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Competing for Order Flow in OTC Markets[*]

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Abstract

We develop a model of a two-sided asset market in which trades are intermediated by dealers and are bilateral. Dealers compete to attract order flow by posting the terms at which they execute trades, which can include prices, quantities, and execution times, and investors direct their orders toward dealers that offer the most attractive terms of trade. Equilibrium outcomes have the following properties. First, investors face a trade-off between trading costs and speeds of execution. Second, the asset market is endogenously segmented in the sense that investors with different asset valuations and different asset holdings will trade at different speeds and different costs. For example, under a Leontief technology to match investors and dealers, per unit trading costs decrease with the size of the trade, in accordance with the evidence from the market for corporate bonds. Third, dealers' implicit bargaining powers are endogenous and typically vary across sub-markets. Finally, we obtain a rich set of comparative statics both analytically, by studying a limiting economy where trading frictions are small, and numerically. For instance, we find that the relationship between trading costs and dealers' bargaining power can be hump-shaped.

1   Introduction

Many assets are traded in over-the-counter (OTC) markets, including government, municipal, and corporate bonds, asset-backed securities, derivatives, and currencies, to name a few. These markets have been growing in size and economic importance, e.g., they play a key role in the implementation of monetary policy (the market for federal funds), the provision of liquidity (markets for repurchase agreements), and the provision of insurance services (markets for derivatives). A distinguishing feature of OTC markets is that there is no centralized exchange: trades are bilateral and agents are free to trade at any mutually agreeable terms.[1] Moreover, trades in OTC markets are typically intermediated by dealers who maintain a two-sided market, simultaneously buying and selling securities on behalf of investors (see, e.g., Duffie, 2012; Li and Schürhoff, 2012).

A recent literature, building off of the framework developed by Duffie, Gârleanu, and Pedersen (2005, henceforth DGP), has developed theoretical models that capture several important features of OTC markets, and has used these models to gain a better understanding of the factors that determine prices, liquidity (e.g., bid-ask spreads, execution times), the volume of trade, and allocations. These models typically formalize OTC markets as highly decentralized trading venues in which bilateral meetings between investors and dealers occur at random time intervals, and the terms of trade are determined according to ex post bargaining. While this formalization has generated a number of novel insights, many OTC markets are not as opaque as these models suggest. In particular, in many OTC markets dealers post (and commit) to certain terms of trade, and more importantly these terms of trade play an important allocative role. As Duffie (2012, pg. 4) writes:

> In some dealer-based OTC markets, especially those with active brokers, a selection of recently quoted or negotiated prices is revealed to a wide range of market participants, often through electronic media such as Reuters, Bloomberg, or MarkitPartners. For other OTC markets, such as those for U.S. corporate and municipal bonds, regulators have mandated post-trade price transparency through the publication of an almost complete record of transactions shortly after they occur. (...)
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> In some active OTC derivatives markets, such as the market for credit default swaps, clients of dealers can request "dealer runs," which are essentially lists of dealers' prospective bid and offer prices on a menu of potential trades. Dealers risk a loss of reputation if they frequently decline the opportunity to trade near these indicative prices when contacted soon after providing quotes for a dealer run.

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[1]The failure of the law of one price in OTC markets is well documented; see, e.g., Green, Hollifield, and Schürhoff (2007).

Motivated by these observations, our objective is to describe an asset market with the following characteristics: (i) Trades between investors and dealers are bilateral and time consuming; (ii) dealers compete to attract order flows by posting publicly (and committing to) the terms at which they execute trades, which can include prices, quantities, and execution times; and (iii) investors strategically direct their orders toward dealers that offer the most attractive terms of trade. By developing a model of intermediation and trade with these characteristics, our framework provides novel predictions about the prices and fees that different types of investors pay, the quantities they trade, and the frequency with which they readjust their portfolios.

The starting point of our analysis is a model of decentralized exchange in which investors hold endogeneous asset positions and receive periodic idiosyncratic shocks that affect their private valuations for the asset. In order to rebalance their asset holdings, investors have to contact dealers who have access to a competitive inter-dealer asset market. The description so far is similar to the one in Lagos and Rocheteau (2009, henceforth LR) that generalizes DGP. The crucial departure from the previous literature is the manner in which we model price formation: we take our equilibrium concept to be that of "competitive search," as first introduced by Moen (1997) in the context of the labor market.

In this setup there is free entry of dealers who post contracts specifying a quantity that they will buy or sell on the inter-dealer market, along with a fee for doing so. Investors with *private valuations* for the asset observe these contracts and direct their order flow to the contract of their choosing.[2] Then, orders take time to be executed, with an expected execution time that is decreasing in the dealer-to-investor ratio for this particular contract.

We provide a general characterization of steady-state equilibria and we establish existence. In equilibrium, dealers design contracts to respond to investors' trading needs, and investors subsequently sort themselves across posted contracts according to their private valuations and current asset holdings. Therefore, more heterogeneity across investors leads to a larger number of submarkets, where dealers offer a given contract and average execution time. We show that dealers posting higher intermediation fees tend to fill orders more rapidly, so that investors face a trade-off between intermediation fees and execution times. Investors who enjoy large gains from readjusting their asset holdings trade faster but pay larger intermediation fees. This result is in accordance with a variety of evidence on execution quality in asset markets. For instance, Boehmer (2005) finds that "high execution costs are systematically associated with fast execution speed, and low costs are associated with slow execution speed. This relationship holds both across markets and

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[2]As we discuss later in the text, a technical advantage of modeling trade according to competitive search with posting is that the analysis can easily accommodate private information. This is not the case when trade is modeled according to random search with ex post bargaining, as it is well known that characterizing the outcome of a bargaining game with private information can be quite cumbersome.

across order sizes." Similarly, Battalio, Hatch, and Jennings (2003) find that orders obtain better trade prices on the NYSE but faster executions at Trimark Securities, a Nasdaq dealer. Boehmer, Jennings, and Li (2007) establish that brokers' routing decisions take into account the disclosure of quality execution across trading centers: markets with low trading costs and fast fills were able to attract more orders.

In order to gain additional insights and to compare our model more squarely with the existing literature, we analyze two special cases in detail. First, we consider an environment in which agents have one of two private valuations (high and low) and their asset holdings lie in the set $\{0, 1\}$. These restrictions make each investor's choice of asset holdings trivial and allow us to focus on the execution speeds that prevail in equilibrium. This version of the model is also directly comparable to DGP. Second, we return to our general model with asset holdings in $\mathbb{R}_+$ and an arbitrary number of preference shocks, but we adopt a Leontief specification for the expected execution time. This trading technology generates average execution delays that are constant across active sub-markets. As a result we can focus on the implications of our equilibrium concept for investors' endogenous asset holdings, and we can compare allocations and prices to those of the LR economy that also has constant contact rates between investors and dealers.

In the first special case, with restricted asset holdings and two utility types, we characterize the asset price, trading fees, and execution times that prevail in the unique equilibrium. As in Vayanos and Weill (2008) and Weill (2008), this equilibrium can be solved in closed form when search frictions are small, allowing for a large set of comparative statics. In the random matching model with ex post bargaining of DGP, buyers and sellers contact dealers at the same speed and the excess order flow fails to get executed. In our competitive search model, by contrast, dealers can be allocated unevenly across the two sides of the market so that buyers and sellers trade at different speeds. The asset price adjusts so as to equalize order flows on both sides of the market and all orders are executed. Our model also has different implications for the bid-ask spread than the ones obtained under random matching. For instance, bid-ask spreads increase with the dealers' exogenous bargaining power in DGP, while in our model the relationship between intermediation fees and the sellers' share of the surplus (or implicit bargaining power) is hump-shaped. Similarly, the price discount, which indicates the rate at which the asset price falls relative to the Walrasian benchmark due to search frictions, is non-monotonic with the dealers' bargaining power.

The second special case we consider is when asset holdings are unrestricted and the matching technology is of the Leontief form, i.e., there is strict complementarity between the measure of dealers and the measure of orders to be executed. We characterize the equilibrium when the dealers' entry cost is small, which serves two purposes. First, the analysis is tractable in this region of the parameter space, and hence we can study how asset demand, bid-ask spreads, and trading

volume respond to changes in the economic environment. Second, the equilibrium asset holdings and the inter-dealer prices are the same as in LR when, in their model, the exogenous bargaining power of dealers is equal to zero. Therefore, this class of equilibria is amenable to a direct comparison with the existing literature. Our analysis reveals that the predictions of the competitive search model differ markedly from the earlier literature in one dimension: the relationship between trading costs and trade sizes. In earlier models with random matching and ex post bargaining, such as that of LR, the trading cost per unit traded increases with the size of the trade. In contrast, under competitive search, per-unit trading costs decrease with trade sizes.[3] This prediction of our model is in line with the evidence documented by Schultz (2001), Edwards, Harris, and Piwowar (2007), and Green, Hollifield, and Schürhoff (2007) for over-the-counter markets. Therefore, the structure of the market helps explain the empirical relationship between trading costs and trade sizes.[4]

Related literature.   Our paper is part of a long tradition in the market micro-structure literature, starting at least with Ho and Stoll (1983), studying competition between dealers who serve outside customers. The description of the asset market that involves search and pairwise meetings between dealers and investors, where dealers have access to a competitive inter-dealer market, is based on DGP and LR. Other papers in this literature include Weill (2007), Gavazza (2011), and Lagos, Rocheteau, and Weill (2011), to name a few. There are also other search-theoretic models of financial intermediation with price-posting. Spulber (1996) describes an environment with competing price-setting dealers. However, in contrast to our analysis, the contract posted by dealers (bid and ask prices) can be observed only after a time-consuming search.[5] For a more thorough literature review of the search-theoretic approach to over-the-counter markets, see Rocheteau and Weill (2011).

Search-and-matching models offer a natural platform to study the interactions of investors trading at different speed. Some papers in the literature study asset pricing when speed differences are exogenously given, e.g., Feldhütter (2012) and Neklyudov (2012). Others are explicit about the trade-off between execution speed and trading costs. An early contribution is the sequential search model of block trading by Burdett and O'hara (1987). More recent work includes the models of Melin (2012) and Praz (2012), in which investors can trade simultaneously in a Walrasian and an OTC market, or that of Pagnotta and Philippon (2011), in which investors can choose between competing trading platforms that offer different execution speeds. We study the choice of exe-

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[3]The reason is that dealers reap a constant fraction of the surplus in LR, and the surplus is convex in trade size. In our competitive search model with Leontief technology, by contrast, the price-setting mechanism implies that investors pay a fixed trading cost to trade any quantity.

[4]See Zhang (2012) for a different explanation in a model with *ex post* bargaining under asymmetric information.

[5]Hall and Rust (2003) extend Spulber (1996) by introducing a second type of middlemen called market makers. Each market maker posts publicly observable prices while posted prices of middlemen are not observable.

cution speed using a different approach, based on the the notion of competitive search, initially developed by Moen (1997). Mortensen and Wright (2002) and Sattinger (2003) interpret this equilibrium notion as one where competing brokers or market makers set up markets and charge entry fees to participants. However, in contrast to our model, the brokers in these models do not contribute to the matching of orders on both sides of the market and do not intermediate trades. Weill (2007, Appendix IV) offers some preliminary analysis of a competitive search equilibrium without free entry of dealers; instead, he assumes that the matching technology is concave in the measure of investors and does not depend on the measure of dealers. Rocheteau and Weill (2011, p. 272) apply competitive search to a simple model of an over-the-counter market where trades are not intermediated by dealers and asset holdings are restricted to $\{0, 1\}$.

Competitive search has also been recently applied to asset markets by Guerrieri and Shimer (2012, 2013) and Chang (2012), in order to study the impact of multidimensional asymmetric information (about common and private values) on liquidity in OTC markets. To do so, these authors build on Guerrieri et al. (2010) and consider "one-sided" competitive search models with $\{0, 1\}$ asset holdings, in which buyers post contracts to attract privately informed sellers. See, also, Inderst and Müller (2002) for the study of a market with durable goods under adverse selection and competitive search. While we deal only with asymmetric information about private values (see, also, Faig and Jerez, 2006), we explicitly characterize an equilibrium in a two-sided market, where prices are posted by dealers who simultaneously buy and sell assets from investors, and we are able to remove asset-holding restrictions.[6] Finally, competitive search has been applied to markets for real assets, such as housing, by Albrecht et al. (2013), Diaz and Jerez (2013), Lester et al. (2013), and Stacey (2012). These models do not have dealers making two-sided markets.

2 Environment

Time is continuous and goes on forever. The economy is populated with two types of infinitely-lived agents: a unit measure of investors and a large measure of dealers. Both types of agents discount the future at rate $r > 0$. There is one long-lived asset in fixed supply $A \in \mathbb{R}_+$. There is also a perishable good, the numéraire, which is produced and consumed by all agents.

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[6]Watanabe (2013) also utilizes a model of competitive search to analyze the role of middlemen in asset markets. In his model, middlemen are assumed to have a greater capacity to store inventory than ordinary sellers. In contrast to our assumption of a two-sided asset market, he assumes that middlemen can acquire these inventories—that is, they can purchase assets from sellers—in a frictionless market, before posting prices and selling to buyers in a market with search frictions.

Preferences. The instantaneous utility function of an investor is $u_i(a)+c$, where $a \in \mathbb{R}_+$ denotes the investor's asset holdings, $c \in \mathbb{R}$ denotes the net consumption of the numéraire good (with $c < 0$ if the investor produces more than he consumes), and $i \in \{1, ..., I\} \equiv \mathcal{I}$ indexes an investor's type, where $1 < I < \infty$. We assume that $u_i(a)$ is continuously differentiable, with $u_i'(a) > 0$, $u_i''(a) < 0$, $u_i'(0) = \infty$ and $u_i'(\infty) = 0$ for all $i \in \mathcal{I}$. We also assume that $u_i'(a) < u_{i+1}'(a)$ for all $a > 0$ and $i < I - 1$, so that investors with higher i have higher demand for the asset.[7] Hence, investors of different types value the services (or dividends) provided by an asset differently.[8]

An investor's preferences change over time according to a Poisson process with arrival rate δ. Conditional on receiving a preference shock, an investor of type i draws a new type $j \in \mathcal{I}$ with probability π_{ij}, where $\sum_{j \in \mathcal{I}} \pi_{ij} = 1$ for all $i \in \mathcal{I}$. These type-switching processes are assumed independent across investors. Unlike investors, dealers receive no utility flow from holding an asset, nor can they hold asset inventories.[9] Their instantaneous utility is simply c, the net consumption of the numéraire good.

Trade. All trades are bilateral and are intermediated by dealers, i.e., they involve one dealer and one investor. Dealers have continuous access to a competitive inter-dealer market in which they can trade on behalf of investors. In order to attract orders from investors, dealers post (and commit to) a publicly observable contract $\sigma = (q, \phi)$ specifying that the dealer will trade a quantity q at the prevailing inter-dealer market price p in exchange for an intermediation fee ϕ. If $q > 0$, the dealer buys the specified amount and delivers it to the investor. If $q < 0$, the dealer acquires the asset from the investor and immediately resells it on the inter-dealer market. There is free entry into the dealer market: dealers can choose to enter by posting a contract at a flow cost $\gamma > 0$, which captures the ongoing costs of advertising their services to investors, maintaining access to the inter-dealer market, and so on.

On the other side of the market, investors observe the contracts that have been posted and submit an order to (at most) one contract. Orders, however, take time to execute. In particular,

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[7]This condition holds, for example, if different preference types are generated by multiplicative utility shocks: $u_i(a) \equiv \varepsilon_i v(a)$ for $\varepsilon_1 < \varepsilon_2 \ldots < \varepsilon_I$.

[8]This can arise for a multitude of reasons. For example, one could think of the asset as a durable good, such as a house, in which case it is natural that agents might be heterogeneous with respect to their valuation of the services provided by a house. There are also a number of reasons why agents might be heterogeneous with respect to their valuation of a financial asset: they can have different levels of risk aversion, financing costs, regulatory requirements, or hedging needs. In addition, the correlation of endowments with asset returns may differ across agents. The current formulation is a reduced-form representation of such differences. For more discussion and examples in which these differences arise endogenously, see, e.g., Duffie, Gârleanu, and Pedersen (2007), Vayanos and Weill (2008), Gârleanu (2009), and Geramichalos and Herrenbrueck (2013).

[9]The assumption that dealers cannot hold assets is of no consequence when analyzing steady-state equilibria, as we do in this paper. See Weill (2007) and Lagos, Rocheteau, and Weill (2011) for dynamic equilibria where dealers hold positions.

if we let θ denote the ratio of dealers to orders at a particular contract, then an investor is served according to a Poisson process with arrival rate $\alpha(\theta)$. Conversely, a dealer who has posted this contract executes orders at rate $\alpha(\theta)/\theta$.[10] We assume that $\alpha(\cdot)$ is continuous, strictly increasing, strictly concave, and satisfies $\alpha(0) = 0$, $\alpha(\infty) = \infty$, and $\alpha(\infty)/\infty = 0$. Note that the strict concavity of $\alpha(\cdot)$ implies that $\alpha(\theta)/\theta$ is strictly decreasing in θ. Finally, we allow investors to withdraw an order at no cost before it has been executed.

3 Equilibrium

In this section we derive conditions for a steady-state equilibrium. This requires describing the optimal entry and posting behavior of dealers, taking as given the order flow they receive from posting any contract and the inter-dealer price; the optimal order submission strategy of investors, taking as given prices and the set of contracts that are available to them; and, finally, a set of conditions ensuring that the market tightness in each active sub-market is consistent with both a stationary distribution across investor types and market clearing.

3.1 Definition

Let Σ denote the set of all possible contracts and $\Sigma^\star$ the set of contracts offered in equilibrium. We assume that an investor can always choose not to send any order, so that $\mathbf{0} = (0,0) \in \Sigma^\star$. The dealer-to-investor ratio, or market tightness, prevailing in a sub-market for contract $\sigma \in \Sigma$ is denoted by $\Theta(\sigma)$. The function $\Theta(\sigma)$ will be defined for all possible contracts, not only those offered in equilibrium. The set of all possible investors' types, (i,a), is $\mathcal{N} \equiv \mathcal{I} \times \mathcal{A}$. For simplicity we let $\mathcal{A} = \mathbb{R}_+$ but our formulation and proofs remain identical if there are asset-holding restrictions. We denote the support of the equilibrium stationary distribution of investors' types by $\mathcal{N}^\star \subseteq \mathcal{N}$.

Dealers. The profits of a dealer who posts a contract $\sigma \in \Sigma$ are denoted by $\Pi(\sigma)$ and solve:

$$r\Pi(\sigma) = -\gamma + \frac{\alpha\left[\Theta\left(\sigma\right)\right]}{\Theta\left(\sigma\right)}\phi.$$

The first term on the right side is the flow cost of posting a contract, while the second term is the expected fee received by a dealer, i.e., the Poisson rate at which a dealer receives the order, $\alpha\left[\Theta(\sigma)\right]/\Theta(\sigma)$, times the fee paid by the investor upon execution of the order, ϕ. Therefore, the

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[10]One can think of α as derived from a standard constant-return-to-scale matching function, $m(d,o)$, that specifies the number of matches between d dealers and o orders. In that case, $\alpha(\theta) \equiv m(d,o)/o = m(\theta,1)$ where $\theta \equiv d/o$.

zero-profit condition of dealers can be written

$$\Pi(\sigma) \leq 0, \text{ with equality if } \sigma \in \Sigma^\star \text{ and } \sigma \neq \mathbf{0}. \tag{1}$$

The expected profits of a dealer are zero in any active sub-market, $\sigma \in \Sigma^\star$, and they are non-positive in inactive sub-markets, $\sigma \in \Sigma \backslash \Sigma^\star$, since otherwise posting a contract in that sub-market would be profitable.

Investors. Let $V_i(a, \sigma, \theta)$ denote the expected lifetime payoff of an investor of type $(i, a) \in \mathcal{N}^\star$ who sends an order for contract $\sigma = (q, \phi)$ given market tightness θ:

$$V_i(a, \sigma, \theta) = \frac{u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_{ij} V_j^\star(a) + \alpha(\theta) \left[V_i^\star(a + q) - pq - \phi \right]}{r + \delta + \alpha(\theta)}. \tag{2}$$

The first term in the numerator is the investor's utility flow; the second term is the investor's expected continuation payoff, conditional on switching types, which occurs with Poisson intensity δ; and the third term is the continuation payoff when his order is executed, which occurs with Poisson intensity $\alpha(\theta)$. The denominator is the effective discount rate. This investor's maximum attainable utility, which we denote by $V_i^\star(a)$, can be written

$$V_i^\star(a) = \sup_{\sigma \in \Sigma^\star} V_i \left[a, \sigma, \Theta(\sigma) \right].$$

Market tightness. We adopt the convention that $\Theta(\mathbf{0}) = 0$. For $\sigma \neq \mathbf{0}$, we assume that:

$$\Theta(\sigma) = \inf \left\{ \theta \geq 0 : V_i(a, \sigma, \theta) > V_i^\star(a) \text{ for some } (i, a) \in \mathcal{N}^\star \right\}, \tag{3}$$

and $\Theta(\sigma) = \infty$ if this set is empty. This definition captures the idea that, if a dealer posts a contract σ, then the investors who value it most will direct their order flow to this contract until they are indifferent between this contract and their best alternative. Put differently, if the market tightness for contract σ were greater than $\Theta(\sigma)$, then there would be a positive measure of investors who would benefit strictly from this contract. Orders sent by these investors would further reduce tightness until it is equal to $\Theta(\sigma)$.[11]

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[11]Note that we are taking the infimum over all θ that create a *strict* utility improvement. This is important because otherwise $\Theta(\sigma) = 0$ for all σ: those investors who have no gains from trade, $V_i^\star(a) = V_i(a, \mathbf{0}, 0)$, would flow into the sub-market in the expectation that they won't be able to trade anyway, and thus $\theta = 0$.

Market clearing and the distribution of investors' states. Let $n_i(da)$ denote the steady-state measure of investors over the set of types, $\mathcal{N}$. The measure $n_i(da)$ must satisfy the following two identities:

$$\sum_{i \in \mathcal{I}} \int_{\mathcal{A}} n_i(da) = 1 \tag{4}$$

$$\sum_{i \in \mathcal{I}} \int_{\mathcal{A}} a n_i(da) = A. \tag{5}$$

Equation (4) imposes that the measures of investors add up to one. Equation (5) ensures that investors hold the entire asset supply and that the inter-dealer market clears.

The order submission strategy of an investor of type (i, a) can be represented by a probability measure, $\lambda(d\sigma \mid i, a)$, over some support

$$\Sigma_i^\star(a) \subseteq \arg \max_{\sigma \in \Sigma^\star} V_i[a, \sigma, \Theta(\sigma)].$$

Then, in a steady-state equilibrium, $n_i(da)$ must satisfy

$$\delta \sum_{j \in \mathcal{I}} n_j(da) \pi_{ji} + \int_{\mathcal{A}} \int_{\Sigma_i^\star(a')} \lambda(d\sigma \mid i, a') \alpha[\Theta(\sigma)] \mathbb{I}_{\{a' + q = a\}}$$

$$= \delta n_i(da) + n_i(da) \int_{\Sigma_i^\star(a)} \lambda(d\sigma \mid i, a) \alpha[\Theta(\sigma)]. \tag{6}$$

The first line of (6) represents the inflow of investors, while the second line represents the outflow. On both lines the first term represents the flow due to type switching, and the second term is the flow due to trade.

Definition 1. *A competitive search equilibrium is a list composed of an inter-dealer market price, p, a set of open sub-markets, $\Sigma^\star$, a market tightness function, $\Theta(\sigma)$, a collection of value functions, $V_i^\star(a)$, an order submission strategy $\lambda(d\sigma \mid i, a)$ with support $\Sigma_i^\star(a)$, and a measure on the set of investors' types, $n_i(da)$, with support $\mathcal{N}^\star \subseteq \mathcal{A} \times \mathcal{I}$, satisfying (1)-(6).*

3.2 Characterization

As is typically true in competitive search models, there is a dual formulation of the problem according to which an investor's expected utility is maximized with respect to trade size, q, intermediation fee, ϕ, and market tightness, θ, subject to dealers' zero-profit condition, $[\alpha(\theta)/\theta]\phi = \gamma$. This dual

problem can be represented conveniently by the following flow Bellman equation:

$$rV_i^\star(a,p) = u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_{ij} \left[V_j^\star(a,p) - V_i^\star(a,p) \right]$$

$$+ \max_{q \geq -a, \theta, \phi} \left\{ \alpha(\theta) \left[V_i^\star(a+q,p) - pq - \phi - V_i^\star(a,p) \right] \right\}$$

$$\text{subject to } \frac{\alpha(\theta)}{\theta} \phi = \gamma.$$

If a utility-maximizing contract, (q, ϕ, θ), was not offered in a candidate equilibrium, a dealer would have a profitable deviation by offering the same contract with a slightly higher fee, thereby attracting the type of investors for whom this contract is optimal. Substituting the zero profits into the Bellman equation, we obtain

Proposition 1. *In any equilibrium, for all $(i,a) \in \mathcal{N}^\star$, investors' value functions solve*

$$V_i^\star(a,p) = \max_{(q \geq -a, \theta) \in \mathcal{Q} \times \mathbb{R}_+} \frac{u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_{ij} V_j^\star(a,p) + \alpha(\theta) \left[V_i^\star(a+q,p) - pq \right] - \gamma\theta}{r + \delta + \alpha(\theta)}. \quad (7)$$

Moreover, there exists a unique collection $\{V_i^\star(a,p)\}_{i \in \mathcal{I}}$, defined over $S = \mathcal{I} \times \mathcal{A} \times (0,\infty)$, that satisfy (7). Each $V_i^\star(a,p)$ is continuous in (a,p) and strictly increasing in a.

The proposition offers an alternative representation of the Bellman equations, which is useful for at least two reasons. First, it allows us to solve for investors' values at each point in the support of the distribution of investors' types, $\mathcal{N}^\star$, without having to characterize $\Sigma^\star$, the set of contracts offered in equilibrium. Second, it shows that our model is observationally equivalent to one in which investors directly choose their order-execution intensity, $\mu \equiv \alpha(\theta)$, but have to incur a convex cost $\gamma\alpha^{-1}(\mu)$. Put differently, the outcome of competition and free entry is that dealers act "as if" they knew investors' private utility type and were making an optimal search intensity decision on their behalf. [12]

Next, we proceed to establish the existence of an equilibrium. First, note that the asset holdings maximizing (7) all belong to $\arg\max_{a'} \{V_i(a',p) - pa'\}$, so they are independent of a, the investor's current asset holdings.[13] For simplicity, let us assume for now that this program has a

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[12]Note, however, that the present competitive search model is not equivalent to a random search model with endogeneous search intensities and bargaining. Indeed, under ex post bargaining, the choice of search intensities generates externalities that are not internalized by the pricing mechanism. In contrast, the outcome of a model with endogeneous search intensities but competitive price posting would typically be constrained efficient. For a model with endogeneous search intensities under ex post bargaining and competitive price posting, see Lagos and Rocheteau (2007).

[13]If zero is an optimal market tightness in (7), then optimal asset holdings can take any value on the real line, so they don't depend on a either.

unique maximizer, which we denote by a_i.[14] We look for a stationary equilibrium in which the support of the distribution of types is $\mathcal{I} \times \{a_1, \ldots, a_I\}$. This is intuitive since, at his first opportunity to trade, an investor chooses asset holdings in the support $\{a_1, \ldots, a_I\}$, and then continues to keep his holdings in this support forever after. Let $\theta_i(a_j)$ be the maximizer of the Bellman equation, (7), when the investor's type is (i, a_j). The inflow-outflow equations for the steady-state distribution, $n_i(a_j)$, are

$$0 = \delta \sum_{k \in \mathcal{I}} n_k(a_j)\pi_{ki} - \delta n_i(a_j) - \alpha[\theta_i(a_j)]n_i(a_j), \text{ if } i \neq j; \tag{8}$$

$$0 = \delta \sum_{k \in \mathcal{I}} n_k(a_i)\pi_{ki} - \delta n_i(a_j) + \sum_{k \in \mathcal{I}} \alpha[\theta_i(a_k)]n_i(a_k), \text{ if } i = j; \tag{9}$$

$$1 = \sum_{(i,j) \in \in \mathcal{I}^2} n_i(a_j). \tag{10}$$

Since finite-state Markov chains have at least one ergodic distribution, it follows that this system of equations has at least one solution (see Lemma 4 in the Appendix). Given any solution, we define the aggregate demand as $\mathcal{D}(p) = \sum_{i,j} n_i(a_j)a_j$.

Proposition 2. *There exists some $p \in \mathbb{R}_+$ such that the $A \in \mathcal{D}(p)$.*

The inter-dealer price identified in Proposition 2 is the basis of a competitive search equilibrium. The first part of the proposition is proved using the Intermediate Value Theorem. Having found a candidate market-clearing price, the second part of the proposition requires constructing the rest of the equilibrium objects in such a way that all conditions of Definition 1 are satisfied. The construction goes as follows. Proposition 2 delivers the equilibrium price, investors' optimal asset holdings and market tightnesses, $\{a_i\}_{i \in \mathcal{I}}$ and $\{\theta_i(a_j)\}_{(i,j) \in \mathcal{I}^2}$, and the distribution of types $\{n_i(a_j)\}$. From these conditions, along with the free-entry condition, (1), we define trade sizes and intermediation fees:

$$q_i(a_j) = a_i - a_j \tag{11}$$

$$\phi_i(a_j) = \gamma \frac{\theta_i(a_j)}{\alpha[\theta_i(a_j)]}. \tag{12}$$

The order submission strategy of an investor in state (i, a_j) is to direct his order flow toward the sub-market $\sigma_i(a_j)$.[15] The set $\Sigma^\star$ is then made up of all $\sigma_i(a_j)$. The value functions are defined as

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[14]Our proof in the appendix deals with the general case in which there may be multiple maximizers.

[15]Note that the equilibrium price equates aggregate demand and supply, but does not explicitly equate "buy" and "sell" order flows in the inter-dealer market. As we show in Appendix A.3, this latter condition is actually implied by the steady-state conditions (8) and (9).

the solution of the Bellman equation:

$$V_i^\star(a) = \max_{k,\ell \,\in\, \mathcal{I}^2} V_i\left[a, \sigma_k(a_\ell), \theta_k(a_\ell)\right],$$

where $V_i(a, \sigma, \theta)$ is defined as in (2). Note that, by construction, for $(i, a_j) \in \mathcal{N}^\star$, $V_i^\star(a_j)$ coincides with the solution of our auxiliary Bellman equation (7) evaluated at (i, a_j). Finally, we let the market tightness associated with any contract $\sigma \neq \mathbf{0}$ be defined as in equation (3).

To conclude this section we recapitulate some key properties of the competitive search equilibrium of an OTC market. First, from the free-entry condition of dealers, in any active sub-market

$$\frac{\alpha(\theta)}{\theta}\phi = \gamma.$$

From the dual formulation of a competitive search equilibrium, this condition implies that investors face a trade-off between the cost at which they can readjust their asset holdings, ϕ, and the order execution time, $1/\alpha(\theta)$. As suggested by Boehmer (2005) and others, this trade-off seems relevant in practice across trading venues. Second, from the first-order condition of the dual problem and the free-entry condition of dealers,

$$\frac{\theta_i(a_j)\alpha'[\theta_i(a_j)]}{\alpha[\theta_i(a_j)]}\left[V_i^\star(a_i, p) - p(a_i - a_j) - V_i^\star(a_j, p)\right] = \frac{\gamma\theta_i(a_j)}{\alpha[\theta_i(a_j)]} = \phi_i(a_j).$$

The dealer's fee is a fraction $\theta\alpha'(\theta)/\alpha(\theta)$ of the match surplus, where this fraction is equal to the elasticity of the matching function. This corresponds to the Hosios (1990) condition for efficiency in markets with search frictions. Relative to models with ex post bargaining, such as those of DGP and LR, where dealers' bargaining power is exogenous, here the dealers' share of the match surplus is endogenous and equal to the contribution of dealers to the matching process. This means that the externalities associated with dealers' entry decision are internalized through the pricing mechanism. An additional consequence of the competitive search formulation is that investors are not subject to a holdup problem when choosing asset holdings; in particular, for a given θ, competition between dealers makes it "as if" investors have an un-intermediated access to the market. Third, the number of active sub-markets will reflect the endogenous heterogeneity across investors since, according to the dual formulation, each investor opens a sub-market that is optimal given his state (i, a_j) with $j \neq i$. Hence, our model offers a rationalization for a high degree of market segmentation.[16]

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[16]Pagnotta and Philippon (2011) argue that "a major feature of the new trading landscape is fragmentation," where market fragmentation corresponds to the phenomenon according to which securities are now traded in multiple markets with different characteristics in terms of quality of execution and trading costs.

Table 1: Parameter Values

Parameter		Value
Asset supply	A	0.75
Discount rate	r	0.05
Elasticity of utility function	σ	0.6
Utility types	$\{\omega_1, \omega_2, \omega_3\}$	$\{0.35, 0.5, 0.65\}$
Type switching intensity	δ	5.5
Transition probabilities	$\{\pi_1, \pi_2, \pi_3\}$	$\{0.0455, 0.9092, 0.0455\}$
Matching intensity	λ	500
Elasticity of matching function	ε	0.9

To illustrate some of the main features of a steady-state equilibrium, we parameterize and solve the model for the simple case of $I = 3$. We adopt isoelastic specifications for the utility and the matching functions, $u_i(a) = \omega_i \frac{a^{1-\sigma}}{1-\sigma}$ and $\alpha(\theta) = \lambda\theta^\varepsilon$. Given the purpose of this exercise, the parameter values (reported in Table 1) are not calibrated to any particular moments, but instead are chosen to yield implications relatively similar to those of Duffie, Gârleanu, and Pedersen (2007) for trading speeds, prices, fees, and allocations.

Table 2 reports some statistics related to the joint distribution of preference types and asset holdings, execution speeds, and fees. More specifically, the first row reports the fraction of investors with preference shock i who have asset holdings a_1, a_2, and a_3, for $i \in \{1, 2, 3\}$; the second row reports the average amount of time it takes each of these types of agents to trade; and the third and fourth rows, respectively, report the total fee and the fee per unit traded (both as a fraction of the equilibrium price) that each type of agent pays. Consider, for example, agents of type $i = 1$. Those with asset holdings a_3 have larger gains from contacting a dealer, relative to those with asset holdings a_2, and hence trade in a sub-market in which trades are executed faster. This affects the steady-state joint distribution of preferences and asset holdings: since they trade more quickly, on average, there are fewer agents in state $i = 1$ with asset holdings a_3 than there are with asset holdings a_2. Notice that this effect on the steady-state distribution—and its subsequent effects on asset prices, trading volume, misallocation, and turnover–is absent in a framework in which meetings between a randomly selected investor and a dealer occur at a constant rate. In addition to creating dispersion in execution times, heterogeneity in the desire to trade also produces dispersion in trading fees. In particular, those agents with more incentive to trade pay higher fees in order to trade more quickly. However, taking into account that these agents trade a larger quantity of the asset, the fee per unit is actually falling with the quantity traded in this example, as in the data. We discuss this last point in Section 4.2 and contrast it with the predictions of existing models.

(Type i, Asset holding a_j)	$(1, a_1)$	$(1, a_2)$	$(1, a_3)$	$(2, a_1)$	$(2, a_2)$	$(2, a_3)$	$(3, a_1)$	$(3, a_2)$	$(3, a_3)$
Fraction of type i holding a_j	0.942	0.057	0.001	0.001	0.997	0.002	0.000	0.06	0.94
Trading time (days)	∞	3	0.5	1.5	∞	1.72	0.5	3	∞
Total trading fee (bp)	0	1.3	1.7	1.48	0	1.43	1.7	1.4	0
Trading fee per unit (bp)	0	3.1	1.7	3.2	0	2.7	1.7	2.7	0

Table 2: Statistics

4 Liquidity and prices in two special cases

In the remainder of the paper we focus on two special cases: (i) The case where asset holdings are restricted to $\mathcal{A} = \{0, 1\}$ and the set of investors' private valuations is $\mathcal{I} = \{\ell, h\}$, but the matching function is general; and (ii) the case where asset holdings are unrestricted in $\mathcal{A} = \mathbb{R}_+$ and the set of private valuations is $\mathcal{I} \subset \mathbb{N}$, but the matching function is Leontief. The first case will allow us to focus on order flows, speeds of execution, and trading costs across markets, taking as given asset positions. The second case will generate speeds of execution that are constant across sub-markets, allowing us to focus on the endogeneous distribution of asset holdings.

4.1 Order flows, speeds of execution, and trading costs

In this section we consider a setting with restricted asset holdings, $\mathcal{A} = \{0, 1\}$, and two utility types, $\mathcal{I} = \{\ell, h\}$, with respective utility flows for the asset, $u_\ell < u_h$.[17] Without loss of generality, we assume that $\pi_{\ell,h} = \pi_\ell$ and $\pi_{h,\ell} = \pi_h$. First, we will provide a characterization of equilibrium objects and show existence and uniqueness. Then, we derive analytical comparative statics for the limit economy with small search frictions. Finally, for large frictions, we offer comparative statics by way of a numerical example.

Bellman equations. Let $\Delta V_i \equiv V_i(1) - V_i(0)$ denote the reservation value of the asset by an investor of type $i \in \{\ell, h\}$, i.e., the expected utility of owning the asset minus the expected utility

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[17]In order to extend our existence proof to the case with restricted asset holdings, we make two additional assumptions on the finite grid $\mathcal{A} \subset \mathbb{R}_+$. First, to ensure that investors can hold all the asset supply, we assume that $\max \mathcal{A} > A$. Second, to keep prices bounded we assume that $u_1(\min \mathcal{A})$ is bounded away from $-\infty$.

of being without the asset. From Proposition 1, ΔV_h solves the following flow Bellman equation:

$$r\Delta V_h = u_h + \delta\pi_\ell \left(\Delta V_\ell - \Delta V_h\right) \quad + \quad \max_{\theta \geq 0}\left\{\alpha(\theta)\left(p - \Delta V_h\right) - \gamma\theta\right\}$$
$$- \quad \max_{\theta \geq 0}\left\{\alpha(\theta)\left(\Delta V_h - p\right) - \gamma\theta\right\}. \qquad (13)$$

The right side of (13) decomposes the flow reservation value into four terms. The first term, u_h, is the flow value of owning the asset. The second term captures the fact that, with intensity $\delta\pi_\ell$, a high-valuation investor switches to a low type, in which case his reservation value drops from ΔV_h to ΔV_ℓ. The third and fourth terms are, respectively, the flow values of searching to sell and buy. The flow value of searching to sell enters the Bellman equation with a positive sign because this option is available to the investor only if he already owns the asset. Therefore, it raises the net utility of owning an asset. By contrast, the flow value of searching to buy enters with a negative sign because this option is only available to investors who do not own the asset.

Similarly, the Bellman equation for ΔV_ℓ is

$$r\Delta V_\ell = u_\ell + \delta\pi_h \left(\Delta V_h - \Delta V_\ell\right) \quad + \quad \max_{\theta}\left\{\alpha(\theta)\left(p - \Delta V_\ell\right) - \gamma\theta\right\}$$
$$- \quad \max_{\theta}\left\{\alpha(\theta)\left(\Delta V_\ell - p\right) - \gamma\theta\right\}. \qquad (14)$$

Lemma 1. *In any equilibrium $\Delta V_h > \Delta V_\ell$ and $p \in (\Delta V_\ell, \Delta V_h)$.*

The first result of Lemma 1 states that high-valuation investors have strictly higher reservation values than low-valuation investors, which follows from the assumption that their utility flow from holding the asset is strictly larger. The second result states that the equilibrium price must lie between the reservation value of high and low types, since otherwise the market would not clear. A consequence of these two results is that, in equilibrium, low-valuation investors have strict incentives to sell and high-valuation investors have strict incentives to buy.

The properties reported in Lemma 1 also hold in DGP, except for one key difference. In DGP, the inter-dealer market price is, generically, at a corner, $p \in \{\Delta V_\ell, \Delta V_h\}$. This result arises because buyers and sellers contact dealers at the same rate, so that p must adjust to make investors on the long side of the market indifferent between trading and not. If $\pi_h > A$, then buyers are on the long side and $p = \Delta V_h$. Conversely, if $\pi_h < A$, then sellers are on the long side and $p = \Delta V_\ell$. Such prices, $\{\Delta V_\ell, \Delta V_h\}$, cannot be the basis of an equilibrium under competition for order flows: indeed, if investors on one side of the market were indifferent between trading or not at the inter-dealer price, i.e., $p = \Delta V_i$ for some $i \in \{\ell, h\}$, then these investors would not be willing to pay any intermediation fee and dealers would have no incentives to make a market for such investors. Therefore, under competitive search the price has to lie strictly between ΔV_ℓ and ΔV_h, so that

dealers can break even with both types and the market can clear.[18]

With this result in mind, we define $S_h \equiv \Delta V_h - p$ as the total surplus generated by a purchase, i.e., the difference between the reservation value of a buyer, ΔV_h, and the reservation value of a dealer, p. The intermediation fee is set so as to share the total surplus between the buyer, who receives $S_h - \phi_h$, and the dealer, who receives ϕ_h. Symmetrically, let $S_\ell \equiv p - \Delta V_\ell$ denote the total surplus generated by a sale. The purchase and sale surpluses solve the following pair of Bellman equations:

$$rS_h = u_h - rp - \delta\pi_\ell(S_h + S_\ell) - \max_\theta \{\alpha(\theta)S_h - \gamma\theta\} \tag{15}$$
$$rS_\ell = rp - u_\ell - \delta\pi_h(S_h + S_\ell) - \max_\theta \{\alpha(\theta)S_\ell - \gamma\theta\}. \tag{16}$$

To solve this system of equations for a given p, we define:

$$\Gamma(S) = (r+\delta)S + \max_\theta \{\alpha(\theta)S - \gamma\theta\}.$$

The function, $\Gamma(S)$, is twice continuously differentiable, strictly increasing, strictly convex, and satisfies $\Gamma(0) = 0$ and $\Gamma(\infty) = 0$. Using the function $\Gamma(S)$, the system (15)-(16) can be written

$$\Gamma(S_h) + \Gamma(S_\ell) = u_h - u_\ell \tag{17}$$
$$\Gamma(S_h) = u_h - rp + (1 - \delta\pi_\ell)S_h - \delta\pi_\ell S_\ell. \tag{18}$$

Equations (17) and (18) implicitly define two strictly decreasing, strictly concave, and continuously differentiable functions $S_h = F(S_\ell)$ and $S_h = G(S_\ell)$, respectively. The slopes of these functions are

$$F'(S_\ell) = -\frac{r + \delta + \alpha(\theta_\ell)}{r + \delta + \alpha(\theta_h)} < G'(S_\ell) = -\frac{\delta\pi_\ell}{r + \delta\pi_\ell + \alpha(\theta_h)},$$

where θ_ℓ and θ_h are the maximizers of the programs defining $\Gamma(S_\ell)$ and $\Gamma(S_h)$, i.e.,

$$\alpha'(\theta_i)S_i = \gamma \ \text{ for } i \in \{\ell, h\}. \tag{19}$$

The ranking of these derivatives implies that if a solution to $F(S_\ell) = G(S_\ell)$ exists, then it must be unique. It also implies that $F(S_\ell)$ crosses $G(S_\ell)$ from above at any intersection, as illustrated in Figure 1. Using these results, we now address the issue of existence.

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[18]A related result occurs in the model of Gavazza (2011). There is a continuous distribution of private valuations of the asset across investors, and participation in the market is costly. As a result, there are two thresholds for private valuations: one above which agents participate as buyers and one below which they participate as sellers. Participation requires that agents get a positive surplus from a trade. So if trades were intermediated by dealers, as in DGP, the inter-dealer market price would satisfy the same condition as in our model.
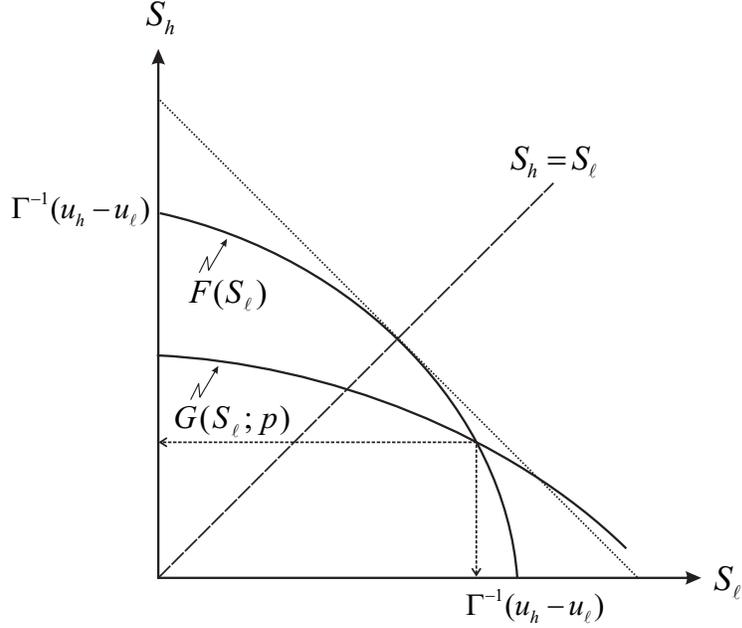
Figure 1: The determination of S_ℓ and S_h given p.

Lemma 2. *There exist $p_\ell < p_h$ such that the system (15)-(16) has a strictly positive solution, $[S_\ell(p), S_h(p)]$, if and only if $p \in (p_\ell, p_h)$. This solution is unique. Moreover, $S_\ell(p)$ is strictly increasing in p, $S_h(p)$ is strictly decreasing in p, and $S_\ell(p_\ell) = S_h(p_h) = 0$.*

According to Lemma 2, for a solution to exist the price must be neither too high nor too low, so that both buy orders and sell orders generate strictly positive surpluses. Moreover, the surplus from selling the asset is increasing in the price $p \in (p_\ell, p_h)$ while the surplus from buying the asset is decreasing in p. Recall, from (19), that there is less entry of dealers on a given side of the market if the corresponding trading surplus is lower. Thus, θ_h decreases with p while θ_ℓ increases with p.

Market clearing. We now solve for the distribution of investors across types and ensure that the asset market clears. First, since $p \in (\Delta V_\ell, \Delta V_h)$ from Lemma 1, the equilibrium buy- and sell-order flows must be equal:

$$\alpha(\theta_h)n_h(0) = \alpha(\theta_\ell)n_\ell(1). \tag{20}$$

The left side is the buy-order flow originating from $(h, 0)$-type investors—investors with a high valuation for the asset but who do not own it—and the right side is the sell-order flow originating from $(\ell, 1)$-type investors—investors who own the asset but have a low valuation for it. We derive

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the measure of buyers, $n_h(0)$, from the following steady-state condition:

$$\delta \pi_h n_\ell(0) = \delta \pi_\ell n_h(0) + \alpha(\theta_h) n_h(0). \tag{21}$$

The left side is the inflow coming from $(\ell, 0)$-investors who change type. The first and second terms on the right side are the outflows of type $(h, 0)$-investors who switch to low type with intensity $\delta \pi_\ell$ or who trade with intensity $\alpha(\theta_h)$, respectively. Similarly, the steady-state equation for the measure of $(\ell, 0)$-type investors is

$$\delta \pi_\ell n_h(0) + \alpha(\theta_\ell) n_\ell(1) = \delta \pi_h n_\ell(0),$$

which, from (20), can be rewritten

$$\delta \pi_\ell n_h(0) + \alpha(\theta_h) n_h(0) = \delta \pi_h n_\ell(0). \tag{22}$$

Equations (21)-(22) can be interpreted as defining the ergodic distribution of a two-state Markov chain, for which $(h, 0)$-type investors transition to type $(\ell, 0)$ at Poisson rate $\delta \pi_\ell + \alpha(\theta_h)$, and $(\ell, 0)$-type investors transition to type $(h, 0)$ at Poisson rate $\delta \pi_h$. Together with the market-clearing condition, $n_h(0) + n_\ell(0) = 1 - A$, these steady-state conditions imply that

$$n_h(0) = (1 - A) \frac{\delta \pi_h}{\delta + \alpha(\theta_h)}. \tag{23}$$

By symmetry,

$$n_\ell(1) = A \frac{\delta \pi_\ell}{\delta + \alpha(\theta_\ell)}. \tag{24}$$

Thus, the equality of order flows, (20), can be rewritten as

$$(1 - A) \frac{\delta \pi_h \alpha(\theta_h)}{\delta + \alpha(\theta_h)} = A \frac{\delta \pi_\ell \alpha(\theta_\ell)}{\delta + \alpha(\theta_\ell)}. \tag{25}$$

Recall from Lemma 2 that θ_h decreases with p while θ_ℓ increases with p. Therefore, one easily sees that the left side of (25) is strictly decreasing and equal to zero at $p = p_h$, while the right side is strictly increasing and equal to zero at $p = p_\ell$. Consequently, there exists a unique price $p \in (p_\ell, p_h)$ equating buy- and sell-order flows. Combining these observations with Lemma 1 we obtain the following result.

Proposition 3. *There exists a unique competitive search equilibrium.*

The remaining equilibrium objects can be found using the same steps as in Section 3.2. In

particular, the trading fees are

$$\phi_h = \frac{\theta_h \alpha'(\theta_h)}{\alpha(\theta_h)} S_h$$

$$\phi_\ell = \frac{\theta_\ell \alpha'(\theta_\ell)}{\alpha(\theta_\ell)} S_\ell.$$

One recognizes here the Hosios (1990) condition for efficiency in economies with search frictions. The trading fee that a dealer charges for a buy-order, ϕ_h, is equal to the dealer's contribution to the sale surplus, as defined by the elasticity of the matching function. This elasticity determines the share of the trade surplus that a dealer appropriates, and hence it measures the implicit bargaining power of dealers when they trade. Note that, unless the matching function has constant elasticity (such as in the Cobb-Douglas matching function) or unless $\theta_\ell = \theta_h$, the implicit bargaining power of dealers will differ depending on whether they trade with buyers or sellers. This asymmetry arises because intermediation on both sides of the market implies that there are two separate matching processes: one between dealers and buyers, and one between dealers and sellers. As a result the Hosios condition must apply separately to each one of them.

Comparative statics near the Walrasian limit.   In order to derive analytical comparative statics, it is useful to study the limiting equilibrium as search frictions vanish. To do this, we first assume that investors contact dealers at intensity $\lambda\alpha(\theta)$, and then we drive the search efficiency parameter, λ, to $+\infty$. We focus on the case $\pi_h > A$, so that high-valuation investors are on the long side of the market (a similar analysis applies to $\pi_h < A$). In this case, there are more high-valuation agents than assets. Hence, in the frictionless benchmark, the price adjusts so that high-valuation investors are indifferent between buying or not, and the price and allocations are

$$p^\star = \frac{u_h}{r}, \quad n_h^\star(0) = \pi_h - A, \quad \text{and} \quad n_\ell^\star(1) = 0.$$

The intermediation fees are implicitly equal to zero. Our main proposition characterizes equilibrium objects near this Walrasian limit.

Proposition 4. *Let the matching technology be $\lambda\alpha(\theta)$ and assume that $\pi_h > A$. As $\lambda \to \infty$, the price, intermediation fees, and the measures of buyers and sellers admit the following first-order*

approximation:

$$p = p^\star - \frac{\delta \pi_\ell s_\ell^\star}{\lambda r} + o\left(\frac{1}{\lambda}\right) \tag{26}$$

$$\phi_h = o\left(\frac{1}{\lambda}\right) \tag{27}$$

$$\phi_\ell = \frac{\theta_\ell^\star \alpha'(\theta_\ell^\star)}{\alpha(\theta_\ell^\star)} \frac{s_\ell^\star}{\lambda} + o\left(\frac{1}{\lambda}\right) \tag{28}$$

$$n_h(0) = n_h^\star(0) + \frac{A\delta\pi_\ell}{\lambda\alpha(\theta_\ell^\star)} + o\left(\frac{1}{\lambda}\right) \tag{29}$$

$$n_\ell(1) = n_\ell^\star(1) + \frac{A\delta\pi_\ell}{\lambda\alpha(\theta_\ell^\star)} + o\left(\frac{1}{\lambda}\right), \tag{30}$$

where $o(1/\lambda)$ is such that $\lim_{\lambda\to\infty} \lambda o(1/\lambda) = 0$, while $s_\ell^\star = \lim_{\lambda\to\infty} \lambda S_\ell$ and $\theta_\ell^\star = \lim_{\lambda\to\infty} \theta_\ell$ jointly solve $\max_\theta \{\alpha(\theta)s_\ell^\star - \gamma\theta\} = u_h - u_\ell$. The market tightness for buyers satisfies $\lim_{\lambda\to\infty}\theta_h = 0$, and the purchase surplus satisfies $\lim_{\lambda\to\infty} \lambda S_h = 0$.

As $\lambda \to \infty$, the different components of the competitive search equilibrium converge toward their Walrasian counterparts. Sellers, who are on the short side of the market, trade almost instantly: market tightness converges to some strictly positive limit, $\theta_\ell^\star$, so that the average execution time, $1/\lambda\alpha(\theta_\ell)$, converges to zero. Buyers, who are on the long side, trade with a non-zero asymptotic execution time. This sharp asymmetry in execution times is necessary to keep the buy- and sell-order flows balanced in the limit.

Proposition 4 allows for analytical comparative statics of various equilibrium objects of interest. For example, an increase in the gains from trade, $u_h - u_\ell$, causes the asymptotic sale surplus, $s_\ell^\star$, to increase. As a result, the asymptotic price discount, $\lim \lambda(p^\star - p)/p^\star$, increases because buyers expect that they will lose more utility upon switching to a low type. The increase in the sale surplus represents a profit opportunity for dealers. By free entry, the supply of intermediation services, $\theta_\ell^\star$, increases, which implies that investors can trade faster. As a result, the asymptotic measures of buyers and sellers, $n_h(0)$ and $n_\ell(1)$, decrease. By the same token, when $\theta_\ell^\star$ increases, intermediaries receive fewer trading opportunities on average, so the zero-profit condition requires that intermediation fees, $\phi_\ell^\star = \lim \lambda\phi_\ell$, go up.

An increase in the flow entry cost of dealers, γ, makes dealers more reluctant to post contracts and hence the asymptotic tightness, $\theta_\ell^\star$, decreases. As a result, the measures of buyers and sellers increase, the asymptotic sale surplus $s_\ell^\star$ increases, and so does the price discount. If the elasticity of the matching function is constant—e.g., the Cobb-Douglas case—intermediation fees increase.

In response to an increase in the intensity of switching to the low valuation, $\delta\pi_\ell$, buyers an-

ticipate that they may switch type soon after purchasing the asset. This lowers their willingness to pay for the asset, so the inter-dealer price decreases. Moreover, when investors switch to the low valuation more quickly after purchasing the asset, the asymptotic measure of sellers, $n_\ell(1)$, increases.[19]

Next, we consider the relationship between the price discount and the elasticity of the matching function. For this we focus on the special case of a Cobb-Douglas matching function.

Corollary 1. *Suppose* $\alpha(\theta) = \theta^\eta$*. Then,*

$$\theta_\ell^\star = \frac{\eta}{1-\eta}\frac{\Delta u}{\gamma} \quad and \quad s_\ell^\star = \left(\frac{\gamma}{\eta}\right)^\eta \left(\frac{u_h - u_\ell}{1-\eta}\right)^{1-\eta}.$$

Thus, the asymptotic market tightness for sellers, $\theta_\ell^\star$*, is an increasing function of* η *and the asymptotic seller's surplus,* $s_\ell^\star$*, is a hump-shaped function of* η*. As a consequence, the seller's asymptotic trading fee and the asymptotic price discount are both hump-shaped functions of the matching elasticity,* η*.*

To gain intuition for the effect of the elasticity, η, note that the equation defining $s_\ell^\star$ can be written as

$$u_\ell + \max_\theta \left\{-\gamma\theta + \alpha(\theta)s\right\} = u_h = rp^\star.$$

This equation is an asymptotic indifference condition. It states that the asymptotic sale surplus, $s_\ell^\star$, must adjust so that a seller with a trading opportunity is indifferent between continuing search (on the left side) and selling his asset (on the right side).

One sees that, when $\eta \to 0$, then $\alpha(\theta) = \theta^\eta$ converges to 1 if $\theta > 0$ and to zero if $\theta = 0$. That is, an arbitrarily small market tightness, θ, is sufficient to attain the maximum search intensity of 1. It is thus intuitive that, in equilibrium, market tightness is approximately zero and dealers incur almost no contract posting cost, $\alpha(\theta)$ is approximately one, and the sale surplus is approximately equal to $u_h - u_\ell$.

When $\eta \to 1$, then the search technology becomes approximately linear. The seller's surplus must then converge to the search cost γ: if it were larger, the utility of continuing search would be unbounded, and if it were smaller it would be zero. Note that, to keep the utility of continuing to search strictly positive, the market tightness must go to infinity.

We can now discuss comparative statics with respect to η. By the envelope condition, the derivative of the utility of continuing to search with respect to η is equal to $s_\ell^\star \log(\theta_\ell^\star)(\theta_\ell^\star)^\eta$, which

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[19]One may have the impression that Proposition 4 implies the counterintuitive result that $n_h(0)$ increases with π_ℓ. This, in fact, is not the case because π_ℓ also enters the leading term in the expansion: $n_h^\star(0) = \pi_h - A = 1 - \pi_\ell - A$. This leading term dictates that an increase in π_ℓ decreases $n_h(0)$, as intuition suggests.
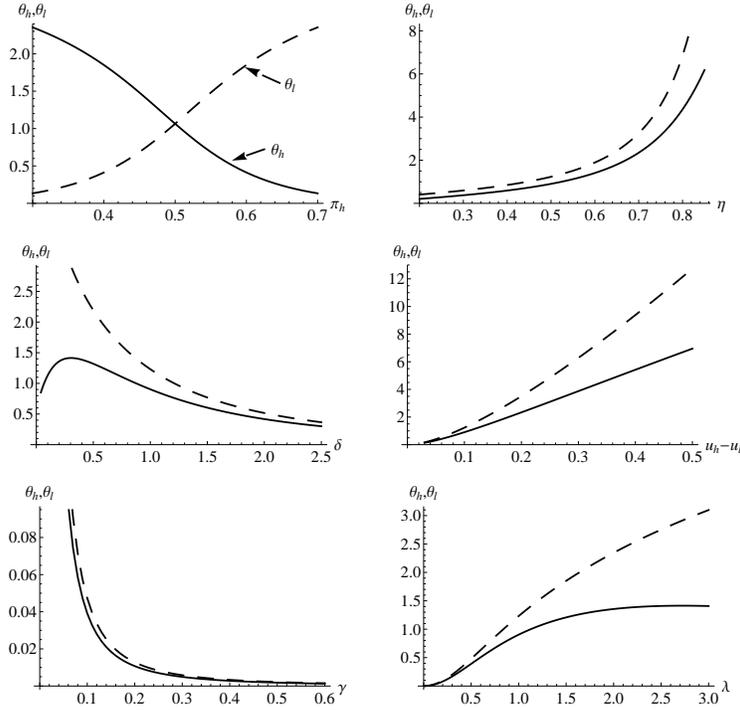
Figure 2: Numerical comparative statics for the market tightness.

is negative for small $\theta_\ell^\star$ and positive for large $\theta_\ell^\star$. Graphically, an increase in elasticity "rotates" the search technology around the point $(1,1)$. It makes it less efficient for small θ and more efficient for large θ.

Now assume that η is small. Then in equilibrium θ is small, implying that an increase in η will reduce search efficiency and, by the indifference condition, increase the sale surplus, $s_\ell^\star$. Vice versa, when η is large, then in equilibrium θ is very large, implying that an increase in elasticity, η, will increase search efficiency and hence reduce the sale surplus, $s_\ell^\star$.

Comparative statics when frictions are large: A numerical example. We investigate equilibria when search frictions are large through a numerical example. We adopt a Cobb-Douglas matching function, $\alpha(\theta) = \lambda\theta^\eta$, normalize $u_h = 1$, and set $u_\ell = 1 - \kappa$. We take the following parameter values: $r = 0.01$, $\delta = 1$, $\pi_h = 0.52$, $\gamma = 0.015$, $\lambda = 1$, $\eta = 0.5$, $\kappa = 0.1$, and $A = 0.5$. The six panels of Figures 2, 3, and 4 illustrate how market tightness, dealers' intermediation fees, and the composition of buyers and sellers in the market, respectively, respond to changes in fundamentals. The plain curves in each panel plot the variables associated with the buyers' sub-market, θ_h, ϕ_h, and $n_h(0)$, while the dashed curves plot variables associated with the sellers' sub-market, θ_ℓ, ϕ_ℓ, and $n_\ell(0)$.
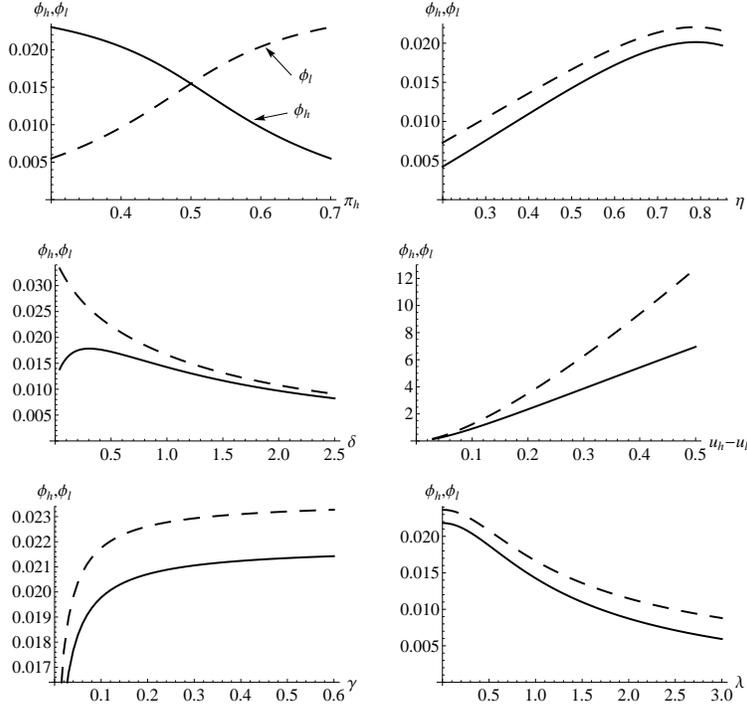
Figure 3: Numerical comparative statics for the intermediation fees.

For given market tightness, as π_h increases, the number of buyers in the market, $n_h(0) = (1-A)\delta\pi_h/[r+\delta+\lambda(\theta_h)^\eta]$, increases while the number of sellers, $n_\ell(1) = A\delta\pi_\ell/[r+\delta+\lambda(\theta_\ell)^\eta]$, decreases. In order for buyers' and sellers' order flows to be equal, i.e., $\alpha(\theta_h)n_h(0) = \alpha(\theta_\ell)n_\ell(1)$, θ_h must decrease and θ_ℓ must increase. This means that the speed of execution of a buy-order will fall while the speed of execution of a sell-order will increase. When $\pi_h = A = 1/2$, the market is symmetric and hence $\theta_h = \theta_\ell$, i.e., execution delays are the same on both sides of the market. Intermediation fees, $\phi_h = \gamma(\theta_\ell)^{1-\eta}/\lambda$ and $\phi_\ell = \gamma(\theta_h)^{1-\eta}/\lambda$, follow a similar pattern as market tightness.

The parameter η represents both the contribution of dealers in the matching process and the share of the match surplus received by dealers, $\phi_h = \eta S_h$ and $\phi_\ell = \eta S_\ell$. As η increases dealers have a higher contribution in the matching process and hence it is optimal to raise market tightness in both sub-markets. Intermediation fees are determined by the free-entry condition $\frac{\alpha(\theta)}{\theta}\phi = \gamma$. For given α, the fact that θ increases implies that dealers' order flow, $\alpha(\theta)/\theta$, decreases. In order to cover their entry cost, dealers must receive higher fees. But for given θ, a higher η means that congestion effects are smaller (e.g., when $\eta = 1$ there is no congestion effect for dealers, i.e., $\alpha(\theta)/\theta$ is independent of θ). As a result of these two opposite effects, intermediation fees can vary in a non-monotonic fashion with η.
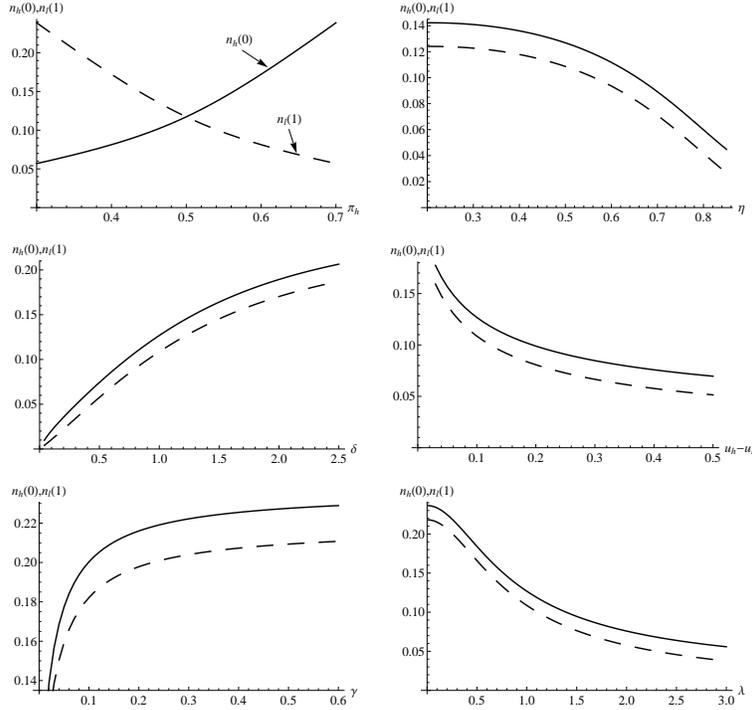
Figure 4: Numerical comparative statics for the measures of buyers and sellers.

The parameter δ in the left panel of the second row represents the frequency of the preference shocks that lead investors to reallocate their portfolios. If δ is very large, investors are likely to receive preference shocks before they can get access to dealers. As a result, there is a low demand for intermediation services, and both market tightness and intermediation fees are low. At the other extreme, for a given market tightness, if δ is very low then most investors have time to meet a dealer before being hit by their next preference shock. Given that $\pi_h = 0.52 > A = 0.5$, $n_h(0)$ remains positive while $n_\ell(1)$ goes to 0. For the order flows to be equal, θ_h must become small and θ_ℓ must become large. The intermediation fees follow a similar pattern as market tightness. In particular, the intermediation fee in the buyers' sub-market, ϕ_h, is a hump-shaped function of δ.

From the right panel in the second row, we see that market tightness and fees are increasing with the size of the gains from trade, $u_h - u_\ell$. The bottom panels allow us to determine how reduced frictions, more efficient matching, or lower entry costs for dealers affect market outcomes. As γ decreases, or as λ increases, market tightness increases and intermediation fees decrease.

Finally, in Figure 5 we plot the price discount of the asset, defined as $(p^* - p)/p^* = 1 - rp$. The findings are consistent with the ones obtained in Proposition 4. In particular, the price discount is a hump-shaped function of the dealer's contribution to the matching process, η. It goes to 0 as γ tends to 0 or as λ goes to infinity.
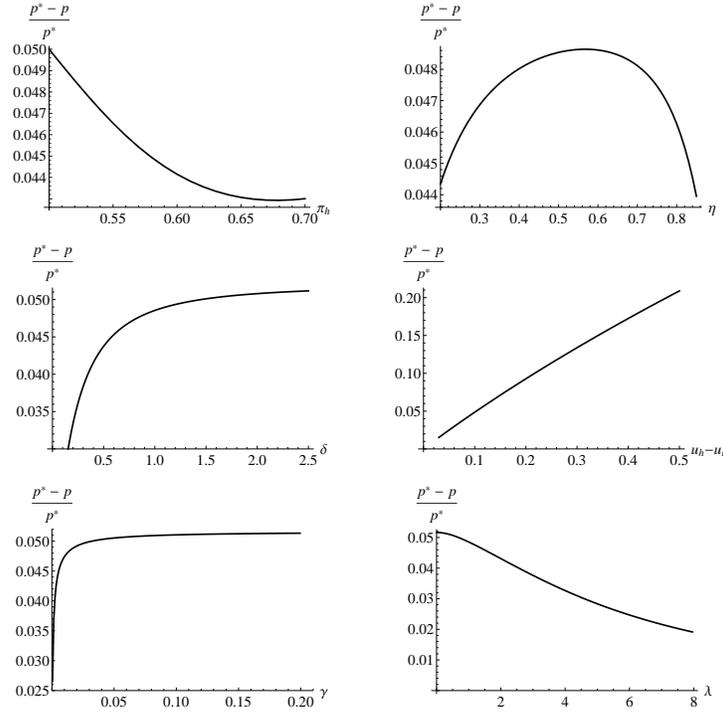
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Figure 5: Numerical comparative statics for the price discount of the asset.

4.2 Asset holdings, market fragmentation, and trading costs

In the previous section, we restricted investors' asset holdings in order to focus on investors' choices of trading speeds and trading costs. In this section we adopt a special matching technology so that market tightness and average execution speeds are equal across active sub-markets, and we focus on the equilibrium distribution of asset holdings.

More specifically, we adopt the same environment as the one described in Section 2 but we suppose that $\alpha(\theta) = \mu \min\{1, \theta\}$, with $\mu > 0$, so that there is a strict complementarity between the number of dealers and the number of orders processed. One interpretation of this technological assumption is that, if $\theta > 1$, each order must be matched to an individual dealer, and the average time for a dealer to process an order is $1/\mu$. If $\theta < 1$, there are not enough dealers to handle all of the orders. In this case, orders are randomly assigned to dealers, so that each order is matched to a dealer with probability θ.[20] For simplicity, we also assume that $\pi_{i,j} \equiv \pi_j$ for all $i, j \in \mathcal{I}^2$, with $\sum_{j \in \mathcal{I}} \pi_j = 1$, so that preference shocks are i.i.d. across investors.

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[20]It should be noted that this matching technology does not satisfy the assumptions imposed earlier: it is not everywhere strictly increasing and continuously differentiable, and $\alpha'(0) = \mu < \infty$. In Appendix A.9, we verify that the properties of equilibrium reported in Section 3 remain true under this specification.

Active sub-markets. When an investor of type i with asset holdings a contacts a dealer and chooses a new portfolio a', the utility gain is

$$\Lambda_i(a', a) = V_i^\star(a') - V_i^\star(a) - p(a' - a),$$

less any fee charged by the dealer. Therefore, as in the benchmark model, an investor of type i with asset holdings a chooses a sub-market that solves

$$\max_{\theta, a', \phi} \{\mu \min\{1, \theta\} [\Lambda_i(a', a) - \phi]\}$$

subject to the free-entry condition of dealers

$$\mu \min\left\{1, \frac{1}{\theta}\right\} \phi \leq \gamma, \text{ with equality if } \theta > 0.$$

Substituting the constraint into the objective, the optimal choice of θ for a type i investor with asset holdings a is

$$\theta_i(a) = \begin{cases} 0 & < \\ [0, 1] & \text{if} \quad \max_{a'} \Lambda_i(a', a) = \gamma/\mu. \\ 1 & > \end{cases} \tag{31}$$

In words, given the prevailing asset price p, an investor of type i with asset holdings a will rebalance his portfolio in equilibrium if the (maximized) gain from doing so exceeds the expected cost incurred by a dealer to process the transaction, which is simply the flow cost γ multiplied by the average time to execute the order $1/\mu$. Note that dealers' entry generates no congestion on other dealers—their matching rate is μ—as long as $\theta \leq 1$. As a result, active sub-markets will typically have $\theta_i(a) = 1$. In each of these active sub-markets, a contract is posted with a fee,

$$\phi_i = \phi \equiv \frac{\gamma}{\mu}, \tag{32}$$

which ensures that dealers earn zero profits, along with a quantity $q_i(a) = a'_i - a$, where

$$a_i = \arg\max_{a' \geq 0} \Lambda_i(a', a). \tag{33}$$

Taken together, the results above imply that the Bellman equation for an investor of type i with

asset holdings a can be rewritten as

$$rV_i^\star(a) = u_i(a) + \delta \sum_k \pi_k[V_k^\star(a) - V_i^\star(a)] + \mu \max \left\{ 0, \max_{a'} \left[\Lambda_i(a', a) - \frac{\gamma}{\mu} \right] \right\}.$$

From the viewpoint of an investor, it is as if trading opportunities arrive at rate μ, and they can choose whether or not to trade at a cost $\phi = \gamma/\mu$. In this sense, our model is formally equivalent to a model of trade with fixed adjustment costs, as in Lo et al. (2004), though the costs of rebalancing one's portfolio arise endogenously in our model.

As in most models with adjustment costs, the equilibrium in our environment could involve "inaction regions," where an investor of type i who holds a_j for $j \neq i$ chooses not to submit a trade order because the fee required to compensate a dealer for executing this order exceeds the expected gains to the investor. In what follows, we will focus on equilibria in which investors who hold their optimal portfolio *always* rebalance their asset holdings after receiving a preference shock. As we establish below, this allows us to sidestep the calculation of value functions for all $a \in \mathbb{R}_+$, and instead characterize equilibrium asset holdings using a simple variational argument.

Optimal asset holdings.   Before characterizing the asset demand functions in our candidate equilibrium, we sketch the logic of our approach (formal proofs are in Appendix A.9). Let $N_i \subset \mathbb{R}_+$ denote the *inaction region* for an investor of type i, i.e., the set of values of a such that an investor of type i with asset holdings in this set would choose not to trade. Given γ/μ sufficiently small, we can establish that $\bigcap_{i \in \mathcal{I}} N_i = \emptyset$, so that an investor of type i with $a \in N_i$ will surely trade if he receives a preference shock $j \neq i$. Given this behavior, there exists a unique $a_i \in N_i$ that maximizes the payoffs of a type i investor.

To characterize these optimal asset holdings, note that a_i is a solution to (33), for each $i \in \mathcal{I}$, where

$$rV_i^\star(a) = u_i(a) + \delta \sum_k \pi_k[V_k^\star(a) - V_i^\star(a)] \tag{34}$$

if $a \in N_i$ and

$$rV_i^\star(a) = u_i(a) + \delta \sum_k \pi_k[V_k^\star(a) - V_i^\star(a)] + \mu \max_{a'} \left[\Lambda_i(a', a) - \frac{\gamma}{\mu} \right] \tag{35}$$

otherwise. One can manipulate (34) and (35) to see that, for all a in the inaction region N_i, the net benefit to a type i investor from acquiring portfolio a, $V_i(a) - pa$, is a positive affine transformation

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of

$$\frac{(r + \mu) [u_i(a) - rpa_i] + \delta \sum \pi_k [u_k(a) - rpa]}{(r + \mu)(r + \delta) - \delta\mu\pi_j}. \tag{36}$$

Since any optimal asset holding must belong to the inaction region N_i, it follows that a_i must satisfy the first-order condition for $V_i(a) - pa$ to be maximized:

$$\frac{(r + \mu) u_i'(a_i) + \delta \sum \pi_k u_k'(a_i)}{r + \mu + \delta} = rp. \tag{37}$$

Note that this equation uniquely determines the optimal asset holding for an investor of type i, a_i. Studying the left side of (37), we see that the demand of an investor of type i depends on his current marginal utility and his expected future marginal utility, weighted by the parameters r, μ, and δ. In particular, an investor places more weight on his current value of marginal utility when he is impatient, when orders are executed quickly, and when he expects to stay in his current state for a long time.[21]

Moreover, if $u_i'(a_i) > \sum_k \pi_k u_k'(a_i)$, then $\partial a_i/\partial\mu > 0$; that is, the investor's demand for the asset increases as trading frictions are reduced. Intuitively, when an investor's current marginal utility is relatively high, an increase in trading speed will cause him to take a larger position now since it will be easier for him to unwind this position in the event of an adverse preference shock. Conversely, if $u_i'(a_i) < \sum_k \pi_k u_k'(a_i)$, i.e., if the investor's marginal utility in the current state is below average, then $\partial a_i/\partial\mu < 0$. More generally, as in Lagos and Rocheteau (2009), faster trading causes investors to take more extreme positions, while slower trading gives agents incentives to choose a more moderate portfolio that will not be too far from the optimal after any preference shock. Finally, as μ goes to infinity, the optimal portfolio tends to the value of a_i that solves $u_i'(a_i) = rp$, which is the portfolio choice that would prevail in a competitive market where all trades can be executed instantaneously.

The steady-state distribution and market-clearing conditions. Given the results above, asset holdings in the candidate equilibrium can be described by a list $\{a_i\}_{i=1}^{I}$. Letting n_{ij} denote the

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[21]The asset demand functions characterized in (37) are closely related to the asset demand functions derived in LR, where investors must bargain with dealers over the quantity to be traded and the corresponding intermediation fee. In particular, our asset demand functions correspond to those of LR *when dealers have zero bargaining power*. Interestingly, LR show that investors' portfolio decisions are socially inefficient whenever dealers have positive bargaining power. Hence, the competitive search mechanism studied in this paper corrects the inefficiencies that arise from an environment with random search and bargaining; intuitively, competition between dealers for order flow ensures that investors receive exactly their marginal gain from readjusting their portfolio.

measure of agents of type i with asset holdings a_j, the steady-state distribution must satisfy

$$\delta \pi_i \sum_{k \in \mathcal{I}} n_{kj} - \delta n_{ij} - \mu n_{ij} = 0, \text{ for } i \neq j \tag{38}$$

$$\delta \pi_i \sum_{k \in \mathcal{I}} n_{ki} + \mu \sum_{k \neq i} n_{ik} - \delta n_{ii} = 0, \text{ for } i = j \tag{39}$$

$$\sum_{(i,j) \in \mathcal{I}^2} n_{ij} = 1. \tag{40}$$

The first term in equation (38) represents the inflow of agents into state ij, with $i \neq j$, which only occurs because of type-switching. The second two terms in equation (38) represent the outflow from state ij, which occurs because of both type-switching (the second term) and trade (the third term). Equation (39) is the steady-state condition for investors holding the optimal portfolio. In this case, inflow can occur either when investors successfully rebalance their portfolio or when they (luckily) receive a preference shock that corresponds to their current asset holdings. Outflow, of course, occurs because of type-switching.

Solving, we find that $\sum_j n_{ij} = \sum_j n_{ji} = \pi_i$, so that

$$n_{ij} = \frac{\delta \pi_i \pi_j}{\mu + \delta}, \quad \text{for } j \neq i, \tag{41}$$

$$n_{ii} = \frac{\delta \pi_i^2 + \mu \pi_i}{\mu + \delta} \quad \text{for } j \neq i. \tag{42}$$

Note that the distribution of probabilities across states is symmetric, $n_{ij} = n_{ji}$. Also, $\partial n_{ij} / \partial \mu < 0$ and $\partial n_{ij} / \partial \delta > 0$ if $j \neq i$, while $\partial n_{ii} / \partial \mu > 0$ and $\partial n_{ii} / \partial \delta < 0$: the measure of investors who are matched to their desired portfolio increases with the speed of execution and decreases with the arrival rate of preference shocks.

Turning now to the determination of the asset price in the competitive inter-dealer market, we note that market clearing requires $\sum_{i,j} n_{ij} a_i = A$. Using the fact that $\sum_j n_{ij} = \pi_i$, this condition reduces to

$$\sum_i \pi_i a_i = A. \tag{43}$$

Equilibrium and comparative statics. An equilibrium such that $\theta_i(a_j) = 1$ for all $i \neq j$ can thus be defined as a list $\{a_i\}$, $\{n_{ij}\}$, ϕ, and p that satisfy (32), (37), and (41)-(43). The individual portfolio choices (a_j) in (37) depend on p, the equilibrium price in the inter-dealer market. The distribution of investors over portfolios and preference types is given by (41) and (42). Given these individual demands, the market-clearing condition (43) determines a unique price. Finally, the

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choice of $\theta_i(a_j)$ described in (32) is optimal provided that γ/μ is sufficiently small.

Using these equilibrium conditions, we can explore the model's implications for prices, trading activity, and allocations. For example, although we have disentangled the asset price from the intermediation fee, one can still compute the *effective price* that an investor pays (or receives) per unit of the asset he buys (or sells). Investors with asset position a_i who trade quantity $a_j - a_i$ through a dealer, pay (or receive, if $a_j - a_i$ is negative)

$$\hat{p}_{ij} = p + \frac{\phi}{a_j - a_i}$$

per unit of the asset. The difference between the prices at which investors buy and sell is sometimes treated as a measure of market liquidity. Notice that if $a_j - a_i > 0$, then

$$\hat{p}_{ij} - \hat{p}_{ji} = \frac{2\phi}{a_j - a_i} = \frac{2\gamma}{\mu(a_j - a_i)} > 0. \tag{44}$$

So for this typical "round trip" transaction, investors trade at a higher effective price when they buy than when they sell. One can notice from (44) that the effective spread decreases with the size of the order, which is in accordance with the evidence documented by Schultz (2001) and Edwards et al. (2007) for over-the-counter markets.[22]

A decrease in γ, the operating costs for dealers, also causes a decrease in the bid-ask spread. The effects of a change in μ are less obvious. On the one hand, an increase in μ has a direct effect in reducing $\phi = \gamma/\mu$. On the other hand, a change in μ also affects equilibrium asset holdings and hence the quantity traded, $a_j - a_i$. However, since the distribution of asset holdings spreads out as μ increases, the quantity of assets traded in many individual trades tends to increase (see Lagos and Rocheteau (2009)); in the case of $I = 2$, for example, it can also be checked that $|a_2 - a_1|$ increases with μ, so the bid-ask spread unambiguously decreases with μ.

The model also has implications for trading volume, which is defined as follows. The flow of investors who can readjust their portfolios per unit of time is μ. A fraction n_{ij} of these investors readjust their portfolio from a_i to a_j so that the quantity they trade is $|a_j - a_i|$. Thus, the total *volume of trade* is

$$\mathcal{V} = \frac{\mu}{2} \sum_{i,j} n_{ij} |a_j - a_i|. \tag{45}$$

An increase in μ has three distinct effects on trade volume. First, the measure of investors in any individual state $(i,j) \in I^2$ who can readjust their portfolios increases, which tends to increase trade

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[22]Similarly, Green, Hollifield, and Schürhoff (2007) document that dealers earn lower average markups on larger trades in the market for municipal bonds. By estimating a bargaining model they find that dealer's bargaining power is substantial and it decreases in trade size.

volume. Second, the proportion $1 - \sum_i n_{ii}$ of agents who are mismatched to their portfolio—and hence the fraction of agents who *wish to trade*—decreases, which tends to reduce trade volume. Finally, as discussed above, the distribution of asset holdings spreads out, which tends to increase the quantity of assets traded in many individual trades, and hence increase volume. Given (41) and (45), it is easy to check that the first two effects combined lead to an increase in $\mathcal{V}$. Therefore, when the third effect is also positive the total volume of trade unambiguously increases with μ.

We can also study some limiting cases. First, as $\mu \to \infty$, so that trades are executed instantaneously, we see from (32) that intermediation fees go to 0. Moreover, from (37), the individual demand for the asset converges to the Walrasian demand, $u'_j(a_j) = rp$, and from (41)-(42) all investors hold their desired portfolios. The same is true as $\delta \to 0$, since type never changes. Finally, as $\gamma \to 0$, the intermediation fee $\phi \to 0$. However, a decrease in γ will only have an effect on asset prices and allocations if it causes a change in the set of active sub-markets; if the initial value of γ was sufficiently small, so that $\theta_i(a_j) = 1$ for all $i \neq j$, then a decrease in γ has no effect on the allocation of assets across agents or on asset prices.

5   Conclusion

We have developed a model of a two-sided asset market where trades are intermediated by dealers and involve a time-consuming matching process. In contrast to the description of OTC markets by DGP and LR, prices are not determined by ex post bargaining but are posted by dealers who compete to attract order flows. This description fits some OTC markets where prices, trading costs (e.g., bid-ask spreads), and execution times are made public, allowing dealers to achieve commitment through reputation.

We have shown existence of equilibrium, and uniqueness for some special cases, and we have characterized some key properties of equilibrium outcomes. First, under competitive search, investors face a trade-off between trading costs and speeds of execution. Investors who would gain the most from readjusting their asset holdings will trade faster and at a higher cost. Second, the asset market is endogenously segmented in the sense that investors in different states will trade at different speeds and different costs. In the simple case where buyers and sellers are homogeneous (see, e.g., DGP) speeds of execution of buy and sell orders as well as trading fees can be asymmetric. Third, dealers' bargaining power is endogenous and is linked to the contribution of dealers to the matching process. As a result, dealers' bargaining powers can vary across sub-markets. For instance, under a specification for the matching function where investors' orders and dealers are strict complements, we found that dealers' market power is smaller when trades are larger, in the sense that intermediation fees per unit of asset traded decrease with the size of the trade.

We have illustrated the tractability of the model by generating a rich set of comparative statics. So far we have only considered the case of private information regarding investors' private valuations for the asset. The use of competitive search as an equilibrium concept should also allow for introducing other forms of private information problems, such as adverse selection (Guerrieri, Shimer, and Wright, 2010) or moral hazard (Li, Rocheteau, and Weill, 2012). It should also make the analysis of aggregate shocks tractable, e.g., due to the block-recursivity property documented by Menzio and Shi (2011) in a different context. We leave these extensions for future research.

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A Proof

A.1 Proof of Proposition 1

We start by noting that $V_i(a, \sigma, \theta)$ can be written

$$V_i(a, \sigma, \theta) = V_i(a, \mathbf{0}, 0) + \frac{\alpha(\theta)}{r + \delta + \alpha(\theta)} \left[V_i^\star(a + q) - pq - \phi - V_i(a, \mathbf{0}, 0) \right], \tag{46}$$

where

$$V_i(a, \mathbf{0}, 0) = \frac{u_i(a) + \delta \sum_{j=1}^I \pi_{ij} V_j^\star(a)}{r + \delta}.$$

Equation (46) shows that the value of submitting an order for contract σ when the market tightness is θ is the sum of two terms: the first term is the no-trade utility, $V_i(a, \mathbf{0}, 0)$, and the second term is proportional to the trading surplus, $V_i(a + q) - pq - \phi - V_i(a, \mathbf{0}, 0)$. The constant of proportionality, $\frac{\alpha(\theta)}{r + \delta + \alpha(\theta)} \leq 1$, captures the time discounting of trading delays associated with tightness θ. The next result is then immediate.

Lemma 3. *In any equilibrium, $V_i^\star(a) \geq V_i(a, \mathbf{0}, 0)$. Moreover the function $\theta \mapsto V_i(a, \sigma, \theta)$ has the following properties:*

1. *If $V_i^\star(a + q) - pq - \phi \leq V_i(a, \mathbf{0}, 0)$, then $V_i(a, \sigma, \theta)$ is decreasing in θ.*

2. *If $V_i^\star(a + q) - pq - \phi > V_i(a, \mathbf{0}, 0)$, then $V_i(a, \sigma, \theta)$ is strictly increasing in θ.*

The first part of the lemma is clear: since investors can always choose $\mathbf{0}$, their maximum attainable utility must be at least equal to $V_i(a, \mathbf{0}, 0)$. The second part of the lemma simply asserts that, if there are strict gains from trading σ, $V_i^\star(a + q) - pq - \phi > V_i(a, \mathbf{0}, 0)$, then investors have a strict preference for higher tightness, because it leads to smaller trading delays. Otherwise, they prefer not to send orders for σ.

Now turning to the proof of Proposition 1, we first show that the right-hand side of (7) is an upper bound for $V_i^\star(a)$. Indeed, for $\sigma = \mathbf{0}$, $V_i(a, \mathbf{0}, 0)$ is equal to the right-hand side of (7) evaluated at $\theta = 0$. For $\sigma \in \Sigma^\star$ but different from $\mathbf{0}$, the dealer's zero-profit condition writes $\alpha \left[\Theta(\sigma) \right] \phi = \gamma \Theta(\sigma)$. Thus:

$$V_i(a, \sigma, \Theta(\sigma)) = \frac{u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_{ij} V_j^\star(a) + \alpha[\Theta(\sigma)] \left[V_i^\star(a + q) - pq \right] - \gamma \Theta(\sigma)}{r + \delta + \alpha[\Theta(\sigma)]},$$

which is clearly less than the right-hand side of (7). Thus, taking the sup over all $\sigma \in \Sigma^\star$, we find $V_i^\star(a)$ is less than the right-hand side of (7).

Toward a contradiction, suppose that $V_i^\star(a)$ is strictly less than the right-hand side of (7). Then, there

must be some $(\hat{\theta}, \hat{q})$ such that

$$V_i^\star(a) < \frac{u_i(a)+\delta\sum_{j\in\mathcal{I}}\pi_{ij}V_j^\star(a)+\alpha(\hat{\theta})\left[V_i^\star(a+\hat{q})-p\hat{q}\right]-\gamma\hat{\theta}}{r+\delta+\alpha(\hat{\theta})}$$

$$\Longleftrightarrow \quad V_i^\star(a) < \frac{u_i(a)+\delta\sum_{j\in\mathcal{I}}\pi_{ij}V_j^\star(a)+\alpha(\hat{\theta})\left[V_i^\star(a+\hat{q})-p\hat{q}-V_i^\star(a)\right]-\gamma\hat{\theta}}{r+\delta}. \tag{47}$$

Note, using the lower bound $V_i(a,\sigma,\theta) \geq V_i(a,\mathbf{0},0)$ in inequality (47), that $\hat{\theta} > 0$. Consider, then, the contract $\hat{\sigma} = (\hat{q},\hat{\phi})$, where $\hat{\phi} = \frac{\gamma\hat{\theta}}{\alpha(\hat{\theta})} + \varepsilon$, for some ε small enough so that $V_i(a,\hat{\sigma},\hat{\theta}) > V_i^\star(a)$. Using again the lower bound in inequality (47), one sees that $V_i^\star(a+\hat{q}) - p\hat{q} - \hat{\phi} > V_i^\star(a) \geq V_i(a,\mathbf{0},0)$. Thus, by Lemma 3, $V_i(a,\hat{\sigma},\theta)$ is strictly increasing in θ. Since it is also continuous, it then follows that $\Theta(\hat{\sigma}) < \hat{\theta}$. Given that $\alpha(\theta)/\theta$ is decreasing, we find that $\Pi(\hat{\sigma}) > 0$, which contradicts dealers' zero-profit condition.

Finally, for the second part of the proposition, we apply the contraction mapping to the auxiliary Bellman equation (7). Consider some arbitrary lower and upper bounds for the price, $0 < \underline{p} < \overline{p}$. If asset holdings are unrestricted, let $\overline{a}$ and $\underline{a}$ be the solutions to $u_I'(a) = r\underline{p}$ and $u_1'(a) = r\overline{p}$, respectively. Since $u_i'(a) \leq r\underline{p}$ for all $a \geq \overline{a}$ and all i, and $u_i'(a)(p) \geq r\overline{p}$ for all $a \leq \underline{a}$ and all i and all $p \in [\underline{p},\overline{p}]$, one can show that an investor will always find it optimal to choose holdings $a \in [\underline{a},\overline{a}]$. If asset holdings are restricted, we let $\underline{a} = \min\mathcal{A}$ and $\overline{a} = \max\mathcal{A}$: by assumption, asset holdings always lie in the interval $[\underline{a},\overline{a}]$. To derive bounds on the choice of θ, note that, up to a positive constant of proportionality, the left derivative with respect to θ of the equation to be maximized on the right-hand side of (7) is equal to

$$-(r+\delta)\gamma + \left[\alpha'(\theta^-)\theta - \alpha(\theta)\right]\gamma + \alpha'(\theta^-)(r+\delta)\left[V_i^\star(a+q) - pq - \frac{u_i(a) + \sum\pi_{ij}V_j^\star(a)}{r+\delta}\right].$$

The first term is strictly negative and independent of θ. The second term is negative because $\alpha(\theta)$ is concave. In the third term, the square bracket is bounded independently of θ and $p \in [\underline{p},\overline{p}]$.[23] Given than $\alpha'(\infty) = 0$, this implies that there exists some $\overline{\theta}$ such that, for all $\theta > \overline{\theta}$, all $p \in [\underline{p},\overline{p}]$, and all $(i,a) \in \mathcal{I} \times [\underline{a},\overline{a}]$, the left-derivative of equation (7) is strictly negative. Therefore, an investor will always find it optimal to choose a market tightness $\theta \in [0,\overline{\theta}]$.

Let $S = \mathcal{I} \times [\underline{a},\overline{a}] \times [\underline{p},\overline{p}]$ when assets are unrestricted, and let $S = \mathcal{I} \times \mathcal{A} \times [\underline{p},\overline{p}]$ when assets are restricted. Let $\mathcal{C}(S)$ be the space of bounded, continuous functions $f : S \to \mathbb{R}$ equipped with the sup norm.

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[23]Indeed $V_i(a) \leq \max_{i\in\mathcal{I}}\frac{u_i(\overline{a})}{r} + \overline{p}\overline{a}$, so that the maximum attainable utility is less than the present value of the maximum utility flow from holding the maximum quantity of assets, plus the time-zero value of selling the maximum quantity of assets at the maximum price. To see this, note that the inter-temporal value of an investor has two terms. The first term is the expected present value of utility flows, $u_i[a(t)]$, net of search cost, $-\gamma\theta(t)$, which is clearly less than $\frac{u_I(\overline{a})}{r}$. The second term is the expected present value of the benefits of selling minus the costs of buying assets at the random contact times $0 < T_1 < T_2 < \dots$. For given realization of contact times, this present value can be written $a(0)pe^{-rT_1} + \sum_{n=1}^{\infty}pa(T_n)\left[e^{-rT_{n+1}} - e^{-rT_n}\right]$. Indeed, the investor resells $a(T_n)$ at time T_{n+1} after buying it at time T_n. Clearly, the first term is less than $\overline{p}\overline{a}$, and all terms in the infinite sum are negative.

Notice that $\mathcal{C}(S)$ is a complete normed vector space. The right-hand side of (7) defines an operator T:

$$T(V_i)(a, p) = \max_{(a', \theta) \in [\underline{a}, \bar{a}] \times [0, \bar{\theta}]} \left[\frac{u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_{ij} V_j(a) + \alpha(\theta)[V_i(a') - p(a' - a)] - \gamma \theta}{r + \delta + \alpha(\theta)} \right]. \quad (48)$$

If $V \in \mathcal{C}(S)$, then, from the Theorem of the Maximum $T(V) \in \mathcal{C}(S)$. See Theorem 3.6 in Stokey and Lucas (1989). Furthermore, T is monotonic and $T(f + k)(a, i) \leq T(f)(a, i) + \frac{\delta + \alpha(\bar{\theta})}{r + \delta + \alpha(\bar{\theta})} k$ for all $k \geq 0$ and all $f \in \mathcal{C}(S)$. Therefore, from Blackwell's Theorem, T is a contraction on $\mathcal{C}(S)$ and it has a unique fixed point in $\mathcal{C}(S)$. See Theorem 3.3 in Stokey and Lucas (1989). Finally, if $V_i(a, p)$ is a strictly increasing function then $T(V_i)(a, p)$ is also strictly increasing so that the unique fixed point of T is increasing. But since $u_i(a)$ is strictly increasing, the Bellman equation (7) implies that $V_i(a)$ is the sum of a strictly increasing function and an increasing function, and so is strictly increasing as well. Finally, since we considered arbitrary lower and upper bounds, $\underline{p}$ and $\bar{p}$, and since $\bar{a} \to \infty$ as $\underline{p} \to 0$, and $\underline{a} \to 0$ as $\bar{p} \to \infty$, the properties extend by continuity to the domain $S = \mathcal{I} \times \mathcal{A} \times (0, \infty)$.

A.2 Proof of Proposition 2

We first note that, by Theorem of the Maximum, the correspondence $(a_i(a, p), \theta_i(a, p))$ maximizing (48) is compact valued and upper hemi continuous. Moreover, one sees that $a_i(a, p)$ is independent on a: it is equal to $\arg\max_{a' \in [\underline{a}, \bar{a}]} V_i(a', p) - pa'$ if $0 \notin \theta_i(a, p)$ and to $[\underline{a}, \bar{a}]$ otherwise. With this in mind, let

$$\mathcal{M}(p) \equiv \left\{ (a_i(p), \theta_j(a_i(p), p)), (i, j) \in \mathcal{I}^2 \right\}.$$

The correspondence $\mathcal{M}(p)$ contains the collection of candidate equilibrium asset holdings if the price is p. It is clearly compact valued and is easily shown to be upper hemi continuous. As explained in the text, we look for an equilibrium in which the distribution of types has a finite support. To deal with potential non-convexity arising if the Bellman equation has more than one maximizer, we partition the population of investors into $G \geq 2$ groups of size μ^g, for some μ^g in the G-dimensional simplex, Δ^G. We assume that each investor in group g optimally keeps its asset holdings in the support $\{a_1^g, a_2^g, \ldots, a_I^g\}$ where, for all i and g, a_i^g is maximizing (7). If the investor has utility type i but asset holding a_j^g, $j \neq i$, then he sends its order for a contract with a market tightness $\theta_i(a_j^g)$ maximizing (7). Therefore, the inflow-outflow equations for the steady-state distribution, $n_i^g(a_j)$, of group g write:

$$\text{if } i \neq j \quad : \quad 0 = \delta \sum_{k \in \mathcal{I}} n_k(a_j^g) \pi_{ki} - \delta n_i^g(a_j^g) - \alpha[\theta_i(a_j^g)] n_i(a_j^g) \quad (49)$$

$$\text{if } i = j \quad : \quad 0 = \delta \sum_{k \in \mathcal{I}} n_k(a_i^g) \pi_{ki} - \delta n_i(a_j^g) + \sum_{k \in \mathcal{I}} \alpha[\theta_i(a_k^g)] n_k(a_i^g) \quad (50)$$

$$\mu^g = \sum_{(i,j) \in \in \mathcal{I}^2} n_i(a_j^g). \quad (51)$$

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The first step is to establish the following result.

Lemma 4. *The system of steady-state equations (49)-(51) has at least one solution.*

Proof. The argument is standard. The system (8)-(9) characterizes the ergodic distributions of the continuous time Markov chain induced by the type switching and trading process. Denote the transition intensities by $\lambda_{k,\ell}$, and let $\lambda_k \equiv \sum_\ell \lambda_{k,\ell}$. The steady-state equations can be rewritten:

$$\forall \ell: \quad \lambda_\ell n_\ell = \sum_k n_k \lambda_{k,\ell} \quad \Longleftrightarrow \quad \forall \ell: \quad \lambda_\ell n_\ell = \sum_k \lambda_k n_k \frac{\lambda_{k,\ell}}{\lambda_k} \quad \Longleftrightarrow \quad \pi = \pi Q,$$

where $\pi_k \equiv \lambda_k n_k$ and $Q_{\ell,k} = \frac{\lambda_{\ell,k}}{\lambda_\ell}$. Thus, a steady-state distribution can be found by solving for an ergodic distribution, π, of the discrete time Markov chain with transition probabilities Q. Since this discrete time Markov chain has a finite state space, it follows from Theorem 11.1 of Stokey and Lucas (1989) that such an ergodic distribution exists. ∎

Equipped with this result, we can define the aggregate demand correspondence as follows. In matrix form, the system of steady-state equations of group g, (49)-(51), can be written as $\Gamma(m^g)n^g = b$, where the entries of $\Gamma(m^g)$ are continuous in a_i^g and $\theta_i(a_j^g)$. Then, we let

$$\mathcal{D}(p) \equiv \left\{ \sum_{g,i,j} \mu^g n_i(a_j^g) a_j^g, \text{ for } \mu^g \in \Delta^G, m^g = (a_i, \theta_i(a_j)) \in \mathcal{M}(p), \text{ and } n^g \text{ s.t. } \Gamma(m^g)n^g = b \right\}.$$

Now let us turn to the proof of Proposition 2. By Lemma 4, the aggregate demand correspondence is non-empty. It is convex by construction. To see that it is compact-valued, consider any sequence of elements of $\mathcal{D}(p)$ generated by some sequences μ_k^g, m_k^g, and n_k^g. Because group measures, asset holdings, and steady-state measures are all bounded, we can extract convergence subsequences $\mu_{k\ell}^g$, $m_{k\ell}^g$, and $n_{k\ell}^g$. Since the simplex and $\mathcal{M}(p)$ are compact, it follows that $\lim \mu_{k\ell}^g \in \Delta^G$ and $\lim m_{k\ell}^g \in \mathcal{M}(p)$. Since $\Gamma(m)$ is continuous, it follows that $\Gamma(\lim m_{k\ell}^g) \lim n_{k\ell}^g = b$. Therefore, the aggregate excess demand generated by $\lim \mu_{k\ell}^g$, $\lim m_{k\ell}^g$, and $\lim n_{k\ell}^g$ belongs to $\mathcal{D}(p)$, and we are done proving compactness. Finally, a similar reasoning establishes that $\mathcal{D}(p)$ is upper hemi continuous. Now, we note that the aggregate demand goes to zero as p goes to infinity, and to infinity as p goes to zero. An application of the Intermediate Value Theorem (easily extended for upper hemi continuous, compact, and convex-valued correspondences) establishes the claim of the proposition.

The last step is to verify the equilibrium condition for the candidate equilibrium objects shown in the text after Proposition 2. Without loss of generality at this stage of the analysis, let us assume that $G = 1$ so that we can simplify notations and drop the "g" subscript everywhere. We first show that $\Theta[\sigma_i(a_j)] = \theta_i(a_j)$. First, we prove that

$$V_i(a_j, \sigma_i(a_j), \theta_i(a_j)) > V_i(a_j, \mathbf{0}, 0).$$

By construction, we have a weak inequality. Suppose, toward a contradiction, that $V_i(a_j, \sigma_i(a_j), \theta_i(a_j)) = V_i(a_j, \mathbf{0}, 0)$. Expressing this equality using (46) and the dealer's zero-profit condition, we have

$$\frac{\alpha[\theta_i(a_j)]}{\theta_i(a_j)} \left[V_i(a + q_i(a_j)) - pq_i(a_j) - V_i(a, \mathbf{0}, 0) \right] = \gamma.$$

Now the right-derivative at $\theta = 0$ of the right-hand side of (48) is equal to

$$\frac{\alpha'(0) \left[V_i(a + q_i(a_j)) - pq_i(a_j) - V_i(a, 0) \right] - \gamma}{r + \delta} > 0$$

using the equality we derived just above and noting that $\alpha'(0) > \alpha(\theta)/\theta$ by strict concavity. Hence, the right-hand side of the auxiliary Bellman equation (48) cannot be maximized at $\theta = 0$, which is a contradiction. Thus, we are in case 2 of Lemma 3, $V_i(a_j, \sigma_i(a_j), \theta_i(a_j))$ is strictly increasing, $\theta_i(a_j) = \inf\{\theta \geq 0 : V_i(a_j, \sigma_i(a_j), \theta) > V_i^\star(a_j)\}$ and so is greater than $\Theta(\sigma_i(a_j))$.

Suppose, toward a contradiction, that the inequality is strict. Then, by definition of $\Theta(\sigma)$, there exists some $\theta \in (0, \theta_i(a_j))$ and some (k, ℓ) such that $V_k(a_\ell, \sigma_i(a_j), \theta) > V_k^\star(a_\ell)$. Using the definition of $V_k(a, \sigma, \theta)$, we obtain that

$$\frac{u_k(a_\ell) + \delta \sum_{m \in \mathcal{I}} \pi_{km} V_m^\star(a_\ell) + \alpha(\theta) \left[V_k^\star(a_\ell + q_i(a_j)) - pq_i(a_j) \right] - \gamma\theta \frac{\alpha(\theta)}{\theta} \frac{\theta_i(a_j)}{\alpha(\theta_i(a_j))}}{r + \delta + \alpha(\theta)} > V_k^\star(a_\ell).$$

Since $\alpha(\theta)/\theta$ is decreasing, it follows that the above inequality remains strict when we subtract $\gamma\theta$ instead of $\gamma\theta \frac{\alpha(\theta)}{\theta} \frac{\theta_i(a_j)}{\alpha[\theta_i(a_j)]}$, which contradicts that $V_k^\star(a)$ solves the auxiliary Bellman equation (48).

Finally, we verify the zero-profit conditions of dealers. Suppose there is $\sigma = (q, \phi)$ such that $-\gamma + \frac{\alpha[\Theta(\sigma)]}{\Theta(\sigma)} \phi > 0$. Then $\Theta(\sigma) < \infty$ and there is some type (i, a_j) and some tightness θ such that $V_i(a_j, \sigma, \theta) > V_i^\star(a_j)$ and $-\gamma + \frac{\alpha(\theta)}{\theta} \phi > 0$. Consider then the contract $\hat{\sigma}$, with $\hat{q} = q$ and $\hat{\phi}$ chosen such that $\gamma = \frac{\alpha(\theta)}{\theta} \hat{\phi}$. Because $\hat{\phi} < \phi$, we have that $V_i(a_j, \hat{\sigma}, \theta) > V_i^\star(a_j)$, which can be written

$$\frac{u_i(a_j) + \delta \sum \pi_{ik} V_k^\star(a_k) + \alpha(\theta) \left[V_i^\star(a_j + \hat{q}) - p\hat{q} \right] - \gamma\theta}{r + \delta + \alpha(\theta)} > V_i^\star(a_j),$$

contradicting that $V_i^\star(a)$ solves the Bellman equation (48).

A.3 Market clearing in the inter-dealer market

In this section, we confirm that the equilibrium conditions described in Definition 1 ensure that the inter-dealer market clears, i.e., that

$$\sum_{i,j} \alpha\left[\theta_i(a_j)\right] n_i(a_j)(a_i - a_j) = 0. \tag{52}$$

To start, summing across all $i \in \mathcal{I}$ and $j \neq i$, we can use (8) to get

$$\sum_{i \in \mathcal{I}} \sum_{j \neq i} \alpha \left[\theta_i(a_j) \right] n_i(a_j) a_j = \delta \sum_{i \in \mathcal{I}} \sum_{j \neq i} a_j \left[\sum_{k \in \mathcal{I}} \pi_{ki} n_k(a_j) - n_i(a_j) \right]. \tag{53}$$

Adopting the convention that $\theta_i(a_i) = 0$, the left-hand side of (53) is equal to

$$\sum_{i,j} \alpha \left[\theta_i(a_j) \right] n_i(a_j) a_j. \tag{54}$$

Meanwhile, the right-hand side of (53) can be written

$$\delta \sum_{k \in \mathcal{I}} \sum_{j \in \mathcal{I}} n_k(a_j) a_j \sum \pi_{ki} - \delta \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} n_i(a_j) a_j - \sum_{i \in \mathcal{I}} \left\{ \delta \left[\sum_{k \in \mathcal{I}} n_k(a_i) \pi_{ki} - n_i(a_i) \right] \right\} a_i$$

$$= -\sum_{i \in \mathcal{I}} \left\{ \delta \left[\sum_{k \in \mathcal{I}} n_k(a_i) \pi_{ki} - n_i(a_i) \right] \right\} a_i \tag{55}$$

$$= \sum_{i,j} \alpha \left[\theta_i(a_j) \right] n_i(a_j) a_i, \tag{56}$$

where the first equality follows from $\sum_i \pi_{ki} = 1$, while the second equality follows from (9). Hence, (54) is equal to (56), which ensures that (52) is satisfied.

A.4 Proof of Lemma 1

Let, for $i \in \{\ell, h\}$, $\theta_i(0) \equiv \arg\max_\theta \{\alpha(\theta) (\Delta V_i - p) - \gamma\theta\}$ and $\theta_i(1) \equiv \arg\max_\theta \{\alpha(\theta) (p - \Delta V_i) - \gamma\theta\}$. We obtain the inequalities:

$$r\Delta V_h \geq u_h + \delta\pi_\ell (\Delta V_\ell - \Delta V_h) + \alpha[\theta_\ell(1)] (p - \Delta V_h) - \gamma\theta_\ell(1) - \alpha[\theta_h(0)] (\Delta V_h - p) - \gamma\theta_h(0)$$

$$r\Delta V_\ell \leq u_\ell + \delta\pi_h (\Delta V_h - \Delta V_\ell) + \alpha[\theta_\ell(1)] (p - \Delta V_\ell) - \gamma\theta_\ell(1) - \alpha[\theta_h(0)] (\Delta V_\ell - p) - \gamma\theta_h(0).$$

Taking the difference between the two, all the search costs cancel out and we obtain

$$\Delta V_h - \Delta V_\ell \geq \frac{u_h - u_\ell}{r + \delta + \alpha [\theta_h(0)] + \alpha [\theta_\ell(1)]} > 0,$$

which establishes the first part of the claim.

For the second part, assume toward a contradiction that $p \geq \Delta V_h$, so that $p > \Delta V_\ell$. Then, it must be the case that $\theta_h(1) \geq 0$, $\theta_\ell(1) > 0$, and $\theta_h(0) = \theta_\ell(0) = 0$. The inflow-outflow equations for $n_h(1)$ and $n_\ell(1)$

become

$$\delta\pi_h n_\ell(1) = \delta\pi_\ell n_h(1) + \alpha[\theta_h(1)]n_h(1)$$

$$\delta\pi_\ell n_h(1) = \delta\pi_h n_\ell(1) + \alpha[\theta_\ell(1)]n_\ell(1),$$

implying, after some manipulations, that

$$\frac{\delta\pi_\ell}{\delta\pi_\ell + \alpha[\theta_h(1)]}\frac{\delta\pi_h}{\delta\pi_h + \alpha[\theta_\ell(1)]}n_\ell(1) = n_\ell(1),$$

so that $n_\ell(1) = n_h(1) = 0$. Thus, the market cannot clear, which is a contradiction. Symmetrically, if $p \le \Delta V_\ell$, one finds that $n_\ell(0) = n_h(0) = 0$ so that $n_\ell(1) + n_h(1) = 1 > A$, which also contradicts market clearing.

A.5   Proof of Lemma 2

The function $F(S_\ell)$ is defined over $[0, \Gamma^{-1}(u_h - u_\ell)]$ and the function $G(S_\ell)$ is defined over $\left[0, \frac{u_h - rp}{\delta\pi_\ell}\right]$. Both functions are zero at the upper bound of their domain. Thus, solving the system of equations (15)-(16) boils down to solving the one-equation-in-one-unknown problem $F(S_\ell) = G(S_\ell)$. We have shown in the text that, if a solution exists, then the function $F(S_\ell)$ must cross the function $G(S_\ell)$ from above. Therefore, for a strictly positive solution to exist, it is necessary and sufficient that $F(S_\ell)$ is above $G(S_\ell)$ at zero, and is eventually below $G(S_\ell)$ for S_ℓ large enough in the intersection of their domains. The first condition can be written

$$
\begin{aligned}
F(0) > G(0) &\iff \Gamma[F(0)] > u_h - rp \text{ and } \Gamma[F(0)] = u_h - u_\ell \\
&\iff \delta\pi_h F(0) < rp - u_\ell \text{ and } \Gamma[F(0)] = u_h - u_\ell \\
&\iff u_h - u_\ell < \Gamma\left(\frac{rp - u_\ell}{\delta\pi_h}\right) \\
&\iff p > p_\ell \text{ s.t. } u_h - u_\ell = \Gamma\left(\frac{rp_\ell - u_\ell}{\delta\pi_h}\right).
\end{aligned}
\tag{57}
$$

The equivalence on the first line follows from the fact that, in equation (18), the function $\Gamma(S)$ is strictly increasing. The equivalence on the second line follows from simple manipulations, using the definition of $\Gamma(S)$. The equivalence on the third and fourth lines follow from the fact that $\Gamma(S)$ is strictly increasing in S. One sees easily using equations (15)-(16) that the price p_ℓ is such that $S_\ell = 0$: in other words, it is the lowest price at which a low-valuation investor is willing to sell his asset, knowing that he'll be able to buy at this price later when he'll turn into a high type.

Next we note that, by definition, $F(S_\ell)$ and $G(S_\ell)$ are zero at the upper bound of their respective domains. Thus, in order for $G(S_\ell)$ to be below $F(S_\ell)$ for S_ℓ large enough in the intersection of their domains, it is necessary and sufficient that the upper bound of the domain of $G(S_\ell)$ is below the upper

bound of the domain of $F(S_\ell)$. This can be written as

$$\Gamma^{-1}(u_h - u_\ell) > \frac{u_h - rp}{\delta \pi_\ell} \quad \Longleftrightarrow \quad u_h - u_\ell > \Gamma\left(\frac{u_h - rp}{\delta \pi_\ell}\right)$$

$$\Longleftrightarrow \quad p < p_h \text{ s.t. } u_h - u_\ell = \Gamma\left(\frac{u_h - rp_h}{\delta \pi_\ell}\right). \tag{58}$$

As before, one sees easily that the price p_h is such that $S_h = 0$: it is the highest price at which a high type is willing to buy.

Finally, we observe that $p_\ell < p_h$. Indeed, from their definitions we have that:

$$\frac{u_h - rp_h}{\delta \pi_\ell} = \frac{rp_\ell - u_\ell}{\delta \pi_h} \Rightarrow rp_h \pi_h + rp_\ell \pi_\ell = \pi_h u_h + \pi_\ell u_\ell.$$

It thus follows that

$$p_h > p_\ell > 0 \quad \Longleftrightarrow \quad r\pi_\ell(p_h - p_\ell) > 0 \Longleftrightarrow rp_h > \pi_h u_h + \pi_\ell u_\ell$$

$$\Longleftrightarrow \quad \Gamma\left(\frac{u_h - \pi_h u_h - \pi_\ell u_\ell}{\delta \pi_\ell}\right) > u_h - u_\ell$$

$$\Longleftrightarrow \quad \Gamma\left(\frac{u_h - u_\ell}{\delta}\right) > u_h - u_\ell.$$

One easily verifies that $\Gamma(S/\delta) > S$, so this inequality holds.

A.6   Proof of Proposition 3

The only thing to show is that the buy-order flow, on the left-hand side of (25), is strictly decreasing in the price, and that the sell-order flow, on the right-hand side of (25), is strictly increasing in the price. For this we first note that, since $\alpha(\theta)$ is increasing, it follows that the left-hand side is an increasing function of θ_h and the right-hand side is a decreasing function of θ_ℓ. Now the first-order condition $\alpha'(\theta_h)S_h = \gamma$ implies that θ_h is an increasing function of S_h, and thus a decreasing function of p, with $\theta_h = 0$ when $p = p_h$. Likewise, the first-order condition for θ_ℓ implies that θ_ℓ is an increasing function of S_ℓ and thus an increasing function of p with $\theta_\ell = 0$ when $p = p_\ell$.

A.7   Proof of Proposition 4

Asymptotic expansion for the price.   Let $s_i \equiv \lambda S_i$ for $i \in \{\ell, h\}$. With this notation, the surplus equation (17) can be written as

$$\frac{r + \delta}{\lambda} s_h + f(s_h) + \frac{r + \delta}{\lambda} s_\ell + f(s_\ell) = u_h - u_\ell,$$

where $f(s) \equiv \max_\theta \{\alpha(\theta)s - \gamma\theta\}$. Note that $f(s)$ is strictly increasing, with $f(0) = 0$ and $f(\infty) = \infty$. It thus follows that s_h and s_ℓ are both bounded by the solution of $f(s) = u_h - u_\ell$. Therefore, as λ goes to infinity, (s_h, s_ℓ) must have at least one accumulation point, $(s_h^\star, s_\ell^\star)$. Since $\pi_h > A$, it follows from the market-clearing condition that $\theta_h < \theta_\ell$ and so from the first-order condition that $s_h < s_\ell$. Thus $s_h^\star \leq s_\ell^\star$. Suppose that $s_h^\star > 0$. Then the market tightness solving $\alpha'(\theta)s_h^\star = \gamma$ is strictly positive, and going to the $\lambda \to \infty$ limit in the market-clearing condition leads to $\pi_h = A$, which contradicts our assumption that $\pi_h > A$. Therefore $s_h^\star$ and $s_\ell^\star$ satisfy

$$
\begin{aligned}
s_h^\star &= 0 \\
f(s_\ell^\star) &= u_h - u_\ell.
\end{aligned}
$$

Thus, (s_h, s_ℓ) has a unique accumulation point, which must be its limit. To obtain a first-order approximation for p we use the second equation (18) to get that

$$
\frac{r + \delta\pi_\ell}{\lambda} s_h + f(s_h) = u_h - rp - \delta\pi_\ell \frac{s_\ell}{\lambda}. \tag{59}
$$

Since $s_h \to 0$ as $\lambda \to \infty$, the first term on the left-hand side is $o\left(\frac{1}{\lambda}\right)$. To analyze the second term, note from the market-clearing condition that $\lambda\alpha(\theta_h)$ must have a positive limit, equating order flows when sellers can contact the market infinitely fast. Letting this limit be λ_h we can write

$$
f(s_h) = \left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right] s_h - \gamma\alpha^{-1}\left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right].
$$

The first term on the left-hand side is $o\left(\frac{1}{\lambda}\right)$ since $s_h \to 0$ when $\lambda \to \infty$. Because $\alpha^{-1}(x)$ is convex, the second term can be bounded by

$$
\begin{aligned}
\alpha^{-1}\left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right] &\leq \left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right] [\alpha^{-1}]'\left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right] \\
&= \left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right] \frac{1}{\alpha' \circ \alpha^{-1}\left[\frac{\lambda_h}{\lambda} + o\left(\frac{1}{\lambda}\right)\right]}.
\end{aligned}
$$

Clearly, because of the Inada condition $\alpha'(0) = \infty$, the upper bound is $o\left(\frac{1}{\lambda}\right)$. We conclude that $f(s_h)$ is $o\left(\frac{1}{\lambda}\right)$. Finally, we note that $\frac{s_\ell}{\lambda} = \frac{s_\ell^\star}{\lambda} + o\left(\frac{1}{\lambda}\right)$.

Asymptotic expansion of the intermediation fee. Consider first the intermediation fee for a seller:

$$
\lambda\phi_\ell = \frac{\theta_\ell\alpha'(\theta_\ell)}{\alpha(\theta_\ell)}\lambda S_\ell \to \frac{\theta_\ell^\star\alpha'(\theta_\ell^\star)}{\alpha(\theta_\ell^\star)}s_\ell^\star,
$$

as claimed in the proposition. Let us turn next to the intermediation fee for buyers:

$$\lambda \phi_h = \frac{\theta_h \alpha'(\theta_h)}{\alpha(\theta_h)} \lambda S_h \to 0$$

since the elasticity is bounded above by one and $\lim_{\lambda \to \infty} \lambda S_h = 0$.

Asymptotic expansion of the distribution of types. From equation (24), given that $\theta_\ell \to \theta_\ell^\star$, we have that

$$\lim_{\lambda \to \infty} \lambda n_\ell(1) = \frac{A \delta \pi_\ell}{\alpha(\theta_\ell^\star)}$$

as claimed. To obtain an asymptotic expansion of $n_h(0)$, plug the equality of order flows, (20), into the equation for $n_h(0)$, (23). In doing so, keep in mind that, for the asymptotic expansion, all the $\alpha(\theta)$ are multiplied by the search efficiency parameter λ. We then obtain

$$n_h(0) = \frac{\delta \pi_h (1 - A)}{\delta + \frac{\lambda \alpha(\theta_\ell) n_\ell(1)}{n_h(0)}} \iff \delta n_h(0) + \lambda \alpha(\theta_\ell) n_\ell(1) = (1 - A) \delta \pi_h$$

$$\iff \delta \left[n_h(0) - (\pi_h - A) \right] = -\lambda \alpha(\theta_\ell) n_\ell(1) + (1 - A) \delta \pi_h - \delta(\pi_h - A)$$

$$\iff n_h(0) - (\pi_h - A) = \frac{\delta A \pi_\ell}{\lambda \alpha(\theta_\ell)} = \frac{\delta A \pi_\ell}{\lambda \alpha(\theta_\ell^\star)} + o\left(\frac{1}{\lambda}\right),$$

where the last line follows after substituting in expression (24) for $n_\ell(1)$. This establishes the claim.

A.8 Proof of Corollary 1

With a Cobb-Douglas matching function, the asymptotic seller surplus, $s_\ell^\star$, and the asymptotic search intensity, $\theta_\ell^\star$, solve the system of equations:

$$\theta^\eta s - \gamma \theta = u_h - u_\ell$$
$$\eta \theta^{\eta-1} s = \gamma.$$

The second equation implies that $\eta \theta^\eta s_\ell^\star = \gamma \theta$. Together with the first equation, this implies that $(1 - \eta)\theta^\eta s = u_h - u_\ell$. Dividing through by the second equation, the "s" cancel out and we obtain the expression for $\theta_\ell^\star$. The expression for $s_\ell^\star$ follows. Clearly, $\theta_\ell^\star$ is increasing in η. As for $s_\ell^\star$ we have:

$$\log\left(s_\ell^\star\right) = \eta \log\left(\frac{\gamma}{\eta}\right) + (1 - \eta)\log\left(\frac{u_h - u_\ell}{1 - \eta}\right) \implies \frac{d \log(s_\ell^\star)}{d\eta} = \log\left(\frac{\gamma(1 - \eta)}{\eta(u_h - u_\ell)}\right),$$

which is strictly decreasing, goes to plus infinity when $\eta \to 0$ and to minus infinity when $\eta \to 1$.

A.9   Formal proofs of the statements in Section 4.2

In this section we characterize an equilibrium for the economy described in Section 4.2, when the posting cost, γ, is small.

Equilibrium price.   We guess and verify that, in this case, the equilibrium price in LR is an equilibrium price in our environment as well.

Let $V_i^\star(a, \gamma)$ be the solution of the Bellman equation (7) when the contract posting cost is γ and the price is equal to the equilibrium price in LR. By an application of the contraction mapping theorem, this function is continuous in (a, γ). In particular, $V_i^\star(a, 0)$ coincides with the value function in LR. We know the following result from their paper.

Lemma 5. *In LR, the value net of purchasing cost, $V_i^\star(a, 0) - pa$, is an increasing and affine transformation of*

$$K_i(a) = \frac{(r + \mu)u_i(a) + \delta \sum_{j \in \mathcal{I}} \pi_j u_j(a)}{r + \mu + \delta} - rpa. \tag{60}$$

Note that $K_i(a)$ is strictly concave and it achieves its unique maximum at the asset holding prevailing in LR's equilibrium.

$$N_i(\gamma_1, \gamma_2) \equiv \left\{ a \ : \ \max_{a'} \left\{ V_i^\star(a', \gamma_1) - pa' \right\} - \frac{\gamma_2}{\mu} < V_i^\star(a, \gamma_1) - pa \right\}.$$

Lemma 6. *The inaction regions $N_i(\gamma, \gamma)$ are disjoint for all γ small enough.*

Proof. Note that LR's net values, $V_i^\star(a, 0) - pa$, are strictly concave so that the inaction regions $N_i(0, \gamma_2)$ are open intervals around the optimal asset holding, $\arg\max_{a'} V_i^\star(a', 0) - pa'$, shrinking monotonically to LR's optimal asset holding when $\gamma_2 \to 0$. Consider then some γ_2 and ε small enough so that all the $N_i(0, \gamma_2 + \mu\varepsilon)$ are disjoint. Noting that $\max_{a'} \left\{ V_i^\star(a', \gamma) - pa' \right\} - V_i^\star(a, \gamma) + pa$ is continuous in (a, γ), there exists some γ_1 such that for all $\gamma_1' \leq \gamma_1$ and all $a \in [\underline{a}, \overline{a}]$:

$$\max_{a'} \left\{ V_i^\star(a', \gamma_1') - pa' \right\} - V_i^\star(a, \gamma_1') + pa > \max_{a'} \left\{ V_i^\star(a', 0) - pa' \right\} - V_i^\star(a, 0) + pa - \varepsilon.$$

In particular, for $a \notin N_i(0, \gamma_2 + \mu\varepsilon)$, we have $\max_{a'} \left\{ V_i^\star(a', 0) - pa' \right\} - \frac{\gamma_2}{\mu} - \varepsilon \leq V_i^\star(a, 0) - pa$. Plugging this into the right-hand side of the above inequality and rearranging, we obtain

$$\max_{a'} \left\{ V_i(a', \gamma_1') - pa' \right\} - \frac{\gamma_2}{\mu} \geq V_i(a, \gamma') - pa.$$

Hence, $a \notin N_i(\gamma_1', \gamma_2)$. By contrapositive, for all $\gamma_1' \leq \gamma_1$, we have that $N_i(\gamma_1', \gamma_2) \subseteq N_i(0, \gamma_2 + \mu\varepsilon)$. Since the sets $N_i(0, \gamma_2 + \mu\varepsilon)$ are disjoint by construction, and since the sets $N_i(\gamma_1', \gamma_2')$ are decreasing in γ_2', it

follows that $N_i(\gamma_1', \gamma_2')$ are disjoint for all $\gamma_1' \leq \gamma_1$ and $\gamma_2' \leq \gamma_2$. Letting $\gamma = \min\{\gamma_1, \gamma_2\}$ we obtain that the sets $N_i(\gamma, \gamma)$ are disjoint for all $\gamma' \leq \gamma$. ∎

Next, following the argument in the text, we find that when γ is small enough, the net value $V_i(a) - pa$ is an increasing affine transformation of $K_i(a)$. Since the optimal asset holding must belong to the *interior* of the inaction region, it must satisfy $K_i'(a) = 0$, which implies in turn that it must coincide with LR's optimal asset holdings. Since the distribution of type must also coincide with that of LR, which satisfies market clearing by construction, it follows that LR's price is an equilibrium price in our economy as long as γ is small enough.

Constructing a competitive search equilibrium. The last thing to do is to verify that the LR price, which we already know clears the market, is the basis of a competitive search equilibrium. We note that, for γ small enough, there are always strict gains from trade when $i \neq j$ and so $\theta_i(a_j) = 1$. The equilibrium objects are then defined in a manner similar to the paragraph following Proposition 3:

$$q_i(a_j) = a_i - a_j, \quad \phi_i(a_j) = \frac{\gamma}{\mu}, \text{ and } \quad \theta_i(a_j) = 1.$$

We let

$$V_i^\star(a) = \max_{k,\ell} V_i\left[a, \sigma_k(a_\ell), \theta_k(a_\ell)\right],$$

where $V_i(a, \sigma, \theta)$ is defined as in equation (2). We will adopt a slightly different definition for market tightness:

$$\Theta(\sigma) = \inf\left\{\theta \geq 0 : V_i(a, \sigma, \theta) \geq V_i^\star(a) \text{ and } V_i^\star(a) > V_i(a, \mathbf{0}, 0) \text{ for some } i \in \mathcal{I}\right\}.$$

In the case when $\alpha(\theta)$ is strictly increasing and strictly concave, this definition is equivalent to the one we used before. It is stronger in the present Leontief case.

We proceed to verify that these candidate equilibrium objects form a competitive search equilibrium. We first show that $\Theta[\sigma_i(a_j)] = \theta_i(a_j)$. First, we note that there are strict gains from trade in LR so, by the continuity arguments already used above, there must be strict gains from trade in our setup when γ is small enough. That is,

$$V_i(a_j, \sigma_i(a_j), \theta_i(a_j)) > V_i(a_j, \mathbf{0}, 0).$$

Given that $V_i(a_j, \sigma_i(a_j), \theta_i(a_j)) = V_i^\star(a_j)$, the definition of $\Theta(\sigma)$ implies that $\Theta[\sigma_i(a_j)] \leq \theta_i(a_j)$. Suppose, toward a contradiction, that the inequality is strict. Then, by definition of $\Theta(\sigma)$, there exists some $\theta \in (0, \theta_i(a_j))$ and some (k, ℓ) such that $V_k(a_\ell, \sigma_i(a_j), \theta) \geq V_k^\star(a_\ell) > V_i(a_\ell, \mathbf{0}, 0)$. Since $V_k(a_\ell, \sigma_i(a_j), \theta) > V_i(a_\ell, \mathbf{0}, 0)$, one sees easily that $\theta \mapsto V_k(a_\ell, \sigma_i(a_j), \theta)$ is strictly increasing in $\theta \in (0, \theta_i(a_j))$. This implies that $V_k(a_\ell, \sigma_i(a_j), \theta_i(a_j)) > V_k^\star(a_\ell)$, which is a contradiction since $V_k^\star(a_\ell)$ is the maximum attainable

utility for this investor.

Finally, we verify the zero-profit conditions of dealers. For this, we first note that, in the Leontief case, the alternative representation of Proposition 1 continues to hold (one can verify that proof goes through almost identically), and so $V_i^\star(a_j)$ solves the Bellman equation (48). Now suppose that there is $\sigma = (q, \phi)$ such that $-\gamma + \frac{\alpha[\Theta(\sigma)]}{\Theta(\sigma)}\phi > 0$. Then $\Theta(\sigma) < \infty$ and there is some type (i, a_j) and some tightness θ such that $V_i(a_j, \sigma, \theta) \geq V_i^\star(a_j) > V_i(a, \mathbf{0}, 0)$ and $-\gamma + \frac{\alpha(\theta)}{\theta}\phi > 0$. Consider then the contract $\hat{\sigma}$, with $\hat{q} = q$ and $\hat{\phi}$ chosen such that $\gamma = \frac{\alpha(\theta)}{\theta}\hat{\phi}$. Because $\hat{\phi} < \phi$, we have that $V_i(a_j, \hat{\sigma}, \theta) > V_i^\star(a_j)$, which can be written

$$\frac{u_i(a_j) + \delta \sum \pi_{ik} V_k^\star(a_k) + \alpha(\theta)\left[V_i^\star(a_j + \hat{q}) - p\hat{q}\right] - \gamma\theta}{r + \delta + \alpha(\theta)} > V_i^\star(a_j),$$

contradicting that $V_i^\star(a_j)$ solves the Bellman equation (48).