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Abstract

This paper proves that the standard quasi-geometric discounting model used in dynamic consumer theory and political economics does not possess Markov perfect equilibria (MPE) with continuous decision rules, if there is a strictly positive lower bound on wealth. It is shown that these discontinuities imply that the decision-maker strictly prefers lotteries over next period's assets. An extension with lotteries is presented, and the existence of an MPE with continuous decision rule is established. The models with and without lotteries are numerically compared, and some appealing properties of the lottery-enhanced model are noted.

Keywords: quasi-geometric, quasi-hyperbolic, time consistency, Markov perfect equilibrium, debt limit, continuous solutions, lotteries

JEL Codes: C73, D11, D90, E21, H63, P16

1 Introduction

Intertemporal preferences with quasi-geometric (or quasi-hyperbolic) discounting have been proposed for studying optimal national savings policy with imperfect altruism (Phelps and Pollak (1968)), for studying the savings behavior of individuals who value commitment to a consumption plan (Laibson (1997)) and for studying optimal growth outside of the strait-jacket of geometric discounting (Barro (1999)). In addition, quasi-geometric discounting may arise endogenously in models of collective decision-making. Examples include models of political turnover and disagreement (Alesina and Tabellini (1990) and Persson and Svensson (1989)), models of majority-based legislative decision-making (Battagliani and Coate (2008)), and nonunitary models of household decision-making (Hertzberg (2012)).[1]

Decision problems with quasi-geometric discounting must be analyzed as a dynamic game because of the implied time inconsistency of intertemporal preferences (Strotz (1956)). This paper is motivated, in part, by a puzzle regarding the computation of Markovian equilibria of this game. Consider a decision-maker (henceforth DM) with an infinite planning horizon, constant endowment, and facing a constant gross interest rate equal to the inverse of the (long-run) discount factor between any two future consecutive periods, β. The (short-run) discount factor between the current period and the next is $\delta\beta$ ($\delta < 1$). In this situation, the "present bias" introduced by δ should cause DMs to persistently dissave. Indeed, for models in which a closed-form Markovian solution can be found, the solution displays continuous (and smooth) dissaving behavior. However, when the equilibrium of the same model is computed on a grid using value (and policy) function iteration, the decision rule found has multiple points of discontinuity, and these points are typically stationary points as well (i.e., steady states with no dissaving). This is true even when the analytical solution is fed in as the initial guess.

The first contribution of this paper is to show that any Markovian decision rule of this stationary environment with CRRA preferences must be discontinuous, if the DM's net wealth

[1] Several recent models with political frictions (Aguiar and Amador (2011) and Azzimonti (2011)) feature versions of "present-bias" characteristic of quasi-geometric discounting.

(i.e., the present value of endowments less debt) cannot fall below a strictly positive value.[2] This has two implications: First, the known continuous (and smooth) analytical solutions for stationary environments work because the DM's wealth is allowed to get arbitrarily close to zero. Second, it explains why computed solutions feature discontinuities even when there is a known continuous and smooth analytical solution. Since value function iteration is done on a grid, the method imposes a de facto lower bound on wealth. Thus, the computation yields a discontinuous solution because the model being computed has discontinuous solutions only.[3]

The second contribution is motivated by the fact that a positive lower bound on wealth is a natural assumption in many applications, for instance, when the DM is an individual facing a borrowing constraint. The discontinuity of Markovian decision rules is, then, an intrinsic property of the equilibrium. However, it is shown that these discontinuities always reflect nonconcave segments of the continuation value function and, consequently, actuarially fair lotteries (over next period's asset choice) raise ex-ante welfare of the DM.[4] Based on this, an extension of the standard quasi-geometric model to lotteries is presented. The extension has several important consequences. First, Markovian decision rules for consumption and *mean* asset choice (i.e., the expected value of next period's asset level) are now continuous and monotone functions of wealth. Second, Markovian decision rules for the lottery-enhanced

[2]Krusell and Smith (2003) present an algorithm for constructing a continuum of discontinuous Markovian decision rules for the neoclassical growth model. In the linear case, their construction requires that the gross interest rate strictly exceed $1/\beta$ and, thus, does not apply to the environment of this paper. Furthermore, they do not prove (or suggest) that discontinuities are a necessary feature of Markovian decision rules.

[3]In the past, the occurrence of discontinuous (computed) solutions when smooth solutions were expected was interpreted as an instance of multiple equilibria. Perhaps guided by this assessment, researchers have been content to restrict attention to parameter values for which discontinuous solutions ("pathologies") do not arise (Laibson, Repetto, and Tobacman (1998)) or have adopted methods other than finite-state value function iteration to locate smooth solutions (Krusell, Kuruscu, and Smith (2002), Judd (2004), Maliar and Maliar (2005)).

[4]The fact that time-inconsistent preferences may imply nonconcave value functions was shown in the working paper version of Luttmer and Mariotti (2007) for a three-period exchange economy (STICERD, Discussion Paper TE/03/446, January 2003). Since actuarially fair lotteries are equivalent to fair-value gambles (Cole and Prescott (1997)), this result points to an underappreciated implication of quasi-geometric discounting. Behavioral justifications of quasi-geometric discounting invariably point to the use of commitment strategies by individuals to "bind" their future selves. But individuals also willingly take risks (gamble to get out of poverty, for instance), and such behavior is an implication of quasi-geometric discounting as well.

model exist under general conditions.[5] Third, Markovian decision rules can be computed with relative ease because the decision rule for mean asset choice can be computed directly (without having to keep track of the lotteries underlying these choices).

The final contribution is a numerical comparison of the lottery-enhanced model with the model without lotteries. Three findings emerge from this comparison. First, for the model for which a closed-form decision rule is known, the lottery-enhanced model solution closely tracks the closed-form solution for asset levels that are far from the debt limit. In contrast, the solution of the standard model on the same grid features a discontinuous decision rule with very different choices. Thus, the lottery-enhanced model can be a good approximation of the model without a positive lower bound on wealth for a (large) range of asset levels.

Second, i.i.d. shocks to endowments can serve to almost concavify the continuation value function, provided the shocks are volatile enough and/or the present bias is small. In such situations, the decision rule from the model without lotteries is seemingly continuous and identical to the decision rule from the lottery-enhanced model (lotteries are effectively superfluous if the continuation value function is almost concave). This finding is in line with Harris and Laibson's (2001) limit result that when endowments are drawn each period from a distribution with continuous density, the Markovian decision rule is continuous in current wealth, provided δ is sufficiently close to 1.

Third, the two models give different answers to an important policy question: Is there an ideal debt limit from the perspective of a current DM with no inherited debt? The answer from the lottery-enhanced model is intuitive: Because of the present bias, there is an ideal debt limit that allows the current DM to indulge its preference for current consumption (and to which it goes immediately). In contrast, there are many debt limits that deliver roughly the same utility to the current DM in the standard model. These debt limits share the property that the location of the *first* stationary point (where DMs will eventually settle) is roughly the same for all of them (the lottery-enhanced model has a unique stationary point

[5] With time-inconsistent preferences, existence of a Markov equilibrium is not guaranteed (Peleg and Yaari (1973)). The existence proofs for the stationary environment given in Laibson (1996) employs a guess-and-verify method that hinges on the lower bound on wealth being zero and the utility function being CRRA.

at the debt limit).

The paper is organized into three parts. The first part, Section 2, presents and analyzes the standard stationary quasi-geometric discounted model with a positive lower bound on wealth. The second part, Section 3, gives the extension to lotteries. The third part, Sections 4–6, presents the numerical comparisons of the model with and without lotteries. The proofs of all lemmas and theorems appear in Appendix A.

2 The Standard Quasi-Geometric Discounting Model

We study a canonical discrete-time, infinite-horizon, intertemporal choice problem with quasi-geometric discounting.[6] The DM has a constant stream of endowment y and a time-additive utility function with a per-period utility function $U(c) = c^{1-\gamma}/(1-\gamma), \gamma > 0$ and $c \geq 0$. The (long-run) discount factor between any two future consecutive periods is $\beta < 1$, while the discount factor between the current and next period is $\delta\beta$, $\delta < 1$. The DM has access to a financial market, in which it can save or borrow subject to a debt limit $\underline{b} < 0$. The price of a unit bond or deposit is β. Importantly, $y + (1-\beta)\underline{b} = \kappa > 0$, which implies that the DM can sustain a strictly positive level of consumption at the debt limit. We follow convention (Laibson (1997) and Krusell and Smith (2003)) and focus on pure-strategy Markov perfect equilibria (MPE), i.e., equilibria in which the current choice of assets is a function of inherited assets.[7]

Let $W(b)$ be the *continuation value* to a DM starting next period with assets b. Then, the current DM's decision problem is:

$$\max_{b' \geq \underline{b}} U(c) + \delta\beta W(b') \tag{1}$$
$$\text{s.t. } c = y + b - \beta b' \geq 0.$$

[6]Under some conditions, the problem is isomorphic to a model of government expenditure choice in which two parties with different per-period utility functions switch power with equal probability. This equivalence is shown in Appendix B.

[7]See Maskin and Tirole (2001) for a general definition of MPE.

Definition 1 *A Markov perfect equilibrium (MPE) is a pair* $\{a(b), W(b)\}$ *such that* $a(b)$ *solves (1) given* $W(b)$ *and* $W(b)$ *solves* $W(b) = U(y + b - \beta a(b)) + \beta W(a(b))$ *given* $a(b)$*.*

Definition 2 *A stationary point (steady state) of an MPE is a* b *such that* $a(b) = b$*.*

Given that the interest rate is equal to the long-run discount rate, the decision rule that maximizes the value of $W(b)$ for each b is $a(b) = b$. For this rule, $c = y + (1 - \beta)\, b$, and $W(b)$ is $U(y + (1 - \beta)\, b)/\,(1 - \beta)$. If discounting were geometric, this is also the rule the current DM would choose. However, with $\delta < 1$, $a(b) = b$ cannot be supported as an MPE: For the associated $W(b)$, the DM would strictly prefer to choose a $b' < b$ for any $b > \underline{b}$. This illustrates the time inconsistency introduced by quasi-geometric discounting: The current DM would like *future* DMs to follow $a(b) = b$ but would prefer to dissave now itself.

The present bias due to $\delta < 1$ implies that there cannot be any savings in any MPE. If $a(b) > b$ for any b, the DM could get strictly higher utility by choosing b this period, and let the next period's DM do the saving (i.e., let next period's DM choose $a(b) > b$). Although the current DM dislikes saving in the current and future periods, it dislikes it more in the current period, given $\delta < 1$, and so would prefer to postpone the saving behavior.

Theorem 1 *In any MPE,* $a(b) \leq b$*, and* $a(b)$ *is increasing in* b*.*

An important implication is that the debt limit is a steady state of any MPE.

Corollary 1 $\underline{b}$ *is a steady state of any MPE.*

Since $a(b) \leq b$ implies $c(b) \geq y + (1 - \beta)b$, recalling the definition of κ gives a second useful corollary:

Corollary 2 *In any MPE,* $c(b) \geq \kappa > 0$*.*

2.1 Discontinuity of MPE Decision Rules

The main goal of this section is to establish that any MPE decision rule $a(b)$ will fail to be continuous on the entire domain. We begin with a result that states that there is a right neighborhood of $\underline{b}$ in which, in any MPE, it is optimal for the DM to choose $b' = \underline{b}$. Let b_{1M} satisfy:

$$U'(y + b_{1M} - \beta\underline{b}) = \delta U'(y + (1-\beta)\underline{b}). \tag{2}$$

Since $U'(c)$ is strictly decreasing in c and $\delta < 1$, $b_{1M} > \underline{b}$. Then, we have:

Theorem 2 *In any MPE, $a(b) = \underline{b}$ for $b \in (\underline{b}, b_{1M}]$.*

To understand why the theorem holds, consider the DM's optimal action if it could bind future DMs to follow any action it wanted and there is no debt limit. Since $W(b)$ is maximized with $a(b) = b$, the DM would want future DMs to choose $a(b) = b$ and would, therefore, choose b' so that $U'(y + b - \beta b') = \delta U'(y + (1-\beta)b')$. This "full-commitment" program is not available to the DM, except when $b = b_{1M}$: In this case, the DM can indulge its present bias and choose $\underline{b}$, and the debt limit ensures that all future DMs will set $a(\underline{b}) = \underline{b}$. When $b < b_{1M}$, the DM would like to choose b' lower than $\underline{b}$, but the debt limit prevents that and its next best option is to choose $\underline{b}$. Notice that the lower is δ, the wider the neighborhood is as the desire to dissave is stronger.

The next lemma further characterizes an MPE under the *assumption* that the decision rule is continuous. If $a(b)$ is continuous, there must be an $\epsilon > 0$, such that for all $b \in (b_{1M}, b_{1M} + \epsilon]$, $a(b)$ is in $[\underline{b}, b_{1M}]$. For such b, the DM's optimal decision will satisfy

$$U'(y + b - \beta b') = \delta U'(y + b' - \beta\underline{b}), \tag{3}$$

since the DM must expect the next period's DM to choose $\underline{b}$ (Theorem 2). Using the parametric form for U, the above FOC implies that for $b \in (b_{1M}, b_{1M} + \epsilon)$,

$$a(b) = \frac{\left(-\left(1-\delta^{1/\gamma}\right) y + \delta^{1/\gamma} b + \beta \underline{b}\right)}{1+\delta^{1/\gamma}\beta}. \quad (4)$$

Denote the r.h.s. of the above equation by $h(b)$. Note that $h(b_{1M}) = \underline{b}$, and $h(b)$ is strictly increasing in b. Therefore, there is a $b_{2M} > b_{1M}$ for which $h(b_{2M}) = b_{1M}$. Hence, for all $b \in (b_{1M}, b_{2M})$, $h(b) \in (\underline{b}, b_{1M})$. This fact strongly suggests that if $a(b)$ is continuous, it must coincide with $h(b)$ for $b \in (b_{1M}, b_{2M})$. The lemma verifies this conjecture.

Lemma 1 *In any MPE with continuous $a(b)$, $a(b) = h(b)$ for all $b \in (b_{1M}, b_{2M})$.*

Since $h(b)$ is continuous, the following holds:

Corollary 3 *In any MPE with continuous $a(b)$, $a(b_{2M}) = h(b_{2M}) = b_{1M}$.*

Thus far, $a(b)$ has been pinned down over $[\underline{b}, b_{2M}]$ under the assumption that $a(b)$ is continuous. Notice that the function has a kink at b_{1M}. The slope of $a(\cdot)$ below b_{1M} is 0, and the slope above b_{1M} is $\delta^{1/\gamma}/\left(1+\delta^{1/\gamma}\beta\right)$. This kink in the decision rule implies that at $b = b_{2M}$, the DM would be strictly better off choosing a $b' > b_{1M} = h(b_{2M})$. To understand why, consider a DM with $b = b_{2M}$. If it chooses a $b' \leq b_{1M}$, it knows that next period's DM will choose $\underline{b}$. But if the DM chooses $b' \in (b_{1M}, b_{2M})$, it knows that by increasing its asset choice it can increase the asset choice of next period's DM at the rate $\delta^{1/\gamma}/\left(1+\delta^{1/\gamma}\beta\right)$. Now recall that from the current DM's perspective, future DMs save too little (the current DM would like future DMs to hold on to their wealth). Given this, the option to increase the savings of next period's DM is valuable to the current DM, and it would strictly prefer to choose $b' > b_{1M}$. This, however, contradicts the continuity of $a(b)$. Thus:

Theorem 3 *In any MPE, $a(b)$ cannot be continuous for all b.*

It is worth emphasizing that Theorem 3 establishes only that $a(b)$ cannot be continuous everywhere. It does not establish that $a(b)$ must be discontinuous at b_{2M} since that implication was derived under the *assumption* that $a(b)$ is continuous, in particular, that it is continuous on $(b_{1M}, b_{2M}]$. But that need not be the case. Indeed, the next section gives conditions under which an MPE has a discontinuity in (b_{1M}, b_{2M}).

An implication of Theorem 3 is that $W(b)$ will fail to be continuous as well. The reason is that, in any MPE, the fact that consumption is bounded below by $\kappa > 0$ implies that the equilibrium payoff at b, $V(b) = U(y + b - \beta a(b)) + \delta\beta W(a(b))$ must be continuous in b. A continuous $V(b)$ and a discontinuous $a(b)$ imply that $W(b)$ must be discontinuous somewhere in its domain.

Theorem 4 *In any MPE, $V(b)$ is strictly increasing and continuous in b and $W(b)$ cannot be continuous for all b.*

2.2 Discontinuous Decision Rule and Multiple Steady States

The fact that MPE decision rules are not continuous everywhere leads to a new channel through which debt levels other than $\underline{b}$ can be sustained as a steady state.

We characterize one type of nondebt-limit steady state that might arise. To do so, it is convenient to define $\Omega_2(b) : [\underline{b}, \infty) \to \mathbb{R}$ as:

$$\Omega_2(b) = \max_{b'} \left\{ U\left(y + b - \beta b'\right) + \delta \left[\beta U\left(y + b' - \beta \underline{b}\right) + \beta^2 \frac{U\left(y + (1-\beta)\,\underline{b}\right)}{(1-\beta)} \right] \right\}$$
$$\text{s.t. } \underline{b} \leq b' \leq b_{1M}$$

and $\Omega_3(b) : [\underline{b}, \infty) \to \mathbb{R}$ as:

$$\Omega_3(b) = \max_{b'} U\left(y + b - \beta b'\right) +$$
$$\delta \left[\beta U\left(y + b' - \beta h\left(b'\right)\right) + \beta^2 U\left(y + h(b') - \beta \underline{b}\right) + \beta^3 \frac{U\left(y + (1-\beta)\,\underline{b}\right)}{(1-\beta)} \right]$$
$$\text{s.t. } b_{1M} \leq b' \leq b_{2M}.$$

Then, $\Omega_2(b)$ is the payoff to the DM if it chooses $b' \in [\underline{b}, b_{1M}]$ and knows that the next period's

DM will choose $\underline{b}$, and $\Omega_3(b)$ is the payoff to the DM if it chooses $b' \in [b_{1M}, b_{2M}]$ and believes that next period's DM will choose b' according to the function $h(b)$ (and, therefore, knows that the DM in the period after next will choose $\underline{b}$). Then, our candidate for a nondebt-limit steady state is a b^* such that

$$\left(1 + \frac{\beta\delta}{1-\beta}\right) U(y + (1-\beta)\, b^*) = \Omega_2(b^*) \geq \Omega_3(b^*). \tag{5}$$

The left-most expression is the steady-state payoff for b^*. This payoff must equal the payoff from going to the debt limit in two periods, and it must weakly dominate the payoff from going to the debt limit in three periods. Such a b^* may not exist, but if it does, then:

Theorem 5 *If $b^* \in (\underline{b}, \infty)$ satisfying (5) exists, then $b^* \in (b_{1M}, b_{2M})$. If $a^*(b)$ is the decision rule of an MPE of the environment where b is restricted to be in $[b^*, \infty)$, then:*

$$a(b) = \begin{cases} \underline{b} & \textit{for } b \in [\underline{b}, b_{1M}] \\ h(b) & \textit{for } b \in (b_{1M}, b^*) \\ b^* & \textit{for } b = b^* \\ a^*(b) & \textit{for } b \in (b^*, \infty) \end{cases} \tag{6}$$

is an MPE decision rule.

Some aspects of this theorem are noteworthy. First, the $h(b)$ is the same function as in equation (4), but now it applies over the smaller interval $[\underline{b}, b^*)$. And since $h(b) < b_{1M}$ for $b < b_{2M}$ and $b^* > b_{1M}$, $a(b)$ features an upward jump (discontinuity) at b^*. The discontinuity is the key as to why b^* can be sustained as a steady state. The DM does not indulge its preference for current consumption because it knows that if it dissaved slightly today, next period's DM will dissave substantially. Since future DMs dissave too much from the current DM's perspective, this behavior is not desirable from the current DM's perspective. Thus, the current DM weakly prefers to choose b^* rather than to dissave.

Second, the theorem asserts that any MPE decision rule of the environment in which asset choices are restricted to lie in $[b^*, \infty)$ can be grafted onto the first three rows of equation (5) to constitute an MPE decision rule. This is possible because for $b \leq b^*$, the DM would prefer to choose b^* over anything greater than b^* regardless of what $a(b)$ is for $b > b^*$, and for $b > b^*$, the DM would prefer to choose b^* over anything less than b^* regardless of what $a(b)$ is for $b < b^*$.

Third, the existence of a nondebt-limit steady state opens up the possibility of other such steady states. For a DM with $b > b^*$, b^* effectively serves as a hard debt limit. Suppose, then, there exists $b^{**} > b^*$ that satisfies a condition analogous to (5) with b^* replaced by b^{**} and $\underline{b}$ replaced by b^* in the definitions of b_{1M}, b_{2M}, Ω_2, and Ω_3. Then, b^{**} can be supported as a steady state without altering the status of b^* as a steady state. This process can (potentially) be carried out many times and explains why the finding of multiple steady states is common in computations (note that steady-state b's need not be negative, i.e., the "debt limits" could simply be a lower bound on asset holdings).

2.3 Discontinuous Decision Rule and Benefit of Lotteries

The continuity of the payoff function $V(b)$ (Theorem 4) implies that at any point of discontinuity of $a(b)$, the DM is indifferent between two pairs of consumption and continuation asset levels $\{c(b'_1), b'_1\}$ and $\{c(b'_2), b'_2\}$ where $b'_1 \neq b'_2$ and so $c(b'_1) \neq c(b'_2)$. Such indifference generates a demand for lotteries.

Any risk-neutral intermediary would be willing to engage in a lottery in which it offers a loan of $\beta\,(\lambda b'_1 + (1-\lambda)\, b'_2)$ and, in the next period, demands b'_1 with probability λ and b'_2 with probability $(1-\lambda)$. For the DM, this lottery provides a *certain* consumption of $\lambda c(b'_1) + (1-\lambda)\, c\,(b'_2)$ and a *random* continuation value with an expectation of $\lambda W(b'_1) + (1-\lambda)\, W\,(b'_2)$. By the strict concavity of $U(\cdot)$, $U(\lambda c(b'_1) + (1-\lambda)\, c\,(b'_2)) + \lambda\delta\beta W(b'_1) + (1-\lambda)\,\delta\beta W\,(b'_2) > \lambda[U(c(b'_1)) + \delta\beta W(b'_1)] + (1-\lambda)\,[U(c\,(b'_2) + \delta\beta W\,(b'_2)] = V(b)$ and the lottery strictly dominates following either of the two deterministic paths. Thus:

Theorem 6 *Let $\{a(b), W(b)\}$ be an MPE. Then, fair-priced lotteries strictly dominate the* *associated pure strategies at any point where $a(b)$ is discontinuous.*

Another way to understand the value of lotteries is the following. Suppose, again, that at b, the DM is indifferent between two pairs of consumption and continuation asset levels $\{c(b_1'), b_1'\}$ and $\{c(b_2'), b_2'\}$. Since the choice set of the DM is convex, $b' = \lambda b_1' + (1-\lambda)\, b_2'$ is available to the DM at b and, so, must be weakly dominated by b_1' or b_2'. Then, $V(b) \geq U(\lambda c(b_1') + (1-\lambda)\, c\,(b_2')) + \delta\beta W(\lambda b_1' + (1-\lambda)\, b_2')$. But (as shown above) concavity of U implies, $U(\lambda c(b_1') + (1-\lambda)\, c\,(b_2')) + \lambda\delta\beta W(b_1') + (1-\lambda)\, \delta\beta W\,(b_2') > V(b)$. Together, these imply $\lambda W(b_1') + (1-\lambda)\, W\,(b_2') > W(\lambda b_1' + (1-\lambda)\, b_2')$. Thus, $W(\cdot)$ is strictly convex between b_1' and b_2'. Lotteries allow the DM to replace convex segments of $W(\cdot)$ by that segment's concave upper envelope, leading to higher (expected) continuation values for a given level of current consumption.

We close this section with a comment on the difficulty of computing equilibria of the standard model. The issue is that the functions $a(\cdot)$ and $W(\cdot)$ cannot be accurately interpolated by standard techniques as both functions will have one or more points of discontinuity and the locations of these points are not known in advance. On the other hand, if the space $[\underline{b}, \infty)$ is approximated by a grid, there is no assurance that the equations for $a(b)$ and $W(b)$ will have a solution on the grid. This computational awkwardness of the quasi-geometric model is a second motivation for developing the model with lotteries.

3 Quasi-Geometric Discounting Model with Lotteries

We extend the environment of the previous section to permit lotteries over next period's level of assets. We characterize equilibria for this lottery-enhanced environment, establish the existence of at least one equilibrium, and discuss the key steps in the computation of an equilibrium.

Both the proof of existence of an equilibrium and its computation require that we restrict ourselves to a bounded choice set. Henceforth, we will assume that the set of possible asset choices belongs to the closed interval $[\underline{b}, \bar{b}]$, where $\underline{b} \geq -y/(1-\beta)$. If $\underline{b}$ is equal to its lower bound, we require that U be well defined over $c \geq 0$. Otherwise, it is sufficient that U be well defined on $c > 0$. In either case, we will assume that $U(c)$ is strictly increasing, strictly concave, and differentiable on the relevant domain.

Let $\mathscr{B}$ denote the Borel σ-algebra generated by the set $[\underline{b}, \bar{b}] \subset \mathbb{R}$. Let ϕ denote a finite measure on the measurable space $([\underline{b}, \bar{b}], \mathscr{B})$ and let Φ denote the set of all such measures. Let $\mathscr{W}$ be the set of all bounded functions defined on $[\underline{b}, \bar{b}]$ that are measurable with respect to $\mathscr{B}$. Given a continuation value function $W(b) \in \mathscr{W}$, the DM solves the following dynamic program:

$$V(b) = \sup_{\phi \in \Phi} U\left(y + b - \beta \int b' \phi(db')\right) + \delta\beta \int W(b')\phi(db') \tag{7}$$
$$\text{s.t.} \int \phi(db') = 1 \text{ and } y + b - \beta \int b'\phi(db') \geq 0.$$

Definition 3 *A Markov perfect equilibrium with lotteries (MPEL) is a pair $\{\phi(b', b), W(b)\} \in \Phi \times \mathscr{W}$ such that the (probability) measure $\phi(b', b)$ solves (7) given $W(b)$, and $W(b)$ solves*

$$W(b) = U\left(y + b - \beta \int b'\phi(db', b)\right) + \beta \int W(b')\phi(db', b),$$

given $\phi(b', b)$.

The probability measure in the definition makes it seem that an MPEL is a challenging equilibrium object to characterize and compute. However, the optimization problem can be reformulated so that $\phi(b', b)$ does not appear explicitly. To do this, consider the problem of choosing the best lottery with a given expected value B; namely,

$$w(B; W) = \sup_{\phi \in \Phi} \int W(b)\phi(db) \tag{8}$$
$$\text{s.t.} \int \phi(db) = 1 \text{ and } \int b\phi(db) = B.$$

Since $W(b) \in \mathscr{W}$, $w(B; W)$ is well defined, and a lottery attaining the supremum generally exists.[8] Then, the following characterization $w(B;W)$ holds:

Lemma 2 *For any $W \in \mathscr{W}$, $w(B;W) : [\underline{b}, \bar{b}] \to \mathbb{R}$ is the concave upper envelope of $W(b)$, i.e., $w(B;W)$ is the least concave function that majorizes $W(b)$.*

The optimization problem (7) can now be restated as follows:

$$V(b) = \max_{B'} U(y + b - \beta B') + \delta\beta w(B';W) \tag{9}$$
$$\text{s.t. } B' \in [\underline{b}, \bar{b}] \text{ and } y + b - \beta B' \geq 0.$$

Then, we have the following equivalent definition of an MPEL:

Definition 4 *An MPEL is a pair $\{A(b), W(b)\}$ such that $A(b)$ solves (9) given $W(b)$, and $W(b)$ solves $W(b) = U(y + b - \beta A(b)) + \beta w(A(b);W)$ given $A(b)$, where $w(\cdot;W)$ is the concave upper envelope of $W(b)$ (equivalently, $w(\cdot;W)$ solves (8)).*

This definition parallels the definition of an MPE in the standard quasi-geometric discounting model. The main differences are that $w(\cdot;W)$ replaces $W(b)$ in the optimization problem and, correspondingly, the r.h.s. of the recursion for $W(b)$ has $w(\cdot;W)$. Lotteries are implicit in these replacements but otherwise do not make an explicit appearance.

3.1 Characterization

The concavity of $w(\cdot;W)$ implies well-behaved decision rules for any admissible $W(b)$. This fact makes the lottery-enhanced model analytically and computationally facile.

Theorem 7 *For any $W \in \mathscr{W}$, $A(b)$ and $c(b)$ are continuous and increasing in b, and $A(b)$ is Lipschitz with constant $1/\beta$.*

[8] For this class of problems, it is generally sufficient to optimize over the space of (atomic) measures that assign mass to as many points as there are constraints — in this case, two (see, for instance, Glashof and Gustafson (1983) and Hornstein and Prescott (1993)). If the choice set is restricted to two-point lotteries, it is easy to show that a lottery attaining the supremum exists.

The analog of Theorem 1 holds here as well. However, since the bound applies to $A(b)$, *realized* b' (given $A(b)$) may exceed b.

Theorem 8 *In any MPEL, $A(b) \leq b$.*

A steady state now is a debt level b such that the DM chooses b with probability 1. Since a feasible lottery cannot put any probability mass on values of $b < \underline{b}$, the analog of Corollary 1 holds:

Corollary 4 *$\underline{b}$ is a steady state of any MPEL.*

The analog of Corollary 2 also holds.

Corollary 5 *In any MPEL, $c(b) \geq y + (1 - \beta)b \geq \kappa > 0$.*

One important difference between the standard model and the model with lotteries is that, in any MPEL, $W(b)$ is both continuous and monotone (but not generally concave).

Theorem 9 *In any MPEL, $W(b)$ is continuous and increasing in b.*

3.2 Existence

Let $\mathscr{C}$ be the space of continuous functions defined on $[\underline{b}, \bar{b}]$. Let $\|f\| = \sup_{\underline{b} \leq b \leq \bar{b}} |f(b)|$ be the norm on $\mathscr{C}$. Then $\mathscr{C}$ is a complete normed vector (Banach) space. Let $\mathscr{F} = \{f \in \mathscr{C} : f(b) \in [\underline{b}, \bar{b}]\}$. Then $\mathscr{F}$ is a nonempty, bounded, closed, and convex subset of $\mathscr{C}$.

Lemma 3 *For every $A \in \mathscr{F}$, there exists a unique $W(b; A) \in \mathscr{C}$ that solves $W(b; A) = U(y + b - \beta A(b)) + \beta w(A(b); W(b; A))$. Furthermore, $W(b; A)$ is continuous in A.*

Then:

Theorem 10 *An MPEL exists.*

3.3 Computation

The model is computed via discretization. Let $\{b_1, b_2, \ldots, b_N\} \subset [\underline{b}, \bar{b}]$ be the chosen grid. Suppose that $\{W^k(b_i)\}_{i=1}^N$ is the W function delivered by the $(k-1)$th iteration. By Lemma 7, $w(B; W^k)$ is the concave upper envelope of $\{W^k(b_i)\}_{i=1}^N$. Viewed as N pair of points in $\mathbb{R}^2$, the concave upper envelope of $\{W^k(b_i)\}_{i=1}^N$ is the upper convex hull of these points. Several algorithms exist to compute convex hulls. The one used here is Andrew (1979), which goes as follows:

Let $P^k \subseteq \{b_i, W^k(b_i)\}_{i=1}^N$ be the points that constitute the upper convex hull, with the elements in ascending order by b. P^k is built up recursively. Suppose the recursion is at the stage in which the last element of P^k is $(b_j, W^k(b_j))$. Consider the point $(b_{j+1}, W^k(b_{j+1}))$. If the line segment connecting $(b_j, W^k(b_j))$ to $(b_{j+1}, W^k(b_{j+1}))$ does not make a clockwise turn relative to the line segment connecting the last two elements of P^k, drop $(b_j, W^k(b_j))$. If, after the drop, P^k contains at least two elements, compare the segment connecting the new last element of P^k to $(b_{j+1}, W^k(b_{j+1}))$ with the line segment connecting the new last two elements of P^k. If the former does not make a clockwise turn relative to the latter, drop the (new) last element of P^k. Continue this way until either the line segment connecting the last element of P^k to $(b_{j+1}, W^k(b_{j+1}))$ does make a clockwise turn relative to the line segment connecting the last two elements of P^k or there is only one element in P^k. At this point, include $(b_{j+1}, W^k(b_{j+1}))$ in P^k and proceed to $(b_{j+2}, W^k(b_{j+2}))$ and repeat these steps. The recursion begins with $P^k = \{(b_1, W^k(b_1)), (b_2, W^k(b_2))\}$ and finishes when $(b_N, W^k(b_N))$ is included in P^k.

Given $\{P^k\}$, $A(b_i; W^k)$ is computed as follows. Suppose that P^k has $J \in \{2, \ldots, N\}$ elements. Denote these by $\{p_j, W^k(p_j)\}_{j=1}^J$. Let $\{s_j\}_{j=1}^{J-1}$, be the slopes between the points j and $j+1$, $j = 1, 2, \ldots, J-1$. By construction, $s_1 > s_2 > \ldots > s_{J-1}$. Then:

1. Evaluate $U'(y + b_i - \beta b_1)$ (the marginal utility of current consumption if $B' = b_1$). If $U'(y + b_i - \beta b_1) \geq \delta s_1$, set $A(b_i; W^k) = b_1$ and stop.

2. Otherwise, continue from $j = 2$ though $J - 1$ until either of the conditions below is satisfied:

   (a) If $U'(y + b_i - \beta b_j) \geq \delta s_{j-1}$ and $U'(y + b_i - \beta b_{j-1}) \leq \delta s_{j-1}$, solve for $\hat{B} \in [p_{j-1}, p_j]$ such that $U'(y + b_i - \beta \hat{B}) = \delta s_{j-1}$, set $A(b_i; W^k) = \hat{B}$ and stop.[9]

   (b) If $U'(y+b_i-\beta b_j) < \delta s_{j-1}$ and $U'(y+b_i-\beta b_j) > \delta s_j$, set $A(b_i; W^k) = b_j$ and stop.

3. If the above conditions are not satisfied for any $j < J$, then for $j = J$ check condition (a) above and, if true, set $A(b_i; W^k)$ accordingly. If false, set $A(b_i; W^k) = b_J$.

Given $\{A(b_i; W^k), W^k\}$, set $W^{k+1}(b_i) = (1 - \zeta)W^k(b_i) + \zeta X^k(b_i)$, where

$$
\begin{aligned}
X^k(b_i) &= U(y + b_i - \beta A(b_i; W^k)) \\
&+ \beta \begin{cases} W^k(b_1) & \text{if } A(b_i; W^k) = b_1, \\ W^k(p_{j-1}) + s_{j-1}(A(b_i; W^k) - p_{j-1}) & \text{if } A(b_i; W^k) \in (p_{j-1}, p_j] \end{cases}
\end{aligned}
$$

and $\zeta \in (0, 1)$ is some chosen "relaxation" parameter. From an initial W^0, the iterations continue until $\max_i |X^k(b_i) - W^k(b_i)| < \epsilon$, for some chosen ϵ (by this point, $\max_i |A(b_i; W^k) - A(b_i; X^k)|$ is generally less than ϵ as well).

4 Smooth and Discrete Solutions

The decision rules from three models with $U(c) = \ln(c)$ are computed and compared. In the first model, there is no strictly positive lower bound on wealth and a closed-form differentiable (smooth) solution is available.[10] In the second model, the asset space is a uniformly spaced

[9] If $\hat{B} \in (p_{j-1}, p_j)$, $A(b_i; W^k)$ is being assigned a value that is "off the grid." What this means is that the DM is choosing the grid p_{j-1} with probability $[p_j - \hat{B}]/[p_j - p_{j-1}]$, and the grid p_j with probability $[\hat{B} - p_{j-1}]/[p_j - p_{j-1}]$. Recall our earlier assertion that only two-point lotteries are needed to attain $w(B; W)$.

[10] As shown in Krusell, Kuruscu, and Smith (2002), the decision rule is

$$a(b) = \left[\frac{R\delta\beta}{(1 - \beta + \delta\beta)} - 1\right]\left(\frac{Ry}{R - 1}\right) + \frac{R\delta\beta}{(1 - \beta + \delta\beta)} b,$$

grid on $[-0.25, 0.10]$. The third model has the same grid as the second model, but lotteries are permitted. In all models, $y = 1$, $\delta = 0.98$, $R = 1.05$, and $\beta = 1/R$.

The top panel of Figure 1 plots the decision rule from the standard model solved on the grid. It confirms that the decision rule is discontinuous with multiple stationary points. The bottom panel plots the closed-form solution along with the two discrete approximations (the 45-degree line is omitted to reduce clutter). For the model with lotteries, we plot $A(b)$, the expected value of next period's assets. When b is away from the debt limit, the closed-form solution and $A(b)$ are quite close. As b approaches the lower bound of the grid, the two decision rules diverge as $A(b)$ goes to the debt limit (its unique stationary point). In contrast, the behavior of a discrete model without lotteries is very different: There are many absorbing values of b', corresponding to all the different nondebt-limit steady states.

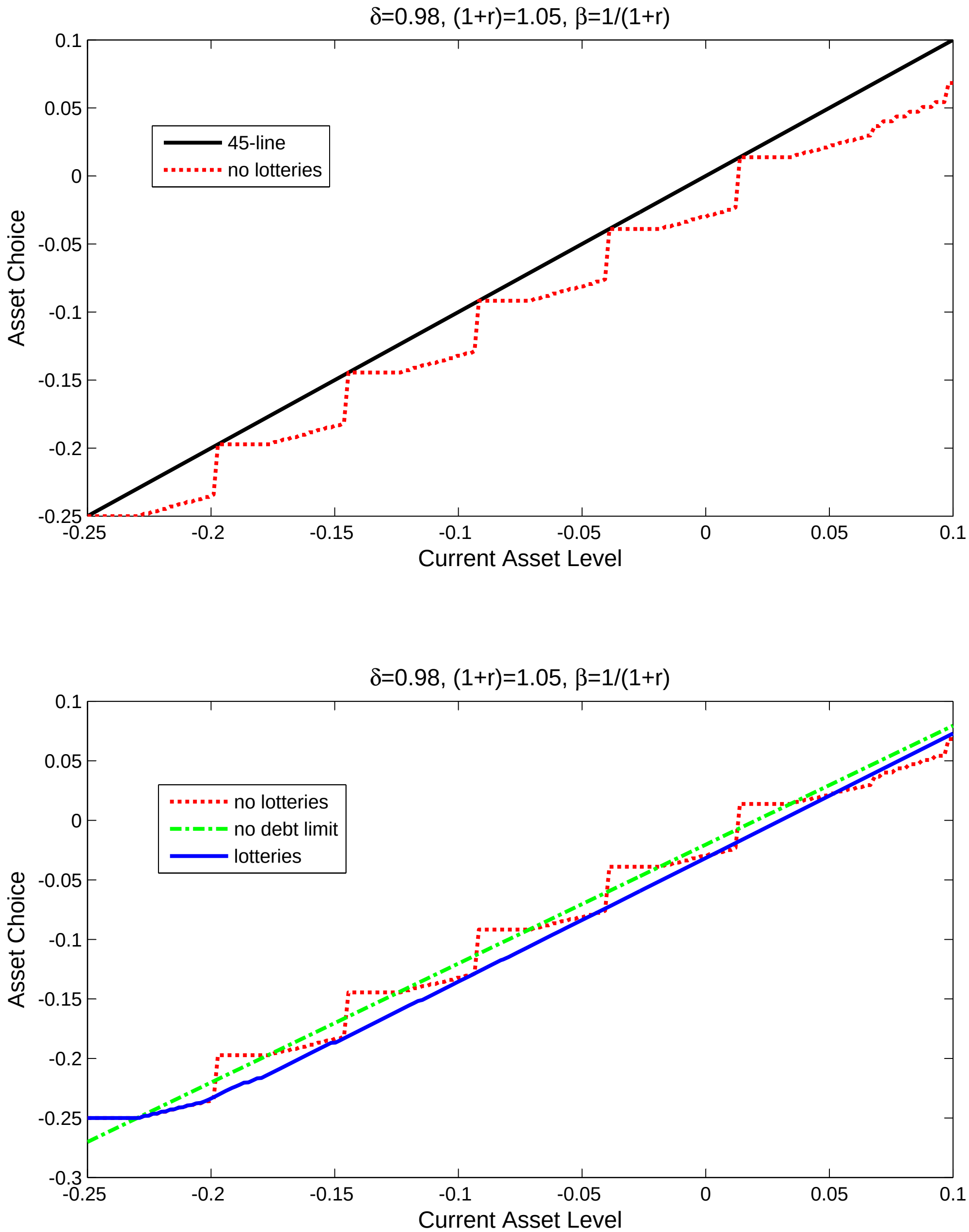
A point to stress is that the closed-form solution cannot be recovered via standard finite-state value/policy function iteration. That solution requires the debt limit to be the negative of the present value of endowments. If $\underline{b}$ is set to this limit, its choice would imply zero consumption and an unboundedly low $V(\underline{b})$. If $\underline{b}$ is set to a slightly larger number, Theorem 3 applies, and the iterations generate a discontinuous decision rule. This fact might explain why locating smooth solutions for quasi-geometric discounting models is often challenging: If the computational method implicitly imposes a lower bound on b *that can actually be chosen*, a smooth solution might fail to exist.

5 Lotteries and Shocks

This section extends the model with lotteries to include shocks to endowments. There are two motivations for this: First, to show that uncertainty can be easily incorporated into the model with lotteries, and, second, to investigate the extent to which shocks can substitute for lotteries in generating continuous decision rules.

where R is the gross interest rate. If $R\beta$=1, $a(b) < b$ and $\lim_{t\to\infty}[a^t(b) - [R/(R-1)]y] = 0$ for any b.

Figure 1


δ=0.98, (1+r)=1.05, β=1/(1+r)
Asset Choice
Current Asset Level
45-line
no lotteries
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
δ=0.98, (1+r)=1.05, β=1/(1+r)
Asset Choice
Current Asset Level
no lotteries
no debt limit
lotteries
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1

Let $F(z, y)$, y, $z \in Y \subset \mathbb{R}$ denote the CDF of next period's endowment z, given the current period's endowment y. Let $W_y(b)$ be the continuation value of a DM that starts next period with assets b, conditional on current endowment y. Then,

$$w(B; W_y) = \sup_{\phi \in \Phi} \int W_y(b)\phi(db) \tag{10}$$

$$\text{s.t.} \int \phi(db) = 1 \text{ and } \int b\phi(db) = B,$$

$$A_y(b) = \operatorname*{argmax}_{B'} U(y + b - \beta B') + \delta\beta w(B'; W_y) \tag{11}$$

$$\text{s.t. } B' \in [\underline{b}, \bar{b}] \text{ and } y + b - \beta B' \geq 0,$$

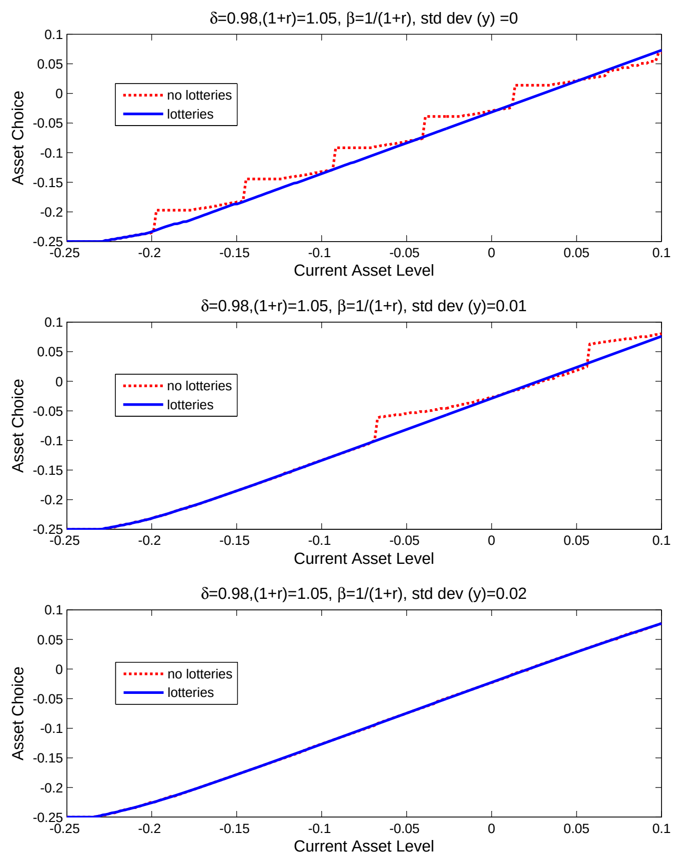
$$W_y(b) = \int [U(z + b - \beta A_z(b)) + \beta w(A_z(b); W_z)] F(dz, y), \text{ for all } y \in Y. \tag{12}$$

An MPEL is a collection $\{A_y(b), W_y(b)\}$, $y \in Y$, that satisfies (10)–(12). The main modification, thus, is that in the first-stage optimization problem (10), there is a $W_y(b)$ for each y. If Y is a discrete set, (11) is solved for each y and the program loops over the finite collection $\{A_y(b), W_y(b)\}_{y \in Y}$ until every component function converges. The proof of existence can be straightforwardly modified to handle this case.[11]

Figure 2 displays the decision rules with and without lotteries. The parameter values for β, R, and δ are the ones used in the previous section. The top panel reproduces the decision rules for the model with and without lotteries displayed earlier for $y = 1$. The other two panels feature i.i.d. endowment shocks and, for these, we display the decision rules for $y = 1$ (its mean value). In the middle panel, σ_y (standard deviation of y) is 0.01. The decision rule without lotteries continues to display jumps, although the jumps are fewer than in the stationary model. Also, the two decision rules coincide (virtually) for a range of debt values. The bottom panel displays the case where $\sigma_y = 0.02$. Now, jumps disappear altogether, and the two decision rules are seemingly identical. These results confirm that sufficient randomness in (future) endowments has the effect of making the continuation value function almost concave. If the continuation value function is almost concave, lotteries should make

[11] When the shocks are i.i.d., we may take $x = y + b$ as the state and solve for $W(x)$ and $A(x)$ ($a(x)$ in the standard model). Except for taking expectations over future y, the model is identical to the stationary one.

Figure 2


δ=0.98,(1+r)=1.05, β=1/(1+r), std dev (y) =0
Asset Choice
Current Asset Level
no lotteries
lotteries
δ=0.98,(1+r)=1.05, β=1/(1+r), std dev (y)=0.01
Asset Choice
Current Asset Level
no lotteries
lotteries
δ=0.98,(1+r)=1.05, β=1/(1+r), std dev (y)=0.02
Asset Choice
Current Asset Level
no lotteries
lotteries

almost no difference to equilibrium decision rules and that is what we find.

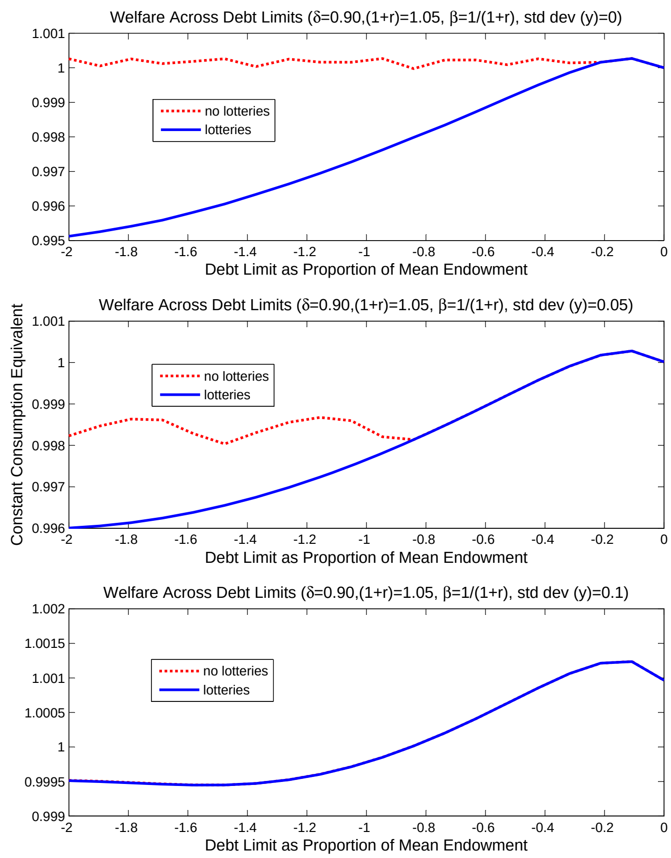
Lest these findings give the impression that shocks are a good substitute for lotteries, two cautionary facts are worth noting. First, there is an important difference in the convergence properties of the two models. As mentioned earlier, when the model without lotteries is solved on a grid, there is no assurance that a (pure-strategy) solution exists. In fact, for our choices of σ_y, the model failed to converge (even though it converged for the stationary case). In Figure 2, we simply displayed the decision rule at the end of 3,000 iterations. In contrast, the model with lotteries converged (in both value and policy functions) to within a tolerance of 10^{-6} in the same number of iterations, regardless of whether σ_y is 0, 0.01, or 0.02. Second, the presence or absence of nonconcavities in the continuation value function (of the model without lotteries) is sensitive to seemingly small differences in δ. For instance, if σ_y is held at 0.02 but δ is lowered to 0.96, jumps reappear.

6 Lotteries and the Ideal Debt Limit

This section compares the implications of the standard model and the model with lotteries for a substantive policy issue. Whether we are concerned with borrowing arrangements for individuals or for countries, there are many policies whose ultimate effect is to alter debt limits. The question we address in this section is: What is the ideal debt limit for the current DM, conditional on starting with no assets?

The top panel of Figure 3 plots $V(0)$ of the current DM in the two models (β and δ are set at their previous values) as $\underline{b}/\text{mean}(y)$ varies from 0 to -2. In the top panel, y is constant at 1. For the model with lotteries, welfare initially increases with $\underline{b}$ but then declines. This accords with intuition: If $\delta = 1$, the DM is indifferent between different $\underline{b}$ since it strictly prefers to set $c = y$. However, with $\delta < 1$, the current DM would prefer to consume more in the current period and a constant amount in all future periods. This can be accomplished by a $\underline{b}$ that is exactly large enough to support the additional consumption desired in the initial period. Further increases in $\underline{b}$ lower welfare because the current DM foresees that it

Figure 3


Welfare Across Debt Limits (δ=0.90,(1+r)=1.05, β=1/(1+r), std dev (y)=0)
1.001
1
0.999
0.998
0.997
0.996
0.995
no lotteries
lotteries
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
Debt Limit as Proportion of Mean Endowment
Welfare Across Debt Limits (δ=0.90,(1+r)=1.05, β=1/(1+r), std dev (y)=0.05)
Constant Consumption Equivalent
1.001
1
0.999
0.998
0.997
0.996
no lotteries
lotteries
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
Debt Limit as Proportion of Mean Endowment
Welfare Across Debt Limits (δ=0.90,(1+r)=1.05, β=1/(1+r), std dev (y)=0.1)
1.002
1.0015
1.001
1.0005
1
0.9995
0.999
no lotteries
lotteries
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
Debt Limit as Proportion of Mean Endowment

will end up consuming too much in the near future and too little in the far future (when the debt limit is reached). The shape of $V(0)$ as a function of $\underline{b}$ is, thus, an inverted-U.

For the model without lotteries, the relationship between welfare and the debt limit is not as intuitive. As in the model with lotteries, welfare declines as $\underline{b}$ is increased beyond the ideal debt limit, but it then rises again. The reason for this odd behavior is that $\underline{b}$ affects the location of the *first* nondebt-limit stationary point (steady state) with debt, where the DM ends up in the long run. As the debt limit expands, the distance between this stationary point and the ideal debt limit fluctuates and, so, welfare fluctuates with $\underline{b}$ as well. The middle panel in Figure 3 shows that the relationship between $V(0)$ and $\underline{b}$ features fluctuations over some range of $\underline{b}$ even when there is substantial volatility in endowments ($\sigma_y = 0.05$). Although these fluctuations occur well past the ideal debt limit, its presence can be relevant if low debt limits are infeasible for some other reason. The bottom panel shows that the relationship between $V(0)$ and $\underline{b}$ is the same for the two models if endowments are sufficiently volatile ($\sigma_y = 0.10$), as the decision rules are now virtually identical.

7 Conclusion

This paper shows that Markovian decision rules of the canonical quasi-geometric discounting model are necessarily discontinuous if there is a strictly positive lower bound on wealth. This finding sheds light on why conventional grid-based computational methods seem to perform poorly for this model. The paper proposed a reformulation of the decision problem in terms of lotteries. The reformulation restores the continuity of Markovian decision rules and, consequently, has the dual benefits of assuring the existence of an equilibrium as well as computational ease.

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A Proofs of Lemmas and Theorems

Proof of Theorem 1

$a(b) \leq b$: (by contradiction). Assume that $a(b) > b$. We will show that the DM can get higher utility by choosing b today when it knows that next period's choice will be $a(b) > b$. If it chooses $a(b)$ this period, it gets

$$U\left(y + b - \beta a(b)\right) + \beta\delta W\left(a(b)\right). \tag{13}$$

If it chooses b this period, it gets

$$U\left(y + b - \beta b\right) + \beta\delta[U\left(y + b - \beta a(b)\right) + \beta W\left(a(b)\right)]. \tag{14}$$

The difference in payoff between the deviation strategy (14) and the putative equilibrium strategy (13) is

$$\Delta = U\left(y + b - \beta b\right) - U\left(y + b - \beta a(b)\right) + \beta\delta U\left(y + b - \beta a(b)\right) - \beta\delta\left(1 - \beta\right) W\left(a(b)\right). \tag{15}$$

We will show that this difference is strictly positive when we replace $W(a(b))$ by an upper bound and, therefore, it will be true for $W(a(b))$ as well.

From today's perspective, the maximum lifetime utility the DM can get starting with b tomorrow is

$$\widetilde{W}(b) = \max_{b', b'', b''', \ldots} U(y + b - \beta b') + \beta U(\overline{y} + b' - \beta b'') + \beta^2 U(y + b'' - \beta b''')....$$

If the DM gets to choose each element of the sequence b', b'', b''', ..., it will choose them such that consumption will be constant. Therefore,

$$\widetilde{W}(b) = \frac{U\left(y + (1 - \beta)b\right)}{1 - \beta}. \tag{16}$$

Since $W(b)$ cannot possibly be more than $\widetilde{W}(b)$, a lower bound on Δ is obtained if we replace $W(a(b))$ in (15) by $\widetilde{W}(a(b))$. This lower bound is given by

$$U(y+b-\beta b) - U(y+b-\beta a(b)) + \beta\delta U(y+b-\beta a(b)) - \beta\delta U(y+(1-\beta)a(b)). \quad (17)$$

Next, by the definition of $\widetilde{W}(.)$ and strict concavity of U, we have

$$\frac{U(y+b-\beta b)}{1-\beta} > U(y+b-\beta a(b)) + \beta\frac{U(y+(1-\beta)a(b))}{1-\beta}$$
$$\Rightarrow U(y+b-\beta b) > U(y+b-\beta a(b)) + \beta\left[U(y+(1-\beta)a(b)) - U(y+b-\beta a(b))\right].$$

Replacing the first term in (17) by the r.h.s. of the above inequality, we obtain another (even weaker) lower bound on Δ:

$$(1-\delta)\beta\left[U(y+(1-\beta)a(b)) - U(y+b-\beta a(b))\right]. \quad (18)$$

Since $a(b) > b$ and $(1-\delta)\beta > 0$, equation (18) is strictly positive. Therefore, the gain from choosing b at b when the policy calls for choosing $a(b) > b$ is strictly positive. Hence, in any MPE $a(b) \leq b$.

$a(b)$ is increasing in b: Suppose $b_1 > b_2$. If $a(b_1)$ is not feasible when b is b_2, then $a(b_2)$ must be less than $a(b_1)$ and we are done. Suppose then that $a(b_1)$ is feasible when b is b_2. By optimality, $U(y+b_2-\beta a(b_2)) + \delta\beta W(a(b_2) \geq U(y+b_2-\beta a(b_1)) + \delta\beta W(a(b_1))$. Since $a(b_2)$ is clearly feasible when b is b_1, we also have $U(y+b_1-\beta a(b_1)) + \delta\beta W(a(b_1)) \geq U(y+b_1-\beta a(b_2)) + \delta\beta W(a(b_2))$. Adding up these two inequalities and rearranging gives

$$U(y+b_1-\beta a(b_1)) - U(y+b_2-\beta a(b_1)) \geq U(y+b_1-\beta a(b_2)) - U(y+b_2-\beta a(b_2)).$$

Since $b_1 > b_2$, strict concavity of U implies $a(b_2) \leq a(b_1)$.

■

Proof of Theorem 2

We will show that, for $b \in (\underline{b}, b_{1M}]$, choosing $\underline{b}$ gives strictly higher utility than any other feasible choice of b'. Define the equilibrium payoff from choosing b', given b, as

$$\Phi(b, b') = U\left(y + b - \beta b'\right) + \delta\beta W(b')$$

and define

$$\widetilde{\Phi}(b, b') = U\left(y + b - \beta b'\right) + \delta\beta\widetilde{W}(b'). \tag{19}$$

Since $\widetilde{W}(b) \geq W(b)$, it follows that

$$\widetilde{\Phi}(b, b') \geq \Phi(b, b'). \tag{20}$$

Next, consider the program

$$\max_{b' \geq \underline{b}} \widetilde{\Phi}(b, b') = \max_{b' \geq \underline{b}} \left\{ U\left(y + b - \beta b'\right) + \delta\beta \left[\frac{U(y + (1 - \beta)b')}{(1 - \beta)} \right] \right\},$$

where we have replaced $\widetilde{W}(b)$ by the equivalent expression in (16). The FOC for the optimal choice of b' is

$$-U'\left(y + b - \beta b'\right) + \delta U'(y + (1 - \beta)b') \leq 0.$$

For $b' = \underline{b}$, this condition holds with an equality when $b = b_{1M}$ (this follows from the definition of b_{1M}), and it holds with a strict inequality for when $b \in [\underline{b}, b_{1M})$. Then, from the strict concavity of U, it follows that

$$\widetilde{\Phi}(b, \underline{b}) > \widetilde{\Phi}(b, b') \text{ for } b \in (\underline{b}, b_{1M}] \text{ and } b' > \underline{b}. \tag{21}$$

Next, by Corollary 1 we know that in any MPE $a(\underline{b}) = \underline{b}$ and, therefore, $W(\underline{b}) = \widetilde{W}(\underline{b})$.

Hence,

$$\Phi(b,\underline{b}) = \widetilde{\Phi}(b,\underline{b}). \tag{22}$$

Finally, combining (20), (21), and (22) gives

$$\Phi(b,\underline{b}) = \widetilde{\Phi}(b,\underline{b}) > \widetilde{\Phi}(b,b') \geq \Phi(b,b') \text{ for } b \in (\underline{b}, b_{1M}] \text{ and } b' > \underline{b}.$$

Thus, for $b \in (\underline{b}, b_{1M}]$, choosing $\underline{b}$ gives strictly higher utility than any other feasible choice regardless of $W(b)$. Hence, $a(b) = \underline{b}$ for $b \in (\underline{b}, b_{1M}]$. ■

Proof of Lemma 1

By Theorems 1 and 2, $a(b) = \underline{b}$ for $b \in [\underline{b}, b_{1M}]$. To prove the rest, we will guess that when $b \in (b_{1M}, b_{2M})$, the optimal choice of b' is obtained from a maximization problem in which the DM optimally chooses b' from the set $[\underline{b}, b_{1M}]$. By Theorems 1 and 2, the DM can predict that, for any b' in this set, future DMs will choose $\underline{b}$. Therefore, the guess, which we denote as $h(b)$, solves

$$h(b) = \underset{b'}{\operatorname{argmax}} \left\{ U\left(y + b - \beta b'\right) + \delta \left[\beta U\left(y + b' - \beta \underline{b}\right) + \beta^2 \frac{U\left(y + (1-\beta)\,\underline{b}\right)}{(1-\beta)} \right] \right\} \tag{23}$$

s.t.

$$\underline{b} \leq b' \leq b_{1M}.$$

Since (23) is a strictly concave problem with a convex constraint set, $h(b)$ exists, and the FOC for b' is both necessary and sufficient to characterize $h(b)$. Taking into account the corner solutions, the FOC is

$$-U'\left(y + b - \beta b'\right) + \delta U'\left(y + b' - \beta \underline{b}\right) \begin{cases} \leq 0 & \text{if } b' = \underline{b} \\ = 0 & \text{if } \underline{b} < b' < b_{1M} \\ \geq 0 & \text{if } b' = b_{1M}. \end{cases}$$

Using the definitions of b_{1M} and b_{2M}, we may verify that, for b in the open interval (b_{1M}, b_{2M}), the FOC is satisfied with equality. The parametric form for U then gives

$$h(b) = \frac{\left(-\left(1-\delta^{1/\gamma}\right)y + \delta^{1/\gamma}b + \beta\underline{b}\right)}{1+\delta^{1/\gamma}\beta}, b \in (b_{1M}, b_{2M}).$$

Note that the postulated $a(b)$ is consistent with the maintained assumption that $a(b)$ is continuous in b, since $\lim_{b\downarrow b_{1M}} h(b) = \underline{b}$ (the quickest way to see this is to observe that the condition that defined b_{1M} is the same as the FOC from which $h(b)$ is derived). Note also that $h(b)$ is strictly increasing in $b \in (b_{1M}, b_{2M})$ with $\lim_{b\uparrow b_{2M}} h(b) = h(b_{2M}) = b_{1M}$. Hence, the guess is also consistent with Theorem 1.

Next, we will verify that our guess is the only solution for $b \in (b_{1M}, b_{2M}]$ if $a(b)$ is to be continuous. For this, it is sufficient to verify that $h(b)$ is the correct guess for $b \in (b_{1M}, b_{2M})$: If $a(b)$ is indeed equal to $h(b)$ over the open interval, then by the assumed continuity of $a(b)$, $a(b_{2M})$ must equal $\lim_{b\uparrow b_{2M}} h(b) = h(b_{2M}) = b_{1M}$.

To proceed, suppose there is a $\hat{b} \in (b_{1M}, b_{2M})$ for which $a(\hat{b}) \neq h(\hat{b})$. Two cases are possible. (i) Suppose that $\underline{b} \leq a(\hat{b}) \leq b_{1M}$. Then we know that the DM must believe that all future asset choices must be $\underline{b}$ and, in that case, we know from (23) that $h(\hat{b})$ is the optimal choice. Since $a(\hat{b}) \neq h(\hat{b})$, the supposition (i) must be false. (ii) Suppose that $a(\hat{b}) > b_{1M}$. Since $a(b_{1M}) = \underline{b}$ and A is continuous, there must exist some $\bar{b} \in (b_{1M}, \hat{b})$ for which $a(\bar{b}) = b_{1M}$. Again, for such a choice, the DM must believe that all future asset choices must be $\underline{b}$. Given these beliefs, however, $h(\bar{b})$ is the optimal choice, so $a(\bar{b}) = h(\bar{b})$. But $h(b) < b_{1M}$ for $b \in (b_{1M}, b_{2M})$. Since $a(\bar{b})$ cannot be both equal to and less than b_{1M}, supposition (ii) must also be false. As (i) and (ii) exhaust all possible cases, we conclude that there cannot exist any $\hat{b} \in (b_{1M}, b_{2M})$ for which $a(\hat{b}) \neq h(\hat{b})$. Therefore, for any MPE with continuous $a(b)$, $a(b)$ must satisfy the form stated in the lemma for $b \in (b_{1M}, b_{2M}]$. ■

Proof of Theorem 3

(By contradiction) Suppose that a continuous $a(b)$ exists. Now, consider a DM with $b = b_{2M}$ that is contemplating choosing $b' \in [b_{1M}, b_{2M}]$. By Lemma 1, the DM can predict what future DMs will choose for every $b' \in [b_{1M}, b_{2M}]$. Using these predictions, we can formulate this choice problem as follows:

$$\max_{b'} U\left(y + b_{2M} - \beta b'\right) +$$
$$\delta\left[\beta U\left(y + b^{'} - \beta a\left(b'\right)\right) + \beta^2 U\left(y + a(b') - \beta \underline{b}\right) + \beta^3 \frac{U\left(\overline{y} + (1-\beta)\,\underline{b}\right)}{(1-\beta)}\right] \tag{24}$$
s.t.
$$b_{1M} \leq b' \leq b_{2M}.$$

Observe that for $b' \in [b_{1M}, b_{2M}]$, $a\left(b'\right) \in [\underline{b}, b_{1M}]$ and, so, $a(a(b')) = \underline{b}$. We will show that the right-hand derivative of the objective function at $b' = b_{1M}$ is strictly positive. Note that the right-hand and left-hand derivatives of $a(b)$ are different at b_{1M}, but in this program, we are only considering b' choices that are at least as large as b_{1M}. The derivative of the objective function is

$$-U'\left(y + b_{2M} - \beta b'\right) + \delta\left[U'\left(y + b^{'} - \beta a\left(b'\right)\right)\left(1 - \beta a'\left(b'\right)\right) + \beta U'\left(y + a(b') - \beta \underline{b}\right) a'\left(b'\right)\right].$$

We know that $a'\left(b'\right) = \frac{\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}$ for $b' \in [b_{1M}, b_{2M})$. So, the FOC becomes

$$-U'\left(y + b_{2M} - \beta b'\right) + \delta\left[U'\left(y + b^{'} - \beta a\left(b'\right)\right)\left(1 - \frac{\beta\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}\right) + \beta U'\left(y + a(b') - \beta \underline{b}\right)\frac{\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}\right].$$

Evaluating this at $b' = b_{1M}$ (and recognizing that $a(b_{1M}) = \underline{b}$) yields

$$-U'\left(y + b_{2M} - \beta b_{1M}\right) + \delta\left[U'\left(y + b_{1M} - \beta \underline{b}\right)\left(1 - \frac{\beta\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}\right) + \beta U'\left(y + \underline{b} - \beta \underline{b}\right)\frac{\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}\right]. \tag{25}$$

By the definition of b_{1M},

$$U'\left(y + b_{1M} - \beta \underline{b}\right) = \delta U'\left(y + \underline{b} - \beta \underline{b}\right).$$

Substituting this into (25) gives

$$-U'\left(y+b_{2M}-\beta b_{1M}\right)+U'\left(y+\underline{b}-\beta\underline{b}\right)\delta\left(\frac{\delta+\beta\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}\right). \tag{26}$$

By the definition of b_{2M},

$$U'\left(y+b_{2M}-\beta b_{1M}\right)=\delta^{2}U'\left(y+\underline{b}-\beta\underline{b}\right). \tag{27}$$

But observe that since $\delta<1$,

$$\delta<\frac{\delta+\beta\delta^{1/\gamma}}{1+\delta^{1/\gamma}\beta}.$$

Therefore, equation (27) implies that the expression in (26) is strictly positive. Hence, the DM can obtain strictly higher utility with a $b'>b_{1M}$. ■

Proof of Theorem 4

$V(b)$ is strictly increasing in b: It follows from the fact that a DM with higher b can always replicate the choice of a DM with lower b (so, continuation value will be the same) but obtain strictly higher current consumption (U is strictly increasing in c).

V is continuous: Suppose that $V(b)$ is not continuous at $\hat{b}$. We know from the monotonicity of V that $\lim_{b\uparrow\widehat{b}}V(b)<V(\widehat{b})$ or $V(\widehat{b})<\lim_{b\downarrow\widehat{b}}V(b)$. Consider the first case. Since $c(\widehat{b})\geq\kappa>0$ (Corollary 2) for $b\in(\hat{b}-\kappa,\hat{b})$, $y+b-\beta a(\hat{b})>0$ and, so, $a(\hat{b})$ is feasible. Therefore, the DM can obtain $U(y+b-\beta a(\hat{b}))+\beta\delta W(a(\hat{b}))$. Since U is continuous in c, $U(y+b-\beta a(\hat{b}))+\beta\delta W(a(\hat{b}))$ is continuous in $b\in(\hat{b}-\kappa,\hat{b})$. Since $V(b)\geq U(y+b-\beta a(\hat{b}))+\beta\delta W(a(\hat{b}))$ in this range, it follows that $\lim_{b\uparrow\widehat{b}}V(b)\geq V(\hat{b})$. Hence, the first case is not possible. Next, consider the second case. Pick $\hat{b}+\mu$, $\mu<\kappa$. By Corollary 2, $c(\hat{b}+\mu)\geq\kappa$. Therefore, $c(\hat{b}+\mu)-\mu\geq\kappa-\mu>0$. Since $c(\hat{b}+\mu)-\mu=y+\hat{b}-\beta a(\hat{b}+\mu)$, $a(\hat{b}+\mu)$ is feasible at $\hat{b}$ and delivers $U(y+\hat{b}-\beta a(\hat{b}+\mu))+\beta\delta W(a(\hat{b}+\mu))$. And, since $\lim_{\mu\to0}U(y+\hat{b}-\beta(\hat{b}+\mu))+\beta\delta W(a(\hat{b}+\mu))=\lim_{b\downarrow\widehat{b}}V(b)$ and $\lim_{b\downarrow\widehat{b}}V(b)>V(\widehat{b})$, there must exist some μ^{*} such that $a(\hat{b}+\mu^{*})$ is feasible for $\hat{b}$ and provides more utility than the choice of $a(\hat{b})$. This contradicts the optimality of $a(\hat{b})$. Hence,

the second case is not possible either. Therefore, $V(b)$ is continuous for all b. ■

$W(b)$ cannot be continuous for all b: Let $\hat{b}$ be a point at which $a(b)$ is discontinuous. Select a sequence $\{b_n\}$ converging to $\hat{b}$ for which $\lim_n a(b_n) = a_1 \neq a(\hat{b})$. By definition, $W(b_n) = U(y + b_n - \beta a(b_n)) + \beta W(a(b_n))$ and $W(\hat{b}) = U(y + \hat{b} - \beta a(\hat{b})) + \beta W(a(\hat{b}))$. Assume, to get a contradiction, that $W(b)$ is continuous. Then, $\lim_n W(b_n) = W(\hat{b})$ and $\lim_n W(a(b_n)) = W(a_1)$. Using these expressions for $W(b_n)$ and $W(\hat{b})$ and the fact that U is continuous gives $U(y + \hat{b} - a_1) + \beta W(a_1) = U(y + \hat{b} - a(\hat{b})) + \beta W(a(\hat{b}))$, or

$$U(y + \hat{b} - \beta a_1) - U(y + \hat{b} - \beta a(\hat{b})) = \beta[W(a(\hat{b})) - W(a_1)]. \tag{28}$$

Next, by continuity of V and U and the assumed continuity of W, $V(\hat{b}) = \lim_n U(y + b_n - \beta a_n(b_n)) + \delta\beta W(a(b_n)) = U(y + \hat{b} - \beta a(\hat{b})) + \delta\beta W(a(\hat{b}))$, we obtain

$$U(y + \hat{b} - \beta a_1) - U(y + \hat{b} - \beta a(\hat{b})) = \delta\beta[W(a(\hat{b})) - W(a_1)] \tag{29}$$

Since $a_1 \neq a(\hat{b})$, both (28) and (29) cannot simultaneously be true for $\delta < 1$. Hence, $W(b)$ cannot be continuous for all b.

The proof of Theorem 5 requires several lemmas.

Lemma 4 $\Omega_3'(b) - \Omega_2'(b) \geq 0$.

Proof. Denote the optimal decision rule for $\Omega_3(b)$ as $a_3(b)$ and the optimal decision rule for $\Omega_2(b)$ as $a_2(b)$. Then $\Omega_k'(b) = U'(y + b - \beta a_k(b))$ for $k = 2, 3$. This follows from the envelope theorem when the corresponding FOC holds as an equality. When the FOC does not hold as an equality, it follows from the fact that $a_k(b)$ then does not change with b. By the definition of the choice sets, $a_3(b) \geq a_2(b)$ and, hence, $\Omega_3'(b) - \Omega_2'(b) \geq 0$. ■

We can now give the proof of the first part of Theorem 5.

Proof of Part 1 of Theorem 5: If b^* exists, $b^* \in (b_{1M}, b_{2M})$.

Proof. To show $b^* < b_{2M}$, we will first show that $\Omega_3(b_{2M}) - \Omega_2(b_{2M}) > 0$. Note that $\Omega_2(b_{2M})$

is the value of the program (23) with $b = b_{2M}$, and $\Omega_3(b_{2M})$ is the value of the program in (24) with $b = b_{2M}$. By Theorem 3, $\Omega_3(b_{2M}) - \Omega_2(b_{2M}) > 0$. Since $\Omega_3'(b) - \Omega_2'(b) \geq 0$, it follows that, if b^* exists, $b^* < b_{2M}$ in order for $\Omega_2(b^*) \geq \Omega_3(b^*)$.

To show that $b^* > b_{1M}$, we show that $\left(1 + \frac{\beta\delta}{1-\beta}\right) U(y + (1-\beta)\, b) < \Omega_2(b)$ for $b \in (\underline{b}, b_{1M}]$, which implies that the equality condition in (5) cannot be satisfied in this range. Consider the following problem for $b \in (\underline{b}, b_{1M}]$:

$$\Phi(b) = \max_{b' \geq \underline{b}} U\left(y + b - \beta b'\right) + \delta\beta \left[\frac{U(y + (1-\beta)b')}{(1-\beta)}\right]. \tag{30}$$

Then, $\Phi(b) = \Omega_2(b)$ since in both problems the optimal choice is $\underline{b}$ and at that choice the two values are the same. Next, observe that in program (30), $b' = b$ was available but not chosen. It follows from strict concavity of U that $\Omega_2(b) > \left(1 + \frac{\beta\delta}{1-\beta}\right) U(y + (1-\beta)\, b)$ for $b \in (\underline{b}, b_{1M}]$. Therefore, if $b^* > \underline{b}$ exists, b^* must strictly exceed b_{1M}. ■

To establish the second part of Theorem 5, two additional lemmas are needed.

Lemma 5 *Let $\Omega_3^R(b)$ be the value of the program that defines $\Omega_3(b)$ but with the choice set restricted to $[b_{1M}, b^*)$. Then, for $b \in (b_{1M}, b^*]$, $\Omega_3^R(b) = \Omega_3(b)$.*

Proof. The key here is that $a_3(b)$ is constrained at its lower bound until some $\tilde{b} > b_{1M}$ and beyond that point $a_3(b)$ rises at a slope less than 1. Therefore, regardless of whether $\tilde{b}$ is to the left or right of b^*, the optimal choice in $\Omega_3(b)$ for $b \in [b_{1M}, b^*]$ is strictly less than b^*.

Let $\tilde{b}$ be the value of b for which $a_3(b) = b_{1M}$ and the constraint that $b' \geq b_{1M}$ just ceases to bind. In this case, the FOC condition for optimality of $a_3(b)$ holds with an equality and gives

$$-U'\left(y + \tilde{b} - \beta b_{1M}\right) + U'\left(y + \underline{b} - \beta\underline{b}\right)\left(\delta\frac{\delta + \beta\delta^{1/\gamma}}{1 + \beta\delta^{1/\gamma}}\right) = 0. \tag{31}$$

Next, we claim that the l.h.s. of (31) is strictly less than 0 at $\tilde{b} = b_{1M}$. From the definition of b_{1M}, we have that the first term on the l.h.s. of (31), after we substitute b_{1M} for $\tilde{b}$, is

$$U'\left(y+b_{1M}-\beta b_{1M}\right)=\left(\frac{\left(1-\beta+\beta\delta^{1/\sigma}\right)}{\delta^{1/\sigma}}\right)^{-\sigma}U'\left(y+\underline{b}-\beta\underline{b}\right).$$

Since

$$(1-\beta+\beta\delta^{1/\sigma})^{-\sigma}>1>[\delta+\beta\delta^{1/\sigma}]/[1+\beta\delta^{1/\sigma}],$$

the l.h.s. of (31) is negative for $\tilde{b}=b_{1M}$. Now observe that, from the concavity of U, the l.h.s. of (31) is strictly increasing in $\tilde{b}$ and, hence, the expression will be 0 for some $\tilde{b}>b_{1M}$.

Next, note that in the region where the upper and lower constraints on b' do not bind

$$a_3'(b)=\frac{\left(\delta\frac{\delta+\beta\delta^{1/\sigma}}{1+\beta\delta^{1/\sigma}}\right)^{1/\sigma}}{\left[\frac{\delta^{1/\sigma}}{1+\delta^{1/\sigma}\beta}+\beta\left(\delta\frac{\delta+\beta\delta^{1/\sigma}}{1+\beta\delta^{1/\sigma}}\right)^{1/\sigma}\right]}$$

which we can verify is strictly less than 1.

Hence, ${\Omega_3}^R(b)=\Omega_3(b)$ for $b\in(b_{1M},b^*]$. ■

Lemma 6 *Let $\Omega_2^R(b)$ be the value of the program that defines $\Omega_2(b)$ but with the choice set restricted to $b'\in[\underline{b},b_{1M})$. Then, for $b\in[\underline{b},b^*]$, $\Omega_2^R(b)=\Omega_2(b)$.*

Proof. Observe that

$$a_2(b)=\begin{cases}\underline{b} & b\in[\underline{b},b_{1M}]\\ h(b) & b\in(b_{1M},b_{2M})\\ b_{1M} & b\in[b_{2M},\infty).\end{cases}$$

From the above, we see that $a_2(b)<b_{1M}$ for all $b<b_{2M}$. Since $b^*<b_{2M}$, the restriction of b' to $[\underline{b},b^*]$ is not binding for $b\in[\underline{b},b^*]$. Therefore, $\Omega_2^R(b)=\Omega_2(b)$. ■

We are now ready to give the proof of the second part of Theorem 5.

Proof of Part 2 of Theorem 5: If $a^*(b)$ is the decision rule of an MPE of the environment where b is restricted to be in $[b^*, \infty)$, then

$$a(b) = \begin{cases} \underline{b} & \text{for } b \in [\underline{b}, b_{1M}] \\ h(b) & \text{for } b \in (\underline{b}, b^*) \\ b^* & \text{for } b = b^* \\ a^*(b) & \text{for } b \in (b^*, \infty) \end{cases}$$

is an MPE decision rule.

Proof. For $b \in [\underline{b}, b_{1M}]$, $a(b) = \underline{b}$ in any MPE by Corollary 1 and Theorem 2.

For $b \in (b_{1M}, b^*)$, we need to show that the DM prefers $b' = h(b)$. We show this by considering four mutually exclusive alternatives: (i) $b' \in [\underline{b}, b_{1M})$, which would imply payoff $\Omega_2^R(b)$, (ii) $b' \in [b_{1M}, b^*)$, which would imply payoff $\Omega_3^R(b)$, (iii) $b' = b^*$, and (iv) $b' > b^*$. First, we show that (i) dominates (ii). Lemmas 4, 5, and 6 imply $\Omega_3^{R\prime}(b) - \Omega_2^{R\prime}(b) = \Omega_3'(b) - \Omega_2'(b) \geq 0$. By (5), $\Omega_2(b^*) \geq \Omega_3(b^*)$. Therefore, $\Omega_3^R(b) < \Omega_2^R(b)$ for $b \in (b_{1M}, b^*)$. Second, we show that (i) dominates (iii). For this, define $\Omega_{b^*}(b) = U(y+b-\beta b^*) + \frac{\beta\delta}{1-\beta} U(y+(1-\beta)\, b^*)$. We know that $\Omega_{b^*}'(b) = U'(y+b-\beta b^*)$ and $\Omega_2'(b) = U'(y+b-\beta h(b))$. For the region $b \in [b_{1M}, b^*)$, $h(b) < b^*$ and, so, $\Omega_{b^*}'(b) > \Omega_2'(b)$. By (5), $\Omega_{b^*}(b^*) = \Omega_2(b^*)$ so $\Omega_{b^*}(b) < \Omega_2(b)$ for $b \in [b_{1M}, b^*)$. Finally, we show that (iii) dominates (iv). By an argument similar to that given in Theorem 2, we may show that, for all $b \leq b^*$, $U(y+b-\beta b^*) + \delta\beta\widetilde{W}(b^*) > U(y+b-\beta b') + \delta\beta\widetilde{W}(b')$ for $b' > b^*$. Since b^* is a steady state, $U(y+b-\beta b^*) + \delta\beta W(b^*) > U(y+b-\beta b') + \delta\beta W(b')$ for all $b' > b^*$. Hence, $h(b)$ is the optimal choice for $b \in (b_{1M}, b^*)$.

For $b = b^*$, we showed above that the DM will never choose $b' > b^*$. If the DM chooses something less than b^*, the best it can do is given by $\Omega_2^R(b^*)$, which is the same as $\Omega_2(b^*)$ by Lemma 6 (as $\Omega_2(b^*) \geq \Omega_3(b^*)$). And, by (5), we know that $\Omega_2(b^*) = \Omega_{b^*}(b^*)$. Therefore the DM is indifferent between choosing b^* or something less.

For $b \in (b^*, \infty)$, we need to show that it is still optimal to choose $a^*(b)$. Since b^* is a steady state of the restricted as well as the unrestricted environments (and therefore has the same continuation values in both environments) and a^* is an MPE of the restricted environment, it is sufficient to show that a DM with $b > b^*$ will never choose $b' < b^*$, in which case $a^*(b)$ will continue to describe the equilibrium strategies for the unrestricted environment. At $b = b^*$, $U(y+b-\beta b^*)+\delta\beta W(b^*) \geq U(y+b-\beta b')+\delta\beta W(b')$ for all $b' < b^*$. Differentiating with respect to b gives $U'(y+b-\beta b^*) > U'(y+b-\beta b')$. Therefore for all $b > b^*$, $U(y+b-\beta b^*)+\delta\beta W(b^*) > U(y+b-\beta b')+\delta\beta W(b')$ for $b' < b^*$. ■

Proof of Theorem 6

Proof. Let $\hat{b}$ be a point of discontinuity of $a(b)$. Select a sequence $\{b_n\}$ converging to $\hat{b}$ such that $\lim_n a(b_n) = \bar{a} \neq a(\hat{b})$. Note that since consumption is above κ for all b, we can use the same argument as in Theorem **??** to establish that $\bar{a}$ is feasible at $\hat{b}$. Now consider the lottery, where $\bar{a}$ is chosen with probability $\lambda \in (0,1)$ and $a(\hat{b})$ with probability $(1-\lambda)$. The expected present value of this lottery is $\beta[\lambda\bar{a}+(1-\lambda)a(\hat{b})]$. If financial intermediaries are risk neutral, the lottery is feasible because $\bar{a}$ and $a(\hat{b})$ are both individually feasible at $\hat{b}$. The payoff to the DM from this lottery is $U(y+\hat{b}-\beta(\lambda\bar{a}+(1-\lambda)a(\hat{b})])+\delta\beta(\lambda W(\bar{a})+(1-\lambda)W(a(\hat{b}))]$. Since $V(b)$ is continuous in b, $\lim_n V(b_n) = U(y+\hat{b}-\beta\bar{a})+\delta\beta W(\bar{a}) = V(\hat{b})$. Also, $V(\hat{b}) = U(y+\hat{b}-\beta a(\hat{b}))+\delta\beta W(a(\hat{b}))$. Therefore, each of the component pure strategies gives the same payoff. Hence, for $\lambda \in (0,1)$, $\lambda[U(y+\hat{b}-\beta\bar{a})+\delta\beta W(\bar{a})]+(1-\lambda)\left[U(y+\hat{b}-\beta a(\hat{b}))+\delta\beta W(a(\hat{b}))\right] = U(y+\hat{b}-\beta a(\hat{b}))+\delta\beta W(a(\hat{b}))$. Now, observe that the l.h.s. of the preceding equality can be expressed as $\lambda U(y+\hat{b}-\beta\bar{a})+(1-\lambda)U(y+\hat{b}-\beta a(\hat{b}))+\delta\beta[\lambda W(\bar{a})+(1-\lambda)W(a(\hat{b}))]$. By strict concavity of U, $U(y+\hat{b}-\beta[\lambda\bar{a}+(1-\lambda)a(\hat{b})]) > \lambda U(y+\hat{b}-\beta\bar{a})+(1-\lambda)U(y+\hat{b}-\beta a(\hat{b}))$. Therefore for $\lambda \in (0,1)$, $U(y+\hat{b}-\beta[\lambda\bar{a}+(1-\lambda)a(\hat{b})])+\delta\beta[\lambda W(\bar{a})+(1-\lambda)W(a(\hat{b}))] > U(y+\hat{b}-\beta a(\hat{b}))+\delta\beta W(a(\hat{b}))$. Hence, the lottery is strictly preferable to the (pure strategy) equilibrium decision. ■

Lemma 7 *For any $W \in \mathscr{W}$, $w(B;W) : [\underline{b},\bar{b}] \to \mathbb{R}$ is the concave upper envelope of $W(b)$, i.e., $w(B;W)$ is the least concave function that majorizes $W(b)$.*

Proof. We will first prove the $w(B;W)$ is concave. Consider B_1, B_2 both elements of $[\underline{b},\bar{b}]$. Let ϕ_k, $k = 1,2$, be the lotteries that attain $w(B_k;W)$. By convexity of the constraint set, the probability measure defined by the compound lottery $\lambda\phi_1 + (1-\lambda)\phi_2$ is feasible for $\lambda B_1 + (1-\lambda)B_2$, where $\lambda \in (0,1)$, and delivers $\lambda w(B_1;W) + (1-\lambda)w(B_2;W)$. Hence $w(\lambda B_1 + (1-\lambda)B_2;W) \geq \lambda w(B_1;W) + (1-\lambda)w(B_2;W)$.

To prove the second part, we must show that if $g(b) \geq W(b)$ and $g(b)$ is concave then $g(B) \geq w(B;W)$. Observe that replacing $W(b)$ with $g(b)$ in (8) gives $w(B;g) \geq w(B,W)$. Since $g(b)$ is concave, we must have $w(B,g) = g(B)$. Therefore, $g(B) \geq w(B;W)$.

Proof of Theorem 7

Continuity: In (9), B' is being chosen from a convex set and the objective function is strictly concave in B' because U is strictly concave in c and w is concave in B'. It follows from the Theorem of the Maximum (Stokey and Lucas (1989), Theorem 3.6, p. 62) that $A(b)$ is continuous in b. The continuity of $c(b)$ follows from the continuity of U and the continuity of $A(b)$.

Monotonicity: The proof that $A(b)$ is increasing in b is essentially the same as the proof that $a(b)$ is increasing in b given in Theorem 1 and is therefore not repeated here. To show that $c(b)$ is increasing in b, we exploit the fact that $w(b;W)$ is concave. Concavity implies that the left-hand and right-hand derivatives of $w(B;W)$ with respect to B, denoted $w'_-(B;W)$ and $w'_+(B;W)$, respectively, exist for all interior points and $w'_+(B;W) \leq w'_-(B;W)$. Furthermore, optimality of $A(b)$ implies

$$\delta w'_+(A(b);W) \leq U'(y+b-\beta A(b)) \leq \delta w'_-(A(b);W). \tag{32}$$

Take $b_1 < b_2$. If $A(b_1) = A(b_2)$, then the result is obvious. If $A(b_2) > A(b_1)$, then from (32) we have $U'(y+b_2-\beta A(b_2)) \leq \delta w'_-(A(b_2))$ and $U'(y+b_1-\beta A(b_1)) \geq \delta w'_+(A(b_1))$. From the concavity of w, we have $w'_-(A(b_2)) \leq w'_+(A(b_1))$. Hence, $U'(y+b_2-\beta A(b_2)) \leq w'_-(A(b_2);W) \leq w'_+(A(b_1);W) \leq U'(y+b_1-\beta A(b_1))$, which implies that $c(b_1) \leq c(b_2)$.

Lipschitz: Since $c(b)$ is increasing in b, $y+b_1-\beta A(b_1) \geq y+b_0-\beta A(b_0)$ for $b_1 > b_0$ and $y+b_1-\beta A(b_1) \leq y+b_0-\beta A(b_0)$ for $b_1 < b_0$. For the first case, this implies $b_1-b_0 \geq \beta[A(b_1)-A(b_0)]$.

For the second case, this implies $b_1 - b_0 \leq \beta[A(b_1) - A(b_0)]$, or $-(b_1 - b_0) \geq -\beta[A(b_1) - A(b_0)]$. Combining the two cases, we have $1/\beta \geq |A(b_1) - A(b_0)|/|b_1 - b_0|$. ■

Proof of Theorem 8

The steps to establish that $A(b) \leq b$ are essentially the same as in Theorem 1. The only difference is that we need an upper bound for $w\left(A(b); W\right)$ instead of $W(A(b))$. As stated in Theorem 1, $\widetilde{W}(b)$ is the maximum utility that the DM can get with only pure strategies. Since $\widetilde{W}(b)$ is strictly concave in b, the maximum utility the DM can get if it chooses a lottery over b' with expected value of b is also $\widetilde{W}(b)$. The remaining steps are then exactly the same as in Theorem 1. ■

Proof of Theorem 9

The continuity of $W(b)$ follows from the continuity of $w(b; W)$ (a concave function is continuous), the continuity of $c(b)$, and the continuity of U. To prove monotonicity, let $b_1 < b_2$ and suppose, to get a contradiction, $W(b_2) < W(b_1)$. Since $c(b_1) \leq c(b_2)$ and $W(b) = U\left(y + b - \beta A(b)\right) + \beta w(A(b); W)$, it follows that $w(A(b_2); W) < w(A(b_1); W)$. Then $A(b_1) \neq A(b_2)$ and from monotonicity of A it follows that $A(b_2) > A(b_1)$. But this implies that the DM with b_2 will be strictly better off if it chose $A(b_1)$ (which is feasible for it) since that would give the DM strictly higher consumption today and strictly higher continuation utility. This contradicts the optimality of $A(b)$. Therefore, $W(b)$ is increasing in b. ■

Proof of Lemma 3

Fix $A \in \mathscr{F}$ and define the operator T_A as

$$(T_A W)(b) = U(y + b - \beta A(b)) + \beta w(A(b); W). \tag{33}$$

We claim that (i) $(T_A W)(b) \in \mathscr{C}$, (ii) T_A is a contraction map, and (iii) T_A is continuous in A. Given these claims, the first part of the lemma follows from (i) and (ii) and an application of the Contraction Mapping Theorem (Stokey and Lucas (1989), Theorem 3.2, p. 50). The second part follows easily from (iii); for a proof see Hutson and Pym (1980), Theorem 4.3.6, pp. 117-118.

Claim (i): Since $W \in \mathscr{C}$ is continuous, it is measurable with respect to space $([\underline{b}, \bar{b}], \mathscr{B})$. Therefore $w(B, W)$ exists. By Lemma 7, $w(B; W)$ is concave and therefore continuous in B. Since A is continuous in b and U is continuous in c, it follows that $(T_A W)(b) \in \mathscr{C}$.

Claim (ii): We verify Blackwell's sufficiency conditions for a contraction map. Consider $\widehat{W}(b) \geq W(b)$, both elements of $\mathscr{C}$. Clearly, $w(B; \widehat{W}) \geq w(B; W)$. Since A is fixed, it follows that $(T_A \widehat{W})(b) \geq (T_A W)(b)$. Hence $T_A W$ is monotone in W. Next, consider $\widehat{W}(b) = W(b) + \theta$. Then, $w(B; \widehat{W}) = w(B; W) + \beta\theta$. Again, since A is fixed, it follows that $(T_A \widehat{W})(b) = (T_A W)(b) + \beta\theta$. Therefore, T_A is a contraction map with modulus β.

Claim (iii): Fix $W \in \mathscr{C}$. Let $A \in \mathscr{F}$ and let $\{A_n\} \subset \mathscr{F}$ be a sequence such that $\lim_n \|A_n - A\| = 0$. From the continuity of $U(c)$ and $w(B; W)$, the sequence $(T_{A_n} W)(b) = U(y + b - \beta A_n) + \beta w(A_n; W)$ converges point-wise to $(T_A W)(b) = U(y + b - \beta A) + \beta w(A; W)$. We will now show that $\lim_n \|T_{A_n} W - T_A W\| = 0$. Observe that, for any $b \in [\underline{b}, \bar{b}]$,

$$
\begin{aligned}
&|U(y + b - \beta A_n + \beta w(A_n; W) - U(y + b - \beta A - \beta w(A; W)| \\
&\leq |U(y + b - \beta A_n - U(y + b - \beta A| + |\beta w(A_n; W) - \beta w(A; W)| \\
&\leq U'(y + (1 - \beta)\underline{b})|A_n - A| + w'_+(\underline{b}; W)\beta|A_n - A| \\
&\leq U'(y + (1 - \beta)\underline{b})\|A_n - A\| + w'_+(\underline{b}; W)\beta\|A_n - A\|.
\end{aligned}
$$

The first inequality follows from the triangle inequality; the second follows from the fact that U and w are both concave and the lowest consumption level possible is $y + (1 - \beta)\underline{b}$ and the lowest asset level possible is $\underline{b}$; and the third follows from the definition of the norm $\| \cdot \|$. To complete the proof, observe that since b was arbitrary, the chain of inequalities will continue to hold if the first term in the chain is replaced by $\sup_{b \in [\underline{b}, \bar{b}]} |U(y + b - \beta A_n + \beta w(A_n; W) - U(y + b - \beta A - \beta w(A; W)|$. But this would then imply $\|T_{A_n} W - T_A W\| \leq U'(y + (1 - \beta)\underline{b})\|A_n - A\| + w'_+(\underline{b}; W)\beta\|A_n - A\|$. Since $\lim_n \|A_n - A\| = 0$, it follows that $\lim_n \|T_{A_n} W - T_A W\| = 0$. ■

Proof of Theorem 10

Since $W(b;A)$ solves the functional equation (33) for a given $A(b)$, the existence of an MPEL is equivalent to the existence of $A^* \in \mathscr{F}$ such that

$$A^*(b) = \underset{B' \in [\underline{b}, \bar{b}]}{\operatorname{argmax}} U(y + b - \beta B') + \delta\beta w(B'^*)). \tag{34}$$

For any given $A \in \mathscr{F}$, define the operator H as

$$(HA)(b) = \underset{B' \in [\underline{b}, b]}{\operatorname{argmax}} U(y + b - \beta B') + \delta\beta w(B'; W(b;A)).$$

We claim that (i) H is continuous, (ii) $H(\mathscr{F}) \subset \mathscr{F}$, and (iii) the family $H(\mathscr{F})$ is equicontinuous. Recalling that $\mathscr{F}$ is a nonempty, bounded, closed, and convex subset of $\mathscr{C}$, the existence of a solution to (34) follows from an application of the Schauder Fixed Point Theorem (Stokey and Lucas (1989), Theorem 17.4, page 520).

Claim (i): Let $A \in \mathscr{F}$ and let $\{A_n\} \subset \mathscr{F}$ be a sequence such that $\lim_n \|A_n - A\| = 0$. We wish to show that $\lim_n \|(HA_n) - (HA)\| = 0$. For this, it is sufficient to show that $\lim_n \|w(\cdot; W_n) - w(\cdot; W)\| = 0$, where $W_n \equiv W(b; A_n)$ and $W \equiv W(b; A)$. Now, observe that by Lemma 3, there exists N_ϵ such that for all $n > N_\epsilon$, $\|W_n - W\| < \epsilon$. Fix $B \in [\underline{b}, \bar{b}]$ and let ϕ_n^B and ϕ^B be the probability measures that attain $w(B, W_n)$ and $w(B, W)$, respectively. Then,

$$\int (W(b;A_n) - W(b;A))\phi^B(db) \le w(B;W_n) - w(B;W) \le \int (W(b;A_n) - W(b;A))\phi_n^B(db)$$

$$\Rightarrow$$

$$-\int \|W_n - W\|\phi^B \le w(B;W_n) - w(B;W) \le \int \|W_n - W\|\phi_n^B$$

$$\Rightarrow$$

$$-\|W_n - W\| \le w(B;W_n) - w(B;W) \le \|W_n - W\|$$

$$\Rightarrow$$

$$|w(B;W_n) - w(B;W)| \le \|W_n - W\|.$$

Since B was arbitrary, we have $\sup_B |w(B;W_n) - w(B;W)| = \|w(\cdot;W_n) - w(\cdot;W)\| \leq \|W_n - W\| < \epsilon$ for all $n > N_\epsilon$. Hence, $\lim_n \|w(\cdot;W_n) - w(\cdot;W)\| = 0$.

Claims (ii) and (iii): Let $A \in \mathscr{F}$. By Lemma 3, $W(b; A) \in \mathscr{C}$. Since $\mathscr{C} \subset \mathscr{W}$, by Theorem 7, $(HA)(b)$ is continuous and, of course, $(HA)(b) \in [\underline{b}, \bar{b}]$. Therefore, $H(\mathscr{F}) \subset \mathscr{F}$. $(HA)(b)$ is also Lipschitz with constant $1/\beta$ and, so, $H(\mathscr{F})$ is an equicontinuous family. ■

B Equivalence Between the Q-Geometric Discounting and Political Disagreement Models

There are two political parties, denoted by N and S, representing two types of people of equal measure. At any date, one of the types is in power and constitutes the government. Types change power randomly over time, with the probability that the type (or party) currently in power remains in power next period with probability 1/2. The level of revenues available each period is y.

Governments might spend on both their own constituents and the opposing party's constituents. Let (g_N, g_S) denote the spending on the two types of constituents. The utility obtained by the government of each type is

$$u_N(g_N, g_S) = \frac{g_N^{1-\gamma}}{1-\gamma} + \theta \frac{g_S^{1-\gamma}}{1-\gamma} \text{ and } u_S(g_N, g_S) = \theta \frac{g_N^{1-\gamma}}{1-\gamma} + \frac{g_S^{1-\gamma}}{1-\gamma},$$

where $\theta \in [0, 1]$ and $0 < \gamma < 1$. The parameter θ captures the degree of polarization in the economy: If $\theta = 0$, then neither type of government gets any benefit from expenditure directed toward the other type's constituents; if $\theta = 1$, then both types care equally about the expenditure directed toward the other type. The parameter γ controls how rapidly the marginal benefit of government expenditure declines with spending.

Let us suppose that region $k \in \{N, S\}$ is in power and optimally allocates available resources between the two types. Given the utility functions, it follows that $g_{\sim k} = \theta^{1/\gamma} g_k$.

Since nothing depends on the identity of the party in power, we need only distinguish governments by whether or not they are in power. Let g denote the per capita spending on members of the party in power. Then, the period utility of the party in power is

$$u_P\left(g\right) = \varphi_P \frac{g^{1-\gamma}}{1-\gamma}, \quad \text{where } \varphi_P = 1 + \theta^{1/\gamma},$$

and the period utility of the party that is out of power, as a function of g, is

$$u_O\left(g\right) = \varphi_O \frac{g^{1-\gamma}}{1-\gamma}, \quad \text{where } \varphi_O = \theta + \theta^{\frac{1-\gamma}{\gamma}}.$$

Since $\theta \in [0,1]$ and $1 \geq \gamma > 0$, $\varphi_P \geq \varphi_O$, where the equality holds only if $\theta = 1$ or $\gamma = 1$.

We can express the decision problem of the party in power recursively as follows:

$$V_P(b) = \max_{b' \in B} \varphi_P \frac{g^{1-\gamma}}{1-\gamma} + \beta\left[0.5 * \left(V_P(b') + V_O(b')\right)\right]$$

s.t.

$$g = \frac{1}{p}\left[y + b - qb'\right] \geq 0,$$

where $p = 1+\theta^{1/\gamma}$ and q is the price of the bond. Let $a(b)$ denote the solution to this decision problem. Then, the lifetime utility of the party currently out of power is

$$V_O(y,b) = \varphi_O \frac{\left(g(b)\right)^{1-\gamma}}{1-\gamma} + \beta\left[0.5 * \left(V_P(a(b)) + V_O(a(b))\right)\right].$$
$$g(b) = \frac{1}{p}\left[y + b - qa(b)\right].$$

Now observe that

$$
\begin{aligned}
&(V_P(b) + V_O(b)) = (\varphi_P + \varphi_O)\,\frac{(g(b))^{1-\gamma}}{1-\gamma} + [(V_P(a(b)) + V_O(a(b)))] \\
&\Rightarrow \\
&\frac{(V_P(b) + V_O(b))}{(\varphi_P + \varphi_O)} = \frac{g(b)^{1-\gamma}}{1-\gamma} + \beta\left[\frac{(V_P(a(b))) + V_O(a(b)))}{(\varphi_P + \varphi_O)}\right].
\end{aligned}
$$

Define

$$
W(b) = \frac{V_P(b) + V_O(b)}{\varphi_P + \varphi_O} \text{ and } V(b) = (1/\varphi_P)\,V_P(b).
$$

Then,

$$
W(b) = \frac{g(b)^{1-\gamma}}{1-\gamma} + \beta W(a(b)) \tag{35}
$$

and

$$
V(b) = \max_{b' \in B} \frac{g^{1-\gamma}}{1-\gamma} + \beta \frac{0.5\,(\varphi_P + \varphi_O)}{\varphi_P} W(b') \tag{36}
$$

s.t.

$$
g = \frac{1}{p}\,[y + b - qb'] \geq 0.
$$

Aside from the normalizing factor $1/p$, (35) and (36) are identical to the key equations of the quasi-geometric discounting model in the text, with $\delta = 0.5\,(\varphi_P + \varphi_O)\,/\varphi_P \leq 1$ and $q = \beta$.