x) < 0$ when $0 < x < -\phi_l$, and since (13) implies $-\phi_l > (c_h - v_l)/(\bar{v} - c_h)$, (16) must hold. Consequently, $H(\pi) < H(\hat{\pi}) < \bar{v} - v_l$, which proves that $\phi_1 > \phi_2$.

Definition of the Threshold, $\hat{u}_l$. To prove Propositions 3 and 4, we first define the threshold $\hat{u}_l$ for various values of $\phi_l < 0$.

Case 1: $\phi_l \leqslant \phi_2$. The threshold satisfies $\hat{u}_l = \bar{u}_l$, the upper bound of F_l, where $\bar{u}_l$ satisfies

$$\bar{v} - \bar{u}_l = (1-\pi)(\bar{v} - c_h). \tag{17}$$

Case 2: $\phi_2 < \phi_l < \phi_1$. The threshold satisfies

$$v_l + (\hat{u}_l - v_l)\left[1 - \pi + \pi F_l(\hat{u}_l)\right]^{\frac{1}{\phi_l}} = \bar{v} - (1-\pi)(\bar{v} - c_h) \tag{18}$$

where $F_l(\hat{u}_l)$ satisfies (18). As we will see below, in this case, the threshold will be such that $F_l(\hat{u}_l) \in (0,1)$ so that the equilibrium is indeed mixed.

Case 3: $\phi_1 < \phi_l < 0$. The threshold is any value such that $\hat{u}_l < \underline{u}_l$ where the lower bound of the support of F_l satisfies

$$(1-\pi)\left[\mu_l(v_l - \underline{u}_l) + \mu_h \Pi_h(\underline{u}_l, c_h)\right] = \bar{v} - \left[v_l + (1-\pi)^{\frac{1}{\phi_l}}(\underline{u}_l - v_l)\right]. \tag{19}$$

This equation determines the lower bound as the value that equates profits from the worst (separating) menu and the best (pooling) menu, where the best menu is determined as the value of u_l such that $F_l(u_l) = 1$ when F_l is determined by (16).

We now prove that the conjectured equilibria defined implicitly by the thresholds ϕ_1 and ϕ_2 are indeed equilibria.

Lemma 7. *Suppose $\phi_1 \leqslant \phi_l < 0$. There exists an equilibrium with only separating menus.*

Proof. It suffices to ensure that global deviations are unprofitable for buyers since, by construction, the distribution $F_l(u_l)$ ensures no local deviations are profitable. To rule out global deviations, a proof similar to that of Proposition 2 can be used. We show that for a given value of u_h', the profit function is strictly concave in u_l' and, therefore, it must be maximized at $u_l' = U_h^{-1}(u_h')$, since at this value the derivative of the profit function is equal to zero (by construction).

Profits from such a global deviation are given by

$$\mu_l \left(1 - \pi + \pi F_l \left(u_l'\right)\right) \left(\nu_l - u_l'\right) + \mu_h \left(1 - \pi + \pi F_h \left(u_h'\right)\right) \Pi_h \left(u_l', u_h'\right).$$

Since Π_h is linear in u_l', the second derivative of the above function is equal to the second derivative of profits from l type sellers. Using (16), we know that $\left(1 - \pi + \pi F_l \left(u_l'\right)\right) = \kappa \left(u_l' - \nu_l\right)^{-\phi_l}$ for some non-negative constant κ. Therefore, we have

$$\begin{aligned}
\frac{\partial^2}{\partial \left(u_l'\right)^2} \mu_l \left(1 - \pi + \pi F_l \left(u_l'\right)\right) \left(\nu_l - u_l'\right) &= -\mu_l \kappa \frac{\partial^2}{\partial \left(u_l'\right)^2} \left(u_l' - \nu_l\right)^{1-\phi_l} \\
&= -\mu_l \kappa \left(1 - \phi_l\right) \left(-\phi_l\right) \left(u_l' - \nu_l\right)^{-1-\phi_l} < 0.
\end{aligned}$$

Lemma 8. *Suppose $\phi_l \leqslant \phi_2$. There exists an equilibrium with only pooling menus.*

Proof. We first prove that no local deviations in the pooling equilibrium strictly improve profits. Below we demonstrate global deviations are also unprofitable. Recall that in an equilibrium with only pooling menus, the distribution $F_l(u_l)$ satisfies

$$\left(1 - \pi + \pi F_l(u_l)\right) \left(\bar{\nu} - u_l\right) = (1 - \pi) \left(\bar{\nu} - c_h\right), \tag{20}$$

where $\bar{\nu} = \mu_h \nu_h + \mu_l \nu_l$, $U_h(u_l) = u_l$, $F_h(u_l) = F_l(u_l)$, and $\mathrm{Supp}(F_l) = [c_h, \bar{\nu} - (1 - \pi)(\bar{\nu} - c_h)]$. Fix any utility, u_l, interior to the support of F_l and consider a local deviation to the menu $(u_l', u_h') = (u_l, u_l + \varepsilon)$. Profits from such a local deviation satisfy

$$\begin{aligned}
& \mu_l \left(1 - \pi + \pi F_l(u_l)\right) \left(\nu_l - u_l\right) + \mu_h \left(1 - \pi + \pi F_l \left(u_l + \varepsilon\right)\right) \Pi_h \left(u_l, u_l + \varepsilon\right) \\
= \; & \mu_l \left(1 - \pi + \pi F_l(u_l)\right) \left(\nu_l - u_l\right) + \mu_h \left(1 - \pi + \pi F_l \left(u_l + \varepsilon\right)\right) \left[\nu_h - u_l - \varepsilon \frac{\nu_h - c_l}{c_h - c_l}\right].
\end{aligned}$$

If local deviations are unprofitable, this function must be maximized at $\varepsilon = 0$, so that F_l must satisfy

$$\mu_h \pi f_l \left(u_l\right) \left[\nu_h - u_l\right] - \mu_h \left(1 - \pi + \pi F_l \left(u_l\right)\right) \frac{\nu_h - c_l}{c_h - c_l} \leqslant 0.$$

Totally differentiating (20) yields the following relationship between F_l and f_l,

$$\pi f_l(u_l) \left(\bar{\nu} - u_l\right) = \left(1 - \pi + \pi F_l(u_l)\right) \tag{21}$$

so that local deviations are unprofitable if

$$\mu_h \pi f_l \left(u_l\right) \left[\nu_h - u_l\right] - \mu_h \pi f_l(u_l) \left(\bar{\nu} - u_l\right) \frac{\nu_h - c_l}{c_h - c_l} \leqslant 0.$$

Since F_l is continuous in our constructed equilibrium, we may simplify this condition using straightforward algebra as

$$u_l \left(\nu_h - c_h\right) \leqslant \bar{\nu} \left(\nu_h - c_l\right) - \nu_h \left(c_h - c_l\right).$$

Consequently, we see that it suffices to check that this deviation is unprofitable at $\max \mathrm{Supp}(F_l)$. Using $u_l = \bar{v} - (1-\pi)(\bar{v} - c_h)$, simple algebraic manipulations show that this local deviation is unprofitable as long as

$$\frac{\bar{v} - v_l}{(1-\phi_l)(\bar{v} - c_h)} \leqslant 1 - \pi, \tag{22}$$

which is guaranteed by Lemma 6 since $\phi_l \leqslant \phi_2$.

To rule out global deviations, we show that for any value of $u_h' \in \mathrm{Supp}(F_l)$, the profit function is increasing in u_l' for all $u_l' \leqslant u_h'$. Thus, profits are maximized at the pooling menu $u_l' = u_h'$ so that there are no profitable deviations.

Profits associated with any global deviation (u_l', u_h') with $u_l' \leqslant u_h'$ and $u_h' \in \mathrm{Supp}(F_l)$ are given by

$$\mu_l \left(1 - \pi + \pi F_l(u_l')\right) \left(v_l - u_l'\right) + \mu_h \left(1 - \pi + \pi F_l\left(u_h'\right)\right) \Pi_h\left(u_l', u_h'\right).$$

Differentiating, we obtain

$$\begin{aligned}
&\mu_l \pi f_l\left(u_l'\right)\left(v_l - u_l'\right) - \mu_l\left(1 - \pi + \pi F_l(u_l')\right) + \mu_h\left(1 - \pi + \pi F_l\left(u_h'\right)\right)\frac{v_h - c_h}{c_h - c_l} \geqslant \\
&\mu_l \pi f_l\left(u_l'\right)\left(v_l - u_l'\right) - \mu_l\left(1 - \pi + \pi F_l(u_l')\right) + \mu_h\left(1 - \pi + \pi F_l\left(u_l'\right)\right)\frac{v_h - c_h}{c_h - c_l} = \\
&\mu_l \pi f_l\left(u_l'\right)\left(v_l - u_l'\right) - \mu_l \phi_l\left(1 - \pi + \pi F_l(u_l')\right)
\end{aligned} \tag{23}$$

where the inequality follows from the fact that $u_l' \leqslant u_h'$ so that $F_l(u_h') \geqslant F_l(u_l')$. Using (21) to substitute for $\pi f_l(u_l')$, we can write the last line of (23) as

$$\mu_l(1 - \pi + \pi F_l(u_l'))\left[1 + \frac{v_l - \bar{v}}{\bar{v} - u_l'} - \phi_l\right].$$

Since $u_l' \leqslant u_h' \leqslant \max \mathrm{Supp}(F_l)$, the expression in brackets takes its minimum value at $u_l' = \max \mathrm{Supp}(F_l)$ so that

$$1 + \frac{v_l - \bar{v}}{\bar{v} - u_l'} - \phi_l \geqslant 1 + \frac{v_l - \bar{v}}{(1-\pi)(\bar{v} - c_h)} - \phi_l \geqslant 0,$$

where the second inequality follows from (22). This implies that the expression in (23) is positive so that profits are globally maximized at $u_l' = u_h'$ for all $u_h' \in \mathrm{Supp}(F_l)$. ■

Lemma 9. *Suppose $\phi_2 < \phi_l < \phi_1$. There exists a mixed equilibrium.*

Proof. Recall that the threshold $\hat{u}_l$ is such that the constructed equilibrium features pooling contracts for $u_l \in [\min \mathrm{Supp}(F_l), \hat{u}_l]$ and separating menus for $u_l \in (\hat{u}_l, \max \mathrm{Supp}(F_l))$. First, we claim that when $\phi_2 < \phi_l < \phi_1$, then $\hat{u}_l$ is interior, in the sense that $c_h < \hat{u}_l < \bar{u}(\hat{u}_l)$. Second, we prove that no local or global deviations are profitable.

To see that $\hat{u}_l$ is interior, conjecture that $\hat{u}_l > c_h$ (we will verify it later), in which case $\hat{u}_l$ must satisfy[1]

$$\bar{v} - \left\{v_l + (\hat{u}_l - v_l)\left[(1-\pi)\frac{\bar{v} - c_h}{\bar{v} - \hat{u}_l}\right]^{\frac{1}{\phi_l}}\right\} - (1-\pi)(\bar{v} - c_h) = 0. \tag{24}$$

Let $H(\hat{u}_l)$ denote the left-hand side of (24). We will prove that when $\phi_2 < \phi_l < \phi_1$, there are two solutions to $H(\hat{u}_l) = 0$ with $\hat{u}_l > c_h$.

First, observe that one solution to $H(\hat{u}_l) = 0$ is given by

$$\hat{u}_l = \bar{u} = \bar{v} - (1-\pi)(\bar{v} - c_h).$$

[1]Recall that equilibrium profits satisfy $\bar{\Pi} = (1-\pi)(\bar{v} - c_h)$ when the worst menu offered in equilibrium is the pooling, monopsony menu.

This solution coincides with the conjecture that all menus are pooling, and therefore $\bar{u}(\hat{u}_l) = \hat{u}_l$.

We argue that there exists another solution $\hat{u}_l \in (c_h, \bar{u})$. We show this by proving that $H(\cdot)$ is convex, $H(c_h) > 0$, and $H'(\bar{u}) > 0$ so that an additional solution in the interval $(c_h, \bar{u})$ must exist.

Note that

$$H'(u) = -\left[(1-\pi)\frac{\bar{v}-c_h}{\bar{v}-u}\right]^{\frac{1}{\phi_l}} - (u-v_l)\frac{1}{\phi_l}\left[(1-\pi)\frac{\bar{v}-c_h}{\bar{v}-u}\right]^{\frac{1}{\phi_l}-1}(1-\pi)(\bar{v}-c_h)(\bar{v}-u)^{-2}.$$

By differentiating $H'(\cdot)$ and applying extensive algebraic manipulations (available upon request), one can show that $H''(\cdot) \geqslant 0$. Recall that $\bar{u}$ is defined so that $H(\bar{u}) = 0$ and

$$H'(\bar{u}) = -1 - \frac{1}{\phi_l}\frac{\bar{u}-v_l}{\bar{v}-\bar{u}} = H'(\bar{u}) = \frac{1-\phi_l}{\phi_l}\left[1 - \frac{\bar{v}-v_l}{(1-\pi)(1-\phi_l)(\bar{v}-c_h)}\right],$$

where the second equality is obtained by substituting for $\bar{u}$ and rearranging terms. When $\phi_l > \phi_2$, the term in brackets is negative, by Lemma 6, so that $H'(\bar{u}) > 0$. Finally, one can show that $H(c_h)$ satisfies

$$H(c_h) = \frac{1}{(1-\pi)^{\frac{1}{\phi_l}} - (1-\pi)}\left[v_l + \frac{\pi(\bar{v}-v_l)}{(1-\pi)^{\frac{1}{\phi_l}} - (1-\pi)} - c_h\right].$$

From Lemma 6, since $\phi_l < \phi_1 < 0$, the term in brackets is strictly positive, and, since the leading fraction is also positive, we must have $H(c_h) > 0$.

Hence, when $\phi_2 < \phi_l < \phi_1 < 0$, there must exist a solution to $H(\hat{u}_l) = 0$ with $\hat{u}_l \in (c_h, \bar{u})$. When $\hat{u}_l < \bar{u}$, one can show that $F_l(\hat{u}_l) < 1$ when F_l is determined by (18) on the interval $[c_h, \hat{u}_l]$, which confirms the conjecture that $\hat{u}_l$ is the interior of the support of F_l.

We now show that buyers cannot improve their profits by deviating from the constructed mixed allocation. As in Lemma 7 with only separation, the distribution F_l for $u_l \in [\hat{u}_l, \max \mathrm{Supp}(F_l)]$ is chosen to ensure local deviations are not profitable. It remains to show, then, that local deviations are not profitable in the pooling region and that no global deviations are profitable. As in Lemma 8 with only pooling menus, it suffices to ensure that at the upper bound of the pooling region, $\hat{u}_l$, no local deviations are profitable, or

$$\hat{u}_l(v_h - c_h) \leqslant \bar{v}(v_h - c_l) - v_h(c_h - c_l). \tag{25}$$

To prove that (25) holds, first note that since $\phi_2 < \phi_l < \phi_1$, we have $c_h < \hat{u}_l < \bar{u}(\hat{u}_l)$. We now prove that (25) is satisfied at $\hat{u}_l$. Algebra (available upon request) shows that (25) may be written as

$$\hat{u}_l \leqslant \frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l.$$

Since $H(\hat{u}_l) = 0$, if $H\left(\frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l\right) \leqslant 0$ then since $H(\cdot)$ is convex, (25) must be satisfied.

Using the form of $H(\cdot)$ implied by the left-hand side of (24), one can show that

$$H\left(\frac{-\phi_l}{1-\phi_l}\bar{v} + \frac{1}{1-\phi_l}v_l\right) = (\bar{v}-v_l)\left[\frac{\bar{v}-v_l-(1-\pi)(\bar{v}-c_h)}{\bar{v}-v_l} + \phi_l\frac{(1-\phi_l)^{\frac{1}{\phi_l}-1}(1-\pi)^{\frac{1}{\phi_l}}(\bar{v}-c_h)^{\frac{1}{\phi_l}}}{(\bar{v}-v_l)^{\frac{1}{\phi_l}}}\right]. \tag{26}$$

We now show that the term in brackets on the right side of (26) is negative. To simplify notation, define $\xi = (1-\pi)(\bar{v}-c_h)/(\bar{v}-v_l)$ so that the term in brackets can be written compactly as

$$1 - \xi + \phi_l(1-\phi_l)^{\frac{1}{\phi_l}-1}\xi^{\frac{1}{\phi_l}}.$$

Let $G(\xi) = 1 - \xi + \phi_l (1-\phi_l)^{\frac{1}{\phi_l}-1} \xi^{\frac{1}{\phi_l}}$ and observe that for $\xi \leqslant 1/(1-\phi_l)$, we have

$$G'(\xi) = -1 + [(1-\phi_l)\,\xi]^{\frac{1}{\phi_l}-1} \geqslant 0$$

so that for low values of ξ, $G(\xi)$ is an increasing function.

Since $\phi_l > \phi_2$, (13) implies that $\xi < 1/(1-\phi_l)$. Moreover, since $G(1/(1-\phi_l)) = 0$, it must be that $G(\xi) \leqslant G(1/(1-\phi_l)) \leqslant 0$, which ensures the term in brackets in (26) is indeed negative as desired.

To rule out global deviations, one can use the arguments provided in the proofs of Lemmas 7 and 8 in each region of the $\mathrm{Supp}(F_l)$. Since the arguments are exact replicas of the arguments above, we omit them here. ■

A.2.3 Proof of Theorem 2

We begin with a lemma that ensures the marginal distribution F_l is continuous (i.e., it has no mass points) when $\phi_l \neq 0$. We then prove uniqueness of the equilibrium first for $\phi_l > 0$ and then for $\phi_l < 0$. (In Appendix D, we demonstrate uniqueness for $\phi_l = 0$.)

Lemma 10. *If $\phi_l \neq 0$, then F_l is continuous.*

Proof. Recall from Lemma 5 that if F_l has a mass point, then it occurs at $\hat{u}_l = v_l$. As well, from Lemma 3, there must exist a positive measure set $S = \{\hat{u}_l, u_h)\}$ such that each equilibrium menu $(\hat{u}_l, u_h)$ has $\Pi_l = 0$. Let $\underline{u}_h$ denote the lowest value of u_h for which $(\hat{u}_l, u_h)$ belongs to the closure of the set S and let $\bar{u}_h$ denote the highest such value. Without loss of generality, we may assume that $(\hat{u}_l, \bar{u}_h)$ and $(\hat{u}_l, \underline{u}_h)$ belong to S and thus deliver the same profits to a buyer as the equilibrium level of profits.

Consider then the value of two different deviations, $(\hat{u}_l - \varepsilon, \underline{u}_h)$ and $(\hat{u}_l + \varepsilon, \bar{u}_h)$, for a small value of $\varepsilon > 0$, both of which must be feasible. The profits from these deviations are given by

$$\begin{aligned}
\Pi(\hat{u}_l - \varepsilon, \underline{u}_h) &= \mu_h (1 - \pi + \pi F_h(\underline{u}_h))\, \Pi_h(\hat{u}_l - \varepsilon, \underline{u}_h) + \mu_l (1 - \pi + \pi F_l(\hat{u}_l - \varepsilon))\, \varepsilon \\
\Pi(\hat{u}_l + \varepsilon, \bar{u}_h) &= \mu_h (1 - \pi + \pi F_h(\bar{u}_h))\, \Pi_h(\hat{u}_l + \varepsilon, \bar{u}_h) - \mu_l (1 - \pi + \pi F_l(\hat{u}_l + \varepsilon))\, \varepsilon.
\end{aligned}$$

These equalities are valid because F_h does not have a mass point and F_l does not have a mass point for $u_l > v_l$ or $u_l < v_l$. Since F_l is then left or right differentiable at $\hat{u}_l$, we have that

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\Pi(\hat{u}_l - \varepsilon, \underline{u}_h)\right|_{\varepsilon=0} &= -\mu_h (1 - \pi + \pi F_h(\underline{u}_h)) \frac{v_h - c_h}{c_h - c_l} + \mu_l \left(1 - \pi + \pi F_l^{-}(\hat{u}_l)\right) \\
\left.\frac{d}{d\varepsilon}\Pi(\hat{u}_l + \varepsilon, \bar{u}_h)\right|_{\varepsilon=0} &= \mu_h (1 - \pi + \pi F_h(\bar{u}_h)) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left(1 - \pi + \pi F_l^{+}(\hat{u}_l)\right).
\end{aligned}$$

The optimality of menus in S implies that both of these expressions must be non-positive. Since the equilibrium distributions are well-behaved above and below v_l, the equilibrium necessarily exhibits the strict rank-preserving property by Theorem 1 and therefore, $F_l^{-}(\hat{u}_l) = F_h(\underline{u}_h)$ and $F_l^{+}(\hat{u}_l) = F_h(\bar{u}_h)$. As a result, the above inequalities imply that

$$\begin{aligned}
-\mu_h \frac{v_h - c_h}{c_h - c_l} + \mu_l &\leqslant 0 \\
\mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l &\leqslant 0.
\end{aligned}$$

When $\phi_l \neq 0$, one of these is violated. Hence, a profitable deviation exists yielding the necessary contradiction. ■

Case 1: $\phi_l > 0$. As we have shown, any separating equilibrium is uniquely determined. Thus, in order to show the uniqueness of the equilibrium in this case, it remains to show that any equilibrium is separating. To see this, suppose to the contrary that $u_l = u_h$ for some menu offered in equilibrium. Now, consider the following alternative menu $(u_l - \epsilon, u_h)$ for a small and positive value of ϵ. This menu is feasible and the change in the profits for a small value of ϵ is given by

$$\mu_l(1-\pi+\pi F_l(u_l))\epsilon - \mu_h(1-\pi+\pi F_h(u_h))\frac{\nu_h - c_h}{c_h - c_l}\epsilon - \mu_l \pi f_l^-(u_l)(\nu_l - u_l)\epsilon,$$

where f_l^- is the left derivative of F_l at u_l; recall from Appendix A.1.2 that F_l must be differentiable.

Using the definition of ϕ_l and the strict rank-preserving property, we can write the above as

$$\mu_l \phi_l (1-\pi+\pi F_l(u_l))\epsilon - \mu_l \pi f_l^-(u_l)(\nu_l - u_l)\epsilon.$$

The above expression must be positive: $\phi_l > 0$, F_l and $f_l^-(u_l)$ are weakly positive, and $u_l > \nu_l$ since $u_l = u_h \geqslant c_h > \nu_l$ where $c_h > \nu_l$ by the lemons assumption. Therefore, this alternative menu is a profitable deviation that yields the necessary contradiction.

Case 2: $\phi_l < 0$**.** To prove the equilibrium characterized in Proposition 4 is unique, we first prove that in any equilibrium with $\phi_l < 0$, if $\bar{u} = \max \operatorname{Supp}(F_l)$, then $U_h(\bar{u}) = \bar{u}$ so that the best menu in equilibrium is a pooling menu. Next, we prove that if the equilibrium has a pooling region, the region begins at the lower bound of the support of F_l or ends at the upper bound of F_l. Additionally, if the equilibrium features a separating region, this region must end at the upper bound of the support of F_l. These results imply that any equilibrium must take one of the three forms described in Proposition 4: only separating, only pooling, or mixed. Finally, we show that the necessary conditions for each type of equilibrium to exist are mutually exclusive so that at most one type of equilibrium exists for each region of the parameter space, ensuring our equilibrium is unique for all $\phi_l < 0$. We prove these results in the following sequence of lemmas.

Lemma 11. *If $\phi_l < 0$, then the best equilibrium menu is a pooling menu.*

Proof. Let $\bar{u} = \max \operatorname{Supp}(F_l)$ and suppose for contradiction that $U_h(\bar{u}) > \bar{u}$. Consider a deviation menu with $\left(u_l', u_h'\right) = (\bar{u} + \varepsilon, U_h(\bar{u}))$. Since $U_h(\bar{u}) > \bar{u}$, this menu is incentive compatible and has $F_l(u_l') = F_l\left(u_h'\right) = 1$. This menu increases the buyer's profits relative to the menu $(\bar{u}, U_h(\bar{u}))$ by the amount

$$-\mu_l \varepsilon + \mu_h \frac{\nu_h - c_h}{c_h - c_l} = -\mu_l \phi_l \varepsilon > 0,$$

where the inequality follows from $\phi_l < 0$. This profitable deviation yields the necessary contradiction so that we must have $U_h(\bar{u}) = \bar{u}$. ■

Lemma 12. *If $\phi_l < 0$ and an equilibrium features $[u_1, u_2] \subseteq \operatorname{Supp}(F_l)$ such that $U_h(u_l) = u_l$ for $u_l \in [u_1, u_2]$, then either $u_1 = \min \operatorname{Supp}(F_l)$ or $u_2 = \max \operatorname{Supp}(F_l)$.*

Proof. Suppose toward a contradiction that a pooling interval with $u_1 > \min \operatorname{Supp}(F_l)$ and $u_2 < \max \operatorname{Supp}(F_l)$ exists. Then there must exist intervals sufficiently close to and below u_1 and above u_2, respectively, in which the equilibrium menus feature separation. Since in these intervals, $U_h(u_l) > u_l$ but $U_h(u_1) = u_1$ and $U_h(u_2) = u_2$, we must have $\lim_{u_l \nearrow u_1} U_h'(u_l) \leqslant 1$ and $\lim_{u_l \searrow u_2} U_h'(u_l) \geqslant 1$. In any region with $U_h(u_l) > u_l$, the distribution F_l must also satisfy

$$\frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)} = \frac{-\phi_l}{u_l - \nu_l},$$

since local deviations must be unprofitable. Moreover, in any such region, by the equal profit condition, U_h must satisfy

$$\bar{v} - \mu_l \phi_l u_l - \mu_h \frac{v_h - c_l}{c_h - c_l} U_h(u_l) = \bar{\Pi} \left(1 - \pi + \pi F_l\left(u_l\right)\right)^{-1},$$

where $\bar{\Pi}$ denotes the level of equilibrium profits.

Using these features of the conjectured equilibrium, in the separating regions, $U_h'(u_l)$ satisfies

$$-\mu_l \phi_l - \left(1 - \mu_l \phi_l\right) U_h'\left(u_l\right) = \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_l)} \frac{\phi_l}{u_l - v_l}$$

and so U_h'' satisfies

$$-\left(1 - \mu_l \phi_l\right) U_h''(u_l) = \frac{\bar{\Pi} \pi f_l\left(u_l\right)}{\left[1 - \pi + \pi F_l(u_l)\right]^2} \frac{\phi_l}{u_l - v_l} + \frac{\bar{\Pi}}{1 - \pi + \pi F_l(u_l)} \frac{-\phi_l}{\left[u_l - v_l\right]^2},$$

which implies that U_h is concave when $\phi_l < 0$. However, the existence of the pooling region implies that $U_h'^{+}\left(u_2\right) \geqslant 1 \geqslant U_h'^{-}(u_1)$, which contradicts the concavity of U_h given that $u_1 < u_2$. Hence, either $u_1 = \min \operatorname{Supp}(F_l)$ or $u_2 = \max \operatorname{Supp}(F_l)$. ■

Lemma 13. *If $\phi_l < 0$ and an equilibrium features $[u_1, u_2] \subseteq \operatorname{Supp}(F_l)$ such that $U_h\left(u_l\right) > u_l$ for $u_l \in (u_1, u_2)$, then $u_2 = \max \operatorname{Supp}(F_l)$.*

Proof. Suppose by way of contradiction that an equilibrium features separation ($U_h(u_l) > u_l$) on an interval $[u_1, u_2] \subseteq \operatorname{Supp}(F_l)$ with $u_2 < \max \operatorname{Supp}(F_l)$. Then there must exist a pooling interval $[u_2, \bar{u}]$ for some $\bar{u}$. Since $u_2 > \min \operatorname{Supp}(F_l)$, Lemma 12 implies that $\bar{u} = \max \operatorname{Supp}\left(F_l\right)$. Since the conjectured equilibrium features separation in $[u_1, u_2]$ with $U_h\left(u_l\right) \to u_l$ as $u_l \to u_2$, we must have $U_h'^{-}\left(u_2\right) \leqslant 1$. Since the conjectured equilibrium satisfies

$$\frac{\pi f_l(u_l)}{1 - \pi + \pi F_l(u_l)} = \frac{-\phi_l}{u_l - v_l}$$

on the interval $[u_1, u_2]$, $U_h'(u_2) \leqslant 1$ implies

$$\frac{1}{1 - \mu_l \phi_l} \left[-\mu_l \phi_l + \frac{\bar{\Pi}}{1 - \pi + \pi F_l\left(u_2\right)} \frac{-\phi_l}{u_2 - v_l}\right] \leqslant 1$$

or

$$-\phi_l \bar{\Pi} \leqslant \left[1 - \pi + \pi F_l\left(u_2\right)\right]\left(u_2 - v_l\right).$$

Since $u_2 < \bar{u}$, $F\left(u_2\right) < 1$ so that

$$-\phi_l \bar{\Pi} < u_2 - v_l. \tag{27}$$

Moreover, Lemma 11 ensures that the best equilibrium menu is pooling with utility $\bar{u}$ and, therefore, equilibrium profits satisfy $\bar{\Pi} = \bar{v} - \bar{u}$. Using the fact that $u_2 < \bar{u}$, substituting for $\bar{\Pi}$ in (27), and rearranging terms, we obtain

$$0 < \phi_l - \frac{v_l - \bar{u}}{\bar{v} - \bar{u}}. \tag{28}$$

We will show that (28) implies that a cream-skimming deviation must be a profitable deviation from the best (pooling) menu, yielding the necessary contradiction. Since the conjectured equilibrium features pooling in the interval $[u_2, \bar{u}]$, for u_l in this interval, the equilibrium satisfies

$$(1 - \pi + \pi F_l(u_l))(\bar{v} - u_l) = (1 - \pi)(\bar{v} - \bar{u})$$

so that

$$f_l(u_l) = \frac{1-\pi+\pi F_l(u_l)}{\pi(\bar{v}-u_l)}. \tag{29}$$

Consider then a cream-skimming deviation of the form $(u_l', u_h') = (\bar{u}-\varepsilon, \bar{u})$, which yields profits equal to

$$(1-\pi+\pi F_l(\bar{u}-\varepsilon))\mu_l(v_l-\bar{u}+\varepsilon)+(1-\pi+\pi F_h(\bar{u}))\mu_h\Pi_h(\bar{u}-\varepsilon,\bar{u}). \tag{30}$$

Differentiating (30) with respect to ε and evaluating it at $\varepsilon=0$, we obtain

$$(1-\pi+\pi F_l(\bar{u}))\mu_l-\pi f_l(\bar{u})\mu_l(v_l-\bar{u})-(1-\pi+\pi F_h(\bar{u}))\mu_h\frac{v_h-c_h}{c_h-c_l},$$

which, given that $F_l(\bar{u})=1$ and $f_l(\bar{u})=1/[\pi(\bar{v}-\bar{u})]$, can be written as

$$\mu_l\left[\phi_l-\frac{v_l-\bar{u}}{\bar{v}-\bar{u}}\right]>0, \tag{31}$$

where the inequality follows from (28). Hence, this cream-skimming deviation strictly increases the buyers' profits relative to the conjectured equilibrium level, a contradiction. ■

Since the only possible equilibria when $\phi_l<0$, then, are fully separating (except at the upper bound of the support of F_l), fully pooling, or mixed, we need only prove that only one of these equilibria may exist for any value of ϕ_l. We have already shown in the proof of Proposition 4 that $\phi_2<\phi_1<0$. Recall that a necessary condition for a fully pooling equilibrium is that $\phi_l\leqslant\phi_2$. Hence, there is no fully pooling equilibrium when $\phi_l>\phi_2$. Similarly, a necessary condition for a fully separating equilibrium is that $\phi_l\geqslant\phi_1$ so that when $\phi_l<\phi_1$, no fully separating equilibrium exists. This means that in the interval $\phi_2<\phi_l<\phi_1$, the only possible equilibrium is a mixed equilibrium. Moreover, the threshold in the mixed equilibrium is interior to the support of F_l only if ϕ_l lies between ϕ_2 and ϕ_1. Hence, at most one of these types of equilibria may exist for any value of $\phi_l<0$, proving that the equilibrium described in Proposition 13 is unique. ■

A.3 Proofs from Section 5

A.3.1 Proof of Proposition 5

In a slight abuse of notation, we write welfare as a function of p,

$$W(p,\mu_h)=(1-\mu_h)(v_l-c_l)+\mu_h\left[(1-p)X_1(p)+pX_2(p)\right], \tag{32}$$

where

$$X_n(p)=(v_h-c_h)\int_{c_l}^{\bar{u}(\pi(p))}x_h(u_l)(v_h-c_h)d\left(F_l^n(u_l,\pi(p)\right)$$

and

$$\pi(p)=\frac{2p}{1+p}. \tag{33}$$

Several facts follow immediately from our characterization of equilibrium. First, note that $X_1(0)=X_2(0)=1$ when $\phi_l<0$, which implies immediately that welfare is (weakly) maximized at $p=\pi(0)=0$ in this region of the parameter space. Second, note that $X_1(0)=X_2(0)=0$ when $\phi_l>0$, while $X_n(p)>0$ for all $p\in(0,1]$. Hence, welfare is minimized at $p=\pi(0)=0$ in this region of the parameter space. To show that welfare is maximized at an interior value of π when $\phi_l>0$, we will prove that $\lim_{p\to 1}W_p(p,\mu_h)<0$.

To this end, first note that

$$\frac{1}{\mu_h}W_p(p,\mu_h) = X_2(p) - X_1(p) + pX_2'(p) + (1-p)X_1'(p).$$

In what follows, we will prove a sequence of results:

1. $\lim_{p\to 1} X_2(p) - X_1(p) = 0$;
2. $\lim_{p\to 1}(1-p)X_1'(p) = 0$;
3. $\lim_{p\to 1} pX_2'(p) < 0$.

The first result follows immediately from the fact that $F_l(u_l)$ converges to a degenerate distribution at $p = \pi(1) = 1$. To prove the second result, we first integrate $X_1'(p)$ by parts:

$$\begin{aligned}
\frac{1}{\nu_h - c_h}(1-p)X_1'(p) &= (1-p)\frac{d}{dp}\int_{c_l}^{\bar{u}(\pi(p))} x_h(u_l)d\left(F_l(u_l,\pi(p))\right) \\
&= (1-p)\frac{d}{dp}\left[x_h(\bar{u}_l(\pi(p))) - \int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l)F_l(u_l;\pi(p))du_l\right] \\
&= -(1-p)\int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l)\frac{dF_l(u_l;\pi(p))}{d\pi}\frac{d\pi}{dp}du_l.
\end{aligned}$$

From the definition of $F(u_l)$ in (1), we have

$$\frac{dF_l(u_l;\pi(p))}{d\pi} = -\frac{F_l(u_l;\pi)}{\pi(1-\pi)}.$$

Therefore,

$$\frac{1}{\nu_h - c_h}(1-p)X_1'(p) = \frac{(1-p)}{\pi(1-\pi)}\frac{d\pi(p)}{dp}\int_{c_l}^{\bar{u}(\pi(p))} x_h'(u_l)F_l(u_l;\pi)du_l.$$

Using (33), we obtain

$$\begin{aligned}
\frac{1}{\nu_h - c_h}(1-p)X_1'(p) &= \frac{2}{\pi(2-\pi)}\frac{2}{(1+p)^2}\int_{c_l}^{\bar{u}(\pi(p))} x_h'(u_l)F_l(u_l;\pi)du_l \\
&= \frac{2}{\pi(2-\pi)}\frac{2}{(1+p)^2}\left[x_h(\bar{u}(\pi(p))) - \int_{c_l}^{\bar{u}(\pi(p))} x_h(u_l)dF_l(u_l;\pi)\right].
\end{aligned}$$

Since F_l becomes degenerate as $\pi \to 1$ and $\lim_{p\to 1}\pi = 1$, this final results implies

$$\lim_{p\to 1}(1-p)X_1'(p) = (\nu_h - c_h) \times 2 \times \frac{1}{2} \times 0 = 0.$$

This completes the proof of the second claim above.

To prove the third result, we first integrate $X_2'(p)$ by parts and differentiate:

$$\begin{aligned}
\frac{1}{v_h - c_h} p X_2'(p) &= p \frac{d}{dp} \int_{c_l}^{\bar{u}(\pi(p))} x_h(u_l) d\left(F_l^2(u_l, \pi(p))\right) \\
&= p \frac{d}{dp} \left[x_h(\bar{u}_l(\pi(p))) - \int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l) \left(F_l^2(u_l; \pi(p))\right) du_l \right] \\
&= -p \int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l) \frac{dF_l^2(u_l; \pi(p))}{d\pi} \frac{d\pi}{dp} du_l.
\end{aligned}$$

Since

$$\frac{d}{d\pi} F_l^2(u_l; \pi) = 2F_l(u_l; \pi) \frac{dF_l(u_l; \pi(p))}{d\pi} = -\frac{2}{\pi(1-\pi)} F_l^2(u_l; \pi), \tag{34}$$

we have

$$\begin{aligned}
\frac{1}{v_h - c_h} p X_2'(p) &= \frac{2p}{\pi(1-\pi)} \frac{d\pi(p)}{dp} \int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l) F_l^2(u_l; \pi(p)) du_l \\
&= \frac{2}{(2-\pi)(1-\pi)} \frac{2}{(1+p)^2} \int_{c_l}^{\bar{u}_l(\pi(p))} x_h'(u_l) F_l^2(u_l; \pi(p)) du_l \\
&= \frac{2}{(2-\pi)(1-\pi)} \frac{2}{(1+p)^2} \left[x_h(\bar{u}_l(\pi(p))) - \int_{c_l}^{\bar{u}_l(\pi(p))} x_h(u_l) d\left(F_l^2(u_l; \pi(p))\right) \right].
\end{aligned}$$

To prove the result, we will show that

$$\lim_{\pi \to 1} \frac{1}{1-\pi} \left[x_h(\bar{u}_l(\pi)) - \int_{c_l}^{\bar{u}_l(\pi)} x_h(u_l) d\left(F_l^2(u_l; \pi)\right) \right] < 0.$$

Define $H(\pi)$ as

$$H(\pi) = x_h(\bar{u}_l(\pi)) - \int_{c_l}^{\bar{u}_l(\pi)} x_h(u_l) d\left(F_l^2(u_l; \pi)\right).$$

Since $\lim_{\pi \to 1} H(\pi) = \lim_{\pi \to 1} 1 - \pi = 0$, we will apply L'Hopital's rule:

$$\lim_{\pi \to 1} \frac{H(\pi)}{1-\pi} = -\lim_{\pi \to 1} H'(\pi).$$

Next, using integration by parts, we have

$$H(\pi) = \int_{c_l}^{\bar{u}_l(\pi)} x_h'(u_l) F_l^2(u_l; \pi) du_l.$$

Therefore, using (34), we have

$$\begin{aligned}
H'(\pi) &= x_h'(\bar{u}_l(\pi)) \frac{d\bar{u}_l}{d\pi} + \int_{c_l}^{\bar{u}_l(\pi)} x_h'(u_l) \frac{dF_l^2(u_l; \pi)}{d\pi} du_l \\
&= x_h'(\bar{u}_l(\pi)) \frac{d\bar{u}_l}{d\pi} - \frac{2}{\pi(1-\pi)} \int_{c_l}^{\bar{u}_l(\pi)} x_h'(u_l) F_l^2(u_l; \pi) du_l \\
&= x_h'(\bar{u}_l(\pi)) \frac{d\bar{u}_l}{d\pi} - \frac{2}{\pi} \frac{H(\pi)}{(1-\pi)}.
\end{aligned}$$

Therefore,

$$\lim_{\pi\to 1}\frac{H(\pi)}{1-\pi} = -\lim_{\pi\to 1} x_h'(\bar{u}_l(\pi))\frac{d\bar{u}_l}{d\pi} + 2\lim_{\pi\to 1}\frac{H(\pi)}{1-\pi},$$

so that, rearranging the terms, we have

$$\lim_{\pi\to 1}\frac{H(\pi)}{1-\pi} = \lim_{\pi\to 1} x_h'(\bar{u}_l(\pi))\frac{d\bar{u}_l}{d\pi}.$$

We now prove that $\lim_{\pi\to 1} x_h'(\bar{u}_l(\pi))\frac{d\bar{u}_l}{d\pi} < 0$. Using the fact that

$$x_h(u_l) = \frac{1}{\mu_h(v_h - c_l)}\left\{(1-\mu_h)(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi} - (1-\mu_h)v_l + u_l - \mu_h c_l\right\},$$

we have

$$x_h'(u_l) = \frac{1}{\mu_h(v_h - c_l)}\left(1 - \phi(1-\mu_h)(v_l - c_l)^{1-\phi}(v_l - u_l)^{\phi-1}\right).$$

Next, since $\bar{u}_l(\pi)$ satisfies

$$1 = (1-\pi)\left(\frac{v_l - c_l}{v_l - \bar{u}_l(\pi)}\right)^{\phi},$$

we have

$$v_l - \bar{u}_l(\pi) = (1-\pi)^{\frac{1}{\phi}}(v_l - c_l),$$

which implies

$$\frac{d\bar{u}_l(\pi)}{d\pi} = \frac{1}{\phi}(1-\pi)^{\frac{1}{\phi}-1}(v_l - c_l)$$

and

$$x_h'(\bar{u}_l(\pi)) = \frac{1}{\mu_h(v_h - c_l)}\left(1 - \phi(1-\mu_h)(1-\pi)^{\frac{\phi-1}{\phi}}\right).$$

Combining these results, we find

$$x_h'(\bar{u}_l(\pi))\frac{d\bar{u}_l}{d\pi} = \frac{v_l - c_l}{\mu_h(v_h - c_l)}\left[\frac{(1-\pi)^{\frac{1}{\phi}-1}}{\phi} - (1-\mu_h)\right]$$

and hence

$$\lim_{\pi\to 1} x_h'(\bar{u}_l(\pi))\frac{d\bar{u}_l}{d\pi} = -\frac{(1-\mu_h)(v_l - c_l)}{\mu_h(v_h - c_l)} < 0.$$

■

A.3.2 Proof of Proposition 6

Proof. We start with the form of $W(p, \mu_h)$ given by (32) and express this instead as a function of π. Tedious, but straightforward calculations can be used to show

$$\begin{aligned} W(\pi, \mu_h) &= (1-\mu_h)(v_l - c_l)\left[1 + \frac{(v_h - c_h)}{(v_h - c_l)}\frac{2(1-\pi)}{(2-\pi)}\right] + \mu_h\left[c_h + \frac{(v_h - c_h)}{(v_h - c_l)}(v_l - c_l)\right] \\ &\quad + \frac{(v_h - c_h)}{(v_h - c_l)}\frac{2(1-\pi)^2}{\pi(2-\pi)}(v_l - c_l)\frac{\phi(\mu_h)}{1 - 2\phi(\mu_h)}\left((1-\pi)^{\frac{1-2\phi(\mu_h)}{\phi(\mu_h)}} - 1\right). \end{aligned}$$

Then $W_{\mu_h}(\pi, \mu_h)$ satisfies

$$W_{\mu_h}(\pi, \mu_h) \;=\; \Theta + \frac{(v_h - c_h)(v_l - c_l)}{v_h - c_l} \frac{2(1-\pi)^2}{\pi(2-\pi)} \frac{d}{d\mu_h} \frac{\phi(\mu_h)}{1-2\phi(\mu_h)} \left[(1-\pi)^{\frac{1}{\phi(\mu_h)}-2} - 1\right], \tag{35}$$

where

$$\Theta = c_h - (v_l - c_l) + \frac{v_h - c_h}{v_h - c_l}(v_l - c_l)\frac{\pi}{2-\pi}.$$

We will argue that when π is sufficiently small, then

$$\lim_{\mu_h \to 0} W_{\mu_h}(\pi, \mu_h) < \lim_{\mu_h \to \mu_0} W_{\mu_h}(\pi, \mu_h), \tag{36}$$

where μ_0 is the value of μ_h such that $\phi(\mu_0) = 0$. Inequality (36) implies that the W_{μ_h} must be increasing on an interval of μ_h[2]; that is, W must be convex on an interval of μ_h. In contrast, the inequality above is reversed when π is sufficiently close to 1.

Let

$$M(\pi, \mu_h) = \frac{d}{d\mu_h} \frac{\phi(\mu_h)}{1-2\phi(\mu_h)} \left[(1-\pi)^{\frac{1}{\phi(\mu_h)}-1} - 1\right]$$

and

$$G(\pi) = \lim_{\mu_h \to 0} M(\pi, \mu_h) - \lim_{\mu_h \to \mu_0} M(\pi, \mu_h).$$

Since the term multiplying $M(\pi, \mu_h)$ in (35) is positive, it suffices to show that $G(\pi) < 0$ for π close to 0 and $G(\pi) > 0$ for π close to 1. Note that

$$\lim_{\mu_h \to 0} M(\pi, \mu_h) \;=\; -\left(\frac{v_h - c_h}{c_h - c_l}\right) \frac{\pi + \log(1-\pi)}{1-\pi}$$

and

$$\lim_{\mu_h \to \mu_0} M(\pi, \mu_h) \;=\; \phi'(\mu_0) \left[-1 - \log(1-\pi) \lim_{\phi \to 0} \frac{(1-\pi)^{\frac{1}{\phi}-2}}{\phi}\right] = \left(\frac{v_h - c_h}{c_h - c_l}\right) \frac{1}{(1-\mu_0)^2}.$$

As a result,

$$G(\pi) = -\left(\frac{v_h - c_h}{c_h - c_l}\right) \frac{\pi + \log(1-\pi)}{1-\pi} - \left(\frac{v_h - c_h}{c_h - c_l}\right) \frac{1}{(1-\mu_0)^2}$$

It follows that

$$\lim_{\pi \to 0} G(\pi) = -\left(\frac{v_h - c_h}{c_h - c_l}\right) \frac{1}{(1-\mu_0)^2}$$

and

$$\lim_{\pi \to 1} G(\pi) = +\infty,$$

which completes the proof. ■

A.4 Proofs from Section 6

A.4.1 Proof of Proposition 7

Given $\hat{\pi}^1 = \hat{\pi}^2 \equiv \hat{\pi}$, we can use the analysis of our benchmark model to characterize the unique equilibrium. In particular, substituting $\hat{\pi} = \pi$, Propositions 2 and 3 characterize the equilibrium offer distributions, $\{F_j(u_j)\}$, along with equilibrium profits, which we denote by $\Pi^\star(\pi)$. For any $\phi_l < 1$, equilibrium

[2] It is straightforward to show that $W_{\mu_h}(\pi, \mu_h)$ is continuous.

profits are continuous and strictly decreasing in π, with $\Pi^\star(0) > 0$ and $\Pi^\star(1) = 0$. By assumption, $C'(\pi)$ is a continuous, strictly increasing function with $C'(0) = 0$ and $C'(1) > 0$, so that there is a unique solution to the first-order condition $C'(\pi) = \Pi^\star(\pi)$. ■

A.4.2 Proof of Lemma 5

Consider first the case of $\phi_l \geqslant 0$. In this region of the parameter space, $\pi^\star$ satisfies

$$C'(\pi^\star) - (1-\mu_h)(1-\pi^\star)(\nu_l - c_l) = 0, \tag{37}$$

and hence clearly $\frac{d\pi^\star}{d\mu_h} < 0$.

Next consider the case of $\phi_l < 0$. If $\phi_l \leqslant \phi_1$, where $\phi_1 < 0$ is defined in (8), then $\pi^\star$ satisfies

$$C'(\pi^\star) - (1-\pi^\star)(\mu_h\nu_h + \mu_l\nu_l - c_h) = 0, \tag{38}$$

and hence clearly $\frac{d\pi^\star}{d\mu_h} > 0$. The more difficult case is when $\phi_1 < \phi_l < 0$. In this case, $\pi^\star$ satisfies

$$C'(\pi^\star) - (1-\pi^\star)\left[\mu_l(\nu_l - \hat{u}_l) + \mu_h\Pi_h(\hat{u}_l, c_h)\right] = 0, \tag{39}$$

where $\hat{u}_l$ satisfies $\Upsilon(\hat{u}_l, \mu_h) = 0$, with

$$\Upsilon(\hat{u}_l, \mu_h) = \bar{\nu} - [\nu_l + (1-\pi)g(\mu_h, \pi)(\hat{u}_l - \nu_l)] - (1-\pi)\left[\mu_l(\nu_l - \hat{u}_l) + \mu_h\Pi_h(\hat{u}_l, c_h)\right] \tag{40}$$

and

$$g(\mu_h, \pi) = (1-\pi)^{\frac{1}{\phi_l}-1}.$$

To prove that $\frac{d\pi^\star}{d\mu_h} > 0$, we will show that

$$\Pi^\star(\pi, \mu_h) = (1-\pi)\left[\mu_l(\nu_l - \hat{u}_l) + \mu_h\Pi_h(\hat{u}_l, c_h)\right]$$

is increasing in μ_h and decreasing in π. To prove the first result, note that

$$\frac{\partial\Upsilon}{\partial\hat{u}_l} = -\frac{(1-\pi)^{\frac{1}{\phi_l}}}{\mu_l} + (1-\pi)\phi_l < 0,$$

since we are looking at the region with $\phi_l < 0$, and

$$\begin{aligned}
\frac{\partial\Upsilon}{\partial\mu_h} &= (\nu_h - \nu_l) - (1-\pi)\left[(\hat{u}_l - \nu_l)\left(\frac{\partial g}{\partial\mu_h} + 1\right) + \Pi_h(\hat{u}_l, c_h)\right] \\
&\geqslant (\nu_h - \nu_l) - (1-\pi)\left[(\hat{u}_l - \nu_l) + \Pi_h(\hat{u}_l, c_h)\right] \\
&= \pi(\nu_h - \nu_l) + (1-\pi)(c_h - \hat{u}_l)\left(\frac{\nu_h - c_l}{c_h - c_l}\right) \quad \geqslant 0,
\end{aligned}$$

where we use that $\frac{\partial g}{\partial\mu_h} < 0$ in the first inequality and $c_h \geqslant \hat{u}_l$ in the last. Hence, $\frac{d\hat{u}_l}{d\mu_h} \geqslant 0$ and

$$\frac{\partial\bar{\Pi}^\star}{\partial\mu_h} = (1-\pi)\left[\Pi_h(\hat{u}_l, c_h) + (\hat{u}_l - \nu_l) - \mu_l\frac{d\hat{u}_l}{d\mu_h}\phi_l\right] \geqslant 0.$$

To show that $\Pi^\star$ is decreasing in π, we must show that

$$\frac{1}{\phi_l}(1-\pi)^{\frac{1}{\phi_l}-1}(\underline{u} - \nu_l) - (1-\pi)^{\frac{1}{\phi_l}}\frac{d\hat{u}_l}{d\pi} \leqslant 0,$$

or

$$\frac{d\hat{u}_l}{d\pi} \geqslant \frac{\hat{u}_l - v_l}{\phi_l(1-\pi)}. \tag{41}$$

Solving (40) explicitly for $\hat{u}_l$ yields

$$\begin{aligned}\hat{u}_l &= \frac{\bar{v} - v_l + (1-\pi)^{\frac{1}{\phi_l}} v_l - (1-\pi)\left(\bar{v} - \mu_h \frac{v_h - c_l}{c_h - c_l} c_h\right)}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)} \\ &= v_l + \frac{\mu_h (v_h - v_l) - (1-\pi)\mu_h \frac{(v_l - c_l)(v_h - c_h)}{c_h - c_l}}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)}\end{aligned}$$

so that

$$\frac{d\hat{u}_l}{d\pi} = \frac{\mu_h \frac{(v_l - c_l)(v_h - c_h)}{c_h - c_l}}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)} - (\hat{u}_l - v_l) \frac{-\frac{1}{\phi_l}(1-\pi)^{\frac{1}{\phi_l}-1} + \mu_l \phi_l}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)}.$$

Thus, we have to show that

$$\begin{aligned}&\frac{\mu_h \frac{(v_l - c_l)(v_h - c_h)}{c_h - c_l}}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)} - (\hat{u}_l - v_l) \frac{-\frac{1}{\phi_l}(1-\pi)^{\frac{1}{\phi_l}-1} + \mu_l \phi_l}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)} \geqslant \frac{\hat{u}_l - v_l}{\phi_l(1-\pi)} \\ \Leftrightarrow\ &\frac{\mu_h \frac{(v_l - c_l)(v_h - c_h)}{c_h - c_l}}{(1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)} \geqslant \frac{(\hat{u}_l - v_l)\mu_l(\phi_l - 1)}{\left((1-\pi)^{\frac{1}{\phi_l}} - \mu_l \phi_l (1-\pi)\right)}.\end{aligned}$$

Again, since $\hat{u}_l \geqslant v_l$ and $\phi_l \leqslant 0$, the right-hand side of the inequality above is negative and the left-hand side is positive. This completes the proof.

A.4.3 Proof of Lemma 6

Let

$$X(\pi) = \left[2\pi(1-\pi) + \pi^2\right](\mu_h X_h(\pi)(v_h - c_h) + \mu_l(v_l - c_l)) \tag{42}$$

denote the gains from trade realized in a symmetric equilibrium with $\pi^\star = \pi$. Welfare is defined as

$$W(\tau) = -2Ac(\pi(\tau)) + X(\pi(\tau)), \tag{43}$$

so that

$$W'(\tau) = \left[X'(\pi) - 2Ac'(\pi)\right]\frac{d\pi}{d\tau}. \tag{44}$$

Since

$$\lim_{A \to 0} \pi^\star = 1,$$

and

$$\lim_{\pi \to 1} X(\pi) < 0,$$

there exists $\widetilde{A} > 0$ such that $X'(\pi) < 0$ whenever $A < \widetilde{A}$. Since $\frac{d\pi}{d\tau} < 0$, it follows immediately from (44) that $W'(\tau) > 0$ in this region. ■

A.5 Proofs from Section 7

A.5.1 Poisson Meeting Technology

Since $nP_n = \lambda Q_n$ for all $n \in \mathbb{N}$, we have:

$$Q_n(\alpha) = \frac{e^{-\alpha}\alpha^{n-1}}{(n-1)!}$$

and $Q_0(\alpha) = 1 - \sum_{n=1}^{\infty} Q_n(\alpha) = 0$. From the definition of $\tilde{\pi}$, we have $\tilde{\pi} = 1 - Q_1(\alpha)$ and substituting into (26) implies

$$\begin{aligned} G_l(u_l;\alpha) &= \frac{1}{1-Q_1(\alpha)} \sum_{n=2}^{\infty} Q_n(\alpha) F_l^{n-1}(u_l;\alpha) \\ &= \frac{Q_1(\alpha)}{1-Q_1(\alpha)} \left[e^{\alpha F_l(u_l;\alpha)} - 1\right]. \end{aligned} \tag{45}$$

Next, using the solution to the differential equation (27),

$$1 - \tilde{\pi} + \tilde{\pi} G_l(u_l;\alpha) = (1-\tilde{\pi}) \left(\frac{v_l - u_l}{v_l - c_l}\right)^{-\phi_l},$$

one can show that

$$G_l(u_l;\alpha) = \frac{Q_1(\alpha)}{1-Q_1(\alpha)} \left[\left(\frac{v_l - u_l}{v_l - c_l}\right)^{-\phi_l} - 1\right]. \tag{46}$$

Combining (45) and (46), we obtain

$$F_l(u_l;\alpha) = \frac{-\phi_l}{\alpha} \log\left(\frac{v_l - u_l}{v_l - c_l}\right).$$

Note also that $F_l(\bar{u}_l;\alpha) = 1$ implies

$$\frac{v_l - \bar{u}_l}{v_l - c_l} = e^{-\frac{\alpha}{\phi_l}}, \tag{47}$$

and

$$f_l(u_l;\alpha) = \frac{\phi_l}{\alpha}(v_l - u_l)^{-1},$$

which we use below.

Next, we evaluate the utilitarian welfare measure given the Poisson meeting technology:

$$\begin{aligned} &\sum_{n=1}^{\infty} P_n(\alpha) \left[\mu_h(v_h - c_h) \int x_h(u_l) d(F_l^n(u_l;\alpha)) + \mu_l(v_l - c_l)\right] \\ &= \mu_h(v_h - c_h) \int x_h(u_l) \sum_{n=1}^{\infty} nP_n(\alpha) F_l^{n-1}(u_l;\alpha) f_l(u_l;\alpha) du_l + \mu_l(v_l - c_l) \sum_{n=1}^{\infty} P_n(\alpha) \\ &= \mu_h(v_h - c_h) \int x_h(u_l) \sum_{n=1}^{\infty} nP_n(\alpha) F_l^{n-1}(u_l;\alpha) f_l(u_l;\alpha) du_l + \mu_l(v_l - c_l)(1 - e^{-\alpha}). \end{aligned}$$

Consider

$$\hat{W}(\alpha) = \int x_h(u_l) \sum_{n=1}^{\infty} nP_n(\alpha) F_l^{n-1}(u_l;\alpha) f_l(u_l;\alpha) du_l. \tag{48}$$

Substituting for $nP_n(\alpha)$, and using (26) and (46), we obtain

$$\begin{aligned}
\hat{W}(\alpha) &= \int x_h(u_l)\alpha \sum_{n=1}^{\infty} Q_n(\alpha) F_l^{n-1}(u_l;\alpha) f_l(u_l;\alpha) du_l \\
&= \int x_h(u_l)\alpha \left[Q_1(\alpha) + (1-Q_1(\alpha))G_l(u_l;\alpha)\right] f_l(u_l;\alpha) du_l \\
&= \int x_h(u_l)\alpha Q_1(\alpha) \left(\frac{v_l-u_l}{v_l-c_l}\right)^{-\phi_l} f_l(u_l;\alpha) du_l.
\end{aligned}$$

Substituting for $f_l(u_l)$ and $x_h(u_l)$ and rearranging terms yields

$$\hat{W}(\alpha) = Q_1(\alpha)\frac{\phi_l(v_l-c_l)^{\phi_l}}{\mu_h(v_h-c_l)} \int (v_l-u_l)^{-1-\phi_l} \left[\mu_l(v_l-c_l)^{1-\phi_l}(v_l-u_l)^{\phi_l} - (v_l-u_l) + \mu_h(v_l-c_l)\right] du_l,$$

where the limits of integration are c_l and $\bar{u}_l(\alpha)$. Applying tedious but straightforward calculus to compute the integral yields

$$\hat{W}(\alpha) = e^{-\alpha}\frac{\phi_l}{\mu_h(v_h-c_l)}\left[\mu_l(v_l-c_l)\frac{\alpha}{\phi_l} + \frac{v_l-c_l}{1-\phi_l}\left(e^{-\alpha\frac{1-\phi_l}{\phi_l}} - 1\right)\right] + \frac{v_l-c_l}{v_h-c_l} - e^{-\alpha}\frac{v_l-c_l}{v_h-c_l}.$$

Evaluating welfare as a function of α then implies

$$\begin{aligned}
W(\alpha) =& \mu_h(v_h-c_h)\left[\frac{v_l-c_l}{v_h-c_l} + \frac{\mu_l(v_l-c_l)}{\mu_h(v_h-c_l)}\alpha e^{-\alpha} + \frac{\phi_l}{1-\phi_l}\frac{v_l-c_l}{\mu_h(v_h-c_l)}\left(e^{-\frac{\alpha}{\phi_l}} - e^{-\alpha}\right) - e^{-\alpha}\frac{v_l-c_l}{v_h-c_l}\right] \\
&+ (1-e^{-\alpha})\mu_l(v_l-c_l).
\end{aligned}$$

Differentiating welfare with respect to α and rearranging terms, we obtain

$$W'(\alpha) = e^{-\alpha}\frac{(v_h-c_h)(v_l-c_l)}{v_h-c_l}\left[\mu_l(1-\alpha) - \frac{1}{(1-\phi_l)}e^{-\alpha\frac{1-\phi_l}{\phi_l}} + \frac{\phi_l}{1-\phi_l} + \mu_h + \mu_l\frac{v_h-c_h}{v_h-c_l}\right].$$

Let $H(\alpha)$ denote the term in brackets in the equation above. Since $H(\alpha)$ is a strictly concave function with $H(0) > 0$ and $\lim_{\alpha\to\infty} H(\alpha) = -\infty$, there exists a unique α^* such that for all $\alpha > \alpha^*$, $H(\alpha) < 0$. Hence, for all finite $\alpha > \alpha^*$, $W'(\alpha) < 0$.

A.5.2 Geometric Meeting Technology

For the Geometric meeting technology with $\lambda(\alpha) = \alpha/(1-\alpha)$, we have

$$Q_n(\alpha) = (1-\alpha)^2 n\alpha^{n-1}$$

and $Q_0(\alpha) = 0$. Much as in the Poisson case, one can use (26) and (27) to show

$$F_l(u_l;\alpha) = \frac{1}{\alpha}\left[1 - \left(\frac{v_l-u_l}{v_l-c_l}\right)^{\frac{\phi_l}{2}}\right],$$

so that

$$f_l(u_l;\alpha) = \frac{\phi_l}{2\alpha}\left(\frac{v_l-u_l}{v_l-c_l}\right)^{\frac{\phi_l}{2}}\frac{1}{v_l-u_l}$$

and the upper bound $\bar{u}_l(\alpha)$ satisfies

$$\left(\frac{v_l - \bar{u}_l(\alpha)}{v_l - c_l}\right)^{\frac{\phi_l}{2}} = 1 - \alpha.$$

Next, we evaluate the utilitarian welfare measure given the Geometric meeting technology:

$$\begin{aligned}
&\sum_{n=1}^{\infty} P_n(\alpha)\left[\mu_h(v_h - c_h)\int x_h(u_l)d(F_l^n(u_l;\alpha)) + \mu_l(v_l - c_l)\right] \\
&= \mu_h(v_h - c_h)\int x_h(u_l)\sum_{n=1}^{\infty} nP_n(\alpha)F_l^{n-1}(u_l;\alpha)f_l(u_l;\alpha)du_l + \mu_l(v_l - c_l)\alpha.
\end{aligned}$$

Consider $\hat{W}(\alpha)$, as defined in (48). Using similar steps to those we used above, one can show that

$$\begin{aligned}
\hat{W}(\alpha) &= (1-\alpha)\frac{\phi_l(v_l - c_l)^{\frac{\phi_l}{2}}}{2\mu_h(v_h - c_l)}\int_{c_l}^{\bar{u}_l(\alpha)}(v_l - u_l)^{-1-\frac{\phi_l}{2}}\left[\mu_l(v_l - c_l)^{1-\phi_l}(v_l - u_l)^{\phi_l} - (v_l - u_l) + \mu_h(v_l - c_l)\right]du_l \\
&= \frac{\phi_l(1-\alpha)}{2\mu_h(v_h - c_l)}(v_l - c_l)\left\{\mu_l\alpha\frac{2}{\phi_l} + \frac{1}{1-\phi_l/2}\left[(1-\alpha)^{2/\phi_l - 1} - 1\right] + \mu_h\frac{\alpha}{1-\alpha}\frac{2}{\phi_l}\right\}.
\end{aligned}$$

Therefore, welfare as a function of α is given by

$$\begin{aligned}
W(\alpha) = &\frac{\phi_l(1-\alpha)(v_h - c_h)(v_l - c_l)}{2(v_h - c_l)}\left\{\mu_l\alpha\frac{2}{\phi_l} + \frac{1}{1-\phi_l/2}\left[(1-\alpha)^{2/\phi_l - 1} - 1\right] + \mu_h\frac{\alpha}{1-\alpha}\frac{2}{\phi_l}\right\} \\
&+ \alpha\mu_l(v_l - c_l).
\end{aligned}$$

Differentiating with respect to α and rearranging terms, we obtain

$$W'(\alpha) = \mu_l(v_l - c_l) + \frac{(v_h - c_h)(v_l - c_l)}{(v_h - c_l)}\left[\mu_l(1-2\alpha) - \frac{1}{1-\phi_l/2}(1-\alpha)^{\frac{2}{\phi_l}-1} + \frac{\phi_l/2}{1-\phi_l/2} + \mu_h\right].$$

Note that $W'(0) = \mu_l(v_l - c_l)$ and

$$W'(1) = \mu_l(v_l - c_l)\frac{c_h - c_l}{v_h - c_l} + \frac{(v_h - c_h)(v_l - c_l)}{(v_h - c_l)}\left[\mu_h + \frac{\phi_l/2}{1-\phi_l/2}\right].$$

Since $W'(0) > 0$ and $W'(1) > 0$ and $W'(\alpha)$ is a strictly concave function of α, there exists no $\alpha \in (0,1)$ such that $W'(\alpha) < 0$.

B Constrained Efficiency

In this section, we examine the efficiency properties of equilibrium outcomes. We define the type of seller $i \in [0,1]$ by $\theta_i \in \Theta$, where $\Theta = \{l, h\} \times \{0,1\} \times \{0,1\}$. The first element of θ_i indicates whether the seller has a high- or low-quality good, the second element equals 1 if the seller is matched with buyer 1 and 0 otherwise, and the third element equals 1 if the seller is matched with buyer 2 and 0 otherwise. We let $c : \Theta \to \{c_l, c_h\}$ denote the valuation a seller of type $\theta \in \Theta$ has for her own good, and $v : \Theta \to \{v_l, v_h\}$ denote the buyer's valuation of a good purchased from a seller with type θ. Let $\bar{\theta} : [0,1] \to \Theta$ denote the mapping from sellers to their respective types, with $\bar{\Theta}$ representing the set of all possible mappings, $\bar{\theta}$.

Each buyer's type consists of the set of sellers with whom the buyer is matched. We represent the type of buyer $k \in \mathcal{B} = \{1,2\}$ as a mapping $m^k(i) : [0,1] \to \{0,1\}$. Let $\boldsymbol{m}^k$ denote the mapping $m^k(i)$ and $\mathcal{M}$ denote the set of all possible functions $\boldsymbol{m}^k$.

We model the realization of $\bar{\theta}$ and $\{m^k\}_{k\in\mathcal{B}}$ as the realization of a random variable that is drawn from a known distribution.[3] This ensures that the beliefs of each buyer and seller about the types of other buyers and sellers conditional on knowledge of their own type give rise to well-defined conditional expectations, as discussed in Uhlig (1996).

An allocation is a given by $\left(t_i^k, x_i^k\right)_{k\in\mathcal{B}, i\in[0,1]}$, where $t_i^k \in \mathbb{R}$ is a transfer of numeraire from buyer k to seller i and $x_i^k \in [0,1]$ is the amount of good transferred from seller i to buyer k. An allocation is *feasible* if for all i and k such that $m^k(i) = 0$, the allocation satisfies $t_i^k = x_i^k = 0$ and for all i, $x_i^1 x_i^2 = 0$. The first constraint ensures that transfers of numeraire and goods only occur between matched buyers and sellers, while the second constraint ensures that trade is exclusive.

We consider the class of *direct mechanisms* given by $(\boldsymbol{t}_i^k, \boldsymbol{x}_i^k)_{k\in\mathcal{B}, i\in[0,1]}$, where $\boldsymbol{t}_i^k : \bar{\Theta} \times \mathcal{M}^2 \to \mathbb{R}$ and $\boldsymbol{x}_i^k : \bar{\Theta} \times \mathcal{M}^2 \to [0,1]$.[4]

Constrained Efficiency with Direct Mechanisms. We begin by defining and characterizing incentive compatible direct mechanisms. A direct mechanism is *incentive compatible* if and only if, for all sellers i,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k\in\mathcal{B}} [t_i^k(\theta_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2) + (1 - x_i^k(\theta_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2))c(\theta_i)]\right] \\
&\geqslant \mathbb{E}\left[\sum_{k\in\mathcal{B}} [t_i^k(\hat{\theta}_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2) + (1 - x_i^k(\hat{\theta}_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2))c(\theta_i)]\right] \quad \forall \hat{\theta}_i \in \Theta,
\end{aligned} \tag{49}$$

and, for each buyer $k \in \mathcal{B}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_i [x_i^k(\bar{\theta}, \boldsymbol{m}^k, \boldsymbol{m}^{-k})\nu(\bar{\theta}_i) - t_i^k(\bar{\theta}, \boldsymbol{m}^k, \boldsymbol{m}^{-k})]di\right] \\
&\geqslant \mathbb{E}\left[\int_i [x_i^k(\bar{\theta}, \hat{\boldsymbol{m}}^k, \boldsymbol{m}^{-k})\nu(\bar{\theta}_i) - t_i^k(\bar{\theta}, \hat{\boldsymbol{m}}^k, \boldsymbol{m}^{-k})]di\right] \quad \forall \hat{\boldsymbol{m}}^k \in \mathcal{M},
\end{aligned} \tag{50}$$

where the conditional expectations in (49) and (50) are taken with respect to other agents' types.

Lastly, a direct mechanism satisfies *individual rationality* if and only if for all sellers i,

$$\mathbb{E}\left[\sum_{k\in\mathcal{B}} [t_i^k(\theta_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2) + (1 - x_i^k(\theta_i, \theta_{-i}, \boldsymbol{m}^1, \boldsymbol{m}^2))c(\theta_i)]\right] \geqslant V^s(\theta_i), \tag{51}$$

where $V^s(\theta_i)$ denotes the expected value a seller expects to receive in equilibrium, or,

$$V^s(\theta_i) = \begin{cases} \int \left[t_{\theta_{i,1}}(u_l) + c(\theta_i)(1 - x_{\theta_{i,1}}(u_l))\right] dF_l(u_l) & \text{if } m^1(i)m^2(i) = 0 \\ \int \left[t_{\theta_{i,1}}(u_l) + c(\theta_i)(1 - x_{\theta_{i,1}}(u_l))\right] d(F_l(u_l)^2) & \text{if } m^1(i)m^2(i) = 1 \end{cases},$$

and for each buyer $k \in \mathcal{B}$,

$$\mathbb{E}\left[\int_i [x_i^k(\bar{\theta}, \boldsymbol{m}^k, \boldsymbol{m}^{-k})\nu(\bar{\theta}_i) - t_i^k(\bar{\theta}, \boldsymbol{m}^k, \boldsymbol{m}^{-k})]di\right] \geqslant V^b, \tag{52}$$

where V^b represents the buyer's expected equilibrium value, or

$$V^b = \frac{1}{2-\pi}\sum_{i=l,h} \mu_i \left\{(1-\pi)\int [\nu_i x_i(u_l) - t_i(u_l)]dF_l(u_l) + \frac{\pi}{2}\int [\nu_i x_i(u_l) - t_i(u_l)]d(F_l(u_l)^2)\right\}.$$

[3] A complete description of one way to model this aggregate shock and the resulting expectations is available upon request.

[4] The Revelation Principle applies immediately to this environment so that we may restrict attention to direct mechanisms without loss of generality.

Characterization. We proceed by characterizing the set of mechanisms that maximize the sum of buyers' utilities. First, we simplify the set of incentive constraints. Note that each seller's match type—i.e., whether they are matched with buyer 1, buyer 2, or both—is correlated with the buyers' match types. As a result, it is straightforward to design a direct mechanism in which sellers have no incentives to lie about their match type, and buyers' incentive constraints are slack. This allows us to rewrite mechanisms simply as transfers (of the numeraire and the good) for each of the four types of sellers: those with high- or low-quality goods and those matched with one or two buyers.

Imposing symmetry, we redefine the mechanism as $\{t(i,n), x(i,n)\}$ for $i = \{l, h\}$ and $n = \{1, 2\}$ as the expected transfer and trade by a seller with quality i and n offers. Interim incentive compatibility requires, for each (i, n)

$$t(i,n) + (1 - x(i,n))c_i \geqslant t(\hat{i},n) + (1 - x(\hat{i},n))c_i. \tag{53}$$

Individual rationality of the sellers requires

$$t(i,1) + (1 - x(i,1))c_i \geqslant V^s(i,1) = \int [t_i(u_l) + (1 - x_i(u_l))c_i] \, dF_l(u_l), \tag{54}$$

$$t(i,2) + (1 - x(i,2))c_i \geqslant V^s(i,2) = \int [t_i(u_l) + (1 - x_i(u_l))c_i] \, d(F_l(u_l)^2). \tag{55}$$

Buyers' utility associated with any such mechanism satisfies

$$\frac{2}{2-\pi} \sum_{i=l,h} \mu_i \left[(1-\pi)(\nu_i x(i,1) - t(i,1)) + \frac{\pi}{2}(\nu_i x(i,2) - t(i,2)) \right]. \tag{56}$$

Thus, a constrained efficient allocation is a feasible allocation which maximizes (56) subject to (53)–(55). It is immediate that such an allocation satisfies $x(l,n) = 1$ for $n = 1, 2$ and

$$t(l,n) = t(h,n) + (1 - x(h,n))c_l.$$

That is, constrained efficient allocations do not distort trade for low-quality sellers (matched with either one or two buyers), and the incentive constraint for low-quality sellers must bind. Moreover, the individual rationality constraints for high-quality sellers necessarily bind. If these constraints did not bind, one could decrease the surplus allocated to sellers of high-quality goods by increasing $x(h,n)$ by ϵ and $t(h,n)$ by ϵc_l. Such a perturbation raises aggregate buyers' payoffs by $\epsilon(\nu_h - c_l)$, preserves incentives, and for ϵ small does not violate individual rationality of high-quality sellers.

We now state our main proposition concerning the efficiency of the equilibrium in our environment.

Proposition 1. *If $\phi_l > 0$ or $\phi_l < \phi_2$, where ϕ_2 is defined in Proposition 4, then the equilibrium is constrained efficient. If $\phi_l \in [\phi_2, 0]$, then the equilibrium is constrained inefficient.*

Proof of Proposition 1. We first prove that equilibrium allocation is constrained efficient when $\phi_l > 0$. To start, note that the individual rationality constraint for low-quality sellers must bind, otherwise one can improve buyers' payoffs by reducing transfers to low-quality sellers and adjusting trade with high-quality sellers to preserve incentive compatibility. Since $\phi_l > 0$, such a perturbation raises buyers' utility. Summarizing the results above, when $\phi_l > 0$ the solution to the program described above must satisfy $x(l,n) = 1$,

$$t(l,n) = V^s(l,n), \tag{57}$$

$$t(h,n) + (1 - x(h,n))c_h = V^s(h,n), \quad \text{and} \tag{58}$$

$$t(l,n) = t(h,n) + (1 - x(h,n))c_l. \tag{59}$$

We now show that expected volume of trade in a constrained efficient allocation coincides with expected trade in our equilibrium. It is clear that trade by low-quality sellers is the same (since $x(l,n) = 1$ for $n = 1, 2$). To see that trade by high-quality sellers also coincides, first note that (57)–(59) imply

$$V^s(h,n) - V^s(l,n) = [1 - x(h,n)]\,(c_h - c_l). \tag{60}$$

Using the definition of $V(i,n)$ in (54)–(55), along with the fact that each menu in equilibrium satisfies the low-quality seller's incentive constraint with equality, we have

$$V^s(h,n) - V^s(l,n) = \int [1 - x_h(u_l)]\,(c_h - c_l)\,d\left(F_l(u_l)^n\right). \tag{61}$$

Solving (60)–(61), we see that the volume of trade under the optimal mechanism between buyers and high-quality sellers with n offers is

$$x(h,n) = \int x_h(u_l)\,d\left(F_l(u_l)^n\right). \tag{62}$$

Using (62), similar algebra reveals that the transfers satisfy

$$t(i,n) = \int t_i(u_l)\,d\left(F_l(u_l)^n\right). \tag{63}$$

An immediate consequence of (62) and (63) is that buyers' utility coincides with what they receive in equilibrium, which proves the claim for $\phi_l > 0$.

Consider next the case of $\phi_l < 0$. We claim that equilibrium is constrained efficient if, and only if, $x_h(u_l) = 1$ for all $u_l \in \mathrm{Supp}(F_l)$. To see why, suppose that the equilibrium satisfies

$$\int x_h(u_l)\,d\left(F_l(u_l)^n\right) < 1$$

for some $n \in \{1, 2\}$. We will show that a perturbation of such an allocation is feasible and increases buyers' utility, i.e., the initial allocation cannot be constrained efficient. To do so, consider the mechanism

$$\begin{aligned}
&t(l,n) = V^s(l,n)\\
&x(l,n) = 1\\
&t(h,n) = \int t_h(u_l)\,d\left(F_l(u_l)^n\right)\\
&x(h,n) = \int x_h(u_l)\,d\left(F_l(u_l)^n\right).
\end{aligned}$$

This mechanism satisfies the incentive and individual rationality constraints by construction. Now consider the following perturbation: for some n and $\epsilon > 0$, let

$$\begin{aligned}
&\hat{t}(l,n) = t(l,n) + (c_h - c_l)\epsilon\\
&\hat{x}(l,n) = 1\\
&\hat{t}(h,n) = t(h,n) + c_h\epsilon\\
&\hat{x}(h,n) = x(h,n) + \epsilon.
\end{aligned}$$

We argue that this perturbation remains feasible and strictly increases buyers' utilities. Note that incentive constraints are satisfied since

$$\hat{t}(l,n) = t(l,n) + (c_h - c_l)\epsilon \geqslant t(h,n) + (1 - x(h,n))c_l + (c_h - c_l)\epsilon = \hat{t}(h,n) + (1 - \hat{x}(h,n))c_l.$$

Moreover, this perturbation raises the payoff of low-quality sellers and leaves the payoff of high-quality sellers unchanged. Finally, buyers' payoffs rise since the net impact of this perturbation is given by

$$\left[\mu_h(\nu_h - c_h) - \mu_l(c_h - c_l)\right]\epsilon = -\phi_l\mu_l(c_h - c_l) > 0,$$

where the last inequality follows from $\phi_l < 0$.

The final step of the proof requires showing that a pooling equilibrium—with $x_h(u_l) = 1$ for all $u_l \in \mathrm{Supp}(F_l)$—is constrained efficient. To see why, note that $V^s(l,n) = V^s(h,n)$ for $n \in \{1,2\}$ in any incentive compatible mechanism with full trade. Since the sellers' participation constraint binds, and the total surplus generated by the constrained efficient allocation coincides with that in the equilibrium, the payoff to the buyers must coincide as well.

C General Trading Mechanisms

In our equilibrium construction, we assumed that buyers offer menus consisting of two contracts—one for high-quality sellers and one for low-quality sellers. In this section, we show that this assumption is without loss of generality. In particular, we consider a game where sellers can send arbitrary messages and buyers offer mechanisms that are deterministic and exclusive—but otherwise unrestricted—mapping the seller's message into potential terms of trade.[5] We prove that the distribution of trades in any equilibrium of this more general setting coincides with that of a game with two-point menus. We prove this within the context of our baseline model, where two buyers face a continuum of sellers.

Intuitively, this result essentially shows that it is impossible for a buyer to screen a seller based on her outside offer. To see why, note that screening is possible only when the payoffs from accepting a given contract differ across types. For example, a seller with a low-quality good gets less utility (compared to one with a high-quality good) from accepting a contract that requires her to retain a fraction of the good. However, sellers who differ only in their alternative offers get the *same* utility from accepting a contract; since trading is exclusive, once they accept the terms of a given contract, their outside offer is *irrelevant*. This feature rules out the ability to screen sellers along this dimension.

The proof proceeds in two steps. First, we map our environment into the general framework of Martimort and Stole (2002), hereafter MS. This allows us to apply their "delegation principle," which establishes that any equilibrium of a game with general mechanisms and messages can be achieved by a menu game. Second, we show that equilibrium menus have at most two contracts that are accepted by sellers in equilibrium. Together, these steps imply that a game where buyers offer 2-point menus induces the same equilibrium distribution of trades as a more general game with arbitrary mechanisms and communication.

Step 1. We begin by expressing payoffs and strategies using the notation of MS. A contract is defined by a quantity-transfer pair $d = (x, t)$. The seller's type is given by $\theta = (j, A)$, where $j \in \{l, h\}$ is the quality of her good and $A \subset \{1, 2\}$ is the set of buyers with whom she is matched. Given a pair of contracts offered by the two buyers, $d = \left(d^1, d^2\right)$, the payoff to a seller of type θ is

$$U\left(d;\theta\right) = \max_{i \in A} \quad t^i + \left(1 - x^i\right)c_j. \tag{64}$$

When a seller has access to both of the buyers and is indifferent between the contracts they offer, we assume she randomizes, with each buyer being chosen with equal probability. We denote the seller's

[5] In a deterministic mechanism, the mapping from the seller's message to an offer is a deterministic function. Note, however, that buyers can still randomize over different mechanisms.

contract choice by $s^i(d;\theta)$, where $s^1(d;\theta)+s^2(d;\theta)=1$, so that the buyer's payoff can be written as

$$V^i(d;\theta)=\left(\nu_j x^i - t^i\right)s^i(d;\theta). \tag{65}$$

There is an unrestricted space of messages, denoted $\mathcal{M}$, available to each buyer-seller pair. The strategy space for buyers is the space of all deterministic communication mechanisms. Formally, such a mechanism consists of a mapping $\hat{d}^i : \mathcal{M} \to \mathcal{D}$ from messages to the set of all contracts $\mathcal{D}=[0,1]\times\mathbb{R}_+$. The set of such mechanisms is represented by $\Upsilon=(\mathcal{D})^{\mathcal{M}}$. Each buyer's strategy σ^i, then, is a distribution over the elements of Υ. A seller's strategy is a joint distribution over messages sent to each buyer with whom she is matched. The timing of the game is as follows. First, sellers draw their types. Second, each of the buyers simultaneously offers a mechanism to the sellers with whom they are matched. Third, each seller chooses a message to send to each of the buyers with whom she is matched. These choices then induce (potentially a pair of) contracts, with the resulting payoffs given by (64)–(65).

We can now apply the delegation principle from MS (Theorem 1). It states that the distribution of contracts and trades induced by any Perfect Bayesian Equilibrium in the game with mechanisms can be achieved by a game where buyers post menus of contracts and sellers choose their desired contract. Formally, a menu game is one in which each buyer's strategy is a distribution (possibly random) $\mu \in \Delta\left(2^{\mathcal{D}}\right)$ over all possible menus $z \subset \mathcal{D}$. Facing two menus, a seller of type j proceeds in two steps. First, she chooses a contract from each menu, which is described by a probability distribution $\chi_j(z_1,z_2;\theta)\in\Delta(z_1\times z_2)$ over pairs of contracts $d\in z_1\times z_2$. She then chooses one of the two contracts according to the functions $s^i(\cdot)$ described above.

Step 2. The second step, stated formally in the following result, shows that equilibrium menus cannot contain more than two "active" contracts, i.e., ones that are actually traded in equilibrium.

Proposition 2. *In any equilibrium of the menu game, any menu z has at most two contracts that are chosen by some seller type in equilibrium.*

Proof. Without loss of generality, consider an arbitrary menu z offered by buyer 1 with positive probability in equilibrium, and define $D_j(z)$ as the set of all contracts in that menu that are chosen by a type j seller with positive probability, i.e.,

$$D_j(z)=\{d^1\in z:\ \exists\, d^2\in z'\in \mathrm{Supp}(\mu):\left(d^1,d^2\right)\in \mathrm{Supp}(\chi_j(z,z')), s^1(d^1,d^2;(j,\cdot))>0\}.$$

We will show that $\left|D_j(z)\right|=1$ for $j\in\{l,h\}$. The strategy is to show that all elements in $D_j(z)$ must yield the same utility to type j sellers *and* the same payoffs to the buyer, i.e., for all $(x,t),(x',t')\in D_j(z)$, we must have

$$t+c_j(1-x)=t'+c_j\left(1-x'\right) \tag{66}$$
$$\nu_j x - t = \nu_j x' - t', \tag{67}$$

which implies $(x,t)=(x',t')$. It is easy to see that the two contracts must offer the same utility to the seller; otherwise she cannot choose both from the same menu with positive probability. To show that they must yield the same payoff to the buyer, consider the offer intended for the type l seller. Now, suppose that $(x,t),(x',t')\in D_l(z)$ with $\nu_l x - t > \nu_l x' - t'$. This inequality, combined with (66), implies

$$\nu_l\left(x-x'\right)>t-t'=c_l\left(x-x'\right)\Rightarrow x>x'.$$

As a result,

$$c_h\left(x-x'\right)>c_l\left(x-x'\right)=t-t'\Rightarrow t+c_h(1-x)>t'+c_h\left(1-x'\right).$$

This implies that $(x',t')\notin D_h(z)$. Hence, if we eliminate from z every contract in $D_l(z)$ except the one that delivers the maximum payoff to the buyer from type l sellers, the buyer's payoff strictly increases and the high type seller's choice is not altered. Therefore, if there is more than one element in $D_l(z)$, they must all yield the same profits.

Now suppose there exist $(x,t),(x',t') \in D_h(z)$ such that $v_h x - t > v_h x' - t'$. As before, this implies $x > x'$. We then have

$$t - t' = c_h\left(x - x'\right) > c_l\left(x - x'\right) \Rightarrow t + c_l(1-x) > t' + c_l\left(1 - x'\right).$$

Hence, $(x',t') \notin D_l(z)$. Then, as with type l sellers, eliminating all contracts in $D_h(z)$ that deliver less than the maximum payoff to the buyer is a profitable deviation. This concludes the proof.

D Mass Point Equilibria: The Case of $\phi_l = 0$

Proposition 3. *Suppose $\phi_l = 0$. The unique equilibrium of the game is described by the pair of distribution functions, with $F_l(u_l)$ degenerate at v_l and $F_h(u_h)$ satisfying*

$$(1 - \pi + \pi F_h(u_h))\, \mu_h \Pi_h(v_l, u_h) = (1-\pi)\, \mu_l (v_l - c_l) \tag{68}$$

with $\mathrm{Supp}(F_h) = [c_h, c_h + \pi(v_l - c_l)(v_h - c_h)/(v_h - c_l)]$.

Proof of Proposition 3. To show that the constructed distributions constitute an equilibrium, we show that there are no profitable deviations. In other words,

$$\forall\left(u_h', u_l'\right): \mu_h\left(1 - \pi + \pi F_l\left(u_l'\right)\right) \Pi_h\left(u_l', u_h'\right) + \mu_l\left(1 - \pi + \pi F_l\left(u_l'\right)\right)\left(v_l - u_l'\right) \leqslant (1-\pi)\, \mu_l (v_l - c_l).$$

We consider two cases:

1. $u_h' > \max \mathrm{Supp}(F_h) = \bar{u}_h$: In this case, when $u_l' > v_l$, the profit function is given by

$$\mu_h \Pi_h\left(u_l', u_h'\right) + \mu_l\left(v_l - u_l'\right).$$

Since $\phi_l = 0$, this function is invariant to changes in u_h' and is strictly decreasing in u_l'. Therefore, its value must be less than its value evaluated at $(\bar{u}_h, v_l)$, which gives the equilibrium profits. When, $u_l' \leqslant v_l$, the profits are given by $\mu_h \Pi_h\left(u_l', u_h'\right)$, which is decreasing in u_h', and therefore

$$\mu_h \Pi_h\left(u_l', u_h'\right) + \mu_l (1-\pi)\left(v_l - u_l'\right) < \mu_h \Pi_h\left(u_l', \bar{u}_h\right) + \mu_l (1-\pi)\left(v_l - u_l'\right).$$

Note that the right-hand side of the above inequality is a linear function of u_l', whose derivative is given by

$$\begin{aligned} \mu_h \frac{v_h - c_h}{v_l - c_l} - \mu_l (1-\pi) &= \mu_h \frac{v_h - c_h}{c_h - c_l} - \mu_l + \mu_l \pi \\ &= -\mu_l \phi_l + \mu_l \pi = \mu_l \pi > 0. \end{aligned}$$

Therefore, we must have that

$$\mu_h \Pi_h\left(u_l', \bar{u}_h\right) + \mu_l (1-\pi)\left(v_l - u_l'\right) \leqslant \mu_h \Pi_h(v_l, \bar{u}_h) = (1-\pi)\, \mu_l (v_l - c_l),$$

where the last equality follows from (68).

2. $u_h' \in [c_h, \bar{u}_h]$. In this case, when $u_l' > v_l$, profits are given by

$$\begin{aligned} \mu_h\left(1 - \pi + \pi F_l\left(u_h'\right)\right) \Pi_h\left(u_l', u_h'\right) + \mu_l\left(v_l - u_l'\right) &\leqslant \mu_h\left(1 - \pi + \pi F_l\left(u_h'\right)\right) \Pi_h\left(v_l, u_h'\right) \\ &= (1-\pi)\, \mu_l (v_l - c_l), \end{aligned}$$

where the inequality is satisfied since $u_l' > v_l$ and the last equality follows from (68).

When $u_l' \leqslant v_l$, profits are given by

$$\mu_h \left(1-\pi+\pi F_l\left(u_h'\right)\right) \Pi_h\left(u_l', u_h'\right)+\mu_l(1-\pi)\left(v_l-u_l'\right).$$

This function is linear in u_l' and its derivative is given by

$$\begin{aligned} \mu_h\left(1-\pi+\pi F_l\left(u_h'\right)\right)\frac{v_h-c_h}{c_h-c_l}-\mu_l(1-\pi) &= (1-\pi)\left(\mu_h\frac{v_h-c_h}{c_h-c_l}-\mu_l\right)+\pi F_l\left(u_h'\right)\frac{v_h-c_h}{c_h-c_l} \\ &= \pi F_l\left(u_h'\right)\frac{v_h-c_h}{c_h-c_l} \geqslant 0. \end{aligned}$$

Therefore, it is maximized at $u_l' = v_l$. This establishes that there are no profitable deviations.

To conclude the proof, we show that the equilibrium constructed is the unique equilibrium when $\phi_l = 0$.

In order to show uniqueness of the equilibrium, it is sufficient to show that, in any equilibrium, F_l must be degenerate at v_l. When F_l is degenerate at v_l, from Lemmas 1 and 4, we know that F_h must be continuous and strictly increasing, and therefore it must satisfy (68).

Suppose that $u_l \neq v_l$ exists that belongs to the support of F_l. Then the proof of Lemma 5 can be used to show that for values of $u_l \neq v_l$, F_l must have no flat and mass points and, consequently, equilibrium must exhibit the strict rank-preserving (SRP) property. Now consider any menu for which $u_l < v_l$ and a deviation that increases the value of u_l by a small amount. In this case, F_l is differentiable and we can write the change in profits from such a deviation as

$$\begin{aligned} &\mu_l \pi f_l^+(u_l)(v_l-u_l)-\mu_l(1-\pi+\pi F_l(u_l))+\mu_h\frac{v_h-c_h}{c_h-c_l}(1-\pi+\pi F_h(u_h)) = \\ &\mu_l \pi f_l^+(u_l)(v_l-u_l)-\mu_l\phi_l(1-\pi+\pi F_l(u_l)) > 0 \end{aligned}$$

where in the above f_l^+ is the right derivative of F_l and we have used SRP. The above implies that increasing u_l must be a profitable deviation, which proves the contradiction. The case with $u_l > v_l$ is ruled out in a similar fashion. This concludes the proof. ■

E Additional Extensions and Robustness

In this section, we examine a few additional extensions of our framework, both to ensure the robustness of our results and to demonstrate that our framework is amenable to more applied work. First, we relax our assumption of linear utility to analyze the canonical model of insurance under private information. Second, we allow the degree of competition to differ across sellers of different quality. Third, we incorporate additional dimensions of heterogeneity, including horizontal and vertical differentiation. Lastly, we consider the case of $N > 2$ types of sellers. All proofs are in Section E.5.

E.1 A Model of Insurance

To start, we analyze a canonical model of insurance under private information, along the lines of Rothschild and Stiglitz (1976), and show that our main results—in particular, the structure of equilibrium menus and the nonmonotonicity of welfare with respect to the degree of competition—extend beyond the linear, transferable utility environment.

A unit measure of agents with strictly increasing, strictly concave utility functions $w(c)$ face idiosyncratic income risk.[6] Their income in normal times is y, but they also face the risk of an "accident,"

[6]Note that, in this application, the "buyers" of insurance are the ones with private information. To avoid confusion, we

which reduces their income by d. The accident is observable and contractible, but the probability of its occurrence, denoted $\theta_j,\ \ j \in \{b, g\}$, is private information. A fraction μ_b of agents are of type b and face a higher risk of accident than type g agents, i.e., $\theta_b > \theta_g$. Principals (the insurance providers) are risk-neutral, which implies that gains from trade are strictly positive for both types. The competitive structure is exactly the same as our baseline model: a fraction $1 - \pi$ of agents receive one offer and the remainder receive two.

A contract consists of a premium and a transfer to the agent in the event of an accident. Since trading is exclusive and the accident is observable, we can also think of the contract as directly offering a utility level in the normal and accident states. As before, we consider menus with two contracts, one for each type, i.e., $\mathbf{z} = \left(u_b^n, u_b^a\right), \left(u_g^n, u_g^a\right)$ such that incentive and participation constraints are satisfied for $j \in \{b, g\}$:

$$\begin{aligned}
\left(IC_j\right): &\quad \theta_j u_j^a + \left(1 - \theta_j\right) u_j^n \geqslant \theta_j u_{-j}^a + \left(1 - \theta_j\right) u_{-j}^n, \\
\left(PC_j\right): &\quad \theta_j u_j^a + \left(1 - \theta_j\right) u_j^n \geqslant \theta_j w\left(y - d\right) + \left(1 - \theta_j\right) w\left(y\right).
\end{aligned}$$

To solve for the equilibrium, we follow the same steps as in Section 4. The first step is to obtain the utility representation. It is straightforward to prove that, in all equilibrium menus, type b agents are fully insured and $\left(IC_b\right)$ binds. This allows us to summarize equilibrium menus with a pair of expected utilities, (u_b, u_g), and allocations given by the solution to the following system of equations:

$$u_b = u_b^a = u_b^n, \qquad u_b = \theta_b u_g^a + (1 - \theta_b) u_g^n, \qquad u_g = \theta_g u_g^a + (1 - \theta_g) u_g^n. \tag{69}$$

In a separating menu, the principal offers type g agents less than full insurance: $u_g^a < u_g^n$ such that $\left(IC_b\right)$ binds. Define $C\left(u\right) \equiv w^{-1}\left(u\right)$ to be the principal's cost of providing a utility level u. Note that $C'\left(u\right), C''\left(u\right) > 0$. Then, the objective of the principal is described by (8), where the type-specific profit functions satisfy

$$\begin{aligned}
\Pi_b\left(u_b, u_g\right) &= y - \theta_b d - C\left(u_b\right), \\
\Pi_g\left(u_b, u_g\right) &= y - \theta_g d - \theta_g C\left(u_g^a\right) - (1 - \theta_g) C\left(u_g^n\right).
\end{aligned}$$

Since w is strictly increasing and concave, we can show that

$$\frac{d\Pi_g\left(u_b, u_g\right)}{du_b} > 0, \qquad \text{and} \qquad \frac{d\Pi_g\left(u_b, u_g\right)}{du_g du_b} > 0\,.$$

The first inequality shows the effect of incentives: more surplus to type b agents relaxes their incentive constraint, allowing the principal to earn higher profits from type g agents. The second inequality shows that the marginal benefit of increasing the utility of type g agents rises with the utility offered to type b agents, implying the strict supermodularity of the profit function. In other words, the complementarity that was at the heart of the strict rank-preserving property in the linear model is present in this version as well. Using this property, we can extend the arguments in Proposition 1, implying that the marginal distributions $F_j,\ j \in \{b, g\}$ do not have any flat portions or mass points. Hence, Theorem 1 applies—equilibria are strictly rank-preserving—and can therefore be described by a distribution over utilities to type b agents, $F_b(u_b)$, and a strictly increasing function $U_g(u_b)$. In Appendix E.5.1, we use the methods from Section 4 to derive the system of differential equations that characterize these functions.

Next, we consider the implications of competition for welfare. For brevity, we restrict attention to the region where all menus are separating and do not involve cross-subsidization. In this case, the consumption of type g agents necessarily varies with the state; this imperfect insurance is the analogue of distortions in the quantity traded in the baseline model. The associated resource costs are thus a natural measure of the efficiency losses (relative to a full information benchmark) in this setting. For a

switch to a principal-agent description.

menu offering u_b to type b agents, this loss is given by

$$L(u_b) \quad = \quad C\left(U_g(u_b)\right) - \left[\theta_g C\left(U_g^a(u_b)\right) + (1-\theta_g)\, C\left(U_g^n(u_b)\right)\right] \ , \tag{70}$$

where U_g, U_g^a, and U_g^n are equilibrium policy functions. Average losses in the economy are then

$$\mathcal{L}(\pi) \quad \equiv \quad (1-\pi)\int L(u_b)\,dF_b(u_b,\pi) + \pi\int L(u_b)\,dF_b(u_b,\pi)^2 \ . \tag{71}$$

In Appendix E.5.1, we show, using a numerical example, that L is U-shaped in u_b, which then implies that $\mathcal{L}(\pi)$ is minimized at an interior value of π. Thus, in markets for insurance, increasing competition among providers can be detrimental for welfare.

E.2 Differential Competition Across Types

In our baseline model, we assume that the probability a seller receives one or two offers is the same for both types. In this subsection, we relax this assumption and allow π to vary across types, so that the probability a type j seller is captive is given by $1-\pi_j$. We will show that both the structure of the equilibrium and its normative properties remain largely unchanged, with the caveat that, for some parameter values, the equilibrium distribution has mass points. For brevity, we restrict attention to the $\phi_l > 0$ case, where all equilibrium menus are separating and cross-subsidization does not occur.

When $\pi_h > \pi_l$, the results in Proposition 1 go through unchanged, and thus the distribution functions F_l and F_h have continuous support and no mass points. This implies that the equilibrium satisfies the strict rank-preserving property and all menus attract the same fraction of noncaptive sellers. When $\pi_l > \pi_h$, both distributions still have continuous supports, but F_l has a mass point if π_l is sufficiently large. The following proposition fully characterizes the unique equilibrium for both cases.

Proposition 4. *If* $\frac{1-\pi_l}{1-\pi_h} < 1-\phi_l$*, then the unique equilibrium* F_l *has full mass at* ν_l *and* F_h *is characterized by*

$$(1-\pi_h+\pi_h F_h(u_h))\,\Pi_h\left(\nu_l,u_h\right) = (1-\pi_h)\,\Pi_h\left(\nu_l,c_h\right). \tag{72}$$

If $\frac{1-\pi_l}{1-\pi_h} \geqslant 1-\phi_l$*, then the unique equilibrium* F_l *satisfies*

$$\frac{\pi_l f_l(u_l)}{1-\pi_l+\pi_l F_l(u_l)}\Pi_l(u_l) \quad = \quad 1 - \frac{1-\pi_h+\pi_h F_l(u_l)}{1-\pi_l+\pi_l F_l(u_l)}\left(\frac{\mu_h}{\mu_l}\right)\frac{\nu_h-c_h}{c_h-c_l} \tag{73}$$

and U_h *is determined by the equal profit condition.*

Equation (73) is similar in structure to (16). The key difference is that the right-hand side, which again measures the (net) marginal cost of providing a unit of surplus to the low type, has an additional term that adjusts for the differential probability that an offer is *accepted* by high types relative to low types. Naturally, this probability is small (i.e., the cost is large) when u_l is small and π_h is large.

The construction of equilibrium follows the strategy in Section 4. The ordinary differential equation in (73), with the boundary condition $F_l(c_l) = 0$, can be solved for F_l. Given F_l, the equal profit condition pins down U_h. The properties of the equilibrium—both positive and normative—are also similar to the baseline model. In particular, x_h is nonmonotonic in u_l which, as before, has interesting implications for the relationship between welfare and competition.

Figure 4 illustrates the effects of varying competition for each type separately. The left panel varies π_h, holding π_l fixed, and shows that more competition for high-quality sellers always reduces welfare; intuitively, more surplus to high-quality sellers tightens the incentive constraints and reduces trade. The right panel varies π_l, holding π_h fixed, which has two effects (exactly as in section 5.2). First, it increases surplus to low-quality sellers, which relaxes incentive constraints and increases trade with high-quality sellers. Second, it makes low-quality sellers relatively less attractive to buyers, inducing them to compete

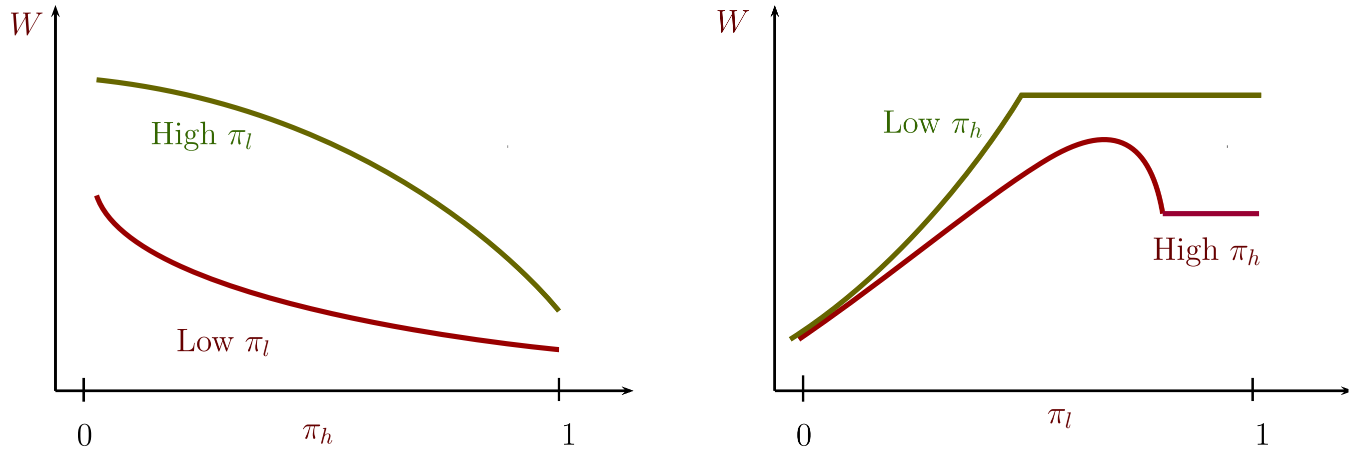


Figure 4: The effect of varying competition on welfare for high- (left panel) and low-quality (right panel) sellers

more aggressively for the high-quality seller, thereby reducing trade. These two competing forces lead to a nonmonotonic relationship between π_l and welfare, provided π_h is sufficiently high.[7]

E.3 Differentiation and Multidimensional Heterogeneity

In this section, in order to enhance the applicability of our framework to applied work, we introduce various types of additional heterogeneity: across buyers, across contracts, and across sellers. In various ways, these generalizations break the stark relationship between a seller's type, the offer she accepts, and the rank of that offer within the distribution of all offers. The cost of these generalizations is some degree of tractability, though we argue that, in most cases, the properties and characterization of equilibria are very similar to the baseline framework. For brevity, we restrict attention to the region of the parameter space where almost all equilibrium menus are separating and not cross-subsidizing.

Horizontal differentiation across buyers. Consider first the possibility that buyers are horizontally differentiated. Specifically, as in the discrete choice model of McFadden (1974), we assume that the payoff to a seller of type i from a contract (x, t) offered by buyer k is

$$u_{ik} \ = \ (1-x)\,c_i + t + \epsilon_k \ = \ u_i + \epsilon_k \, ,$$

where ϵ_k is a buyer-specific preference shock drawn from a continuous distribution H with support $[\underline{\epsilon}, \bar{\epsilon}]$. Note that ϵ is the same for both seller types, so it has no effect on the incentive constraints. Hence, we may once again represent each equilibrium menu by a utility pair (u_l, u_h). A captive seller accepts this menu if u_{ik} is greater than her outside option, c_i, which occurs with probability

$$\tilde{F}_i^c\left(u_i\right) = \int_{c_i-u_i}^{\bar{\epsilon}} dH(\epsilon) = 1 - H\left(c_i - u_i\right) \, . \tag{74}$$

A noncaptive seller of type i accepts this menu if $u_i + \epsilon > \max(u_i' + \epsilon', c_i)$, which occurs with probability

$$\tilde{F}_i^{nc}\left(u_i\right) = \int_{\underline{u}_i}^{\bar{u}_i} \int_{c_i-u_i}^{\bar{\epsilon}} \left(\int_{\bar{\epsilon}}^{u_i+\epsilon-u_i'} dH\left(\epsilon'\right) \right) \quad dH\left(\epsilon\right) \quad dF_i\left(u_i'\right) , \tag{75}$$

[7]When π_h is low, we enter the region with mass points before the second (negative) effect begins to dominate. Since a mass point equilibrium puts full mass at v_l, increasing π_l beyond this point has no effect on welfare.

where F_i is the marginal distribution of utilities offered to type i sellers in equilibrium. Setting $M_i(u_i) = (1-\pi)\tilde{F}_i^c(u_i') + \pi \tilde{F}_i^{nc}(u_i')$, we can write the buyer's problem as

$$\max_{u_l',\, u_h'} \quad \sum_{i \in \{l,h\}} M_i(u_i') \Pi_i(u_l', u_h'). \tag{76}$$

In a separating equilibrium, optimality with respect to u_l requires

$$\frac{m_l(u_l)}{M_l(u_l)}(v_l - u_l) = \phi_l. \tag{77}$$

In other words, the link between the trading probability and the utility offered to the low-quality seller is exactly the same as in our baseline framework, and all of our results go through with respect to the key equilibrium objects M_l and M_h. The only caveat is that recovering the underlying distribution of offers F_l and F_h, which are informative about prices and allocations, typically requires numerical methods.[8]

Horizontal differentiation across contracts. The extension above allows for the possibility that a seller accepts a contract from the "wrong" buyer, i.e., accepts u_i even though a contract $u_i' > u_i$ was available. In this section, we allow for the possibility that a seller accepts the "wrong" contract within a menu, i.e., accepts u_{-i} even though her type is i. In particular, suppose that a fraction δ of low-quality sellers accept the contract intended for a high-quality seller. It is possible to microfound this as a form of "tremble," or as arising from other unmodeled contract features that cause some low-quality sellers to prefer the contract with lower quantity and higher price.[9] For example, the high-price contract might carry other benefits, such as better customer service, that are valued by some low-quality sellers (but not others).

Let $\tilde{v}_h \equiv \frac{\mu_h v_h + \mu_l \delta v_l}{\mu_h + \mu_l \delta}$ be the average value (to the buyer) of goods held by agents who take the contract intended for the high type. We assume that δ is sufficiently small so that $\tilde{v}_h > c_h$. The expected profits of the buyer, conditional on trade, are then given by $\tilde{\Pi}_h(u_l, u_h) = \tilde{v}_h - \left(\frac{\tilde{v}_h - c_l}{c_h - c_l}\right) u_h + \left(\frac{\tilde{v}_h - c_h}{c_h - c_l}\right) u_l$. As in our baseline model, the FOC for u_l and the equal profit condition pin down F_l and U_h:

$$\begin{aligned} \frac{\pi f_l(u_l)}{1-\pi+\pi F_l(u_l)}(v_l - u_l) &= 1 - \frac{\mu_h + \mu_l \delta}{\mu_l(1-\delta)}\left(\frac{\tilde{v}_h - c_h}{c_h - c_l}\right) \equiv \tilde{\phi}_l, & (78) \\ (1-\pi)\mu_l \delta (v_l - c_l) &= (1-\pi+\pi F_l(u_l))\left[\mu_l(1-\delta)(v_l - u_l) + (\mu_h + \mu_l \delta)\tilde{\Pi}_h(u_l, u_h)\right]. & (79) \end{aligned}$$

Note that these equations are very similar to (16)–(17), with $\tilde{\Pi}_h$ and $\tilde{\phi}_l$ replacing Π_h and ϕ_l. Accordingly, the characterization and other results in the preceding sections directly extend.

Vertical differentiation across buyers. Suppose now that sellers attach a higher value to trading with certain buyers, i.e., that the utility of a type i seller from accepting a contract (x, t) from buyer $k \in \{1, 2\}$ is given by $c_i(1-x) + t + B^k$, where $B^1 \equiv B > 0$ and B^2 is normalized to zero.[10] This implies that the cost of delivering utility to sellers is lower for buyer 1 or, equivalently, his profits are higher than those of buyer 2, i.e., $\Pi_i^1(u_l, u_h) = \Pi_i^2(u_l, u_h) + B$. Not surprisingly, in this environment, the equilibrium distribution of menus is also asymmetric. Let $F_i^k(u_i)$, $k \in \{1, 2\}$ denote the marginal distribution of utilities offered by buyer k to type j sellers. In Appendix E.5.3, we characterize an equilibrium in which

[8] The differential equation in (77), along with the equal profit condition and the system of integral equations in (74)–(75) must be solved jointly for F_i, and this system is only analytically tractable under special assumptions on the distribution H.

[9] For simplicity, we make two additional assumptions. First, a captive low-quality seller still chooses the more attractive menu, even when she takes the contract intended for the high-quality seller. Second, we assume that the buyer does not (or cannot) try to use contract terms to separate out these low-quality sellers.

[10] Equivalently, and more consistent with our earlier interpretation, one could imagine a measure of buyers, with a fraction of each type $k \in \{1, 2\}$. The simplification here implies that a noncaptive seller will always have one offer from a type 1 buyer and one from a type 2 buyer, though this could be relaxed.

these distributions satisfy the strict rank-preserving property, except at the lower bound of the support, where F_i^2 has a mass point.[11]

Multidimensional seller heterogeneity. Finally, our baseline framework posits a tight connection between the valuations of the seller and the buyer. While this is a natural assumption when sellers are heterogeneous along a single dimension—asset quality—it is also natural to consider the case in which sellers have heterogenenous preferences as well. A simple way to incorporate this additional heterogeneity into our analysis is to assume that a seller's type is a tuple $(c, \tilde{v})$, with $c \in \{c_h, c_l\}$ denoting the seller's valuation for her asset and $\tilde{v} \in \{\tilde{v}_h, \tilde{v}_l\}$ denoting the buyer's valuation. This allows for the possibility that some high- (low-) quality assets are held by sellers who, for idiosyncratic reasons, have a low (high) valuation for them. In an asset market interpretation, for example, this could arise from heterogeneity in discount rates or liquidity needs. Let μ_{ij} denote the proportion of sellers of type $(c_i, \tilde{v}_j)$. We can show that it is not possible for buyers to separate sellers with the same c but different $\tilde{v}'$s. Let $\mu_i = \sum_j \mu_{ij}$ denote the fraction of sellers with valuation c_i, $i \in \{h, l\}$ and $\nu_i = \frac{\sum_j \mu_{ij}\tilde{v}_j}{\mu_i}$ denote the average value (to the buyer) of the assets held by sellers of type i. Assuming that gains from trade are positive, so that $c_i < \nu_i$, it is easy to see that our analysis of the baseline model goes through exactly. In other words, additional preference heterogeneity changes the interpretation of buyer values in our baseline model, but otherwise leaves the analysis unchanged.

E.4 The Model with Many Types

We now extend our analysis to the case with an arbitrary, finite number of seller types. We focus our attention on equilibria where all offers are separating menus. We do so for two reasons. First, in the case of $N = 2$, this region yields some of the most interesting results—such as the nonmonotonicity of welfare in π—and we want to confirm that these results are true in a more general setting. Second, in the equilibrium with all separating menus, the monotonicity constraints are slack ($x_i < x_{i+1}$), which is the most commonly studied case in the mechanism design literature.[12] We first provide a method for constructing such a separating equilibrium, and then use the constructed equilibrium to demonstrate that the welfare implications from the model with two types extend to the general case of $N > 2$.

Suppose there are $N \geqslant 2$ types, with buyers and sellers deriving utility ν_i and c_i, respectively, per unit from a good of type $i \in \mathcal{N} \equiv \{1, ..., N\}$. The types are ordered so that $\nu_1 < \nu_2 < ... < \nu_N$ and $c_1 < c_2 < ... < c_N$, and there are gains from trading all types of goods, i.e., $\nu_i > c_i$ for all $i \in \mathcal{N}$. The distribution of types is summarized by the vector $(\mu_1, \dots, \mu_N)$, with $\sum_{i \in \mathcal{N}} \mu_i = 1$. As in our benchmark model, sellers (of all types) are privately informed about the quality of their good and receive two offers with probability π and one offer with probability $1 - \pi$.

Equilibrium Properties. The definition of strategies and a (symmetric) equilibrium are identical to those in the model with two types, and hence we omit them for brevity. We begin our analysis, in Lemma 14 below, by establishing that buyers' offers never distort the quantity traded with the lowest type of seller, and that local incentive constraints always bind "upward," i.e., equilibrium offers always leave a type i seller indifferent between his contract and the one intended for type $i+1$. As a result, a buyer's offer can again be summarized by the indirect utilities it delivers to each type $i \in \mathcal{N}$.

Lemma 14. *For almost all equilibrium menus:*

1. *There is full trade with the lowest type, so that $x_1 = 1$, and the local incentive constraints are binding upward, so that*
$$t_i + c_i(1 - x_i) = t_{i+1} + c_i(1 - x_{i+1}) \textit{ for all } i = 1, 2.....N-1;$$

[11] Our analysis requires one additional assumption: a seller who is indifferent between two menus chooses the one offered by buyer 1. The resulting system of differential equations can be solved numerically to obtain the equilibrium distributions.

[12] See, e.g., Fudenberg and Tirole (1991).

2. *Each menu can be summarized by a utility vector* $\mathbf{u} = (u_1, \cdots, u_N)$ *with* $u_i \geqslant c_i \ \forall\, i$ *and*

$$1 \geqslant \frac{u_N - u_{N-1}}{c_N - c_{N-1}} \geqslant \cdots \geqslant \frac{u_2 - u_1}{c_2 - c_1} \geqslant 0, \tag{80}$$

with the corresponding quantities and transfers given by

$$\begin{aligned} &x_1 = 1, \quad x_i = 1 - \frac{u_i - u_{i-1}}{c_i - c_{i-1}}, \quad i = 2, 3.....N \\ &t_1 = u_1, \quad t_i = u_i - \frac{c_i}{c_i - c_{i-1}} \left(u_i - u_{i-1}\right), \quad i = 2, 3.....N. \end{aligned} \tag{81}$$

This proof of Lemma 14 is a direct extension of the proof of Lemma 1, and hence it is omitted for brevity. Given the results, we can recast each buyer's problem in terms of the utility vector $\mathbf{u}$. In particular, given a family of marginal distributions $F_i(u_i)$ for $i \in \mathcal{N}$, each buyer chooses a vector $\mathbf{u}$ to solve

$$\max_{u_i \geqslant c_i} \ \sum_{i=1}^{N} \mu_i \left(1 - \pi + \pi F_i\left(u_i\right)\right) \Pi_i \left(u_{i-1}, u_i\right) \tag{82}$$

subject to the monotonicity constraints in (80), where (in a slight abuse of notation) profits per trade with a seller of quality i are given by

$$\begin{aligned} \Pi_1\left(u_1\right) &= \nu_1 - u_1, \\ \Pi_i\left(u_{i-1}, u_i\right) &= \nu_i - \frac{\nu_i - c_{i-1}}{c_i - c_{i-1}} u_i + \frac{\nu_i - c_i}{c_i - c_{i-1}} u_{i-1}, \qquad \text{for all } i = 2.....N. \end{aligned} \tag{83}$$

The program in (82) resembles a standard mechanism design problem, where the binding incentive constraints are substituted into the profit functions in (83). The monotonicity constraints in (80) are necessary to ensure that local incentive compatibility implies global incentive compatibility.

We now formally define a separating equilibrium, provide a characterization and a method for constructing such equilibria, and then use numerical examples to study their normative properties.

Definition 1. *An equilibrium is separating if the utility vector* $\mathbf{u}$ *associated with any equilibrium menu solves the relaxed problem of maximizing the objective in* (82) *ignoring the monotonicity constraints in* (80).

As a first step, in the conjectured equilibrium, one can use an induction argument to extend Proposition 1, establishing that all the distributions $F_i\left(u_i\right)$ are continuous with connected support. Since the profit function is strictly supermodular, any separating equilibrium must satisfy the strict rank-preserving property. The following proposition summarizes.

Proposition 5. *If* $\phi_1 = 1 - \frac{\mu_2}{\mu_1}\frac{\nu_2 - c_1}{c_2 - c_1} \neq 0$*, then, in any symmetric separating equilibrium,*

1. *For all* $i \in \mathcal{N}$, $F_i\left(\cdot\right)$ *has a connected support and is continuous.*

2. *There exists a sequence of strictly increasing real-valued functions* $\{U_i\left(u_1\right)\}_{i=2}^{N}$ *such that the utility vector associated with any equilibrium menu* $\mathbf{z}$ *satisfies:*

$$\mathbf{u}\left(\mathbf{z}\right) = \left(u_1\left(\mathbf{z}\right), U_2\left(u_1\left(\mathbf{z}\right)\right), U_3\left(u_1\left(\mathbf{z}\right)\right), \cdots, U_N\left(u_1\left(\mathbf{z}\right)\right)\right). \tag{84}$$

As in the model with two types, Proposition 5 greatly simplifies the construction of separating equilibria: it implies that we only need to characterize the distribution of offers to the lowest type,

$F_1(u_1)$, together with the sequence of functions $\{U_i(u_1)\}_{i=2}^{N}$.[13] The equilibrium distribution of utilities can then be derived from the fact that all types have the same ranking across equilibrium menus, i.e., $F_i(U_i(u_1)) = F_1(u_1)$ for all $i = 2, ..., N$.

Equilibrium construction. We now illustrate how to construct a separating equilibrium. Differentiability of the profit function in (82) implies that any separating equilibrium must satisfy

$$\frac{\pi f_i(U_i(u_1))}{1-\pi+\pi F_i(U_i(u_1))}\Pi_1(u_1) = \phi_i \tag{85}$$

$$\frac{\pi f_i(U_i(u_1))}{1-\pi+\pi F_i(U_i(u_1))}\Pi_i(U_{i-1}(u_1), U_i(u_1)) = \phi_i \quad \text{for all } i = 2, ..., N, \tag{86}$$

where ϕ_i, the marginal cost of increasing the utility of a seller of type i, is given by

$$\begin{aligned}
\phi_1 &= 1 - \frac{\mu_2}{\mu_1}\frac{v_2 - c_2}{c_2 - c_1} \\
\phi_i &= \frac{v_i - c_{i-1}}{c_i - c_{i-1}} - \frac{\mu_{i+1}}{\mu_i}\frac{v_{i+1} - c_{i+1}}{c_{i+1} - c_i}, \qquad \text{for all } i = 2, \cdots, N-1 \\
\phi_N &= \frac{v_N - c_{N-1}}{c_N - c_{N-1}}.
\end{aligned}$$

Equation (85) implies that F_1 must satisfy

$$\frac{\pi f_1(u_1)}{1-\pi+\pi F_1(u_1)} = \frac{\phi_1}{v_1 - u_1}. \tag{87}$$

Since the strict rank-preserving property implies that each U_i must satisfy $F_i(U_i(u_1)) = F_1(u_1)$, it must be the case that $U_i'(u_1) f_1(U_i(u_1)) = f_1(u_1)$. Substituting this result into (86) implies that the equilibrium functions U_i must satisfy the set of differential equations:

$$U_i'(u_1) = \frac{\phi_1}{\phi_i}\frac{\Pi_i(U_{i-1}(u_1), U_i(u_1))}{v_1 - u_1} \qquad \text{for all } i = 2, \cdots, N. \tag{88}$$

The system of differential equations (87) and (88) are ordinary first-order differential equations; to complete the characterization, we need only provide the appropriate boundary conditions. As in the two-type model, these conditions depend critically on the marginal costs, $(\phi_1, \ldots, \phi_N)$, and are closely tied to the outcome under monopsony. The following result shows that the solution to a monopsonist's problem can be represented in the form of a threshold type.

Lemma 15. *Let J denote the largest integer $i \in \{1, 2...N\}$ such that*

$$\sum_{i=1}^{J-1} \mu_i \phi_i < 0, \tag{89}$$

with $J = 1$ if $\sum_{i=1}^{k} \mu_i \phi_i > 0$ for all $k \in \{1, 2....N\}$. The solution to a monopsonist's problem is to set $u_i = c_J$ for $i \leqslant J$ and $u_i = c_i$ for $i > J$.

Intuitively, the accumulated marginal cost of trading with the first J types is negative ($\sum_{i=1}^{J-1} \mu_i \phi_i < 0$), so they are pooled. In contrast, for the remaining types, the information rents outweigh the potential

[13] This proposition relies on the assumption that the marginal cost of transfers to the lowest type net of any benefits arising from binding incentive constraints, ϕ_1, is nonzero. As in the two-type case, this assumption is required to show that equilibrium distributions do not have mass points.

gains, so the monopsonist chooses not to trade with them.[14] The next result links this threshold J to the best and worst menu when $\pi > 0$.

Lemma 16. *Let J be as defined in Lemma 15. Then, in any equilibrium, the best menu has $u_i = u_J$ for $i < J$, and the worst menu has $u_i = c_i$ for all $i \geqslant J$.*

To see the intuition, note that the best menu trades with probability 1, i.e., attracts all captive and noncaptive sellers. Therefore, it cannot be profitable for that menu to separate types that a monopsonist finds profitable to pool; if $u_i < u_J$ for some $i < J$, then increasing u_i has no effect on the probability or composition of trades but yields strictly higher profits (because the effective marginal cost of increasing u_i is negative). Similarly, it cannot be profitable for the worst menu to give any surplus to the types that the monopsonist finds optimal to shut out completely; if such a menu offers more than c_i to any type $i > J$, the buyer can raise her profits simply by lowering that utility.

The system of differential equations (87)-(88), along with the boundary conditions described in Lemma 16, describe necessary conditions for any separating equilibrium. By the Picard-Lindelöf theorem, it has a unique solution. In Appendix E.5.4, we provide analytical expressions for this solution. To ensure that this solution is an equilibrium, one need only verify that the monotonicity constraints (80) are satisfied for every $u_1 \in \mathrm{Supp}(F_1)$.

Finally, we solve two numerical examples using the method described above. The two cases both have $N = 4$, but differ in the marginal cost vector, $(\phi_1, ...\phi_N)$.[15] In the first case, $J = 1$, so the monopsonist only trades with the lowest type. In the second case, $J = 2$. We use these cases to demonstrate the robustness of the welfare results in section 5.2. In Figure 5, we plot expected trade for types 2 through 4 (recall that $x_1 = 1$ always) as a function of π. They show a nonmonotonic relationship between expected trade and competition. In the first case (left panel), in which the monopsonist only trades with type 1, trade by all three types is hump-shaped. This is analogous to the case with $\phi_l > 0$ in the two-type model. In the second case (right panel), however, trade with one of the types (type 2) is monotonically decreasing in π. This is similar to the case with $\phi_l < 0$ in the two-type model. In both cases, these patterns imply that ex-ante welfare is maximized at $\pi < 1$.

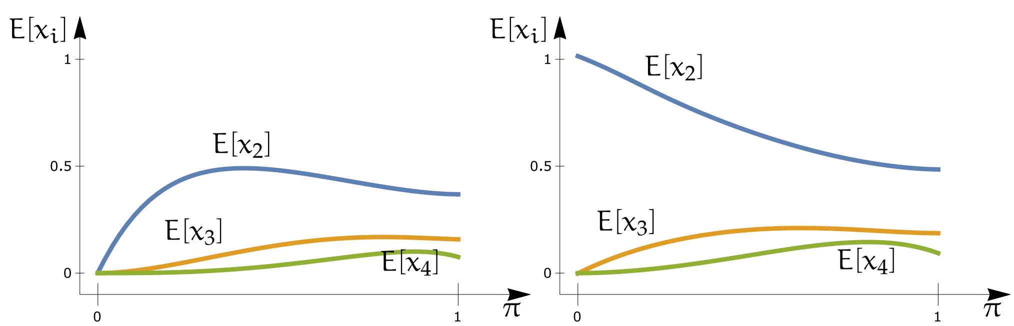


Figure 5: Expected trade and competition when $N = 4$ and $J = 1$ (left panel) or $J = 2$ (right panel).

E.5 Proofs

E.5.1 Construction of Equilibrium for the Insurance Model

The construction of equilibrium follows the logic of Section 4. For brevity, we focus on the region of the parameter space where all equilibrium menus are separating and involve no cross-subsidization.

[14] For brevity, we ignore the non-generic case in which the inequality in (89) is satisfied with equality.

[15] For both cases, we assume a uniform distribution $\mu_i = 0.25$ for all i, with valuations $c_i = 1, 2, 3, 4$ and $v_i = c_i\delta + 0.5$. In case 1, $\delta = 1.2$ and in case 2, $\delta = 1.3$. In each case, we solve the system (87)-(88) and verify that the monotonicity constraints are satisfied.

This obtains when the fraction of type-b agents, μ_b, is sufficiently large. The optimality conditions with respect to u_b and u_g in this case are

$$\frac{\pi f_b(u_b)}{1-\pi+\pi F_b(u_b)}\Pi_b(u_b) = C'(u_b) - \frac{\mu_g}{\mu_b}\left[\frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^a\right)\right] \tag{90}$$

$$\frac{\pi f_g(u_g)}{1-\pi+\pi F_g(u_g)}\Pi_g(u_b,u_g) = \frac{(1-\theta_g)\theta_b}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_b)}{\theta_b-\theta_g}C'\left(u_g^a\right). \tag{91}$$

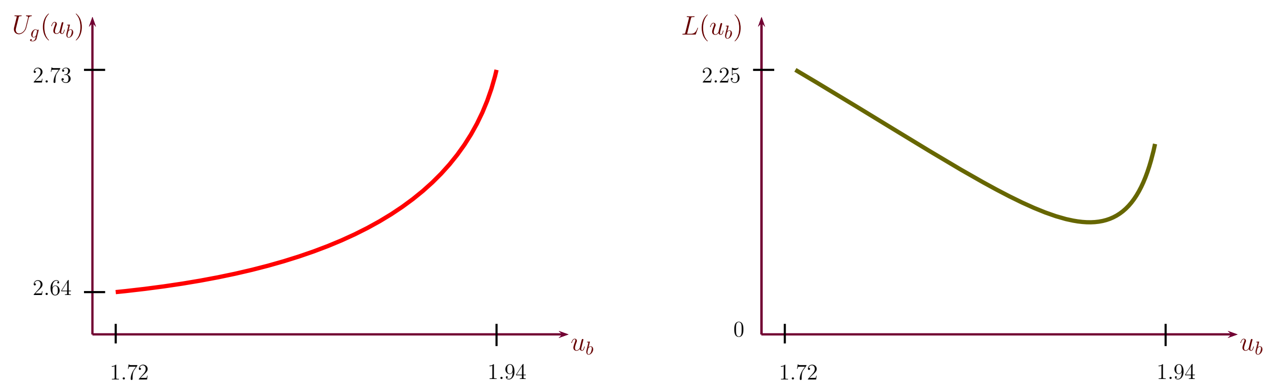
These two differential equations, along with the boundary conditions $F_j(\underline{u}_j) = 0$ with $\underline{u}_j \equiv \theta_j w(y-d) + (1-\theta_j)w(y)$, characterize the equilibrium. Note that these are similar in structure to (16), except that the marginal cost of delivering utility varies with the level of utility (this was constant in the linear model). To solve this system, we make use of the SRP relationship, $F_b(u_b) = F_g(U_g(u_b))$, which implies $f_b(u_b) = f_g(U_g(u_b))U_g'(u_b)$. Dividing the first differential equation by the second and using the SRP identities, we obtain

$$\frac{\Pi_b(u_b)U_g'(u_b)}{\Pi_g(u_b,U_g(u_b))} = \frac{C'(u_b) - \frac{\mu_g}{\mu_b}\left[\frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_g)}{\theta_b-\theta_g}C'\left(u_g^a\right)\right]}{\frac{(1-\theta_g)\theta_b}{\theta_b-\theta_g}C'\left(u_g^n\right) - \frac{\theta_g(1-\theta_b)}{\theta_b-\theta_g}C'\left(u_g^a\right)}, \tag{92}$$

where u_g^n and u_g^a are related to u_b and U_g through (69). Equation (92) is thus an ordinary differential equation in U_g, along with the boundary condition $U_g(\underline{u}_b) = \underline{u}_g$. Note that this does not depend on π. Given U_g, equations (90) – (91) can be solved for the distribution functions.

Given a functional form for the utility function, w, this system can be solved numerically. Figure 6 depicts the solution for the following parameterization: $w(c) = \sqrt{2c}$, $y = 10$, $d = 9$, $\theta_b = 0.9$, $\theta_g = 0.6$, $\mu_g = 0.3$. The left panel plots the equilibrium U_g, while the right panel shows the resource losses associated with imperfect insurance—specifically, the function $L(u_b)$ from (70).

Figure 6: Effect of varying competition



E.5.2 Type-Specific π

Since our proofs that F_h and F_l have no flat regions and F_h has no mass points immediately extend to the case when $\pi_l \neq \pi_h$, we omit them in the interest of brevity. Hence, we begin by analyzing the potential for mass point equilibria; that is, for $F_l(\cdot)$ to feature a mass point—to emerge when $\pi_l \neq \pi_h$.

Proposition 6. *Suppose $\pi_l < \pi_h$. Then $F_l(\cdot)$ does not have a mass point.*

Proof. We prove a profitable deviation exists much as in the case when $\pi_l = \pi_h$. In particular, in any such equilibrium with a mass point, $\Pi_l = 0$ and the following inequalities must hold

$$\begin{aligned} -\mu_h \left(1 - \pi_h + \pi_h F_l^- \left(\hat{u}_l\right)\right) \frac{v_h - c_h}{c_h - c_l} + \mu_l \left(1 - \pi_l + \pi_l F_l^- \left(\hat{u}_l\right)\right) &\leqslant 0 \\ \mu_h \left(1 - \pi_h + \pi_h F_l^+ \left(\hat{u}_l\right)\right) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left(1 - \pi_l + \pi_l F_l^+ \left(\hat{u}_l\right)\right) &\leqslant 0. \end{aligned}$$

Rearranging the above, we must have

$$\frac{1 - \pi_l + \pi_l F_l^- \left(\hat{u}_l\right)}{1 - \pi_h + \pi_h F_l^- \left(\hat{u}_l\right)} \leqslant \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} \leqslant \frac{1 - \pi_l + \pi_l F_l^+ \left(\hat{u}_l\right)}{1 - \pi_h + \pi_h F_l^+ \left(\hat{u}_l\right)}. \tag{93}$$

Since $F_l^+ \left(\hat{u}_l\right) > F_l^- \left(\hat{u}_l\right)$ and $\pi_l < \pi_h$, then we must have that

$$\frac{1 - \pi_l + \pi_l F_l^- \left(\hat{u}_l\right)}{1 - \pi_h + \pi_h F_l^- \left(\hat{u}_l\right)} > \frac{1 - \pi_l + \pi_l F_l^+ \left(\hat{u}_l\right)}{1 - \pi_h + \pi_h F_l^+ \left(\hat{u}_l\right)},$$

which is a contradiction. ■

Proposition 7. *Suppose $\pi_l > \pi_h$. If a mass points exists, then $F_l(v_l) = 1$.*

Proof. First, it is immediate that a mass point cannot exist for any $u_l \neq v_l$. Hence, suppose by way of contradiction that there is a mass on v_l that is not full. Then either $F_l^- \left(v_l\right) > 0$ or $F_l^+ \left(v_l\right) < 1$. Since above and below v_l, the equilibrium features no mass points, the equilibrium must also satisfy the strict rank-preserving property. Let $S = \{(v_l, u_h)\}$ and note that S must have positive measure. Furthermore, the set S must be of the form $\{(v_l, u_h) : u_h \in [\underline{u}_h, \bar{u}_h]\}$. Note that we have, $\bar{u}_h > \underline{u}_h \geqslant c_h > v_l$.

Therefore, in a neighborhood around S, all equilibrium menus should be separating. As a result, they must satisfy the optimality condition with respect to u_l—for values of $u_l \in [v_l - \varepsilon, v_l + \varepsilon] \setminus \{v_l\}$ for small but positive ε (depending on whether mass is above or below v_l):

$$-\mu_l \left(1 - \pi_l + \pi_l F_l \left(u_l\right)\right) + \mu_l \pi_l f_l \left(u_l\right) \left(v_l - u_l\right) + \mu_h \left(1 - \pi_h + \pi_h F_h \left(u_h\right)\right) \frac{v_h - c_h}{c_h - c_l} = 0.$$

Using the SRP property,

$$-\mu_l \left(1 - \pi_l + \pi_l F_l \left(u_l\right)\right) + \mu_l \pi_l f_l \left(u_l\right) \left(v_l - u_l\right) + \mu_h \left(1 - \pi_h + \pi_h F_l \left(u_l\right)\right) \frac{v_h - c_h}{c_h - c_l} = 0.$$

Therefore, if positive mass is above v_l, we must have that

$$\mu_h \left(1 - \pi_h + \pi_h F_l \left(u_l\right)\right) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left(1 - \pi_l + \pi_l F_l \left(u_l\right)\right) > 0,$$

and if it is below,

$$\mu_h \left(1 - \pi_h + \pi_h F_l \left(u_l\right)\right) \frac{v_h - c_h}{c_h - c_l} - \mu_l \left(1 - \pi_l + \pi_l F_l \left(u_l\right)\right) < 0.$$

From above, if mass point is to be an equilibrium property, the inequality (93) must hold:

$$\frac{1 - \pi_l + \pi_l F_l^- \left(v_l\right)}{1 - \pi_h + \pi_h F_l^- \left(v_l\right)} \leqslant \frac{\mu_h}{\mu_l} \frac{v_h - c_h}{c_h - c_l} \leqslant \frac{1 - \pi_l + \pi_l F_l^+ \left(v_l\right)}{1 - \pi_h + \pi_h F_l^+ \left(v_l\right)} < \frac{\pi_l}{\pi_h}. \tag{94}$$

Now suppose that $F_l^+(v_l) < 1$. Then, from the differential equation above,

$$F_l(u_l)\left[\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l\right]-\mu_l\pi_l f_l(u_l)(u_l-v_l)+\mu_h(1-\pi_h)\frac{v_h-c_h}{c_h-c_l}-\mu_l(1-\pi_l)=0.$$

The general solution to the above differential equation is given by

$$F_l(u_l)=A_1(u_l-v_l)^{\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l}{\mu_l\pi_l}}+A_2.$$

Since $\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\pi_l\mu_l}{\mu_l\pi_l}<0$ from (94), the above expression approaches either $\pm\infty$ as u_l approaches v_l from above. Hence, $F_l^+(v_l)<1$ cannot hold.

Now suppose that $F_l^-(v_l) > 0$. Then, similar to above, we must have that

$$F_l(u_l)=A_1(v_l-u_l)^{\frac{\mu_h\pi_h\frac{v_h-c_h}{c_h-c_l}-\mu_l\pi_l}{\mu_l\pi_l}}+A_2.$$

As u_l converges to v_l, the above expression converges to ∞, which is in contradiction with $F_l^-(v_l)<1$. This proves the claim.

■

Proof of Proposition 4. We have already shown a mass point equilibrium, if it exists, must place full mass at v_l. Now, the worst menu in a mass point equilibrium (i.e., the one with the lowest u_h) must set $u_h = c_h$ (otherwise, lowering u_h strictly raises profits). By construction, a function F_h that satisfies (72) ensures equal profits at all points in the support. To rule out other deviations, consider the payoff from offering $u_l' = v_l - \varepsilon,\quad u_h' \in [\underline{u}_h, \bar{u}_h]$. The change in profits (per ε) satisfy

$$\mu_l(1-\pi_l)-(1-\pi_h+\pi_h F_h)\mu_h\frac{v_h-c_h}{c_h-c_l}=\left[1-\frac{(1-\pi_h+\pi_h F_h)}{(1-\pi_l)}\frac{\mu_h}{\mu_l}\frac{v_h-c_h}{c_h-c_l}\right]\mu_l(1-\pi_l).$$

It is sufficient to show that this is negative at the bottom, i.e., when $F_h = 0$, which leads to

$$1-\frac{(1-\pi_h)}{(1-\pi_l)}\frac{\mu_h}{\mu_l}\frac{v_h-c_h}{c_h-c_l}<0 \quad\Rightarrow\quad \frac{1-\pi_l}{1-\pi_h}<1-\phi.$$

To rule out equilibria without mass points, note that, in such an environment, the equilibrium is strictly rank-preserving, so there must be a worst menu, i.e., one with $F_l = F_h = 0$. If it is a pooling menu, then it must offer $u_h = u_l = c_h$. In other words, $\Pi_l = v_l - c_h < 0$. On the other hand, if it is a separating one, it must satisfy the FOC for u_l :

$$\frac{\pi_l f_l}{1-\pi_l+\pi_l F_l}\Pi_l=1-\left(\frac{1-\pi_h}{1-\pi_l}\right)(1-\phi_l)\quad<0\quad\Rightarrow\quad\Pi_l<0$$

i.e., the worst menu in an equilibrium without mass points must necessarily lose money on the low type. But then, the best menu must also lose money, because

$$\Pi_l(\bar{u}_l)=v_l-\bar{u}_l<v_l-\underline{u}_l<0.$$

Now, consider a deviation of the form $(\bar{u}_l-\varepsilon,\bar{u}_h)$ changes profits, relative to $(\bar{u}_l,\bar{u}_h)$, by

$$\mu_l-\mu_h\frac{v_h-c_h}{c_h-c_l}-\mu_l f_l\Pi_l(\bar{u}_l)=\mu_l\phi-f_l\Pi_l(\bar{u}_l)\quad>\quad 0,$$

yielding the desired contradiction. Thus, in a mass point equilibrium, the distribution of u_l is degenerate at v_l, i.e., buyers make zero profits from type-l sellers. A buyer can deviate and offer a lower u_l, but that brings higher profits only from the *captive* $l-$types at the expense of lower profits from both captive and noncaptive $h-$types. When the condition in part (1) of the proposition is satisfied, π_l is sufficiently high or equivalently, the fraction of captive $l-$types is too low to make such a deviation attractive.

E.5.3 Equilibrium with vertical differentiation

Here, we conjecture and characterize an equilibrium with vertical differentiation. We restrict attention to the region of the parameter space where both buyers offer separating contracts without cross-subsidization. First, note that the upper and lower bounds of the distributions of both buyers must coincide, i.e., the distributions of offers by both buyers have the same support. This then implies that F_l^2 has mass of α at its lowest point c_l. To see this, consider the equal profit condition for each buyer (recall that all ties are resolved in favor of buyer 1):

$$\begin{aligned}(1-\pi)(v_l-c_l) &= \Pi(\bar{u}_l,\bar{u}_h)\\(1-\pi+\pi\alpha)(v_l-c_l+B) &= \Pi(\bar{u}_l,\bar{u}_h)+B.\end{aligned}$$

Solving, we obtain $\alpha=\frac{B}{B+v_l-c_l}$. Next, we posit that (i) $U_h^1(u_l)$ is strictly increasing everywhere in the support (ii) $U_h^2(u_l)=c_h$ for $u_l\in[c_l,c_l+s],\quad s\geqslant 0$. In the interval $(c_l+s,\overline{u}_l]$, $U_h^2(u_l)$ is strictly increasing. Formally, the distributions F_j^k satisfy the strict rank-preserving conditions

$$F_l^1(u_l) = F_h^1(U_h^1(u_l)) \qquad u_l\in[\underline{u}_l,\overline{u}_l] \tag{95}$$

$$F_l^2(u_l) = F_h^2(U_h^2(u_l)) \qquad u_l\in(c_{l+s},\overline{u}_l]. \tag{96}$$

The optimality conditions for u_l and u_h for the two buyers yield:

$$\frac{\pi f_l^2(u_l)}{1-\pi+\pi F_l^2(u_l)}\Pi_l^1(u_l) = 1-\frac{\mu_h}{\mu_l}\left(\frac{1-\pi+\pi F_h^2\left(U_h^1(u_l)\right)}{1-\pi+\pi F_l^2(u_l)}\right)\frac{v_h-c_h}{c_h-c_l} \tag{97}$$

$$\frac{\pi f_h^2(u_h)}{1-\pi+\pi F_h^2\left(U_h^1(u_l)\right)}\Pi_h^1\left(u_l,U_h^2(u_l)\right) = \frac{v_h-c_l}{c_h-c_l} \tag{98}$$

$$\frac{\pi f_l^1(u_l)}{1-\pi+\pi F_l^1(u_l)}(v_l-u_l) = 1-\frac{\mu_h}{\mu_l}\left(\frac{1-\pi+\pi F_h^1\left(U_h^2(u_l)\right)}{1-\pi+\pi F_l^1(u_l)}\right)\frac{v_h-c_h}{c_h-c_l} \tag{99}$$

$$\frac{\pi f_h^1(u_h)}{1-\pi+\pi F_h^1\left(U_h^2(u_l)\right)}\Pi_h^2\left(u_l,U_h^2(u_l)\right) = \frac{v_h-c_l}{c_h-c_l}. \tag{100}$$

This system of equations (95) − (100), along with the boundary conditions

$$\begin{aligned}F_l^1(c_l) &= F_h^1(c_h)=0\\F_l^2(c_l) &= \alpha\\F_l^1(\overline{u}_l) &= F_l^2(\overline{u}_l)=1\\F_h^1(\overline{u}_h) &= F_h^2(\overline{u}_h)=1\\(1-\pi)(v_l-c_l) &= \left(1-\pi+\pi F_l^1(c_l+s)\right)(v_l-c_l-s)+(1-\pi)\Pi_h(c_l+s,c_h)\end{aligned}$$

characterize the six unknown functions $F_l^1, F_l^2, F_h^1, F_h^2, U_h^1$, and U_h^2.

E.5.4 Proofs for Extension to N Types

Proof of Lemma 14. This proof is a direct extension of the proof of Lemma 1, and hence is omitted for brevity.

Proof of Proposition 5. To show the strict rank-preserving property, we first show that F_j's are continuous and strictly increasing. The argument for this claim is inductive.

Step 1: F_N is strictly increasing and continuous.

F_N *is strictly increasing.* Suppose, toward a contradiction, that there is an interval $\left[u_N', u_N''\right]$ where F_N is constant and takes a value between 0 and 1. Without loss of generality, we can assume that u_N'' belongs to some contract that is offered in equilibrium. Let one such menu be given by $\mathbf{u}'' = \left(u_1'', \cdots, u_N''\right)$. Given our assumption that the equilibrium is separating, this menu must maximize $\sum_{i=1}^{N} \mu_i \left(1 - \pi + \pi F_i\left(u_i\right)\right) \Pi_i\left(u_{i-1}, u_i\right)$ over the set of menus that are subject to the participation constraints. Now consider a menu given by $\left(u_1'', \cdots, u_{N-1}'', u_N'' - \varepsilon\right)$ for a small ε. Since $u_N'' > u_N' \geqslant c_N$, this menu satisfies the participation constraint. Moreover, this menu keeps the fraction of noncaptive N types constant while increasing profits per N-th type, thus yielding higher profits, a contradiction.

F_N *is continuous.* Suppose, toward a contradiction, that F_N has a mass point at $\hat{u}_N$. Let $\mathbf{u} = \left(u_1, \cdots, u_{N-1}, \hat{u}_N\right)$ be an arbitrary equilibrium menu with its N-th element given by $\hat{u}_N$. Note that we must have $\Pi_N\left(u_{N-1}, \hat{u}_N\right) \leqslant 0$ and $\hat{u}_N = c_N$. The fact that $\Pi_N\left(u_{N-1}, \hat{u}_N\right) \leqslant 0$ is immediate, since otherwise a small increase in $\hat{u}_N$ would attain a higher level of profits. Additionally, if $\hat{u}_N > c_N$, then a small decrease in $\hat{u}_N$ would attain higher profits. Such a change increases profits because either $\Pi_N < 0$—in which case this change decreases the probability that an N type accepts the offer discretely—or $\Pi_N = 0$—in which case this change makes profits per N type strictly positive.

Non-positivity of profits, together with $\hat{u}_N = c_N$, implies that

$$v_N - \frac{v_N - c_{N-1}}{c_N - c_{N-1}} c_N + \frac{v_N - c_N}{c_N - c_{N-1}} u_{N-1} \leqslant 0 \Rightarrow \frac{v_N - c_N}{c_N - c_{N-1}} u_{N-1} \leqslant \frac{v_N - c_N}{c_N - c_{N-1}} c_{N-1} \Rightarrow u_{N-1} \leqslant c_{N-1}.$$

This inequality, together with the participation constraint, $c_{N-1} \leqslant u_{N-1}$, implies that u_{N-1} must equal c_{N-1} and $\Pi_N = 0$. That is, any menu $\mathbf{u}$ with $\hat{u}_N$ as its N-th element must also satisfy $u_{N-1} = c_{N-1}$, so that F_{N-1} must also have a mass point at c_{N-1}. Repetition of this argument implies that any menu containing a mass point at $\hat{u}_N$ must also satisfy $u_j = c_j$, and thus F_j must have a mass point at c_j. However, then a small increase in u_1 from $u_1 = c_1$ must increase profits, as F_1 puts a mass at c_1 and profits from type 1 sellers are positive. This yields the necessary contradiction.

Step 2: If $\{F_k\}_{k=j+1}^{N}$ are strictly increasing and continuous, then F_j must have the same properties.

To prove this claim, we first prove the following lemma:

Lemma 17. *Suppose that, for some $j \leqslant N-1$, the distributions $\{F_k\}_{k=j}^{N}$ are continuous and strictly increasing. Then there exists a sequence of strictly increasing and continuous functions $\left\{U_{k,j}\left(u_j\right)\right\}_{k=j+1}^{N}$ such that for any menu $\hat{\mathbf{u}}$ offered in equilibrium with its j-th element given by $\hat{u}_j$, $\left(\hat{u}_{j+1}, \cdots, \hat{u}_N\right) = \left(U_{j+1,j}\left(\hat{u}_j\right), \cdots, U_{N,j}\left(\hat{u}_j\right)\right)$.*

Proof. We prove this claim by induction. For any value of u_{N-1}, let $U_N^{+}\left(u_{N-1}\right)$ be the set of values of u_N such that equilibrium menus exist with $(N-1)$-th and N-th elements given by $\left(u_{N-1}, u_N\right)$.

We first show that $U_N^{+}\left(u_{N-1}\right)$ is a strictly increasing function. Using exactly the same arguments as in the two-type case, it is straightforward to show that: (*i*) $U_N^{+}\left(u_{N-1}\right)$ must be a strictly increasing correspondence; and (*ii*) if $u, u' \in U_N^+(u_{N-1})$, then $[u, u'] \subseteq U_N^+(u_{N-1})$. These results are direct implications of strict supermodularity of the function $\mu_N(1-\pi+\pi F_N(u_N))\Pi_N(u_{N-1}, u_N)$ and the strict monotonicity of F_N.

Now suppose that for some $\hat{u}_{N-1}$, $U_N^+(\hat{u}_{N-1})$ is a correspondence and so contains an interval given by $[u', u'']$. Then

$$\Pr(u_{N-1} = \hat{u}_{N-1}) = \int_{\{(u_1,\cdots,u_{N-2},\hat{u}_{N-1},u_N)\in \mathrm{Supp}(\Phi)\}} d\Phi \geqslant F_N(u'') - F_N(u') > 0,$$

where the last inequality follows from the fact that F_N is strictly increasing. This inequality implies that F_{N-1} has a mass point at $\hat{u}_{N-1}$, in contradiction with the assumption that F_{N-1} is continuous. Hence, U_N^+ must be a single-valued function.

One can also adapt our arguments from the two-type case to show that $U_N^+(u_{N-1})$ is strictly increasing. If it were constant on an interval, then F_N must have a mass point, contradicting the continuity of F_N. Thus, $U_N^+(u_{N-1})$ is a strictly increasing function and we may write profits from the N-th type as function of u_{N-1} only. Let this function be given by $\Pi_N^+(u_{N-1})$.

Next, let $U_{N-1}^+(u_{N-2})$ be defined in a similar fashion as above. Since the profit function

$$\mu_{N-1}(1-\pi+\pi F_{N-1}(u_{N-1}))\Pi_{N-1}(u_{N-2}, u_{N-1}) + \Pi_N^+(u_{N-1})$$

is strictly supermodular and F_{N-1} and F_{N-2} are strictly increasing and continuous, U_{N-1}^+ must be a strictly increasing, single-valued function. Exact repetition of this argument implies that for all $k \in \{j, \ldots, N-1\}$, U_j^+ is a strictly increasing function. Therefore, we must have that

$$U_{k,j}(\hat{u}_j) = U_k^+\left(U_{k-1}^+\left(\cdots\left(U_{j+1}^+(\hat{u}_j)\right)\right)\right)$$

for all $k \in \{j+1, \ldots, N\}$, and this concludes the proof. ■

We now return to proving step 2 of the induction argument.

F_j *is strictly increasing.* Suppose, by way of contradiction, that F_j has a flat over an interval $[u_j', u_j'']$. Much as in Lemma 5, we prove that if F_j is flat on the interval $[u_j', u_j'']$, then the marginal benefit of delivering one additional unit of surplus to type $j+1$ (incorporating the impact on all types $i > j+1$) changes with $u_j \in [u_j', u_j'']$. This fact allows us to show alternative menus with higher levels of profits than the conjectured equilibrium level must exist.

To see this, first let $U_{j+1}^+(u_j)$ be the correspondence defined in the proof of Lemma 17. By our induction assumption and Lemma 17, profits from types $\{j+1, \ldots, N\}$ can be written as

$$\mu_{j+1}(1-\pi+\pi F_{j+1}(u_{j+1}))\Pi_{j+1}(u_j, u_{j+1}) + \Pi_{j+2}^+(u_{j+1}),$$

where $\Pi_{j+2}^+(u_{j+1})$ are equilibrium profits constructed by applying $U_{k,j+1}$ as defined in Lemma 17. Note that these profits are strictly supermodular in (u_j, u_{j+1}), and, as a result, $U_{j+1}^+(u_j)$ is a strictly increasing correspondence. Additionally, since F_j is flat over the interval $[u_j', u_j'']$, we must have that $U_{j+1}^+(u_j')$ and $U_{j+1}^+(u_j'')$ must have a common element (as in the proof of Lemma 5). Let $\overline{u}_{j+1}$ be this common element.

Let $\mathbf{u}'$ be an equilibrium menu with j-th element given by u_j' and $(j+1)$-th element given by $\overline{u}_{j+1}$ and $\mathbf{u}''$ be an equilibrium menu with j-th element given by u_j'' and $j+1$-th element given by $\overline{u}_{j+1}$. Note that a perturbation of $\mathbf{u}'$ that increases u_j' by a small amount must not increase profits. Similarly, a perturbation of $\mathbf{u}''$ that decreases u_j'' by a small amount must not increase profits. Since F_j is flat on $[u_j', u_j'']$, non-positivity of these two perturbations imply

$$-\mu_j F_j(u_j')\frac{v_j - c_{j-1}}{c_j - c_{j-1}} + \mu_{j+1}F_{j+1}(\overline{u}_{j+1})\frac{v_{j+1} - c_{j+1}}{c_{j+1} - c_j} = 0. \tag{101}$$

As a consequence, profits obtained from any menu $\hat{\mathbf{u}}$, which is the same as $\mathbf{u}'$ except at its j-th element and has j-th element equal to $u_j \in [u_j', u_j'']$, must yield the same profits as $\mathbf{u}'$.

We now show that a perturbation from some such $\hat{\mathbf{u}}$ must strictly increase profits. In particular, consider a perturbation from $\hat{\mathbf{u}}$ that increases $u_{j+1} = \bar{u}_{j+1}$ by a small amount, ε. Since F_{j+1} is strictly increasing and continuous, the change in profits from this perturbation is given by

$$\mu_{j+1} f_{j+1}\left(\overline{u}_{j+1}\right) \Pi_{j+1}\left(u_j, \overline{u}_{j+1}\right) + \mu_{j+1}\left(1 - \pi + \pi F_{j+1}\left(\overline{u}_{j+1}\right)\right) \frac{\nu_{j+1} - c_j}{c_{j+1} - c_j} + \frac{d}{du_{j+1}} \Pi_{j+2}^{+}\left(\overline{u}_{j+1}\right). \tag{102}$$

Since $f_{j+1}\left(\overline{u}_{j+1}\right) > 0$ and Π_{j+1} is linear in u_j, the expression in (102) must be nonzero for some $u_j \in (u_j', u_j'')$. This implies some menu can strictly raise profits above the conjectured equilibrium level and is a contradiction. Thus, F_j cannot have a flat.

F_j *is continuous.* Now suppose that F_j has a discontinuity at $\hat{u}_j$. As in step 1, it must be that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right) \leqslant 0$. There are two possibilities: $\hat{u}_j = c_j$ or $\hat{u}_j > c_j$. If $\hat{u}_j = c_j$, then a straightforward adaptation of the argument in step 1—where we proved F_N is continuous—can be applied to yield a contradiction. Hence, consider the second case with $\hat{u}_j > c_j$. Notice immediately that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right)$ must equal zero, since otherwise a small decrease in $\hat{u}_j$ would strictly increase profits. Since there is a unique value $\hat{u}_{j-1}$ such that $\Pi_j\left(\hat{u}_{j-1}, \hat{u}_j\right) = 0$, if F_j has a mass point at $\hat{u}_j$, F_{j-1} must also have a mass point at some $\hat{u}_{j-1}$. Repeating this argument implies that F_1 must have a mass point, and this mass point must be at ν_1, since $u_1 = \nu_1$ is the unique value such that $\Pi_1(u_1) = 0$.

Let $\mathbf{u} = \left(\nu_1, \ldots \hat{u}_{j-1}, \hat{u}_j, u_{j+1}, U_{j+2,j+1}\left(u_{j+1}\right), \ldots, U_{N,j+1}\left(u_{j+1}\right)\right)$. Since the distribution functions $F_{j+1}, \ldots, F_N$ have no mass points, $U_{j+1}^{+}(\hat{u}_j) = [u_{j+1}', u_{j+1}'']$ for some values u_{j+1}' and u_{j+1}''.

Let $1 \leqslant k \leqslant j$ be the highest index for which $\phi_k \neq 0$; recall, by assumption, $\phi_1 \neq 0$ so that $k \geqslant 1$. Now consider two different perturbations from $\mathbf{u}$, where we perturb elements k through j according to

$$\begin{aligned}
\mathbf{u}^{-} &= \left(\nu_1, \ldots, \hat{u}_{k-1}, \hat{u}_k - \epsilon, \ldots, u_j' - \epsilon, u_{j+1}', U_{j+2,j+1}\left(u_{j+1}'\right), \ldots, U_{N,j+1}\left(u_{j+1}'\right)\right), \\
\mathbf{u}^{+} &= \left(\nu_1, \ldots, \hat{u}_{k-1}, \hat{u}_k + \epsilon, \ldots, u_j' + \epsilon, u_{j+1}'', U_{j+2,j+1}\left(u_{j+1}''\right), \ldots, U_{N,j+1}\left(u_{j+1}''\right)\right).
\end{aligned}$$

For small ϵ, the change in the profits from the above perturbations are, respectively, given by

$$\begin{aligned}
&\mu_k(1 - \pi + \pi F_k^{-}(\hat{u}_k)) \frac{\nu_k - c_{k-1}}{c_k - c_{k-1}} + \mu_{k+1}(1 - \pi + \pi F_{k+1}^{-}(\hat{u}_{k+1})) + \cdots + \mu_j(1 - \pi + \pi F_j^{-}(\hat{u}_j)) \\
&\qquad - \mu_{j+1}(1 - \pi + \pi F_{j+1}^{-}(u_{j+1}')) \frac{\nu_{j+1} - c_{j+1}}{c_{j+1} - c_j}, \\
&-\mu_k(1 - \pi + \pi F_k^{+}(\hat{u}_k)) \frac{\nu_k - c_{k-1}}{c_k - c_{k-1}} - \mu_{k+1}(1 - \pi + \pi F_{k+1}^{+}(\hat{u}_{k+1})) - \cdots - \mu_j(1 - \pi + \pi F_j^{+}(\hat{u}_j)) \\
&\qquad + \mu_{j+1}(1 - \pi + \pi F_{j+1}^{-}(u_{j+1}'')) \frac{\nu_{j+1} - c_{j+1}}{c_{j+1} - c_j}.
\end{aligned}$$

Since the distributions F_i are well behaved above and below each $\hat{u}_i$, the strict rank-preserving property implies $F_i^{-}(\hat{u}_i) = F_{j+1}(u_{j+1}')$, and $F_i^{+}(\hat{u}_i) = F_{j+1}(u_{j+1}'')$ for all values of $i \leqslant j$. We may then write the change in profits from these perturbations, respectively, as

$$\begin{aligned}
&(1 - \pi + \pi F_k^{-}(\hat{u}_k)) \sum_{i=k}^{j} \mu_i \phi_i, \\
&-(1 - \pi + \pi F_k^{+}(\hat{u}_k)) \sum_{i=k}^{j} \mu_i \phi_i.
\end{aligned}$$

Since k is the highest index below j for which $\phi_k \neq 0$, one of the above expressions must be positive.

Therefore, one of the constructed menus increases profits, yielding a contradiction. The claim that equilibrium is strictly rank-preserving then follows immediately from Lemma 17. ■

Proof of Lemma 15. The monopsonist maximizes

$$\mu_1 (v_1 - u_1) + \sum_{i=2}^{N} \mu_i \left[v_i - \frac{v_i - c_{i-1}}{c_i - c_{i-1}} u_i + \frac{v_i - c_i}{c_i - c_{i-1}} u_{i-1} \right] = \sum_{i=1}^{N} \mu_i (v_i - \phi_i u_i)$$

subject to the monotonicity constraint

$$1 \geqslant \frac{u_n - u_{n-1}}{c_n - c_{n-1}} \geqslant \cdots \geqslant \frac{u_{i+1} - u_i}{c_{i+1} - c_i} \geqslant \frac{u_i - u_{i-1}}{c_i - c_{i-1}} ... > 0. \tag{103}$$

Given the linearity in payoffs and constraints, the solution to this problem is a single price offer, i.e., $u_i = c_J$, $i \leqslant J$ and $u_i = c_i$ for $i > J$ for some $J \in \{1, 2...N\}$; see arguments in Myerson (1985) and Samuelson (1984). To see why J must be the *largest* integer such that $\sum_{i=1}^{J-1} \mu_i \phi_i < 0$, suppose otherwise, i.e., $\exists\ k < J$ such that $\sum_{i=1}^{k-1} \mu_i \phi_i < 0$ and the monopsonist sets $u_i = c_k$ for $i \leqslant k$ and $u_i = c_i$ for $i > k$. Then, a deviation which increases all u_i for $i < J$ by ε changes profits by $-\varepsilon \sum_{i=1}^{J-1} \mu_i \phi_i > 0$. ■

Proof of Lemma 16. To show that the best equilibrium menu satisfies $u_i = u_J$ for $i < J$, suppose by way of contradiction that for some $i < J$, $u_i < u_J$. The monotonicity constraint implies $u_J > u_{J-1}$; if $u_J = u_{J-1}$, then we must have $u_i = u_{i-1}$ for all $i < J$. Now, consider an alternative menu that increases all the utilities of types below J by ε. The probability of trade with any type does not change (since this is already the best menu), the change in profits is given by $-\varepsilon \sum_{i=1}^{J-1} \mu_i \phi_i$, which is strictly positive by the definition of J in (89).

To show that the worst equilibrium menu satisfies $u_i = c_i$ for $i \geqslant J$, suppose by way of contradiction that $u_{J+k} > c_{J+k}$ for some $k \geqslant 0$. This inequality, together with repeated application of the monotonicity constraint, implies that $u_i > c_i$ for all $i \leqslant J + k$. Now consider an alternative menu that lowers the utility of all types below and including $J + k$ by ε. This does not change the probability of trade, as the original menu is the worst menu. However, the change in profits from captive types is $\varepsilon \sum_{i=1}^{J+k} \mu_i \phi_i$, which is positive by the definition of J in (89). ■

The Solution to the System of ODEs in (88). The general solution to this system of equations depends on the sign of the profits from the lowest types, $v_1 - u_1$. From (87), this profit is positive when $\phi_1 > 0$, and negative when $\phi_1 < 0$. In what follows, we assume that the sequence $\gamma_i = \frac{v_i - c_{i-1}}{c_i - c_{i-1}} \frac{\phi_1}{\phi_j}$ takes on different values for all $i \geqslant 2$, i.e., $\gamma_i \neq \gamma_j$.[16] We thus have the following general solution:

$$U_i = \sum_{k=0}^{i} a_{k,i} \left(|v_1 - u_1| \right)^{\gamma_k}$$

with

$$\gamma_0 = 0, \gamma_1 = 1,$$

where

$$\begin{aligned} a_{0,i} &= \frac{v_i (c_i - c_{i-1})}{v_i - c_{i-1}} + \frac{v_i - c_i}{v_i - c_{i-1}} a_{0,i-1} \\ a_{k,i} &= \frac{v_i - c_i}{v_i - c_{i-1}} \frac{\gamma_i}{\gamma_i - \gamma_k} a_{k,i-1} \end{aligned}$$

[16] While it is possible to provide the general solution of the ODEs, this assumption greatly simplifies the formulation.

with

$$a_{0,1} = \nu_1$$
$$a_{1,1} = \mathbf{sgn}(\nu_1 - u_1),$$

where **sgn** is 1 if its argument is positive and -1 when its argument is negative.

In the above formulation, the variables $\{a_{i,i}\}_{i=2}^{N}$ are unknown and have to be determined by the boundary conditions in Lemma 16. To do this, for any value of $\underline{u}_1 = \min Supp(F_1)$, we can use equation (87) to solve for F_1, with the boundary condition for $\underline{u}_1$. We can then find the value of $\overline{u}_1$, i.e., the upper bound of the support of F_1, using $F_1(\overline{u}_1) = 1$. We refer to this value as $\tilde{u}_1(\underline{u}_1)$ as a function of $\underline{u}_1$. The boundary conditions then are given by:

$$U_J(\underline{u}_1) = c_J, \cdots, U_N(\underline{u}_1) = c_N$$
$$U_2(\tilde{u}_1(\underline{u}_1)) = \tilde{u}_1(\underline{u}_1), \cdots, U_J(\tilde{u}_1(\underline{u}_1)) = \tilde{u}_1(\underline{u}_1)$$

This is a system of $N - J + 1 + J - 1 = N$ equations with N unknowns given by $a_{i,i}{}_{i=2}^{N}$ and $\underline{u}_1$. Solving this system of equations determines the equilibrium.

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