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A TRACTABLE CIRCULAR CITY MODEL WITH AN APPLICATION TO THE EFFECTS OF DEVELOPMENT CONSTRAINTS ON LAND RENTS

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Abstract

A tractable production-externality-based circular city model in which both firms and workers choose location as well as intensity of land use is presented. The equilibrium structure of the city has either (i) no commuting ("mixed-use" form) or (ii) a central business district (CBD) of positive radius and a surrounding residential ring. Regardless of which form prevails, the intra-city variation in all endogenous variables displays the negative exponential form: $x(r) = x(0)e^{-\phi_x r}$ (where r is the distance from the city center and ϕ_x depends only on preference and technology parameters). An application is presented wherein it is shown that population growth may lead to a *smaller* increase in land rents in cities that cannot expand physically because these cities are less able to exploit the external effect of greater employment density.

Keywords: Land use, density gradients, agglomeration economies, commuting costs
JEL Codes: E10, R

1 Introduction

We present a model of a production-externality-based circular city in which both firms and workers choose location as well as intensity of land use. In our model, the internal structure of the city is endogenous and can take exactly one of two forms. Either the city is of the "mixed-use" form where there is no commuting and workers reside at the same location as the firm they work for or it has a "monocentric city" structure with a circular central business district (CBD) of positive radius and a surrounding residential ring. Regardless of which form prevails, the intra-city variation in all endogenous variables – residential and commercial rents, employment and residential densities, and wages – display (over their relevant domains) the negative exponential form: $x(r) = x(0)e^{-\phi_x r}$, where r is distance from the city center (which is indexed by 0) and ϕ_x depends only on preference and technology parameters.

Our model is a variant of the Lucas & Rossi-Hansberg (2002) circular city model in which the proximity between two firms located at different points in the plane is measured not as the crow flies (as is the case in Lucas & Rossi-Hansberg) but as the sum of the lengths of the rays connecting each point to the city center. With this one change, the equilibrium of their model simplifies dramatically to one with the properties noted above. Our model is also a close relative of what Anas, Arnott & Small (2000) call a *panexponential* city model, defined to be a city in which all rent and density gradients are negative exponentials. While Anas, Arnott and Small offered their definition in the context of the classic monocentric city model (where all employment, by assumption, occurs at the city center), negative exponentiality arises in our model for the more general setting where firms and workers compete for land at each point in the city.

Aside from connecting two ostensibly different theoretical structures, our model makes three other contributions. First, it strengthens the link between urban economic theory and an influential strand of empirical work that began with Clark (1951) and Mills (1969). The first of these early studies established that the negative exponential form gave a remarkably good description of population density in urban areas and the second showed that the same was true for land values in Chicago during most of the 19th century.[1] Since then, the negative exponential (or, equivalently, the log-linear) form has been analyzed extensively in the empirical literature.[2] By providing a structural interpretation to the exponents of e (or, equivalently, the coefficients on distance in the log-linear form), our model sheds light on the determinants of population, employment and land price variation within a city, and why these patterns change over time.

Second, it sharpens our understanding of the equilibrium relationship between the utility deliverable by a city and its population. In the standard monocentric city model, higher population leads to lower utility because there is no explicit benefit for firms to locate together at the center of the city (see for instance, Brueckner (1987)). In contrast, studies that focus on the creation of cities imply that the relationship between utility and population is an inverted-U (Henderson (1974)). These models posit some benefit from agglomeration but assume that firms must locate at the city center and, typically, also assume that intensity of land use by households is exogenously given (see, for instance, the survey by Abdel-Rahman & Anas (2004)). Thus, an important question left open in the existing literature is the conditions under which the inverted-U result is maintained when intra-city land use is analyzed in more detail.[3] We will show that for the inverted-U to emerge, the agglomeration parameter must be bounded above by a quantity that depends on parameters that govern the intensity of land use by businesses and households, and, in the case in which cities have the CBD form, it also requires that communication costs exceed commuting costs.

The third contribution of our paper is to present an application of our model to a substantive economic question. The question is the impact of urban growth controls on urban land price appreciation when the economy is experiencing population growth. Intuition might

[1] Eden & Sclar (1975) and Atack & Margo (1998) establish similar patterns in historical land values for Boston and New York City, respectively. More recently, Glaeser & Kahn (2001) have used the negative exponential form to track the evolution of employment density in U.S. metropolitan areas over the last 50 years or so.

[2] See, for instance, Anas, Arnott & Small (1998) and the references cited therein.

[3] This question was not addressed in Lucas & Rossi-Hansberg (2002), although the structure of their model is certainly suited to answering it.

suggest that land price appreciation will be greater in cities where urban development is more restricted.[4] We show that this intuition ignores the impact of production externalities: Population growth can cause cities that can expand more easily to experience *larger* increases in land prices than cities that cannot expand easily. This effect comes from the fact that a city that can expand more easily benefits more from the production externality. This is always true if cities are of mixed-use form and is true for the CBD form if the externality is more important in production than land.[5]

The paper is organized as follows. In the next section, we describe the basic environment. In section 3, we analyze the internal structure of the city and establish that it can take only one of two forms, depending on technological and preference parameters. In section 4, we analyze the nature of spatial equilibrium when the city is of the mixed-use form. In section 5, we do the same for the CBD city. Section 6 presents an application of the model to the role of urban development constraints on land rents when the economy as a whole experiences population growth. The Appendix gives proofs of some of the more technical results.

2 Environment

Space is modeled as a flat featureless plain extending infinitely in all directions, with an arbitrary point marked off as the center. In polar coordinates, the center is the point $(0,0)$ and all other points have coordinates (r,θ), where r is the length of a straight line connecting the point to the center and θ is the angle this line makes at $(0,0)$. Given that each point in space is physically indistinguishable from any other, it is natural to focus on allocations that are symmetric relative to the center. A location is then described fully by the radius,

[4]This intuition appears to have guided recent work on house price appreciation. Glaeser, Gyourko & Saiz (2008) looked for evidence of a negative relationship between elasticity of housing supply and the amplitude of housing price cycles. They found some supportive evidence but also noted that there are many cities with an elastic supply of housing that also experienced puzzlingly large price increases and subsequent declines. In the same vein, Davidoff (2010) finds a puzzling lack of relationship between supply elasticities and house price movements.

[5]As discussed later in the paper, this case can occur in our model without implying that all production is concentrated in one location.

r, of the circle centered at $(0, 0)$ on which it resides, and there is a continuum of locations for each r (all the points on the circle of radius r).

Utility function of a worker depends on the consumption of the single *numeraire* good avaliable in this economy and on the service flow from land. A worker who resides in location r has utility

$$U = c^{\beta}(r)l(r)^{1-\beta}, \;\; \beta \in (0, 1) \tag{1}$$

where $l(r)$ is the consumption of land in location r and $c(r)$ is consumption at location r.

A firm has a technology to produce the single consumption good. The production function of a firm that uses one unit of land at location s is:

$$Y(s) = Az(s)^{\gamma}n^{\alpha}(s), \;\; \alpha \in (0, 1), \gamma > 0 \tag{2}$$

where $n(s)$ is the number of workers per unit of land at location s, A is a TFP term that is common to all firms in the city, and $z(s)$ is a term – to be defined more precisely below – that captures the efficiency gain that comes from proximity to workers employed by firms in other locations.

A key assumption is that the proximity between any two firms is measured by the sum of the distance of the two firms from the city center. In other words, if one firm is located on a circle of radius r and the other firm is located on a circle of radius s, the distance of the firms to each other is simply $(r + s)$. The assumption that "distance" between two firms is measured by the sum of the lengths to the city center is reasonable if communication between workers in different firms requires travel to a central meeting place and the road system is radial.

Letting $N(s)$ denote the number of workers employed by a firm at location s, the total "production externality" utilized by a firm at location r is:

$$z(r) = \int_0^\infty 2\pi s \exp\left(-\delta\left(r+s\right)\right) N(s) ds.$$

Since $z(0) = \int_0^\infty 2\pi s \exp\left(-\delta s\right) N(s) ds$, the above definition implies

$$z(r) = z(0) \exp\left(-\delta r\right). \tag{3}$$

Thus, irrespective of the distribution of employment across the city, the strength of the production externality decays at the rate δ with distance from the city center. The spatial distribution of employment affects the strength of the production externality at any location only through the $z(0)$ term. This property of the production externality greatly simplifies model mechanics.

There is a technology for commuting. This technology allows workers to commute to any firm that is located on the straight line that connects the worker's residential location to the city center. We follow Anas, Arnott & Small (2000) and Lucas & Rossi-Hansberg (2002) and assume that a worker who resides in location s and commutes to a firm at location r has $\exp(-\kappa|s-r|)$ unit of time to devote to production, where $\kappa > 0$.[6]

There is also a technology for converting land from its natural state into land that can be used by workers and firms. The cost of converting a unit of natural land into developed land is d units of the consumption good.

Finally, following convention, it is assumed that all land in the economy is owned by entities outside of the model. These entities decide whether to convert any given unit of natural land into developed land and then rent the developed land to workers and firms.

[6] As noted in Anas, Arnott & Small (2000), this assumption is key to obtaining an exponentially declining land rent and population density function without making counterfactual assumptions on the structure of preferences for land. Coupled with our assumption regarding how proximity between firms is calculated, we can extend the negative exponential form to commercial rents as well as employment density.

3 On the Internal Structure of the City

In this section, we show that the spatial organization of the city is consistent with either one of two forms, depending on technology and preference parameters. From this we infer that only one of these two forms can occur in equilibrium. This result can be rigorously established by applying the method described in Lucas & Rossi-Hansberg (2002) (section 3, pp. 1453-1462) to the case in which the productivity function $z(r)$ is of the form in 3, namely, $z(0)\exp(-\delta r)$. In the interests of brevity, what we do here is simply derive conditions on parameter values under which the two forms can prevail and show that these conditions are mutually exclusive and exhaust the parameter space.

We will assume that the set of developed locations are all points on and inside of a circle of radius S (this circle defines the city boundary). The question we want to answer is: How is this developed land allocated between commercial and/or residential use? It is customary in urban economic theory to approach land use in terms of bid rent functions (Alonso (1964) and Fujita (1989)). Let $w(r)$ be the market wage at location r. Turning first to firms, let $q_F(r)$ be the maximum rent a firm would be willing to pay for a unit of land at location r. This quantity is simply $Az(r)^{\gamma}n^*(r)^{\alpha} - w(r)n^*(r)$, where $n^*(r)$ is the optimal choice of n conditional on locating at r, and is given by

$$n^*(r) = \left[A\alpha z(r)^{\gamma}/w(r)\right]^{1/(1-\alpha)}. \tag{4}$$

Then,

$$q_F(r) = [(1-\alpha)/\alpha]\left[\alpha A z(r)^{\gamma} w(r)^{-\alpha}\right]^{1/(1-\alpha)}. \tag{5}$$

As is intuitive, the maximum rent a firm is willing to pay depends positively on the location's productivity and negatively on the location's wage.

Turning to households, let $q_H(r, s)$ be the maximum rent a worker would be willing to pay for a unit of land at location r, given that he will work at location s. Conditional on paying $q_H(r,s)$ in rent, a worker's optimal choices of c and l at location r are

$$c^*(r,s) = \beta w(s) \exp(\kappa|s-r|) \text{ and } l^*(r,s) = (1-\beta) w(s) \exp(\kappa|s-r|)/q_H(r,s), \tag{6}$$

and his optimal utility is $\beta^\beta(1-\beta)^{1-\beta} w(s)\exp(\kappa|s-r|) q_H(r,s)^{-(1-\beta)}$. If $\bar{u}$ is the maximum utility a worker can obtain from locating somewhere else, then

$$q_H(r,s) = (1-\beta)\,\beta^{\beta/(1-\beta)} (w(s)\exp(\kappa|s-r|)/\bar{u})^{1/(1-\beta)}. \tag{7}$$

As is intuitive, the maximum rent a worker is willing to pay for land at r depends positively on the wage he earns and negatively on the utility he can get elsewhere.

Consider first the mixed-use case in which firms and their workers co-locate. In this case, the bid rent functions $q_F(r)$ and $q_H(r,r)$ must coincide for all $r \in [0,S]$; otherwise, either firms will outbid workers or workers will outbid firms, and a location will be fully commercial or fully residential. Setting s equal to r in the bid rent function for households, setting the resulting bid rent function equal to the bid rent function for firms, and using the expression in 3 for $z(r)$ implies

$$\begin{aligned} w(r) &= [(1-\alpha)\alpha^{-\alpha}]^{\frac{1-\beta}{1-\alpha\beta}} [\beta^{(1-\beta)}(1-\beta)^\beta]^{-\frac{1-\alpha}{1-\alpha\beta}} A^{\frac{1-\beta}{1-\alpha\beta}} \bar{u}^{-\frac{1-\alpha}{1-\alpha\beta}} z(0)^{\frac{\gamma(1-\beta)}{1-\alpha\beta}} \times \\ &\quad \exp\left(-\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} r\right). \end{aligned} \tag{8}$$

It is evident that wages decline exponentially from the city center, reflecting the fact that the production externality is felt most strongly at the center. However, for this wage profile to be an equilibrium, it must be the case that workers do not have an incentive to commute to a job closer to the city center to take advantage of higher wages. This requires that the rise in wages as a worker commutes toward the center not exceed the loss in working time due to commuting, namely,

$$\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} \leq \kappa. \tag{9}$$

If commuting costs are high (κ is large), if the production externality is weak (γ is small), and if communication between workers in different locations is not too difficult (δ is low), then mixed use of urban land can be sustained in equilibrium.

We now turn to the case in which the city has a CBD structure. In this case, there is an endogenously determined boundary $S_F < S$ such that all $r \in [0, S_F)$ are devoted to production and all $s \in (S_F, S]$ are devoted to residential use and the boundary S_F can be devoted to either use. If there is a central business district, workers must be indifferent between working at different locations within this district. This implies that in the business district the wages must satisfy the condition

$$w(r) = w(0) \exp(-\kappa r) \;\text{ for } r \in [0, S_F]. \tag{10}$$

Substituting this into the expression for $n^*(r)$ and using the expression for $z(r)$ in (3) yields

$$q_F(r) = [(1-\alpha)/\alpha] w(r) n(r) = q_F(0) \exp\left(\frac{\kappa\alpha - \delta\gamma}{1-\alpha} r\right).$$

Given that workers earn the same regardless of where they work, there is no need to keep track of their place of work in order to determine their bid rent for a particular residential location. The maximum rent a worker is willing to pay for land at location r and still get a utility of $\overline{u}$ (using the equilibrium condition above) is:

$$q_H(r) = (1-\beta)\, \beta^{\frac{\beta}{1-\beta}} \left(\frac{w(0)\exp(-\kappa r)}{\overline{u}}\right)^{\frac{1}{1-\beta}}.$$

For CBD structure to be an equilibrium outcome, it must be the case that at the boundary of the CBD, the bid rent curve for firms is steeper than the bid rent curve of workers. This requirement imposes a constraint on the admissable value of κ. Observe that

$$\ln q_H(r) = \ln(1-\beta) + \frac{\beta}{1-\beta} \ln\beta + \left(\frac{1}{1-\beta}\right) \ln\left(\frac{w(0)}{\overline{u}}\right) - \frac{\kappa}{1-\beta} r,$$

which implies that the slope of the worker's bid-rent function at S_F is $[-\kappa/(1-\beta)]q_H(S_F)$. Next, observe that

$$\ln q_F(r) = \ln(q_F(0)) + \left(\frac{\kappa\alpha - \delta\gamma}{1-\alpha}\right) r$$

which implies that the slope of the firm's bid-rent function at S_F is $[(\kappa\alpha-\delta\gamma)/(1-\alpha)]q_F(S_F)$. Since at the boundary of the CBD $q_H(S_F) = q_F(S_F)$, the necessary slope condition boils down to two conditions. First, it must be the case that $\kappa\alpha - \delta\gamma < 0$ (otherwise the bid-rent curve for firms would be rising away from the center) and, second, it must be the case that $-(\kappa\alpha - \delta\gamma)/(1-\alpha) > \kappa/(1-\beta)$. This implies that

$$\kappa < \frac{(1-\beta)\gamma\delta}{(1-\beta\alpha)}. \tag{11}$$

Since both α and β are less than unity, it follows that the factor $(1-\beta)/(1-\beta\alpha)$ is less than 1. Therefore, the above condition implies that $\kappa < \gamma\delta$, which in turn implies that $\alpha\kappa < \gamma\delta$. Therefore, the only condition that κ must satisfy in order for the city to have a CBD is (11). Observe that this condition is the exact complement of the condition (9). This shows that the internal structure of the city can be only one of these two types.

4 The Mixed-Use City

The goal of this section (and the next) is to understand the nature of the equilibrium relationship between the utility deliverable by a city and its population. We will show that for the inverted-U to emerge, the agglomeration parameter γ must be bounded above by a quantity that depends on parameters that govern the intensity of land use by businesses *and* workers.

In the first part of the analysis, we will take the population of the city P and its size S as given and see how employment and residential densities as well as wages and rents in each location are determined. In the second part of the analysis, we will endogenize S, which will depend on the cost of converting undeveloped land into developed urban land.

To begin, observe the expression for $n^*(r)$ along with the expressions for $w(r)$ in (8) and $z(r)$ in (3), pins down the equilibrium spatial profile of employment density as a negative exponential function:

$$n(r) = n(0) \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right). \tag{12}$$

Employment density declines with distance from the city center because labor is less productive farther from the city center, and, notwithstanding the fact that wages are also lower farther from the city center, fewer workers can be profitably employed on a unit of land. The rate of decline is faster the larger is the communication cost parameter δ.

The value of $n(0)$ is determined by invoking the two market clearing conditions that apply in this model. First, there is the labor market clearing condition. Since a firm and its workers co-locate, each location is a "local labor market" and demand and supply have to match for each location. Letting $\theta(s)$ denote the fraction of land that is devoted to production in location r, we can express this requirement as

$$n^*(r)\theta(r) = [1-\theta(r)]/l^*(r). \tag{13}$$

Since $n^*(r) = [\alpha q(r)]/[(1-\alpha)w(r)]$ and $l^*(r) = (1-\beta)w(r)/q(r)$, we find that $\theta(r) = [1-\beta]/[1-\alpha\beta] = \theta$.[7] Thus, the proportion of land devoted to production is constant across all locations in the city, and the level of employment in location r, $N(r)$, is simply $\theta n(r)$.

The second market clearing condition requires that the total number of residents in the

[7]The mixed-use case has also been analyzed in Wheaton (2004) for an exogenously given productivity gradient and exogenously given land use intensities for firms and workers. Wheaton does not impose the local labor market clearing condition (13). Instead, the fraction of land in use by firms (or workers) at any location is determined by the relative magnitude of the rent levels for each use.

city must equal the total population of the city, P, namely,

$$P = \int_0^S 2\pi r[1-\theta]/l^*(r)dr.$$

Using (12) and (13), the above implies

$$P = 2\pi\theta n(0) \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr. \tag{14}$$

Using (14), we can express employment density at the city center as a function of S and P:

$$n(0) = \frac{P}{2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{15}$$

Knowing $N(r)$ also allows us to express the strength of the production externality at the city center in terms of $n(0)$, namely,

$$z(0) = 2\pi \int_0^S r \exp(-\delta r) N(r) ds = 2\pi\theta n(0) \int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr, \tag{16}$$

which then implies

$$z(0) = P \frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{17}$$

Once $z(0)$ and $n(0)$ are determined, rents in each location can be recovered from expression $q(r) = (1-\alpha)Az(r)^\gamma n(r)^\alpha$:

$$q(r) = (1-\alpha)Az(0)^\gamma n(0)^\alpha \exp\left(-\frac{\gamma\delta}{1-\alpha\beta} r\right) \tag{18}$$

Finally, equilibrium residential density is given by

$$1/l(r) = (1-\beta)/[\beta(1-\alpha)]n(0)\exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right). \tag{19}$$

Before proceeding further, it may be helpful to summarize some partial equilibrium results so that the basic mechanisms in the model can be highlighted. Proposition 1 deals with the effect of changes in population, holding fixed the size of the city.

Proposition 1 *(The Effect of Population Size P): Holding S constant, (i) employment and productivity rise proportionately with P, (ii) the elasticity of rents in any location with respect to P is $\alpha+\gamma$, (iii) the elasticity of wage in any location with respect to P is $\alpha+\gamma-1$, and (iv) elasticity of $\bar{u}$ with respect to P is $\beta(\alpha+\gamma)-1$.*

Proof. (i) follows from the fact that both $n(0)$ and $z(0)$ are proportional to P; (ii) follows from (i) and the fact that $q(r) = (1-\alpha)Az(r)^{\gamma}n(r)^{\alpha}$; (iii) follows from (i) and the fact that $w(r) = \alpha Az(r)^{\gamma}n(r)^{\alpha-1}$; (iv) follows from (ii) and (iii) and the fact that $\bar{u} = \beta^{\beta}(1-\beta)^{1-\beta}w(r)q(r)^{-(1-\beta)}$.

Proposition 2 deals with effects of changes in the size of the city, holding city population fixed.

Proposition 2 *(The Effect of City Size S): Holding P constant, (i) employment, productivity, and rents at the city center are decreasing in S, (ii) if $\alpha+\gamma \leq 1$, wages at the city center are decreasing in S, otherwise the effect is ambiguous, (iii) if $\beta(\alpha+\gamma) \leq 1$, $\bar{u}$ is increasing in S, otherwise the effect is ambiguous.*

Proof. Knowing $n(0)$ and $z(0)$ in terms of P and S allows us to express $w(0)$, $q(0)$, and $\bar{u}$

in terms of P and S. For $w(0)$ we substitute (15) and (17) into (4) to obtain

$$w(0) = \alpha A P^{(\alpha+\gamma)-1} \left[\frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr} \right]^{\gamma} \times \left[2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr \right]^{(1-\alpha)}. \quad (20)$$

For $q(0)$, we substitute (20) and (17) into (5) to obtain

$$q(0) = (1-\alpha) A P^{(\alpha+\gamma)} \left[\frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr} \right]^{\gamma} \times \left[2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr \right]^{-\alpha}. \quad (21)$$

For $\bar{u}$ we substitute (15) and (17) into $\beta^\beta (1-\beta)^{1-\beta} w(r) q(r)^{-(1-\beta)}$ to obtain

$$\bar{u} = K A^\beta P^{-(1-\beta(\alpha+\gamma))} \times \quad (22)$$
$$\left[\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right] r\right) dr \right]^{\gamma\beta} \left[2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr \right]^{1-\beta(\alpha+\gamma)},$$

where K is a positive constant.

By Lemma 2 (in the Appendix), $(\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right] r\right) dr)/(\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr)$ is decreasing in S. Given this, the results follow from the expressions given above. ■

So far we have taken S as given. Now we turn to the determination of S. Since it costs d units of the consumption good to convert one unit of undeveloped land into urban land, the equilibrium value of S is simply one that solves:

$$q(S; P) = d, \quad (23)$$

where $q(S; P)$ denotes the rent at the boundary of the city when the population is P. The following proposition establishes monotonicity of rent at the city boundary with respect to the distance of the boundary from the city center. This property implies that a unique value of S solves (23).

Proposition 3 *Given P, the rent at the boundary of the city is strictly decreasing in S with $\lim_{S\to 0} q(S;P) = \infty$ and $\lim_{S\to\infty} q(S;P) = 0$. Furthermore, $q(S;P)$ is increasing in P.*

Proof. : From (18) we have that

$$q(S;P) = q(0)\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right).$$

By Proposition 2, $q(0)$ is decreasing in S. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right)$ is strictly decreasing in S, it follows that $q(S;P)$ is strictly decreasing in S. To establish the limit properties, we use equation (21). As S approaches 0, the term $\int_0^S r\exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right)dr/\int_0^S r\exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right)dr$ in $q(0)$ approaches 1 (this follows from an application of L'Hospital's Rule) and the term $\left[2\pi\theta\int_0^S r\exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right)dr\right]^{-\alpha}$ approaches infinity. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right)$ approaches 1, it follows that $\lim_{S\to 0} q(S;P) = \infty$. Going the other way, as S approaches ∞, all three terms approach 0. Hence, $\lim_{S\to\infty} q(S;P) = 0$. To prove the second part, observe from (21) that, given S, $q(0)$ is increasing in P. Therefore, $q(S;P)$ is increasing in P. ■

Corollary 1 *Given P, there exists a unique S_d that solves $q(S_d;P) = d$.*

Finally, we come to the relationship between U (the utility deliverable by a city) and P when the city boundary adjusts so that the rent at the boundary is d. We will denote this relationship by $U_d(P): \mathbb{R}_{++} \to \mathbb{R}$.

We can think of this function as a composition of two functions: One function is the relationship between S and P, denoted $S_d(P)$, that gives the size of the city when the population is P and rent at the boundary is d. The other function, denoted $U_d(S)$, gives the relationship between utility deliverable by a city of size S, when rent at the boundary is d. Unfortunately, the function $S_d(P)$ is an implicit one, without a closed-form expression. As a result, $U_d(P)$ is also implicitly defined. However, the function $U_d(S)$ does have a closed-form (and easily visualizable) expression. Therefore, in what follows, we first characterize the (implicit) function $S_d(P)$ and then the explicit function $U_d(S)$. The end result will be an implicit characterization of $U_d(P) = U_d(S_d(P))$.

The $S_d(P)$ function is given by the requirement that the rent at the boundary of the city be d. From (21) and (23), it follows that $S_d(P)$ satisfies:

$$d = (1-\alpha)AP^{(\alpha+\gamma)} \left[\frac{\int_0^{S_d(P)} r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr}{\int_0^{S_d(P)} r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr} \right]^{\gamma} \times$$
$$\left[2\pi\theta \int_0^{S_d(P)} r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr \right]^{-\alpha} \left[\exp\left(-\frac{\gamma\delta}{1-\beta\alpha} S_d(P)\right)\right].$$

Proposition 4 *$S_d(P) : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is strictly increasing in P. Furthermore, $\lim_{P\to 0} S_d(P) = 0$ and $\lim_{P\to\infty} S_d(P) = \infty$.*

Proof. There are three square-bracketed terms in which $S_d(P)$ appears. The last term is strictly decreasing in S_d. Ignoring the exponent $-\alpha$, the second term in square brackets is strictly increasing in S_d. Therefore, taking into account the exponent, this term is also strictly decreasing in S_d. Finally, by Lemma 2, the first term in square brackets is also strictly decreasing in S_d. Therefore, the r.h.s. of the above equation is strictly decreasing in S_d. Since the r.h.s. is strictly increasing in P, it follows that $S_d(P)$ must be strictly increasing in P.

To establish the first limiting property, let $\{P_n\}$ be a sequence converging to 0. From the first part, we know that $S_n = S(P_n)$ is a strictly decreasing sequence. Since S_n is bounded below by 0, it follows that $\lim S_n = S_0 \geq 0$ must exist. Suppose, to get a contradiction, that $S_0 > 0$. Then all square-bracketed terms converge to positive numbers. This implies that the r.h.s. converges to 0, which is impossible since the l.h.s. is strictly positive. Therefore, $\lim_{P\to 0} S_d(P) = 0$.

To establish the second limiting property, let $\{P_n\}$ be a sequence diverging to ∞. By the first part of this proposition, $S_n = S(P_n)$ is a strictly increasing sequence. We claim that this sequence is unbounded. Suppose not. Then, since the sequence is strictly increasing and bounded above, it must converge to a limit $S_\infty > 0$. This means that all the terms involving $S_d(P_n)$ converge to positive numbers. Therefore, the r.h.s. diverges to ∞, which is impossible since the l.h.s. is finite. Therefore, $\lim_{P\to\infty} S(P_n) = \infty$. ■

The $U_d(S)$ function also derives from the requirement that rent at the boundary be d, that is from the requirement, $d = q(0)e^{-\frac{\delta\gamma}{1-\alpha\beta}S}$. To obtain this relationship, we use the expression for $q(0) = (1-\alpha)\,Az(0)^\gamma n(0)^\alpha$, which can be expressed in terms of only $n(0)$ using (16). The other equation we will use is that $q_H(0) = q_F(0) = q(0)$. This gives us the condition $(1-\alpha)\,Az(0)^\gamma n(0)^\alpha = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}}\left(\frac{w(0)}{\bar{u}}\right)^{\frac{1}{1-\beta}}$, which again can be expressed in terms of only $n(0)$and $\bar{u}$ by expressing $w(0)$ and $z(0)$ only in terms of $n(0)$and other parameters.

When these two equations in terms of $n(0)$ and $\bar{u}$ are solved, we obtain

$$
\begin{aligned}
&U_d(S) = Kd^{-\frac{(1-\beta(\alpha+\gamma))}{\alpha+\gamma}}A^{\frac{1}{(\alpha+\gamma)}} \times \\
&\exp\left(-\frac{\delta\gamma\,(1-\beta(\alpha+\gamma))}{(1-\alpha\beta)\,(\alpha+\gamma)}S\right) * \left[\int_0^S 2\pi r \exp\left(-\frac{\delta\,(1-\beta\alpha+\gamma\beta)}{1-\beta\alpha}r\right)dr\right]^{\frac{\gamma}{(\alpha+\gamma)}}, \quad (24)
\end{aligned}
$$

where K is a positive constant.

Proposition 5 *If* $1-\beta(\alpha+\gamma) > 0$, $U_d(S)$ *is single-peaked with respect to* S, *with* $\lim_{S\to 0} U_d(S) = \lim_{S\to\infty} U_d(S) = 0$ *and the peak occuring for a strictly positive* S.

Proof. It is clear that $\lim_{S\to 0} U_d(S) = 0$. Furthermore, since $1-\beta(\alpha+\gamma) > 0$, the integral term in square brackets converges to a positive finite number as S diverges to ∞ and the S term converges to 0. Hence, $\lim_{S\to\infty} U_d(S) = 0$.

To determine the shape of this function, it is convenient to examine the $\ln(U_d(S))$:

$$
\begin{aligned}
&\ln(U_d(S)) = D + \\
&\frac{\gamma}{\gamma+\alpha}\ln\left(\int_0^S 2\pi r\exp\left(-\frac{\delta\,(1-\beta\alpha+\gamma\beta)}{1-\beta\alpha}r\right)dr\right) - \left(\frac{\gamma\delta\,(1-\beta\alpha-\gamma\beta)}{(\gamma+\alpha)\,(1-\beta\alpha)}S\right),
\end{aligned}
$$

where D is a constant. Thus, on the logarithmic scale, $U_d(S)$ has a linear component that starts at 0 and declines linearly with S and a component that starts at $-\infty$ and increases at most logarithmically with S. Since the rate of growth of a log function at 0 is infinite, it is clear that $\ln(U_d(S))$ increases at $S = 0$ and reaches a peak at some point $S > 0$. Hence, $U_d(S)$ has the same shape. ■

Corollary 2 *If $1 - \beta(\alpha + \gamma) > 0$, $U_d(P) = U_d(S_d(P))$ is single-peaked with respect to S. Furthermore, $\lim_{P\to 0} U_d(P) = \lim_{P\to\infty} U_d(P) = 0$.*

Proof. Treating $U_d(P)$ as differentiable with respect to P, we have

$$\frac{\partial U_d(P)}{\partial P} = \frac{\partial U_d(S)}{\partial S}\frac{\partial S_d(P)}{\partial P}.$$

Since the second term on the r.h.s. is always positive, the shape of the $U_d(P)$ function is determined by the shape of the $U_d(S)$ function. Hence, $U_d(P)$ is also single-peaked. The limiting properties follow directly from (24) and the limiting properties of $S_d(P)$. ■

It is worth noting that Proposition 5 is consistent with the results in Propositions 1 and 2. Recall that Proposition 1 states that when $1-\beta(\alpha+\gamma) > 0$, utility in the city is decreasing in P and Proposition 2 states that under the same condition utility is increasing in S. Thus, as S increases and the city fills up with people so that the rent at the boundary is d, there are two offsetting forces working on utility obtained by residents of the city. When the city is physically small, the utility enhancing effect of S is stronger than the utility decreasing effect of higher population. Eventually, though, the utility depressing effect of higher population dominates and utility declines with S.

To understand better why utility is increasing when P is low but declining when P is high, it is helpful to think of the case in which the city cannot expand at all. In this case, utility deliverable by the city declines as population increases. With population growth, even if the wages increase (which happens when $\alpha + \gamma - 1 > 0$), utility declines because the increase in wages, and the implied increase in c, is not large enough to compensate for the lower consumption of residential space. This comes from the condition $1 - \beta(\alpha + \gamma) > 0$. In the case in which the city can expand at the cost of d, the city does expand with higher P and allows workers to increase their consumption of land, but only when the city is small. As we see from the equation (15), employment density at the center (which is inversely proportional to the consumption of land per worker at the center) becomes increasingly insensitive to an increase in S as S gets higher. In the limit, equilibrium allocations at the center become similar to the case in which S does not change. Although some people move to the outskirts when S goes up, they form an increasingly small portion of the general population, and, so, this re-shuffling does not affect employment and residential densities in the center much.

The condition $1 - \beta(\alpha + \gamma) > 0$ is our analog of what Fujita, Krugman & Venables (1999) call the "no-black-hole condition." If this condition is violated then, as is evident from the expression of $\ln(U_d(S))$, utility deliverable by the city would be increasing in S. Since $S_d(P)$ is always increasing in P, this, in turn, would imply that utility deliverable by the city would be increasing in P. Under these conditions, the model would imply that all production should take place in one giant city. To rule this out (hence the appellation "no-black-hole condition"), the strength of increasing returns in the model must be bounded above.[8]

5 The CBD City

As in the previous section, we will first explain the determination of employment and residential density functions, and the wage and rent functions for given values of S and P. In the second part, we will explain how S is determined.

To begin, the expression for $n^*(r)$ along with the expression for $w(r)$ in (10) pins down

[8] It is of interest to note that Lucas (2001) and Lucas & Rossi-Hansberg (2002) assume a condition that is stronger, namely, $1 - (\alpha + \gamma) > 0$. Although this condition is also labeled a "no-black-hole condition," it is needed to rule out a different kind of black hole, one in which all firms pile up at 0 (the city center) with each firm using a vanishingly small amount of land but enjoying unboundedly high external effect. This possibility is not a concern for us because the form of the external effect function implies that equilibrium employment density must have the negative exponential form and, hence, productivity at the city center is naturally bounded above by city size and total population, as seen in (17).

the equilibrium spatial profile of employment density as:

$$n(r) = n(0)\exp\left(-\frac{(\delta\gamma-\kappa)}{1-\alpha}r\right) \text{ for all } s \in [0, S_F].$$

Thus, employment density declines with distance from the city center. The rate of decline is faster the larger is the communication cost parameter δ, smaller is commuting cost κ, and smaller is $(1-\alpha)$ (i.e., less important is land in the production function). The rate of decline depends positively on γ also, which comes from the fact that when the external effect is strong, firms have stronger reason to congregate toward the center of the city.

From our discussion of bid rent functions for the CDB, we have that the equilibrium rent function $q(r)$ is given by

$$q(r) = \max\left\{q_F(0)\exp\left(-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}r\right), q_H(0)\exp\left(-\frac{\kappa}{1-\beta}r\right)\right\} \tag{25}$$

The rent function is a continuous function declining in distance from the city center with the speed of decline undergoing a discrete drop at the boundary between the commercial and residential districts. The fact that there must be a discrete drop follows from (11). Observe that at S_F

$$q_F(0)\exp(-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}S_F) = q_H(0)\exp(-\frac{\kappa}{(1-\beta)}S_F). \tag{26}$$

The labor market clearing condition ensures that exactly the right level of labor is available in each work location, taking into account the time lost in commuting to work. In the mixed city case, this condition is determined by $\theta(s)$. In the CBD case, we already know that $\theta(s) = 1$ for $s \in [0, S_F]$ and $\theta(s) = 0$ for $s \in (S_F, S]$. As we will see, what this condition determines now is the boundary of the commercial district, namely, S_F. To develop this condition, note that taking into account the time lost in commuting, the total supply of labor time available at the border of the CBD is $\int_{S_F}^{S} \frac{2\pi r}{l(r)} e^{-\kappa(r-S_F)} dr$. If the employment density at a CBD location r is $n(r)$, the labor time needed at the border of the commercial district to fulfill this demand is $e^{\kappa(S_F - r)} n(r)$. Therefore, the total time needed at the border of the CBD to satisfy total labor demand inside the commercial district is $\int_0^{S_F} 2\pi r n(r) e^{\kappa(S_F - r)} dr$. Equality of labor demand and supply then requires

$$\int_0^{S_F} 2\pi r n(r) \exp\left(\kappa(S_F - r)\right) dr = \int_{S_F}^{S} \frac{2\pi r}{l(r)} \exp\left(-\kappa(r - S_F)\right) dr,$$

which, using the fact that $l(r) = (1-\beta) w(0) e^{-\kappa r} / q_H(r)$ and the expressions for $n(r)$ and $q_H(r)$ derived earlier, simplifies to:

$$n(0)w(0)(1-\beta) \int_0^{S_F} r \exp\left(-\frac{\delta\gamma - \kappa\alpha}{1-\alpha} r\right) dr = q_H(0) \int_{S_F}^{S} r \exp\left(-\frac{\kappa}{(1-\beta)} r\right) dr. \tag{27}$$

Finally, using (26) and $\frac{1-\alpha}{\alpha} n(0) w(0) = q_F(0)$, we obtain:

$$\left[\int_{S_F}^{S} s \exp\left(-\frac{\kappa}{1-\beta} s\right) ds\right] =$$

$$\frac{(1-\beta)}{(1-\alpha)} \alpha \left[\int_0^{S_F} s \exp\left(\frac{\alpha\kappa - \gamma\delta}{1-\alpha} s\right) ds\right] \exp\left(\frac{-\kappa + \delta\gamma + \beta\kappa\alpha - \beta\delta\gamma}{(1-\alpha)(1-\beta)} S_F\right) \tag{28}$$

Observe that this is an equation in which the only endogenous variables are S_F and S. Taking S as given, we have the following proposition regarding the relationship between S_F and S.

Proposition 6 *For each $S > 0$, there exists a unique $S_F(S) \in (0, S)$, where $S_F(S)$ is strictly increasing in S. Furthermore, $\lim_{S \to 0} S_F(S) = 0$ and $\lim_{S \to \infty} S_F(S) = \bar{S}_F > 0$.*

Proof. Given any $S > 0$, the upper bound on κ implies that the r.h.s. of (28) is increasing in S_F. The l.h.s. of (28) is clearly decreasing in S_F. Furthermore, the r.h.s. is 0 for $S_F = 0$, while the l.h.s. is strictly positive, and the r.h.s. is strictly positive for $S_F = S$, while the l.h.s. is 0. Therefore, for each $S > 0$ there is a unique $S_F \in (0, S)$ that ensures that (28) is satisfied. Observe also that as S goes up and S_F does not change, the integral on the l.h.s.

goes up. Since the r.h.s. is increasing in S_F, the equilibrium S_F must be strictly higher. Thus $S_F(S)$ is strictly increasing in S.

To prove the second part, observe that since $S_F(S) < S$ for all S, it must be the case that $\lim_{S\to 0} S_F(S) = 0$. To prove the other limiting result, we will first establish that $\lim_{S\to\infty} S_F(S)$ is bounded above. Let S_n be an increasing sequence diverging to ∞. Let $S_F(S_n)$ be a corresponding sequence of S_F that satisfies (28). Then $S_F(S_n)$ is also a strictly increasing sequence. Suppose that this sequence diverges to ∞. Then, by Lemma 2, we know that

$$\lim_{S_n\to\infty} \int_{S_F(S_n)}^{S_n} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds = 0.$$

But this implies that the l.h.s. of (28) converges to 0, while the r.h.s. diverges to ∞, which is impossible. Hence, $S_F(S_n)$ must be bounded above. Since $S_F(S)$ is strictly increasing, it follows that $\lim S_F(S)$ must converge to some number $\bar{S}_F > 0$. ■

Proposition 6 makes sharp predictions about the size of the CBD relative to the size of the city. In particular, controlling for the physical size of the city, the physical size of the CBD is predicted to *not* depend upon population size or productivity of the city. Also, it predicts that there is an upper bound on CBD size that depends only on technology and preference parameters. Since the monocentric city literature tends to ignore the determination of the size of the CBD, these – potentially testable – implications of the model are noteworthy.

The requirement that the number of residents of the city equal total population P pins down $n(0)$. Specifically, we must have $P = \int_{S_F}^{S} \frac{2\pi r}{l(r)} dr$, where the integrand gives the total number of residents living on each circle of radius r. Observing that $l(r) = (1-\beta)w(0)\exp(-\kappa r)/q_H(0)\exp\left(-\frac{\kappa}{1-\beta}r\right)$, yields

$$P = \frac{q_H(0)}{(1-\beta)w(0)} \int_{S_F}^{S} 2\pi r \exp\left(-\frac{\beta\kappa}{(1-\beta)}r\right) dr.$$

Using (26), the fact that $q_F(0) = (1-\alpha)w(0)n(0)$, and (28), we obtain:

$$n(0) = \frac{P}{\left[2\pi \int\limits_{S_F(S)}^{S} r\left(\exp \frac{-\kappa\beta}{1-\beta} r\right) dr\right]} \frac{\left[\int\limits_{S_F(S)}^{S} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{0}^{S_F(S)} r \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha} r\right) dr\right]}. \tag{29}$$

Knowing $n(0)$ allows us to pin down the external effect at the city center. Observe that $z(0) = n(0) \int_0^{S_F} 2\pi r \exp\left(\frac{\kappa-\delta\gamma}{1-\alpha} - \delta\right) r dr$ and so,

$$z(0) = \frac{P\left[\int\limits_{0}^{S_F} 2\pi r \exp\left(\left(\frac{\kappa-\delta\gamma}{1-\alpha} - \delta\right) r\right) dr\right]}{\left[2\pi \int\limits_{S_F}^{S} r \exp\left(\frac{-\kappa\beta}{1-\beta} r\right) dr\right]} \frac{\left[\int\limits_{S_F}^{S} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{0}^{S_F} r \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha} r\right) dr\right]} \tag{30}$$

Once $n(0)$ and $z(0)$ are known, all other endogenous variables (employment and residential density, wages and rents by location) can be easily determined.

As in the mixed-use case, we provide some partial comparative static results in order to highlight how the model with CBD works.

Proposition 7 *(The Effect of Population Size P): Holding S constant, (i) employment and productivity rise proportionately with P, (ii) the elasticity of rents in all locations with respect to P is $\alpha+\gamma$, (iii) the elastcity of wages with respect to P is $\alpha+\gamma-1$, and (iv) the elasticity of utility deliverable by the city with respect to P is $\beta(\alpha+\gamma)-1$.*

Proof. If S is fixed, then by Proposition 6 S_F is fixed as well. Then (i) follows from the expressions given above for $n(0)$ and $z(0)$; (ii) follows from (i), (25), (26) and $q_F(0) = (1-\alpha)Az(0)^\gamma n(0)^\alpha$; (iii) follows from (i), (ii), and the fact that $w(r) = [\alpha/(1-\alpha)]q_F(r)/n(r)$; and (iv) follows from (ii) and (iii) and the fact that $\bar{u} = \beta^\beta(1-\beta)^{1-\beta} w(S_F) q_H(S_F)^{-(1-\beta)}$ ■

Proposition 8 *(The Effect of City Size S): Holding P constant and provided that $\delta > \kappa$ (i) employment, productivity, and rents at the city center are decreasing in S (ii) if $\alpha+\gamma \leq 1$,* *wages in all locations are increasing in S; otherwise, the effect of S on wages is ambiguous, (iii) if $\beta(\alpha+\gamma) \leq 1$, $\bar{u}$ is increasing in S, otherwise the effect of S on $\bar{u}$ is ambiguous.*

Proof. (i) Lemma 3 in the Appendix establishes that, holding fixed P, $n(0)$ is decreasing in S and, provided $\delta > \kappa$, $z(0)$ is decreasing in S. Since $q_F(0) = (1-\alpha)Az(0)^\gamma n(0)^\alpha$, it follows that rents at the city center are also declining in S. For (ii) we know that $w(0) = \alpha A z(0)^\gamma n(0)^{\alpha-1}$. Using the fact that $z(0) = n(0)\int_0^{S_F} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr$, we have that

$$w(0) = \alpha A \left[\int_0^{S_F} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr\right]^\gamma n(0)^{\alpha+\gamma-1}. \tag{31}$$

Hence, given (i), we have that if $\alpha+\gamma \leq 1$, $w(0)$ is increasing in S. For (iii), note that using 26) and $\bar{u} = \beta^\beta(1-\beta)^{1-\beta} w(S_F) q_H(S_F)^{-(1-\beta)}$, we obtain

$$\bar{u} = KA^\beta e^{\frac{[\delta\gamma-\kappa]-\beta[\delta\gamma-\alpha\kappa]}{1-\alpha}S_F} \left[\int_0^{S_F} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr\right]^{\gamma\beta} n(0)^{-(1-\beta(\alpha+\gamma))}, \tag{32}$$

where K is a positive constant. Since $[\delta\gamma-\kappa]-\beta[\delta\gamma-\alpha\kappa] > 0$ (by the upper bound on κ), it follows that $\bar{u}$ is increasing in S provided $\beta(\alpha+\gamma)) \leq 1$. ■

Now we turn to the determination of S. The following proposition establishes that rent at the boundary is decreasing in the city size, provided the communication cost parameter δ is larger than the commuting cost parameter κ.

Proposition 9 *Given P, $q_H(S)$ is decreasing in S if $\delta > \kappa$. Furthermore, (i) $\lim_{S\to\infty} q_H(S) = 0$ and (ii) $\lim_{S\to 0} q_H(S) = \infty$.*

Proof. See Appendix.

Corollary 3 *Given P, there exists a unique $S_d(P)$ such that $q_H(S_d(P), 0) = d$.*

Now we turn to characterizing the relationship between the utility deliverable by the CBD city and its population. As in the mixed-use city case, we will view this relationship as a composition of two functions: $U_d(P) = U_d(S_d(P))$.

The relationship between S and P when rent at the boundary is d is obtained by setting $d = q_H(S)$. Since $q_H(S) = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S}$ and $q_F(0) = (1-\alpha)Az(0)^\gamma n(0)^\alpha$ from (26), (29), (30) we get

$$
\begin{aligned}
d &= (1-\alpha)AP^{\alpha+\gamma} \times \\
& \left[\left[\frac{\int_0^{S_F(S_d(P))} s \exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right) ds}{\int_0^{S_F(S_d(P))} s \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right) ds} \right] \left[\frac{\int_{S_F(S_d(P))}^{S_d(P)} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds}{\int_{S_F(S_d(P))}^{S_d(P)} s \left(\exp \frac{-\kappa\beta}{1-\beta}s\right) ds} \right] \right]^\gamma \times \\
& \left[\left[\frac{\int_{S_F(S_d(P))}^{S_d(P)} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds}{\int_{S_F(S_d(P))}^{S_d(P)} s \left(\exp \frac{-\kappa\beta}{1-\beta}s\right) ds} \right] \frac{1}{2\pi \left[\int_0^{S_F(S_d(P))} s \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right) ds \right]} \right]^\alpha \times \\
& \exp\left(\frac{\kappa-\gamma\delta-\alpha\kappa\beta+\beta\gamma\delta}{(1-\alpha)(1-\beta)} S_F(S_d(P)) \right) \times \exp\left(-\frac{\kappa}{(1-\beta)} S_d(P) \right).
\end{aligned}
$$

Proposition 10 *If $\delta > \kappa$, $S_d(P) : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is strictly increasing in P. Furthermore, $\lim_{P\to 0} S_d(P) = 0$ and $\lim_{P\to\infty} S_d(P) = \infty$.*

Proof. We will establish that the r.h.s. of the above equation is decreasing in S. The r.h.s. is a product of five positive terms. The last (fifth) term is clearly decreasing in S. From the upper limit on κ, the fourth term is decreasing in S_F and, since $S_F(S)$ is increasing in S, it is also decreasing in S. By Lemma 4 (in the Appendix), the third term is decreasing in S. Finally, also by Lemma 4, the second term is decreasing in S as well. Hence, the r.h.s. is decreasing in S. Since the r.h.s. is strictly increasing in P, it follows that $S_d(P)$ is strictly increasing in P.

To establish the limit properties, consider first a sequence $\{P_n\}$ converging to zero. Then, $S_d(P_n)$ is a decreasing sequence that is bounded below by 0. Therefore, it must converge to some $\underline{S}_d \geq 0$. Suppose, to get a contradiction, that $\underline{S}_d > 0$. Then, all the terms containing S or S_F converge to strictly positive quantitities. Therefore, the r.h.s will converge to 0, which is impossible given that the l.h.s. is strictly positive. To the prove the other limit result, let $\{P_n\}$ be a sequence diverging to infinity. Then, $S_d(P_n)$ is a strictly increasing sequence. Suppose, to get a contradiction, that this sequence is bounded above. Then, the sequence must converge to some strictly positive number $\bar{S}_d > 0$. Then, all the terms containing S or S_F will also converge to strictly positive numbers. Thus, the r.h.s. will diverge to infinity, which is impossible given that the l.h.s. is a finite number d. Hence, $S_d(P_n)$ must also diverge to infinity. ■

To obtain the relationship between utility deliverable by the city and its TFP and size, when rent at the boundary is d, we begin with the requirement that $q_H(S) = d$, which gives that $d = q_H(0)e^{\left(-\frac{\kappa}{1-\beta}S\right)}$. The second equation we utilize is the labor market clearance equation (27). These two equations can be expressed in terms of $n(0)$, $\overline{u}$, S_F, and other parameters. Thus, we obtain

$$U_d(S) = KA^{\frac{1}{(\gamma+\alpha)}} d^{\frac{\beta(\alpha+\gamma)-1}{(\gamma+\alpha)}} \times \left[2\pi \int_0^{S_F(S)} se^{\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)} ds\right]^{\frac{\gamma}{(\gamma+\alpha)}} e^{\left(\kappa \frac{\beta(\alpha+\gamma)-1}{(1-\beta)(\gamma+\alpha)} S\right)} \left[e^{\left(\frac{-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma}{(1-\alpha)(1-\beta)} S_F(S)\right)}\right]^{\frac{\gamma+\alpha-1}{\gamma+\alpha}}, \quad (33)$$

where K is some positive constant.

Proposition 11 *If $\delta > \kappa$ and $1 - \beta(\alpha + \gamma) > 0$, $\lim_{S\to 0} U_d(S) = \lim_{S\to\infty} U_d(S) = 0$.*

Proof. From Proposition 6 we know the $S_F(S)$ converges to zero as S converges to 0, and therefore the integral term in (33) converges to 0. Also, we know that as S diverges to infinity, S_F converges to a finite number, and so behavior of $U_d(S)$ is ultimately dominated by $e^{\kappa \frac{\beta(\alpha+\gamma)-1}{(1-\beta)(\gamma+\alpha)} S}$, which converges to 0 (given our assumption that $\beta(\alpha + \gamma) < 1$). Hence, $\lim_{S\to\infty} U_d(S) = 0$. ■

While it is harder to establish that $U_d(S)$ has a single peak in general (because of the positive dependence of S_F on S), it is likely to have a single peak if $\alpha + \gamma \approx 1$. To see this, observe

$$\begin{aligned}\ln(U_d(S)) &= D + \frac{\gamma}{\alpha+\gamma}\ln\left[2\pi\int_0^{S_F} r\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}r\right)dr\right] + \\ &\quad \kappa\frac{\beta(\alpha+\gamma)-1}{(1-\beta)(\gamma+\alpha)}S + \left(\frac{\gamma+\alpha-1}{\gamma+\alpha}\right)\frac{-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F,\end{aligned}$$

where D is a constant. Notice that the term that is linear in S_F is small when $\alpha+\gamma\approx 1$ and the behavior of $\ln(U_d(S))$ is then determined by two terms, one of which increases at most logarithmically in S_F and another which declines linearly in S. If a unit increase in S leads to a less than unit increase in S_F (which must be true eventually since S_F is bounded above by $\bar{S}_F$), we have the same situation as in the mixed-use case: one component rises at most logarithmically in S, and another component starts at the origin and declines linearly in S. Therefore $\ln(U_d(S))$ must be single-peaked in S, and, thus, $U_d(S)$ must be single-peaked in S as well.

6 Population Growth, Development Constraints, and Land Rents

In this section, we use the model developed in previous sections to explore the effects of urban growth controls on land rents when the economy as a whole experiences population growth. As noted in the introduction, our goal is to assess the notion that locations that have difficulty expanding (due to topography or urban growth controls) will experience a larger increase in the value of land (i.e., rents) as demand for urban land increases. We show that this intuition needs to be modified when production externalities are present: A city that can expand easily benefits more from the production externality, and this effect may end up increasing rents more in that city.

We assume that the economy is composed of two cities and that, initially, these two cities are in a symmetric equilibrium, each with population P. We assume that P is large enough so that the cities are on the downward sloping portion of their $U_d(P)$ curve. This ensures that the initial symmetric equilibrium is a stable one.

We will consider the nature of the new equilibrium when there is an increase in population from P to P'. To keep matters simple, we will assume that there is some restriction that makes it impossible to increase the supply of developed land in city 1 so that city 1 cannot physically grow beyond its current size. There is no such constraint on the expansion of urban land in city 2. We are interested in determining how land rents respond in the two cities.

In the proposition to follow, we establish that if total population rises, city 2 will expand in size. This result hinges on the shape of the $U_d(P)$ curve; in particular, that it is an inverted U. This assumption is satisfied when cities are of the mixed-use form (as shown in Proposition 5, Corollary 2), and it is likely to be satisfied when the cities have the CBD form. Once we know that city 2 will be bigger in size in the new equilibrium, it is straightforward to establish the conditions under which rents will be higher in city 2 relative to city 1. For the mixed-use case, it will turn out that this will always be the case. For the CBD case, it will require that $\gamma > (1 - \alpha)$.

Proposition 12 *Let $\bar{S}$ and $\bar{P}$ be the common size and population of the two cities in the initial equilibrium and let S_2' be the size of city 2 in the new equilibrium, with higher total population. Then $S_2' > S$.*

Proof. First, we will establish that the population in city 1 must be higher in the new equilibrium. Suppose, to get a contradiction, that $P_1' \leq \bar{P}$. Since the size of city 1 is given, it follows from Proposition 1 for mixed-use cities and from Proposition 7 for CBD cities that the city must be delivering as much or higher utility relative to the original equilibrium. However, since overall population is higher, $P_2' > \bar{P}$. Since $U_d(P)$ is downward sloping at $\bar{P}$, city 2 must then be delivering strictly *lower* utility relative to the original equilibrium (it is further along the $U_d(P)$ curve to the right). This contradicts the equilibrium condition of equal utility in the two cities. So, $P_1' > \bar{P}$. Since city 1 cannot expand, it follows, from

Proposition 1 and 7 again, that utility delivered by city 1 in the new equilibrium must be strictly lower than in the original equilibrium. The condition of equal utility implies that the same must be true in city 2. Therefore, given that $U_d(P)$ is declining at $\bar{P}$, $P_2' > \bar{P}$. Then, it follows from Proposition 6 for mixed-use cities and Proposition 10 for CBD cities that $S_2' > \bar{S}$. ■

Having established that the city that can expand will, in fact, expand with population growth, we can now turn to the effects of population growth on land rents. Denote by $q_i'(r)$ the rent in location r in city i in the new equilibrium with higher total population.

Proposition 13 *If cities are mixed-use, $q_2'(r) > q_1'(r)$ for all $r \in [0, \bar{S}]$.*

Proof. We will use $q_H(0) = q_F(0) = q(0)$. This gives us the condition $(1-\alpha)\,Az(0)^{\gamma}n(0)^{\alpha} = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}}\left(\frac{w(0)}{\bar{u}}\right)^{\frac{1}{1-\beta}}$, which again can be expressed in terms of only $n(0)$and $\bar{u}$ by expressing $w(0)$ and $z(0)$ only in terms of $n(0)$and other parameters:

$$n(0) = KA^{\frac{\beta}{1-\beta(\alpha+\gamma)}}\bar{u}^{\frac{-(1-\beta)}{1-\beta(\alpha+\gamma)}}\left[\int_0^S r\exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right)dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}}$$

where K is, again, some positive constant. Observe that this equation must hold in both the expanding as well as the nonexpanding city. By Proposition 12, $S_2' > \bar{S} = S_1'$, so we have that $n_2'(0) > n_1'(0)$. Therefore, it must be the case that $z_2'(0) > z_1'(0)$. Thus, the external effect is stronger in the expanding city. Since the utility of workers must be the same across cities, it follows from (18) that $q_2'(0) > q_1'(0)$. If rents in the city center are higher in city 2 relative to city 1, it follows from (18) again that rents in city 2 must also be higher relative to city 1 for all comparable locations (in terms of distance from the city center). ■

Thus, the result is that when the city is mixed use and there is population growth, the city that cannot expand physically will see a smaller rise in land rents than the city that can expand. Ultimately, the reason for this result is intuitive. The city that can expand absorbs more workers and, therefore, benefits more from the production externality. This means that firms are more productive in the expanding city and are willing to pay more for land. In a mixed-use city, workers compete with firms for land in every location, so residential rents rise relatively more in the expanding city as well.

It is worth noting that since $q_2'(0) > q_1'(0)$, it must be the case that $w_2'(0) > w_1'(0)$ (otherwise workers would get lower utility in the second city). However, whether wages rise or fall with the expansion in population depends on the strength of the production externality relative to the importance of land in the production of goods. We have the following result:

Proposition 14 *If $\gamma > 1 - \alpha$, the expansion in the population will be accompanied by an increase in real wages in both locations. If $\gamma = 1 - \alpha$, real wages remain unchanged in city 1 but increase in city 2. If $\alpha + \gamma < 1$, wages decline in city 1 but may or may not decline in city 2.*

Proof. The proof follows from (20), which can be re-written as

$$w(0) = \alpha A P^{(\alpha+\gamma)-1} \times$$
$$\left[\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr\right]^{\gamma} \left[2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr\right]^{(1-\alpha-\gamma)}.$$

Consider first the case in which $\alpha + \gamma > 1$. In city 2, both P and S are higher and, therefore, $w_2(0)$ will be higher on both counts. In city 1, S remains unchanged, but P is higher, and so, real wages are also higher. If $\alpha + \gamma = 1$, then $w(0)$ depends only on S and, so, it remains unchanged in city 1 but increases in city 2. If $\alpha + \gamma < 1$, then $w(0)$ will be lower in city 1 because P is higher and S is unchanged. In city 2, the effect of higher S and higher P work in opposite directions (the former lowers real wages, while the latter raises it) and, therefore, the effect on $w(0)$ is ambiguous. ■

We now turn to the CBD case. We can establish that commercial rents will always be higher in the city that can expand, but whether residential rents are higher depends on what happens to real wages with the expansion in population. If real wages rise, residential rents in the expanding city will be higher as well.

Proposition 15 *For CBD cities, $q_{F,2}'(r) > q_{F,1}'(r)$ and, if $\gamma > 1 - \alpha$, $q_{H,2}'(r) > q_{H,2}'(r)$.*

Proof. To show that commercial rents will be higher in the expanding city, observe that using (32) we get

$$\frac{n_2'(0)}{n_1'(0)} = \frac{\left[\int_0^{S_{2F}'} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}} e^{-\frac{\kappa(1-\alpha\beta)-\delta\gamma(1-\beta)}{(1-\alpha)(1-\beta(\alpha+\gamma))} S_{2F}'}}{\left[\int_0^{\bar{S}_F} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr\right]^{\frac{\gamma\beta}{1-\beta(\alpha+\gamma)}} e^{-\frac{\kappa(1-\alpha\beta)-\delta\gamma(1-\beta)}{(1-\alpha)(1-\beta(\alpha+\gamma))} \bar{S}_F}}$$

Since $1-\beta(\alpha+\gamma) > 0$, given (11) and Proposition 12, it follows that $n_2'(0) > n_1'(0)$. Hence, $z_2'(0) > z_1'(0)$. Since $q_F(0) = (1-\alpha)Az(0)^\gamma n(0)^\alpha$, it follows that $q_{F,2}'(0) > q_{F,1}'(0)$. To prove the residential rent part, observe that $w(0) = \alpha A z(0)^\gamma n(0)^{\alpha-1}$, which implies $w(0) = \alpha A \left[\int_0^{S_F} 2\pi r e^{\left(\frac{\kappa-\delta\gamma}{1-\alpha}-\delta\right)r} dr\right]^\gamma n(0)^{\gamma+\alpha-1}$. Therefore, if $\alpha+\gamma \geq 1$, $w_2'(0) > w_1'(0)$. Since $q_H(0) = \beta^{\beta/(1-\beta)}(1-\beta)w(0)^{1/(1-\beta)}\bar{u}^{-1/(1-\beta)}$, it follows that $q_{H,2}'(0) > q_{H,1}'(0)$. Therefore, residential rents in city 2 will be higher than residential rents in comparable locations in city 1. ■

Since rent in any location is simply $\max\{q_F(r), q_H(r)\}$, the immediate implication of Proposition 15 is:

Corollary 4 *$q_2'(r) > q_1'(r)$ for all $r \in [0, \bar{S}]$.*

7 Appendix

Lemma 1 *Let $0 \leq s_L < s_U$. If $k_1 < k_2$,*

$$e^{(k_2-k_1)s_L} < \frac{\int_{s_L}^{s_U} s e^{k_2 s} ds}{\int_{s_L}^{s_U} s e^{k_1 s} ds} < e^{(k_2-k_1)s_U}.$$

And, if $k_2 < k_1$,

$$e^{(k_2-k_1)s_U} < \frac{\int\limits_{s_L}^{s_U} se^{k_2 s}ds}{\int\limits_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_L}.$$

Proof. We will establish the first set of inequalities (the proof of the second set is entirely analogous). Turning first to the l.h.s. inequality, observe that $se^{k_2 s} = se^{s_L k_2+(s-s_L)k_2}$and $se^{k_1 s} = se^{s_L k_1+(s-s_L)k_1}$. Multiplying both sides of the latter equation by $e^{(k_2-k_1)s_L}$ yields $e^{(k_2-k_1)s_L}se^{k_1 s} = se^{s_L k_2+(s-s_L)k_1} \leq se^{s_L k_2+(s-s_L)k_2} = se^{k_2 s}$, where the inequality follows because $k_2 > k_1$ and $s - s_L \geq 0$. Furthermore, the inequality is strict for all $s \in (s_L, s_U]$. Therefore, integrating the first and last expressions in the chain with respect to s, we have

$$e^{(k_2-k_1)s_L}\int\limits_{s_L}^{s_U} se^{k_1 s}ds < \int\limits_{s_L}^{s_U} se^{k_2 s}ds.$$

Turning to the r.h.s. inequality, observe that $se^{k_2 s} = se^{s_U k_2+(s-s_U)k_2}$and $se^{k_1 s} = se^{s_U k_1+(s-s_U)k_1}$. Multiplying both sides of the the latter equation by $e^{(k_2-k_1)s_U}$ yields

$$e^{(k_2-k_1)s_U}se^{k_1 s} = se^{k_2 s_U+(s-s_U)k_1} \geq se^{s_U k_2+(s-s_U)k_2} = se^{k_2 s},$$

where the inequality follows since $k_2 > k_1$ and $s - s_U \leq 0$. Furthermore, the inequality is strict for all $s \in [s_L, s_U)$. Therefore, integrating the first and last terms in the chain with respect to s, we have

$$e^{(k_2-k_1)s_U}\int\limits_{s_L}^{s_U} se^{k_1 s}ds > \int\limits_{s_L}^{s_U} se^{k_2 s}ds. \quad \blacksquare$$

Lemma 2 *Let* $0 \leq s_L < s_U$. *Let*

$$\Lambda(s_L, s_U) = \frac{\int_{s_L}^{s_U} s e^{k_2 s} ds}{\int_{s_L}^{s_U} s e^{k_1 s} ds}$$

If $k_1 < k_2$, *then* $\partial \Lambda(s_L, s_U)/\partial s_U > 0$, *and* $\partial \Lambda(s_L, s_U)/\partial s_L > 0$. *And, if* $k_2 < k_1$, $\partial \Lambda(s_L, s_U)/\partial s_U < 0$, *and* $\partial \Lambda(s_L, s_U)/\partial s_L < 0$.

Proof. We begin with the case in which $k_1 < k_2$. Observe that

$$\frac{\partial \ln(\Lambda(s_L, s_U))}{\partial s_U} = \frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} s e^{k_2 s} ds} - \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} s e^{k_1 s} ds}$$

Suppose, to get a contradiction, that $\partial \Lambda(s_L, s_U)/\partial s_U \leq 0$. Then, we must have

$$\frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} s e^{k_2 s} ds} \leq \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} s e^{k_1 s} ds}$$

Or, given that all elements are positive, we have

$$\exp([k_2 - k_1] s_U) = \frac{s_U \exp(k_2 s_U)}{s_U \exp(k_1 s_U)} \leq \frac{\int_{s_L}^{s_U} s e^{k_2 s} ds}{\int_{s_L}^{s_U} s e^{k_1 s} ds}$$

But this contradicts the r.h.s. inequality in Lemma 1. Therefore, $\partial \Lambda(s_L, s_U)/\partial s_U > 0$. Analogous proof can be given for the case in which $k_2 < k_1$. ■

Lemma 3 *Let* $0 < s_L < s_U$ *and* $k > 0$. *Let* $I(s_U, s_L, k) = \int_{s_L}^{s_U} s \exp(-ks) \, ds$. *Then (i)* $\lim_{s_U, s_L \to \infty} I(s_U, s_L, k) = 0$ *and (ii)* $\lim_{s_U \to \infty, s_L \to \underline{s}} I(s_U, s_L, k) = \bar{I} > 0$.

Proof. Observe that

$$\int\limits_{s_L}^{s_U} se^{-ks}ds = \frac{s_U e^{-ks_U} - s_L e^{-ks_L}}{-k} - \frac{e^{-ks_U} - e^{-ks_L}}{k^2}.$$

To prove (i), notice that as s_U and s_L go to infinity, the second term goes to 0, and the first term (on an application of L'Hospital's rule to s/e^{ks}) also goes to 0. To prove (ii), observe that if s_U goes to infinity and s_L converges to $\underline{s}$, then $I(s_U, s_L, k)$ converges to

$$\frac{-\underline{s}e^{-k\underline{s}}}{-k} + \frac{e^{-k\underline{s}}}{k^2} > 0. \quad \blacksquare$$

Lemma 4 *For the CBD city, $n(0)$ is decreasing in S and, provided $\delta > \kappa$, $z(0)$ is also decreasing in S.*

Proof. : Consider first $n(0)$:

$$n(0) = \left[\frac{\int\limits_{S_F}^{S} s\exp\left(-\frac{\kappa}{1-\beta}s\right)ds}{\int\limits_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds}\right]\left[\frac{P}{2\pi\int\limits_{0}^{S_F} s\exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds}\right]$$

By Lemma 2, the first term in square brackets is decreasing in S and S_F, and the second term is evidently decreasing in S_F. Since $S_F(S)$ is an increasing function of S, it follows that $n(0)$ is a decreasing function of S.

Next, consider $z(0)$:

$$z(0) = P\left[\frac{\int_0^{S_F} s\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)ds}{\int\limits_{0}^{S_F} s\exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds}\right]\left[\frac{\int\limits_{S_F}^{S} s\exp\left(-\frac{\kappa}{1-\beta}s\right)ds}{\int\limits_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds}\right].$$

Since $\delta > \kappa$, it follows from Lemma 2 that the first term in square brackets is decreasing in S_F. And, by Lemma 2 again, the second term in square brackets is also decreasing in both

S and S_F. Since S_F is increasing in S, it follows that $z(0)$ is decreasing in S. ■

Proof of Proposition 9

To prove the first part, note that $q_H(S) = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S}$. Since $e^{-\frac{\kappa}{(1-\beta)}S}$ is decreasing in S, we need to show that $q_H(0)$ is decreasing in S. We will show this by first showing that $q_F(0)/q_H(0)$ is increasing in S and then showing that $q_F(0)$ is decreasing in S. To begin, note that $q_F(0)e^{-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}S_F} = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S_F}$, which implies that $q_F(0)/q_H(0) = e^{-\frac{-\kappa+\delta\gamma+\alpha\beta\kappa-\beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F}$. By the upper bound on κ, the r.h.s. of the latter equation is increasing in S_F. Since $S_F(S)$ is increasing in S, it follows that $q_F(0)/q_H(0)$ is increasing in S. Turning to $q_F(0)$, we have that $q_F(0) = (1-\alpha)Az(0)^{\gamma}n(0)^{\alpha}$. By Lemma 4 $n(0)$ is decreasing in S and, if $\delta > \kappa$, $z(0)$ is decreasing in S. This establishes that $q_F(0)$ is decreasing in S. Therefore, $q_H(0)$ is decreasing in S.

We now turn to limiting behavior of $q_H(S)$.

Part (i): $\lim_{S\to\infty} q_H(S) = 0$. Consider

$$q_H(S) = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}}\left(\frac{w(0)\exp(-\kappa S)}{\overline{u}}\right)^{\frac{1}{1-\beta}}.$$

Using (29), (31), and (32), we can express the ratio of $w(0)$ to $\bar{u}$ as

$$\frac{w(0)}{\overline{u}} = KP^{(1-\beta)(\gamma+\alpha)}A^{-1}\left(\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds\right)^{-(1-\beta)(\gamma+\alpha)} \times$$
$$\left(\int_0^{S_F} s\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)ds\right)^{\gamma(1-\beta)} \times$$
$$\exp\left(\frac{(-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma)(\gamma+\alpha-1)}{(1-\alpha)}S_F\right),$$

where K is a positive constant. Given that $\lim_{S\to\infty} S_F(S) = \bar{S}_F$, the last two terms approach finite numbers. And, by Lemma 3, $\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds$ appoaches a strictly positive finite number. Thus, we can conclude that as $S \to \infty$, the ratio $w(0)/\bar{u}$ approaches a finite number as well. Therefore, the limiting behavior of $q_H(S)$ is governed by the limiting behavior of $\exp(-\kappa S)$. Hence, $\lim_{S\to\infty} q_H(S) = 0$.

Part (ii): $\lim_{S\to 0} q_H(S) = \infty$

Since $S > S_F(S), S \to 0$ implies $S_F(S) \to 0$. Then, it is easiest to show that $q_F(0) = (1-\alpha)\, z(0)^\gamma n(0)^\alpha$ goes to infinity, which would imply that $q_H(S)$ goes to infinity also. Turning first to $n(0)$, observe that

$$n(0) = \left[\frac{\int_{S_F}^{S} s\exp\left(-\frac{\kappa}{1-\beta}s\right)ds}{\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds}\right]\frac{P}{2\pi\left[\int_0^{S_F} s\exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds\right]}$$

We know from Lemma 1 that

$$\exp(\kappa S_F) < \frac{\left[\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds\right]}{\left[\int_{S_F}^{S} s\exp\left(-\frac{\kappa}{1-\beta}s\right)ds\right]} < \exp(\kappa S).$$

This implies that as S and S_F converge to 0 (and, so, both $\exp(\kappa S_F)$ and $\exp(\kappa S)$ converge to 1) the term in square brackets converges to 1. We also know that $\left[\int_0^{S_F} s\exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds\right]$ goes to zero as S_F goes to zero, so $n(0)$ goes to infinity as S goes to zero.

Turning next to $z(0)$, we know that

$$z(0) = P\left[\frac{\int_0^{S_F} s\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)ds}{\int_0^{S_F} s\exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right)ds}\right]\left[\frac{\int_{S_F}^{S} s\exp\left(-\frac{\kappa}{1-\beta}s\right)ds}{\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds}\right]$$

We know from Lemma 1, that the two square-bracketed terms converge to 1 as S and S_F goes to zero. That means $z(0)$ converges to P as S goes to zero. It follows that $q_F(0)$ goes to infinity as S goes to zero. Since S is almost zero, there is effectively no depreciation of rents over distance. So, for there to be any residential land, $q_H(S)$ must also diverge to ∞.

■

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