WORKING PAPERS

RESEARCH DEPARTMENT

WORKING PAPER NO. 12-14
DEBT DILUTION AND SENIORITY IN A MODEL OF DEFAULTABLE SOVEREIGN DEBT

Satyajit Chatterjee
Federal Reserve Bank of Philadelphia

Burcu Eyigungor
Federal Reserve Bank of Philadelphia

June 2012

RESEARCH DEPARTMENT, FEDERAL RESERVE BANK OF PHILADELPHIA

Ten Independence Mall, Philadelphia, PA 19106-1574 • www.philadelphiafed.org/research-and-data/

Debt Dilution and Seniority in a Model of Defaultable Sovereign Debt

Satyajit Chatterjee[1] and Burcu Eyigungor

Federal Reserve Bank of Philadelphia

June 2012

[1]Corresponding Author: Satyajit Chatterjee, Research Department, FRB Philadelphia, 10 Independence Mall, Philadelphia, PA 19106. Tel: 215-574-3861. Email: chatterjee.satyajit@gmail.com. The authors thank seminar participants at Florida State University, the University of Rochester, and the University of Wisconsin-Madison for thoughtful comments.

Abstract

An important source of inefficiency in long-term debt contracts is the *debt dilution problem*, wherein a borrower ignores the adverse impact of new borrowing on the market value of outstanding debt and, therefore, borrows too much and defaults too frequently. A commonly proposed remedy to the debt dilution problem is seniority of debt, wherein creditors who lent first are given priority in any bankruptcy or restructuring proceedings. The goal of this paper is to incorporate seniority in a quantitatively realistic, infinite horizon model of sovereign debt and default and examine, both theoretically and quantitatively, the extent to which seniority can mitigate the debt dilution problem.

JEL:

Key Words : Debt Dilution, Seniority, Sovereign Default

1 Introduction

Debt is one of the main vehicles via which firms and sovereign countries finance investment and consumption. While debt payments are typically noncontingent, default on debt is generally always a possibility, and we see both firms and countries defaulting frequently on their debt.

Given the deadweight costs of default, a major focus of the debt literature has been on understanding (and proposing) institutional arrangements that can make debt contracts more efficient so that the value of the firm or utility of the country is maximized. When long-term debt is involved, a major source of inefficiency is the *debt dilution problem*, wherein the borrower ignores the adverse impact on the value of outstanding debt when deciding whether to issue new debt and, therefore, borrows too much and defaults too frequently.[1] In the absence of an institutional arrangement that protects current long-term creditors from losses in value resulting from additional future borrowings, current creditors lend funds at an interest rate that is high enough to cover these anticipated losses (in expectation).[2] This makes long-term debt costly and, ultimately, creates welfare losses for the borrower.

The debt dilution problem has received considerable attention in policy debates surrounding sovereign debt crises. In particular, the possibility of debt dilution (and the attendant higher interest cost of debt) is thought to induce sovereign borrowers to opt for debt structures that are hard to dilute, such as very short-term debt (Sachs & Cohen (1982) and Kletzer (1984)) or debt that cannot be easily restructured so that the costs of default are high and the likelihood of default correspondingly low (Shleifer (2002) and Dooley (2000)). But the tendency toward short-maturity debt exposes the sovereign to the risk of a confidence-driven rollover crisis (Giavazzi & Pagano (1990), Cole & Kehoe (1996), Chatterjee & Eyigungor (2011)), and the tendency toward hard-to-restructure debt makes crises very costly when they happen, perhaps inefficiently so (Bolton & Jeanne (2009)). The debt dilution problem has thus been viewed as an important reason for

[1] Argentina's decision to issue bonds to a wide base of borrowers in the late 1990s is an example of a sovereign choosing to dilute the value of debt in the hands of existing bondholders. Similarly, the decision by Russia and Ukraine to issue short-term debt in the months leading up to default in late 1998 is another example of dilution. In these instances, the possibility of dilution arguably encouraged these countries to take on more debt than they would otherwise and made default more likely.

[2] In practice, certain classes of creditors are generally (but not always!) accorded priority over others, for example, bonds generally get priority over bank loans. But within a particular class of creditors – the class of bondholders, for instance – treatment is generally equal.

emerging market borrowers to be crisis-prone and, in the event of a crisis, experience a costly and protracted period of restructuring (Borensztein et al. (2006)).

In the last half-dozen years or so, a rapidly growing quantitative-theoretic literature has focused on explaining the unique characteristics of emerging market business cycles (Neumeyer & Perri (2005), Aguiar & Gopinath (2006), and Arellano (2008), among others). This literature points to the highly volatile and strongly countercyclical country interest rates as a driver for the pronounced volatility of consumption and countercyclicality of net exports in (open) emerging economies. It builds on the negative relationship between default spreads and real output implicit in the model of defaultable sovereign debt developed in Eaton & Gersovitz (1981). Counterfactual exercises conducted with calibrated models that match debt-to-output ratios and typical maturity length of sovereign debt reveal that if the sovereign could issue debt whose value is protected from capital losses resulting from additional future borrowing, the frequency of default as well as the volatility of interest rates would decline substantially and aggregate welfare would improve (Chatterjee & Eyigungor (2011) and Hatchondo, Martinez & Sosa-Padilla (2011)). Thus, this business cycles literature also points to debt dilution as the key reason emerging economies borrow to the point where default spreads are large and volatile.

In the case in which creditors expect to recover something from the debtor in the event of default, the debt dilution problem can be mitigated if there is an explicit seniority structure on debt. This is the so-called "first-in-time" or "absolute priority" rule that requires that creditors who lent first be paid in full before creditors who lent later are paid. Such a rule makes it harder to dilute the value of existing debt in so far as existing creditors do not have to share payment on the defaulted debt with new creditors.[3] Since a sovereign default is typically followed by a re-structuring of the defaulted debt (wherein creditors are given new debt in place of old debt), imposition of a "first-in-time" rule has been proposed as a (partial) solution to the debt dilution problem (Borensztein et al. (2006), Bolton & Skeel (2004), and Bolton & Jeanne (2009)) as well as

[3] In the corporate finance literature, Fama & Miller (1972) gave an early discussion of how creditors of a firm can protect themselves from dilution by making their loans senior. It is important to note, however, that seniority is not a panacea. For instance, giving seniority to the most recently issued debt might lead to better outcomes if investment decisions are endogenous (Hennessey (2004)) or if there is the possibility of an inefficient default due to coordination difficulties among existing debt holders (Saravia (2010)). Also, Bizer & DeMarzo (1992) show that if the borrower can influence the probability of default through his effort decisions, the ability to borrow sequentially can lead to inefficiently high levels of debt and default even if seniority among creditors is respected.

an aid to an orderly re-structuring of debt following default (Gelpern (2004)).

The goal of this paper is to advance our understanding of the role of seniority in ameliorating the costs of debt dilution by incorporating seniority in a quantitatively realistic model of sovereign debt. We build on the long-term debt and default model presented in Chatterjee & Eyigungor (2011). The country borrows long-term in a competitive bond market and has the option to default on its debt. Following default, with some delay (potentially) the sovereign agrees to service a lower level of debt (debt write-down). Thus, there is payment on defaulted debt in the sense that the old debt is settled with new debt. During the period of default and in the period between default and settlement, the sovereign does not have access to international financial markets and suffers output losses. Within this setup two different market arrangements are analyzed and compared. In the first, in the event of default, all debts are treated equally (the current system); in the second, a "first-in-time rule" is followed.

We make three sets of contributions. First, on the methodological side, we show how the high dimensionality of the borrower's state space resulting from the need to keep track of the time of issuance of a bond can be sidestepped by indexing each bond by its *rank* or seniority at the time of issuance. With this indexation scheme, a single additional continuous state variable is sufficient to keep track of seniority and thereby renders the model computationally tractable.[4]

Second, on the quantitative side, we calibrate the model (in which all debts are treated equally in default) to Argentine facts, specifically, its average debt-to-GDP ratio, the average spread on its debt, the volatility of the spread, and the level of repayment on its debt in the most recent default episode. Then, assuming no prior debt, we investigate how behavior of the model changes

[4]With regard to dimensionality of the state space, there are two separate challenges that need to be met. First, to talk meaningfully about seniority one must allow for long-term debt. If it is assumed that the maximum maturity of any debt is T periods, in an ongoing dynamic setting one has to keep track of at least T *continuous* state variables (namely, the size of obligations that are due in the next T periods) to correctly compute default probabilities. Even for small values of T (say, 3 or 4), the problem becomes computationally intractable. One way to address this dimensionality challenge is to model long-term debt as debt that matures probabilistically, as in Chatterjee & Eyigungor (2011) and Hatchondo & Martinez (2009). If all bonds are treated equally in default, then there is no need to keep track of when a particular bond was issued as all bonds have the same payoff structure going forward. But if bonds that were issued earlier are given priority in default over bonds that were issued later, then it becomes necessary to keep track of when a particular bond was issued in order to determine its payoff in default. In the random maturity context, there may be bonds outstanding that were issued many, many periods ago. Once again, there are many continuous state variables (the quantity of bonds issued at different times in the past) that need to be kept track of to compute prices. The indexation scheme described in the text overcomes this problem by introducing a single state variable (the rank) that, in effect, keeps track of when a bond was issued and hence its priority in the event of default.

when we replace the current system with one where seniority is respected in default. We find that enforcing seniority reduces the frequency of default by more than 40 percent, reduces the mean and volatility of spreads by about 67 percent, and increases the average debt carried by the economy by about 31 percent. The gain in welfare from these changes is just under 2.0 percent of consumption in perpetuity. We also show that this gain is generally decreasing in the level of prior debt – if enforcing seniority requires making prior debt senior to any new debt. Indeed, for sufficiently high levels of prior debt, enforcing seniority can lead to a welfare loss.

Finally, on the theoretical side, we provide conditions under which seniority can completely overcome the debt dilution problem, meaning that the value of outstanding bonds becomes invariant to additional borrowing by the sovereign. Among other things, the key requirement for this to be the case is that the renegotiated debt level cannot be less than the maximum amount of debt the country would be willing to service to avoid default costs. We then describe a simple bargaining environment in which this and other needed conditions are satisfied. Specifically, we show that if renegotiation is costless and lenders as a group have the ability to make a one-time take-it-or-leave-it offer of a debt-writedown to the sovereign, then a dilution-free equilibrium exists.

2 Preferences and Endowments

Time is discrete and denoted $t \in \{0, 1, 2, ...\}$. The sovereign receives a strictly positive endowment x_t each period. The stochastic evolution of x_t is governed by the sum of persistent and transitory components:

$$x_t = y_t + m_t. \tag{1}$$

Here $m_t \in M = [-\bar{m}, \bar{m}]$ is a transitory income shock drawn independently each period from a mean zero probability distribution with continuous cdf $\mu(m)$, and y_t is a persistent income shock that follows a finite-state Markov chain with state space $Y \subset \mathbb{R}_{++}$ and transition law $\Pr\{y_{t+1} = y'|y_t = y\} = F(y, y') > 0,\ y$ and $y' \in Y$.

The sovereign maximizes expected utility over consumption sequences, where the utility from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1 \tag{2}$$

The momentary utility function $u(\cdot) : [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing, strictly concave, and $|u(c)| < U$.

3 Market Arrangements and the Option to Default

The sovereign can borrow in the international credit market with the option to default. We analyze long-term debt contracts that mature probabilistically (Hatchondo & Martinez (2009), Arellano & Ramanarayanan (2010), and Chatterjee & Eyigungor (2011)). Specifically, each unit of outstanding debt matures the next period with probability λ. If the unit does not mature, which happens with probability $1 - \lambda$, it gives out a coupon payment z. We assume that unit bonds are infinitesimally small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer's coupon obligations for the next period will be $z \cdot (1 - \lambda)b$ with certainty and the issuer's payment-of-principal obligations will be λb with certainty.

The option to default means that the sovereign has the right to unilaterally stop servicing its debt obligations – i.e., stop making coupon and principal payments. Default is costly in several ways. First, the sovereign loses access to the international credit market – cannot borrow or save in the period of default. And, following the period of default, the sovereign continues in financial autarky for a random number of periods until some settlement with creditors is reached. Specifically, settlement happens with probability $0 < \xi < 1$ and when it happens the sovereign agrees to a new debt level given by the function $G(y, k)$, where k is the amount of debt defaulted on and y is the persistent component of output in the period of settlement. Second, during its sojourn in financial autarky (the period of default and the random number of periods between default and settlement), the sovereign loses $\phi(y) > 0$ of its persistent component of output, y. Third, the sovereign's transitory component of income drops to $-\bar{m}$ in the period of default.[5]

We make some assumptions about ϕ and G. Regarding ϕ we assume that $y - \phi(y)$ is increasing

[5]This technical assumption is made for the purpose of speeding up computation.

in y and that $y_{\min} - \phi(y_{\min}) - \bar{m} > 0$, which ensures that $y - \phi(y) + m > 0$ for all $(y, m) \in (Y \times M)$.[6] Regarding G we assume that for any $y \in Y$, $G(y, k)$ is increasing in k, i.e., the larger is the amount of defaulted debt (the more negative is k) the larger is the amount of debt issued in the settlement (all else the same).

We assume that there is a single type of bond (z, λ) available in this economy. We assume that the sovereign can choose the size of its debt from a finite set $B = \{b_I, b_{I-1}, \ldots b_2, b_1, 0\}$, where $b_I < b_{I-1} < \ldots < b_2 < b_1 < 0$. As is customary in this literature, we will view debt as negative assets.

Since there is repayment on defaulted debt, we need to specify how this repayment is divided among existing creditors. We will analyze two different market arrangements, one that treats all existing creditors equally and one in which creditors are ranked by seniority and junior claimants receive payments only if senior claimants have been fully compensated. For each market arrangement, we will assume that lenders are risk-neutral and that the market for sovereign debt is competitive.

4 The Model Without Seniority

4.1 Decision Problem of the Sovereign

With this market arrangement, the price of a unit bond will depend only on the current persistent component of output and on the level of outstanding debt. We will denote the price of unit bond by $q(y, b')$.

Consider the decision problem of a sovereign with $b \in B$ and endowments $(y,\, m)$. Denote the sovereign's lifetime utility conditional on repayment by the function $V(y, m, b) : Y \times M \times B \to \mathbb{R}$, its lifetime utility conditional on being excluded from international credit markets by the function $X(y, m, k) : Y \times M \times B \to \mathbb{R}$, and its unconditional (optimal) lifetime utility by the function

[6] In this paper, a function $f(x)$ is *increasing (decreasing) in* x if $x' > x$ implies $f(x') \geq (\leq) f(x)$ and is *strictly increasing (strictly decreasing)* in x if $f(x') > (<) f(x)$.

$W(y, m, b) : Y \times M \times B \to \mathbb{R}$. Then,

$$X(y, m, k) = u(c) + \beta\{[1 - \xi]E_{(y' \, m')|y}X(y', m', k) + \xi E_{(y' \, m')|y}W(y', m', G(y', k)\} \tag{3}$$

s.t.

$$c = y - \phi(y) + m.$$

The sovereign's lifetime utility under financial autarky reflects the fact that it loses $\phi(y)$ of its output and can expect to be let back into the international credit market next period with probability ξ. At re-entry the sovereign starts off with the debt given by $G(y', k)$.

Next,

$$V(y, m, b) = \max_{b' \in B} u(c) + \beta E_{(y' \, m')|y}W(y', m', b') \tag{4}$$

s.t.

$$c = y + m + [\lambda + (1 - \lambda)z]\, b - q(y, b') \left[b' - (1 - \lambda)b\right]$$

The above assumes that the budget set under repayment is nonempty, meaning there is at least one choice of b' that leads to nonnegative consumption. But it is possible that (y, b, m) is such that all choices of b' lead to negative consumption. In this case, repayment is simply not an option and the value of $V(y, m, b)$ is set to $-\infty$.

Finally,

$$W(y, m, b) = \max\{V(y, m, b), X(y, -\bar{m}, b)\}$$

Since W determines both X and V (via equations (3) and (4), respectively), equation (**??**) defines a Bellman equation in W.

We assume that if the sovereign is indifferent between repayment and default it repays and if it is indifferent between two distinct b's it chooses the larger one (i.e., chooses a lower debt level over a higher one). Let $d(y, m, b)$ denote the default decision rule, where $d = 1$ if it is optimal for the sovereign to default and 0 otherwise. Let $a(y, m, b)$ denote the sovereign's choice of b' conditional on repayment.

4.2 Equilibrium

The world one-period risk-free rate r_f is taken as exogenous. Since there is repayment on defaulted debt, there will be a market price of this debt. Let $q_D(y,k)$ denote the market price of a unit defaulted bond when the total number of defaulted bonds outstanding is k and the current fundamental of the economy is y. Then, under competition, the price of a unit bond satisfies the following pricing equation:

$$q(y,b') = E_{(y' m')|y}\left[[1-d(y',m',b')]\frac{\lambda+(1-\lambda)[z+q(y',a(y',m',b'))]}{1+r_f}+d(y',m',b')\frac{q_D(y',b')}{1+r_f}\right]$$

Under competition, the market price of defaulted debt must be such as to ensure zero expected profits for any investor who buys this debt. Let $P(y,m,G(y,k))$ be the per-unit value of the new debt issued in settlement when the state at the time of settlement is (y,m) and the amount of debt issued in settlement is $G(y,k)$. Then,

$$P(y,m,G(y,k)) = \begin{cases} \lambda+(1-\lambda)[z+q(y,a(y,m,G(y,k))] & \text{if } d(y,m,G(y,k))=0 \\ q_D(y,G(y,k)) & \text{if } d(y,m,G(y,k))=1 \end{cases}$$

The value of the debt offered in settlement takes into account the possibility that the sovereign may default again immediately upon settlement, in which case the per-unit value of the new debt will be the per-unit value of defaulted debt when the state is y and the amount of defaulted debt is $G(y,k)$. If the sovereign does not default immediately, then with probability λ the bond matures and returns 1, and with the complementary probability, the bondholder receives the coupon payment and the market value of an unmatured bond. This value depends on the post-settlement state $(y,m,G(y,k))$.

Given this, the market price of a unit of defaulted debt will satisfy the following functional equation:

$$q_D(y,k) = [1-\xi]E_{(y'|y)}\left[\frac{q_D(y',k)}{1+r_f}\right]+\xi E_{(y',m'|y)}\left[\frac{G(y',k)}{k}\frac{P(y',m',G(y',k))}{1+r_f}\right]. \tag{5}$$

Observe that if the quantity $G(y',k)/k$ is less than 1 (which, typically, it will be in an equilibrium), a unit defaulted bond will, upon settlement, become less than a unit of the restructured debt. We can interpret $(k - G(y', k))/k$ as the "haircut" taken by creditors in the settlement process.

It is worth pointing out that the possibility of debt dilution can lead to behavior that blurs the distinction between repayment and default: Rather than defaulting on its debt right away, the sovereign may choose to borrow as much as it can and then default *with certainty* next period. To understand why, suppose that $G(y, k)$ is independent of k. Suppose the sovereign services its current debt b and then issues enough debt, denoted B', such that default is certain next period.[7] What does it gain from this strategy? The sovereign has to pay for the maturing part of the outstanding debt and make coupon payments on debt that does not mature, which is equal to $[\lambda + (1-\lambda)z]\, b$. To offset this, there is the revenue that comes from issuances of new debt equal to $-q(y, B')\,(B' - [1-\lambda]b)$. Since default is certain for B', the outstanding value of its obligations next period is going to be $q_D(y', G(y')) \times G(y')$. Therefore, $q(y, B')B' = E_{y'|y}\left[\frac{q_D(y', G(y')) \times G(y')}{1+r_f}\right]$, which is independent of B'. Thus, the sovereign can maximize revenue from the new issuances by minimizing $q(y, B')(1-\lambda)(-b)$. Now observe that for any $b' \leq B'$, $q(y, b') = E_{y'|y}\left[\frac{q_D(y', G(y')) \times G(y')/b'}{1+r_f}\right]$. Hence, $q(y, b')(1-\lambda)(-b)$ can be minimized by making b' go to -∞ which makes the price of existing debt go to zero. So whenever $E_{y'|y}\left[\frac{q_D(y', G(y'))(-G(y'))}{1+r_f}\right] - [\lambda + (1-\lambda)z]\,(-b)$ ($= \Delta$, say) is positive, the country might prefer to borrow as much as it can and then default for certain next period rather than default on its debt right away.[8] By borrowing as much as it can today, the sovereign dilutes the value of existing debt as much as possible. Put differently, it promises as much of its default payment as possible to investors who buy its new debt. In this way, the sovereign increases its current consumption at the expense of its existing creditors and postpones the costs of default by one period.

Issuing as much debt as possible before defaulting is not optimal if the debt is for one period ($\lambda = 1$) or if there is no repayment following default ($G(y, k) = 0$). In the first case, there is no outstanding debt to dilute and, in the second case, borrowing beyond the point where default is certain means that $q(y, b')b'$ is zero. In either case, the sovereign does not get any extra consumption by this strategy.[9] More generally, this strategy of borrowing as much as possible prior to default

[7]Any b' for which $y_{max} + \bar{m} + [z + (1-\lambda)]b' < 0$ will do since the sovereign will never have enough resources to service this amount of debt.

[8]When $G(y, k)$ is independent of k, the payoff from defaulting in the current period is $X(y, -\bar{m})$, while the payoff from issuing an infinite amount of debt is $u(y + \Delta) + \beta E_{y'|y} X(y', -\bar{m})$. If $\Delta > 0$, defaulting on an infinite amount of debt will generally be better than defaulting on a finite amount of debt.

[9]In Yue (2010) debt recovery after default is of the form $G(y')$ but there is no gain to issuing huge amounts of

is also not optimal if the Δ defined above is negative. This will happen if $(-b)$ is too high or if $E_{y'|y}\left[\frac{q_D(y',G(y'))(-G(y'))}{1+r_f}\right]$ is too low, i.e., if the gain from dilution is not large enough to more than offset the current debt service payments.

5 The Model with Seniority

The straightforward way to keep track of seniority is to introduce the "time-since-issuance" as an additional state variable. But, given the random maturity nature of our bonds, this would require keeping track of many (potentially, infinitely many) continuous states. To circumvent this problem, we propose a novel and computationally tractable way of keeping track of seniority that does not involve keeping track of the "time-since-issuance."

The idea is to assume that every unit bond outstanding has a ranking denoted by s. If there are b units of the debt outstanding, the ranking of any given unit bond is some number $s \in [b, 0]$. The closer s is to 0, the higher is the ranking of the unit bond and higher is its seniority. We continue to assume that b is a member of the finite set B. However, s is a *continuous* variable since it is a member of $[b, 0]$. In effect, we are assuming that the sovereign can issue bonds in chunks $\{b_1, b_2, \ldots, b_I\}$ but within each chunk there is a ranking of unit bonds that compose that chunk.

How does the ranking of a unit bond evolve over time? Suppose that the sovereign has b unit bonds outstanding at the end of any period. Consider a unit bond with rank $s \in [b, 0]$. Since any unit bond has a probability λ of maturing next period, among the bonds that are higher ranked than s there is a fraction λ that will mature. Thus, at the start of next period, there will only be $(1-\lambda)s$ bonds with a rank higher than s. This means that we can preserve the current ranking among all bonds with a rank greater than or equal to s if we reset the rank of each unit bond that survives into the next period to $(1-\lambda)s$. This re-setting rule implies that the unit bond with rank 0 (the most senior unit bond) continues to have rank 0 as long as it survives and any other unit bond's rank approaches 0 at a geometric rate the longer it survives.

If the sovereign issues new bonds in any period (i.e., $b' < (1-\lambda)b$), each member of the mass

debt prior to default because debt is short term. Chatterjee and Eyigungor (2011) and Hatchondo and Martinez (2009) have long-term debt, but there is no repayment following default, so there is no incentive to dilute existing debt prior to default.

of newly issued bonds is assigned a unique rank s in the interval $[b', (1-\lambda)b]$. If the sovereign buys back debt (i.e., $b' > (1-\lambda)b$), then we assume that it is the mass of least senior bonds, namely, the unit bonds with $s \in [(1-\lambda)b, b']$ that are bought back (this assumption is discussed in more detail below).

Since the payoff to the bondholder in the case of default depends on the rank (seniority) of the bond held, the price of a unit bond will now depend on its ranking. Denote the price of unit bond with $s \geq b$ by $q(y, b, s)$. As before, denote the sovereign's lifetime utility by $W(y, m, b)$, its utility under repayment by $V(y, m, b)$, and its utility under exclusion by $X(y, m, b)$. Then,

$$V(y, m, b) = \max_{b' \in B} u(c) + \beta E_{(y' m')|y} W(y', m', b') \tag{6}$$

$$\text{s.t.} \tag{7}$$

$$c = y + m + [\lambda + (1-\lambda)z]\, b + R(y, b, b')$$

where $R(y, b, b')$ denotes the revenue received from changing the asset level from b to b' (it will be positive if new bonds are issued and negative if bonds are bought back) and is given by:

$$R(y, b, b') = \begin{cases} q(y, b', b')\,[(1-\lambda)\,b - b'] & \text{if } (1-\lambda)\,b \leq b' \\ \int_{b'}^{b(1-\lambda)} q(y, b', s)ds & \text{if } (1-\lambda)\,b > b' \end{cases}$$

If the sovereign is buying back bonds, it will attempt to minimize its purchasing cost: We will show later in the paper that a more senior unit bond fetches a higher price than a less senior unit bond, so the sovereign can minimize purchasing costs by purchasing the most junior debt.[10] We also assume that the price of unit bonds bought back is equal to the price of the most junior debt following the buyback, regardless of seniority. The reason for making this assumption is that a bondholder with a unit bond with $s > b'$ who does not sell the bond during the buyback will be in possession of a bond with ranking b' following the buyback. This unit bond can sell for $q(y, b', b')$. Thus, a bondholder with a unit bond $s < b'$ would be unwilling to sell at any price less than $q(y, b', b')$. Furthermore, bondholders who hold unit bonds with ranking $s > b'$ have bonds whose

[10] Since less senior debt will rise in seniority when more senior debt is bought back, whether the sovereign buys senior or junior debt will have no effect on the seniority structure of oustanding debt. Thus, it is always optimal to purchase the most junior debt as it is the cheapest.

price, namely, $q(y, b', s)$, is at least as large as $q(y, b', b')$. Thus, a sovereign that wishes to buy back debt can implement its plans at least cost if it announces that it will buy back the most junior debt at the price $q(y, b', b')$.

Matters are simpler if the sovereign plans on issuing new debt. In this case, the revenue received from the sale is the integral of the revenue received from each unit bond with ranking between $[b', (1-\lambda)b]$.

We continue to assume that in the case of default, the sovereign is excluded from the international capital markets in the period of default and that, following the period of default, the sovereign is let back into the international capital market with probability ξ. At the time of re-entry, the sovereign agrees to service $G(y, k)$ units of debt going forward, where y is the fundamentals of the economy at the time of re-entry/settlement and k is the level of defaulted debt. The ranking of a unit bond in the restructured debt is simply its ranking s in the defaulted debt, provided $s \geq G(y, k)$. If $s < G(y, k)$, the unit bond is not part of the settlement and it ceases to exist.

The expressions for $X(y, m, k)$ and $W(y, m, b)$ remain the same as in the model without seniority.

5.1 Equilibrium

As before, the world one-period risk-free rate is r_f. With seniority, the market price of a unit of defaulted debt will depend on the rank of the unit bond, since more senior bonds are paid off before less senior bonds. Denote by $q_D(y, k, s)$ the market price of a unit of defaulted debt of rank $s \geq k$. Then the price of a unit bond of rank $s \geq b'$ satisfies the following functional equation:

$$
\begin{aligned}
&q(y, b', s) = \\
&E_{(y'\, m')|y}\left[[1 - d(y', m', b')]\frac{\lambda + [1-\lambda][z + q(y', a(y', m', b'), \max\left([1-\lambda]s, a(y', m', b')\right))]}{1 + r_f}\right] \\
&\qquad + E_{(y', m')|y)}\left[d(y', m', b')\frac{q_D(y', b', s)}{1 + r_f}\right]. \qquad (8)
\end{aligned}
$$

Observe that in the event there is repayment the rank of the unit bond is $(1-\lambda)s$ if the bond is not bought back, i.e., $a(y', m', b') < (1-\lambda)s$. If the bond is bought back, then the price the bondholder will receive will be equal to the price of a unit bond with rank $a(y', m', b')$ (as explained above).

When seniority is enforced, payment on defaulted debt in the event of settlement depends on the ranking of the bond. Let $P(y, m, G(y, k), s)$ denote the value of what the bondholder receives on a defaulted bond of rank s when the current state of the economy is (y, m) and the amount of debt issued in settlement is $G(y, k)$. If $s < G(y, k)$ (i.e., the rank of the unit debt is too low to be part of the settlement), then $P(y, m, G(y, k), s) = 0$. If $s \geq G(y', k)$, then

$$\begin{aligned} & P(y, m, G(y, k), s) \\ = & \begin{cases} \lambda + (1-\lambda)[z + q(y, a(y, m, G(y, k)), \max\{(1-\lambda)s, a(y, m, G(y, k))\})] & \text{if } d(y, m, G(y, k)) = 0 \\ q_D(y, G(y, k), s) & \text{if } d(y, m, G(y, k)) = 1. \end{cases} \end{aligned} \tag{9}$$

The value of the payment depends on what happens in the period of settlement. If the sovereign does not default, then with probability λ the unit bond matures and the bondholder receives 1, and with probability $(1-\lambda)$, the bond does not mature and the bondholder receives the coupon payment z and the market value of an unmatured bond of rank $(1-\lambda)s$ if this bond is not bought back. If it is bought back, the bondholder receives the market value of a unit bond with rank $a(y, m, G(y, k))$.[11] If the sovereign defaults immediately upon settlement (which, again, is possible), the bondholder receives defaulted debt of rank s, where the number of bonds in default is now $G(y, k)$.

Putting all this together, the price of a defaulted bond of rank s is given by

$$q_D(y, k, s) = E_{(y', m'|y)} \left[\frac{(1-\xi) q_D(y', k, s) + \xi P(y', m', G(y', k), s)}{1 + r_f} \right]. \tag{10}$$

We give a characterization result regarding the behavior of the equilibrium price schedule with respect to s and confirm our earlier claim that, all else remaining the same, the price of a unit bond is increasing in seniority (or rank) s. The proof of this claim uses properties of contraction maps.

Proposition 1 *The equilibrium pricing function $q^*(y, b', s)$ is increasing in s.*

Proof. See Appendix ■

[11] For simplicity, the coupon payment on the restructured debt is assumed to be the same as the original debt.

6 Seniority and Welfare: The Argentine Case

In this section we explore the quantitative implications, both positive and normative, of introducing seniority. We focus on Argentina, the country most intensively studied in the quantitative sovereign debt literature. For parameter selection we closely follow Chatterjee and Eyigungor (2011). This model assumes that the (momentary) utility function is CRRA with curvature parameter $(1-\gamma)$, the default cost function $\phi(y) = \max\{0, d_0 y + d_1 y^2\}$, and the stochastic process for the persistent component of output (y) follows an AR1 process with parameters $(\rho, \sigma_\epsilon^2)$. The parameter values that are taken directly from this model are displayed in Table 1.

Table 1: Parameters Selected Independently

Parameter	Description	Value
γ	Risk Aversion	2
$\bar{m}$	Bound on m	0.006
σ_m	Standard Deviation of m	0.003
σ_ϵ	Standard Deviation of ϵ	0.027092
ρ	Autocorrelation	0.948503
ξ	Probability of Reentry	0.0385
r_f	Risk-free Return	0.01
λ	Reciprocal of Avg. Maturity	0.05
z	Coupon Payments	0.03

Regarding repayment, we assume that $G(y,k) = 0.3 \times \bar{y}$. This choice of G is motivated by the fact that during the most recent default episode, bondholders received 30 cents on each dollar of defaulted debt and the mean level of debt in Argentina was equal to mean quarterly real GDP (or 25 percent of mean annual GDP). That leaves three other parameters to be calibrated, namely, d_0, d_1, and β. These parameters are selected so that the model can match: (i) an average external debt-to-output ratio of 1.0, which is the average external debt-to-output ratio for Argentina over the period 1993Q1-2001:Q4; (ii) the average default spread over the same period of 0.0815; and (iii) the standard deviation of the spread of 0.0443. Table 2 reports the parameter values that achieve these targets.

Table 2: Parameters Selected Jointly

Parameter	Description	Value
β	Discount Factor	0.94668
d_0	Default Cost Parameter	-0.21790
d_1	Default Cost Parameter	0.30925

Table 3 reports the effects of moving from an environment without seniority to one in which there is seniority.[12] There is a significant increase in the lifetime utility of the sovereign from this move. In the absence of seniority, the value of constant consumption that gives the same average lifetime utility starting from zero debt and zero transitory income shock (the average is computed over the invariant distribution of y) is 1.02. If the sovereign were to issue bonds ranked by seniority, the constant consumption equivalent of new average lifetime utility rises to 1.0391. Thus, in constant consumption equivalents, the sovereign would be willing to pay 1.9 percent of consumption in perpetuity for this arrangement.

Table 3: Welfare, Default, and Debt With and Without Seniority

Statistic	w/o Seniority	w/ Seniority
Average Welfare, $m = 0$, $b = 0$	1.0200	1.0391
Average Spread	0.0817	0.0258
Std. Dev of Spreads	0.0443	0.0145
Default Probability	0.0631	0.0360
Average (Face value of Debt)/Output	1.00	1.31

Where does this gain in welfare come from? One sees that seniority raises the average debt level and, simultaneously, lowers the frequency of default (default probability). Both factors contribute to the increase in welfare. To better understand these effects, we will first analyze why seniority raises the average debt level and then explain why it lowers the default frequency.

[12]We assume that the country reaches a settlement with the same face value of debt regardless of whether seniority is imposed or not. In reality, the nature of restructuring negotiations might depend on whether all creditors are treated equally or if some creditors have seniority over others. The outcome of bargaining between creditors and sovereigns following default is an active area of research with no settled insights. Given this, we chose to focus on differences in welfare while keeping $G(y, k) = 0.3 \times \bar{y}$ the same between the two market arrangements.

As we explained earlier, in the absence of seniority the sovereign may want to promise all repayment on defaulted debt to *new* debtors by diluting outstanding debt as much as possible prior to default. Indeed, in our calibration, this is what occurs in equilibrium.[13] So, just prior to default, the sovereign issues an infinite amount of new debt so as to borrow and consume the value of resources it is going to give to creditors when settlement occurs (in our calibration this quantity is $0.3 \times \bar{y}$). This additional consumption, just prior to default, is not available to the sovereign when seniority is enforced. Thus, even though the output costs of default are the same regardless of whether or not there is seniority, the full cost of default is greater when seniority is enforced. As a result, the level of debt that can be sustained in equilibrium without default is higher with seniority than without. The increase in the level of sustainable debt is welfare improving because the sovereign is impatient and prefers to borrow.[14]

To understand why the frequency of default is lower when seniority is enforced, we need to understand the incentives that the sovereign faces to extend its borrowing into regions where the probability of default is higher. Suppose that the sovereign issues additional debt in the current period, i.e., $b' < (1-\lambda)b$. In the absence of seniority, the revenue from a marginal unit of debt sold is (treating b' as a continuous variable):

$$q(y, b') + \frac{\partial q(y, b')}{\partial b'}[b' - (1-\lambda)b].$$

The term $q(y, b')$ is the revenue from the sale of the marginal unit. But since $q(y, b')$ is increasing in b', this sale decreases the value of all bonds (the ones being currently issued as well as the ones that already exist) by $\frac{\partial q(y,b')}{\partial b'}$. This means that the sovereign loses the amount $\frac{\partial q(y,b')}{\partial b'}[b' - (1-\lambda)b]$ on all the inframarginal sales. The important point to note here is that although the sale of the marginal unit reduces the value of all outstanding debt, the sovereign cares only about the loss on

[13] Since neither the level of debt nor the spread are welldefined for this period, we exclude these observations when computing the average debt and spreads and volatility of spreads in the model without seniority.

[14] There is also another effect that works in the same direction. When there is resettlement of defaulted debt, the cost of default includes the real value of the resettled debt with which the sovereign eventually exits autarky. With seniority, the value of the debt received in settlement is almost the same as its risk-free value. This is because any unit bond with ranking equal to or higher than $0.3 \times \bar{y}$ is essentially protected from default (the only loss suffered is the loss due to delays in making coupon and principal payments during the autarky period following default). Thus the sovereign expects to pay all the coupons and principal payments on the re-settled debt. Without seniority, the sovereign expects to fully dilute the value of re-settled debt when there is default at some point in the future. Thus, the value of resettled debt is lower when seniority is not imposed (by about 3 percent). This is another reason why default is more attractive when seniority is absent.

the inframarginal sales, i.e., the loss imposed on the *new* debt being issued. In effect, the marginal sale expropriates resources from existing creditors by the amount

$$\frac{\partial q(y,b')}{\partial b'}(1-\lambda)b.$$

When seniority is enforced, and assuming again that $b' < (1-\lambda)b$, the revenue gain from the marginal unit of debt sold is

$$q(y,b',b') + \int_{b'}^{b(1-\lambda)} \frac{\partial q(y,b',s)}{\partial b'} ds.$$

The sovereign receives $q(y,b',b')$ from the sale of the marginal unit but loses some revenue on its inframarginal sales because of depreciation of the value of the inframarginal units. Since price of bonds with different rankings will be affected by different amounts from the increase in debt, the loss is the integral term in the expression above.

With seniority, the expropriation term analogous to (**??**) is:

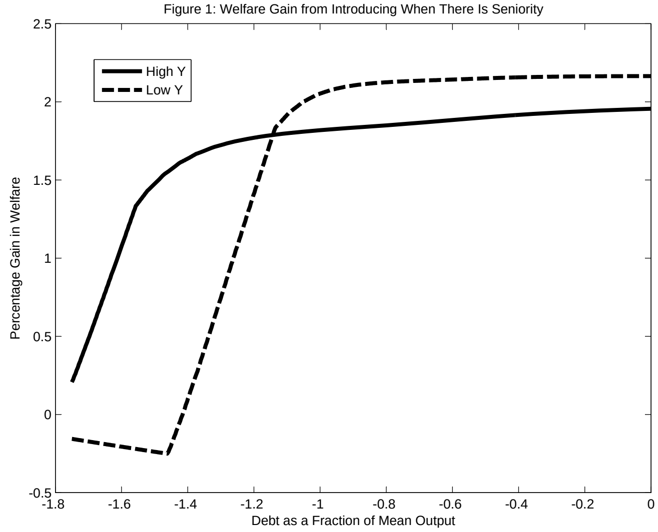
$$\int_{b(1-\lambda)}^{0} \frac{\partial q(y,b',s)}{\partial b'} ds.$$

This is the decline in the value of outstanding debt caused by the sale of the marginal unit of debt. The important point is that the value of the more senior debt is affected much less by new issuances of debt. Indeed, if the sovereign did not spend any time in financial autarky following default, any bond with a ranking above $0.3 \times \bar{y}$ would be fully protected because these bondholders would be paid off in new debt that exactly resembles the defaulted debt. Thus, the value of all bonds that have a ranking higher than $0.3 \times \bar{y}$ is (almost) independent of the total debt of the sovereign. The same logic applies to the value of debt with a ranking less than $0.3 \times \bar{y}$ but close to it. The value of these bonds may also be almost independent of the level of total debt issued because surviving bonds go up in seniority at the geometric rate $(1-\lambda)$. Bonds that currently have a ranking below $0.3 \times \bar{y}$ but close to it will also be protected by the time default comes around.

In this way, seniority protects the value of outstanding bonds from being diluted by the issuance of new debt. The attenuation of the debt dilution problem implies that the loss in value implied by a higher probability of default is borne disproportionately by the new debt that is issued. This in turn means that the revenue collected from new issuances drops more sharply when there is seniority versus when there is not. This discourages the sovereign from extending borrowing into regions where there is a significant probability of default and where both spreads and the frequency of default are lower with seniority. The reduction in the frequency of default is a second source of the welfare gain stemming from seniority.

Could the existence of prior debt negate the welfare gains from introducing seniority? The reason for considering this question is that sovereign debt contracts typically include a *pari passu* clause, which asserts that existing creditors cannot, under any circumstance, be treated any worse than new creditors. This clause implies that seniority can be introduced only if existing creditors are made senior to all new creditors. But if the country already has a lot of debt outstanding, making all that debt senior will result in a large wealth transfer from the sovereign to existing creditors. The loss in welfare from this transfer will cut into the welfare gain the sovereign experiences from enforcing seniority among *new* creditors.

Figure 1: Welfare Gain from Introducing When There Is Seniority



We investigate the effect of prior debt on the welfare gain from introducing seniority. We assume that when seniority is introduced, each outstanding unit bond is randomly assigned a rank between 0 and the $(1-\lambda)b$ and all new issuances get ranks below the lowest-ranked prior debt.[15] Figure 1 shows the welfare gains to Argentina from switching to a seniority regime for different levels of existing debt for two different levels of current output. Note, first, that even when prior debt is 0, the welfare gain from switching to seniority varies with the output level, the gain being higher for lower levels of output. This is intuitive: when output is low, the sovereign has a greater need to borrow, and imposing seniority helps to lower borrowing costs and, therefore, raises welfare. As the level of prior debt increases, the welfare gain from imposing seniority drops as more of the gain goes to existing creditors. It is noteworthy that there continues to be a welfare gain from introducing

[15] An alternative would be to treat all existing bonds as equally senior. This would lead to a common price for existing bonds when seniority is imposed. However, if under either arrangement the sovereign never buys back its existing debt, its behavior going forward will be the same regardless of which alternative is followed. This is because what matters then for the price of new bonds is only that they are junior to existing bonds. Given this, the total value of existing debt will be the same regardless of which arrangement is followed. Since lenders are risk-neutral, they should be indifferent between which arrangement is followed. We follow the first arrangement because it is easier to implement numerically.

seniority for fairly high levels of prior debt. Thus, quantitatively speaking, the presence of prior debt does not appear to be a major hurdle to incorporating a seniority clause in sovereign debt contracts.[16]

7 When Is Seniority a Solution to the Debt Dilution Problem?

In this section, our objective is to determine conditions under which seniority fully solves the debt dilution problem. By "solving the debt dilution problem" we mean that the equilibrium price function q^* has the property that given any y and s, $q^*(y, s, s) = q^*(y, b', s)$ for all $b' < s$. In other words, the price of a unit bond with given seniority s is unaffected by additional borrowing by the sovereign.

We establish two results. First, we show that if there is no delay in reaching a settlement and if optimal decision rules satisfy certain monotonicity conditions, seniority solves the debt dilution problem provided $G(y, k)$ is specified in a particular way. Second, we describe a bargaining environment that delivers all the needed monotonicity conditions and, if renegotiation is costless, plausibly delivers the $G(y, k)$ function that solves the debt dilution problem. This second result extends, to a fully dynamic setting, Bolton and Jeanne's (2009) finding that seniority can solve the debt dilution problem if renegotiation is costless and the creditors have all the bargaining power.

In presenting the theoretical arguments, we will suppress the transitory shock m and assume that the sovereign is not permitted to buy back its debt in the open market. Also, for the second of the two points mentioned above, it is important that the payoff from default be independent of bond prices q. So we take the payoff from default to be some given function $X(y)$.[17]

[16] Initially, the drop in welfare is concave with respect to the level of prior debt, but then it becomes almost linear. The linearly dropping portion of the graphs begins at the point where, in the absence of seniority, the sovereign wants to default. The point where the graphs begin to rise is where the sovereign wishes to default even when seniority is imposed. When there is seniority, the payoff from default is, of course, independent of the debt level, but the payoff from default in the absence of seniority continues to decline with debt. The reason for this is that, as noted earlier, the way the sovereign defaults is to issue new debt up to the maximum debt level possible and then default for sure the following period. Because of this behavior, the payoff from default is no longer independent of the amount of debt being defaulted on because the sovereign services the existing debt prior to defaulting for sure the following period.

[17] For instance, $X(y)$ can be the payoff when the sovereign is permanently excluded from both borrowing and lending internationally (financial autarky).

Then, the utility from repayment, $V(y,b)$, solves the following Bellman equation:

$$\begin{aligned} & V(y,b) = \max_{b' \in B} u(c) + \beta E_{(y'|y)} \max\{V(y',b'), X(y')\} \\ & \text{s.t.} \\ & c = y + m + [\lambda + (1-\lambda)z]\, b + R(y,b,b') \\ & b' \leq (1-\lambda)b. \end{aligned} \tag{11}$$

Taking the existence of $V(y,b)$ as given, it is easy to show that $V(y,b)$ is strictly increasing in b (given $b^0 > b^1$, all choices that are feasible for b^0 are also feasible for b^1 and afford strictly greater consumption). Let

$$b_D(y) = \text{ argmin }_{\{b' \in B\}} \{X(y) \leq V(y,b)\}. \tag{12}$$

Then, $b_D(y)$ is the largest debt level for which repayment is at least as good as default. If the debt level at the start of the period exceeds $b_D(y)$ (i.e., $b < b_D(y)$), the sovereign will default. Then, we have the following result:

Proposition 2 *If $a^*(y,b)$ is increasing in b and $d^*(y,b)$ is decreasing in b, seniority solves the debt dilution problem provided settlement is always reached in the period of default and $G(y,k) = b_D(y)$.*

Proof. See Appendix ■

To gain intuition for this result, consider the following situation. Suppose that the sovereign begins the period with debt b_0 and issues more debt, say, to a level $b_1 < b_0$. We will assume that this additional borrowing increases the probability of default next period. That is, there exists at least one $\hat{y} \in Y$ such that $b_1 < b_D(\hat{y}) < b_0$. We want to examine how the value of an *existing* unit bond with some particular ranking $\tilde{s} \geq b_0$ is affected by the increase in default probability induced by this additional borrowing. Consider what happens to the payoff of this bond if $\hat{y}$ is realized next period. If $\hat{y}$ is realized, there is default and the debt is immediately written down to $G(\hat{y},k)$. Suppose, first, that $G(y,k) > b_D(\hat{y})$. Then, it is possible that $\tilde{s}$ is such that $b_1 < b_D(\hat{y}) < b_0 \leq \tilde{s} < G(\hat{y},k)$.[18] Then, the $\tilde{s}$ unit bond will not be part of the settlement and will receive 0. In this case, the increase in the

[18]This would be the case, for instance, if $G(y,k)$ is close to 0 (there is very little debt offered in the re-structuring).

likelihood default from additional borrowing will lower the value of the $\tilde{s}$ bond. Next, suppose that $G(\hat{y}, k) = b_D(\hat{y})$. Since $b_1 < b_D(\hat{y}) < b_0$ (there is no default for $\hat{y}$ when the debt is b_0) and $\tilde{s} \geq b_0$, the unit bond will be part of the settlement and the bondholder *immediately gets back a new bond that has exactly the same payoff structure as the old bond*. Hence, the increase in the likelihood of default has no effect on the value of the $\tilde{s}$ unit bond.[19] To complete the argument we must also consider what happens if some other value of y is realized next period. Here, we can show that as long as the monotonicity conditions are satisfied, the payoff to the $\tilde{s}$ bond remains unaffected by the additional borrowing (the formal argument is recursive and uses the logic of contraction maps).

We now describe an environment that can throw some light on the determination of $G(y, k)$ and the conditions under which the requirements noted in Proposition 2 will actually hold. Suppose that an indebted sovereign has two options with regard to not servicing its debt. First, it can permanently default, in which case the sovereign does not make any payment ever on the defaulted debt and goes into financial autarky forever. Second, it can agree to a renegotiation of its debt, wherein it agrees to service a lower level of debt, with creditors being serviced in the order of seniority. Renegotiation is potentially costly, with the cost being a one-period proportional output loss of χ, $\chi \geq 0$, but the sovereign maintains access to financial markets even in the period of renegotiation. Assume also that creditors get to make a one-time take-it-or-leave it offer with respect to the renegotiated level of debt. Finally, continue to assume that the sovereign is not permitted to buy back its debt in the open market.

In this environment, if the (y, b) combination is such that permanent default is strictly better than repayment, the sovereign will be tempted to default. However, since creditors do not get anything in the case of a permanent default, they have the incentive to accept a debt writedown. Since creditors have all the bargaining power, it is reasonable to assume that they will propose a debt writedown that is just enough to make the sovereign want to service the lower level of debt rather than default and that the sovereign will accept this proposal.[20] Therefore, whenever the

[19]The situation is different if the debt is renegotiated down to $G(y, k) > b_D(y)$. Then, it is possible that $b_1 < b_D(\hat{y}) < b_0 \leq \tilde{s} < G(y, k)$ and the $\tilde{s}$ unit bond will not be part of the settlement and will receive 0. In this case, the increase in the likelihood of default from additional borrowing *will* lower the value of the bond.

[20]Of course, a debt writedown affects creditors with different rankings differently. As we have already noted, creditors with low enough rankings will not be part of the settlement and will get nothing. Since these creditors get nothing in the case of a permanent default, they ought to be indifferent between agreeing to or not agreeing to the debt writedown if that is their only choice. Note, however, that low-ranked creditors do have an incentive to hold out if they believe they can get transfers from senior creditors as compensation for agreeing to the writedown.

sovereign is tempted to default, there will be an immediate writedown of the debt, and the sovereign will continue on with the lower level of debt and full access to financial markets.

In these circumstances, the dynamic program of the sovereign is as follows:

$$\begin{aligned} & V(y,b) = \max_{b' \in B} u(c) + \beta E_{y'|y} \max\{V(y',b'), V((1-\chi)y', b_D(y';\chi)\} \\ & \text{s.t.} \\ & c \leq y + [\lambda + (1-\lambda)z]b + R(y,b',b) \\ & b' \leq (1-\lambda)b, \end{aligned} \tag{13}$$

where,

$$b_D(y;\chi) = \text{ argmin }_{\{b \in B\}} \{X(y) \leq V((1-\chi)y, b)\}. \tag{14}$$

Then we have the following result:

Proposition 3 *If $\chi = 0$ then, under a mild continuity assumption, a dilution-free equilibrium price function $q^*(y,b',s)$ exists (i.e., for each y and s, $q^*(y,s,s) = q^*(y,b',s)$ for all $b' < s$).*

Proof. See Appendix ■

Proposition 3 relies on some strong assumptions, in particular, that lenders have all the bargaining power as well as the ability to make a one-time take-it-or-leave it offer and also that resettlement/restructuring does not have any resource costs ($\chi = 0$). If the assumption that lenders have all the bargaining power is dropped, but the assumption that bargaining is one-shot is retained, the bargaining problem will resemble the one analyzed in Yue (2010). In this case, the debt has to be written down more for the settlement to be acceptable to the sovereign, i.e., $G(y,k)$ will exceed $b_D(y)$, and seniority will fail to solve the debt dilution problem completely. If the assumption of one-shot bargaining is also dropped in favor of an alternating offer set-up, the bargaining problem will begin to resemble the one analyzed in Benjamin & Wright (2009), who show that both the sovereign and the creditors have an incentive to delay settlement to a more opportune future time.

This "hold out problem" can be avoided if the original debt contract has some form of "collective action clause" that permits senior creditors to force a settlement on all creditors when default is imminent. We will proceed under the assumption that this, in fact, is so.

The possibility of a delay in reaching settlement (more so if the delay is costly in terms of resources) will give an additional reason for $G(y, k)$ to exceed $b_D(y)$ and, therefore, an additional reason for seniority to be less effective in solving the debt dilution problem. How much less effective remains an interesting research question. The welfare gain results for Argentina presented earlier – where both the writedown as well as the delay were considerable – suggest that seniority may be effective in combating the ill-effects of dilution even if the process of reaching settlement takes time and the sovereign retains considerable bargaining power.

References

Aguiar, Mark, and Gita Gopinath. 2006. "Defaultable Debt, Interest Rate and the Current Account." *Journal of International Economics*, 69: 64–83.

Arellano, Cristina. 2008. "Default Risk and Income Fluctuations in Emerging Markets." *American Economic Review*, 98(3): 690–712.

Arellano, Cristina, and Ananth Ramanarayanan. 2010. "Default and Maturity Structure in Sovereign Bonds." Unpublished.

Benjamin, David, and Mark L. Wright. 2009. "Recovery Before Redemption: A Theory of Sovereign Debt Renegotiation." Unpublished.

Bizer, David S., and Peter M. DeMarzo. 1992. "Sequential Banking." *Journal of Political Economy*, 100: 41–61.

Bolton, Patrick, and David Skeel. 2004. "Inside the Black Box: How Should Sovereign Bankruptcy Framework be Structured?" *Emory Law Journal*, 53: 763–822.

Bolton, Patrick, and Olivier Jeanne. 2009. "Structuring and Restructuring Sovereign Debt: The Role of Seniority." *Review of Economic Studies*, 76: 879–902.

Borensztein, Eduardo, Marcos Chamon, Olivier Jeanne, Paolo Mauro, and Jeromin Zettelmeyer. 2006. "Sovereign Debt Structure and Crisis Prevention." *Occasional Paper, International Monetary Fund.*

Chatterjee, Satyajit, and Burcu Eyigungor. 2011. "Maturity, Indebtedness and Default Risk." *American Economic Review*. Forthcoming.

Cole, Harold, and Tim Kehoe. 1996. "A Self-Fulfilling Model of Mexico's 1994-1995 Debt Crisis." *Journal of International Economics*, 41: 309–330.

Dooley, Michael P. 2000. "Can Output Losses Following International Financial Crisis be Avoided?" NBER Working Paper No. 7531.

Eaton, Jonathan, and Mark Gersovitz. 1981. "Debt with Potential Repudiation: Theoretical and Empirical Analysis." *Review of Economic Studies*, 47: 289–309.

Fama, Eugene, and Merton H. Miller. 1972. *The Theory of Finance.* Holt, Rinehart and Winston, NY, USA.

Gelpern, Anna. 2004. "Building a Better Seating Chart for Sovereign Restructurings." *Emory Law Journal*, 53: 1119–1161.

Giavazzi, Francesco, and Marco Pagano. 1990. "Confidence Crisis and Public Debt Management." In *Public Debt Management: Theory and History.* , ed. Rudiger Dornbusch and Mario Draghi, 125–142. Cambridge University Press: Cambridge, MA.

Hatchondo, Juan C., and Leonardo Martinez. 2009. "Long Duration Bonds and Sovereign Defaults." *Journal of International Economics*, 79(1): 117–125.

Hatchondo, Juan C., Leonardo Martinez, and Cesar Sosa-Padilla. 2011. "Debt Dilution and Sovereign Default Risk." *Federal Reserve Bank of Richmond Working Paper 10-08R.*

Hennessey, Christopher A. 2004. "Tobin's Q, Debt Overhang, and Investment." *Journal of Finance*, LIX(4): 1717–1742.

Hutson, Vivian, and John S. Pym. 1980. *Applications of Functional Analysis and Operator Theory.* Academic Press, NY, USA.

Kletzer, Kenneth M. 1984. "Asymmetries of Information and LDC Borrowing with Sovereign Risk." *Economic Journal*, 94(374): 497–307.

Neumeyer, Pablo A., and Fabrizio Perri. 2005. "Business Cycles in Emerging Economies: The Role of Interest Rates." *Journal of Monetary Economics*, 52(2): 345–380.

Sachs, Jeffrey, and Daniel Cohen. 1982. "LDC Borrowing with Default Risk." NBER Working Paper No. 925.

Saravia, Diego. 2010. "On the Role and Effects of IMF Seniority." *Journal of International Money and Finance*, 29: 1024–1044.

Shleifer, Andrei. 2002. "Will the Sovereign Debt Market Survive ?" *American Economic Review*, 93(2): 85–90.

Stokey, Nancy L., and Robert E. Lucas. 1989. *Recursive Methods in Economic Dynamics.* Harvard University Press, MA, USA.

Yue, Vivian Z. 2010. "Sovereign Default and Debt Renegotiation." *Journal of International Economics*, 80(2): 176–187.

A Appendix

In this appendix we give the proofs of Propositions 1 and 2 in the text. In preparation, we state two Lemmas that will be used in the proofs. The first Lemma establishes that the function space needed in our analysis has the requisite completeness property. The second Lemma is a version of Blackwell's sufficiency conditions (for an operator to be a contraction) when the operator takes as inputs vector-valued functions. The proofs are standard but are provided for completeness.

Lemma 1 *Let $W \subset R^L$ and let $f : W \to R^K$ be a vector-valued function defined on W and let $\Phi_K(W)$ be the space of all such functions. Let $F(W) \subset \Phi_K(W)$ be the set of bounded functions for which* $\sup_w \max_k |f^k(w)| \leq B$. *Let $\| f \| = \sup_x \max_k \{|f^1(w)|, |f^2(w)|, \hat{a}e||f^K(w)|\}$. Define the (uniform) metric $\rho(f, g) = \| f - g \|$. Then $(F(W), \| \cdot \|)$ is a complete metric space.*

Proof. It is easy to show that $\| \cdot \|$ satisfies all properties of a metric. Hence, $(F(W), \| \cdot \|)$ is a metric space. To establish completeness of $(F(W), \| \cdot \|)$, we need to show that every Cauchy sequence in this space converges to an element of the space. Let f_n be any sequence in $F(W)$

such that for any $\epsilon > 0$, there exists N_ϵ such that for all $m, n > N_\epsilon$, $\| f_n - f_m \| < \epsilon$. Fixing w and k, this implies that for all $m, n > N_\epsilon$, $|f_n^k(w) - f_m^k(w)| < \epsilon$. By the completeness property of real numbers it follows that $f_n^k(w)$ converges to some real number $f^k(w)$. We will take the function $f(w) = (f^1(w), f^2(w), \cdots, f^K(w))$ as the candidate function to which the sequence f_n converges. Evidently, $f(w) \in F(W)$. To show that $\| f_n - f \| \to 0$ as $n \to \infty$, fix $\epsilon > 0$. By the Cauchy property, there exists N_ϵ such that for all $m, n \geq N_\epsilon$ $\| f_m - f_n \| \leq \epsilon/2$. Then, for any w and k, it follows that for all $m, n \geq N_\epsilon$, $|f_n^k(w) - f_m^k(w)| \leq \epsilon/2$. Hence, for all $m, n \geq N_\epsilon$, $|f_n^k(w) - f^k(w)| \leq |f_n^k(w) - f_m^k(w)| + |f_m^k(w) - f(w)| \leq \epsilon/2 + |f_m^k(w) - f(w)|$. Now, since $f_m^k(w)$ converges to $f(w)$, for m sufficiently far along in the sequence we have $|f_m^k(w) - f(w)| \leq \epsilon/2$. Hence, it follows that for all $m, n \geq N_\epsilon$, $|f_n^k(w) - f^k(w)|\hat{a} \leq \epsilon$. Since both w and k were arbitrary, we have that for all $n \geq N_\epsilon$, $\sup_x \max_k |f^k(w) - f(w)| = \| f_n - f \| \leq \epsilon$. Since ϵ was arbitrary, we have $\| f_n - f \| \to$ as $n \to \infty$. ■

Lemma 2 *Let* $T : (F(W), \| \cdot \|) \to (F(W), \| \cdot \|)$ *be an operator satisfying: (i) (monotonicity)* $f, g \in F(W)$ *and* $f(w) \leq g(w)$, *for all* $w \in W$, *implies* $(Tf)(w) \leq (Tg)(w)$, *for all* $w \in W$ *(here,* $f(w) \leq g(w)$ *means* $f^k(w) \leq g^k(w)$ *for all* $k = 1, 2, \ldots, K$*); (ii) (discounting) there exists some* $\beta \in (0, 1)$ *such that* $[T(f + a)](x) \leq (Tf)(w) + \beta a$ *for all* $f \in F(W)$, $a \in R\hat{a}, \check{S}^K$, $w \in W$ *(here* $(f + a)(w)$ *is the function defined by* $(f + a)(w) = f(w) + a$*). Then* $\| (Tf) - (Tg) \| \leq \beta \| f - g \|$, *i.e.,* T *is a contraction mapping with modulus* β.

Proof. For any f and $g \in F(W)$, $f \leq g + \| f - g \|$ since we are adding to each component of g the largest absolute difference possible between the f and g values for any given component. Then, by properties (i) and (ii), we have $(Tf)(w)\hat{a} \leq (Tg)(w) + \beta \| f - g \|$. Reversing the roles of f and g and using the same logic gives $(Tg)(w) \leq (Tf)(w) + \hat{I}^2 \| f - g \|$. Fix w and k. Then, combining the two inequalities gives $|(Tf)^k(w) - (Tg)^k(w)| \leq \beta \| f - g \|$. Since both w and k were arbitrary, we must have $\sup_w \max_k |(Tf)^k(w) - (Tg)^k(w)| = \| (Tf) - (Tg) \| \leq \beta \| f - g \|$. Hence, T is a contraction with modulus β. ■

Proof of Proposition 1. Let $q^*(y, b\hat{a}e^2, s)$ and $q_D^*(y, k, s)$ be equilibrium pricing functions. Let $d^*(y, m, b)$ and $a^*(y, m, b)$ be the equilibrium default and asset decision rules, respectively. Then, this 4-tuple together satisfy equations (8)-(10). Now, let $W = Y \times B \times [b_I, 0]$ and define $f : W \to R^2$ as $f(w) = (q(y, b', s), q_D(y, k, s))$. Define $F(W) = \{f : 0 \leq f \leq L\}$ as the set of nonnegative functions with upper bound $L = (\lambda + [1-\lambda]z)/(r+\lambda)$. Define $T : (F(W), \|\cdot\|) \to \Phi_2(W)$ as $(Tf)(w) = ((T_1 f)(w), (T_2 f)(w))$, where

$$
\begin{aligned}
&(T_1 f)(w) \\
&= E_{(y'\, m')|y}\left[[1-d^*(y',m',b')]\frac{\lambda+[1-\lambda][z+f^1(y',a^*(y',m',b'),\max\left([1-\lambda]s,a^*(y',m',b')\right))]}{1+r}\right] \\
&\qquad + E_{(y',m')|y)}\left[d^*(y',m',b')\frac{f^2(y',b',s)}{1+r}\right].
\end{aligned}
$$

and

$$
(T_2 f)(w) = E_{(y',m'|y)}\left[\frac{(1-\xi)f^2(y',k,s)+1_{\{s\geq G(y',k)\}}\xi P(f)}{1+r}\right]
$$

where

$$
\begin{aligned}
&P(f) = \\
&\begin{cases} [\lambda+(1-\lambda)][z+f^1(y,a^*(y',m',G(y',k)),\max\{(1-\lambda)s,a^*(y',m',G(y',k))] & \text{if } d^*(y',m',G(y',k))=0 \\ f^2(y',G(y',k),s) & \text{if } d^*(y',m',G(y',k))=1. \end{cases}
\end{aligned}
$$

Then, T satisfies both monotonicity and discounting. To see this, observe that if $g(w) \geq f(w)$, then from inspection it is clear that $(Tg)(w) \geq (Tf)(w)$ and hence T satisfies monotonicity. Next, consider applying T to $f(w)+a$. Then,

$$
\begin{aligned}
(T_1[f+a])(y,b',s) &= (T_1 f)(y,b',s) + E_{(y'\, m')|y}\left\{[1-d^*(y',m',b')]\frac{(1-\lambda)a}{1+r} + d^*(y',m',b')\frac{a}{1+r}\right\} \\
&\leq (T_1 f)(y,b',s) + \left(\frac{1}{1+r}\right)a,
\end{aligned}
$$

and

$$
\begin{aligned}
& (T_2[f+a])(y,b',s) \\
= \;& (T_2f)(y,b',s) + E_{(y'\, m')|y}\left\{ \frac{(1-\xi)a + 1_{\{s\geq G(y',k)\}}\xi \begin{cases} (1-\lambda)]a & \text{if } d^*(y',m',G(y',k))=0 \\ a & \text{if } d^*(y',m',G(y',k))=1. \end{cases}}{1+r} \right\} \\
= \;& (T_2f)(y,b',s) + \frac{(1-\xi)a}{1+r} + \frac{\xi}{1+r}E_{(y'\, m')|y}\left[1_{\{s\geq G(y',k)\}} \begin{cases} (1-\lambda)a & \text{if } d^*(y',m',G(y',k))=0 \\ a & \text{if } d^*(y',m',G(y',k))=1. \end{cases} \right] \\
\leq \;& (T_2f)(y,b',s) + \left(\frac{1}{1+r}\right)a.
\end{aligned}
$$

Hence $(T[f(w)+a])(w) \leq (Tf(w)) + \beta a$, where $\beta = 1/(1+r)$. Therefore, T satisfies discounting. Finally, note that if $f(w) = L$ then $(Tf)(w) \leq L$ and so, by monotonicity, $f \in F(W)$ implies $(Tf)(w) \in F(W)$.

It is also clear from inspection that T preserves monotonicity with respect to s: If $f(y,b',s)$ is increasing in $s \geq b'$, then $(Tf)(y,b',s))$ is increasing in $s \geq b'$. Now, note that the set $F'(W) = \{f \in F(W) : f \text{ is increasing in } s\}$ is a closed subset of $F(W)$. It follows from Lemma 1 and 2 above and Theorem 3.1 and Corollary 1 to Theorem 3.1 of Stokey & Lucas (1989)(pp. 50-52) that there is an unique $f^* \in F'(W)$ such that $(Tf^*)(w) = f^*(w)$. Since $(q^*(y,b',s), q_D^*(y,k,s))$ satisfies equations (8)-(10), $(q^*(y,b',s), q_D^*(y,k,s))$ must be $f^*(w)$. Since $f^* \in F'(W)$, it follows that $q^*(y,b',s)$ and $q_D^*(y,k,s)$ are increasing in s. ■

Proof of Proposition 2. We have to show that for each y and s,

$$q(y,s,s) = q(y,b',s) \text{ for all } b' < s. \tag{15}$$

In the absence of buybacks, the basic equation is

$$q(y,b',s) = \\ E_{y'|y}\left[[1-d^*(y',b')]\frac{\lambda+[1-\lambda][z+q(y',a^*(y',b'),[1-\lambda]s)]}{1+r}+ \\ d^*(y',b')1_{\{s\geq b_D(y')\}}\frac{\lambda+[1-\lambda][z+q(y',a^*(y',b_D(y')),[1-\lambda]s)]}{1+r}\right]. \tag{16}$$

We will use contraction mapping arguments again. Let $W = Y \times B \times [b_I, 0]$ and define $f : W \to R$ as $f(w) = q(y, b', s)$. Define $F(W)$ as the set of nonnegative functions f with bound $L = (\lambda+[1-\lambda]z)/(r+\lambda)$. Define the operator T as the r.h.s of (16), where the decision rules $a^*(y, b)$ and $d(^*(y, b)$ and the function $b_D(y)$ are taken as given. Then $T : (F(W), \|\cdot\|) \to (F(W), \|\cdot\|)$ and T is a contraction map with modulus $1/(1 + r)$.

We will show that if q satisfies (15), then $(Tq)(w)$ satisfies (15) also, i.e., $(Tq)(y, b', s) = (Tq)(y, s, s)$ for all $b' < s$. To this end, fix y and s and consider $b' < s$.

Let y' be such that $d^*(y', s) = d^*(y', b') = 0$ (i.e., y' is a state in which there is repayment for both debt levels). By the no-buyback restriction, $a^*(y', s) \leq (1-\lambda)s$. By the assumed monotonicity of asset decision rule, $a^*(y', b') \leq a^*(y', s)$. By (15), $q(y', a^*(y', b'), [1-\lambda]s) = q(y', a^*(y', s), [1-\lambda]s)$.

Next, suppose y' is such that $d^*(y', s) = 0$ but $d^*(y', b') = 1$. Then, the payoff when debt is s will be $\lambda + [1-\lambda][z + q(y', a^*(y', b'), [1-\lambda]s)]$ and the payoff when debt is b' will be $\lambda + [1-\lambda][z + q(y', a^*(y', b_D(y')), [1-\lambda]s)]$. From the definition of $b_D(y')$, we have that $b_D(y') < s$ (because $V(y', s) > X(y')$ while $V(y', b_D(y')) \leq X(y')$ and V is increasing in b). Hence, by the assumed monotonicity of the asset decision rule, $a^*(y', b_D(y')) \leq a^*(y', s) \leq (1-\lambda)s$. By (15), we have that $q(y', a^*(y', b_D(y')), [1-\lambda]s) = q(y', a^*(y', s), [1-\lambda]s)$.

Next, suppose that y' is such that $d^*(y', s) = d^*(y', b') = 1$. Then, the payoff when debt is s or b' is $\lambda + [1-\lambda][z + q(y', a^*(y', b_D(y')), [1-\lambda]s)]$.

Since $d^*(y, b)$ is decreasing in b these are the only three cases we need to examine.

Putting the results together, we conclude that $(Tq)(y', b', s) = (Tq)(y', s, s)$ for all $b' < s$. To complete the proof, observe that the set $F'(W) = \{f \in F(W) : f \text{ satisfies (15)}\}$ is a closed subset of $F(W)$. It follows (from Theorem 3.1 and Corollary 1 to Theorem 3.1 of Stokey & Lucas (1989) again) that $f^* \in F'(W)$. Hence, it must be true that $q^*(y, b', s)$ satisfies (15). ■

We now turn to the proof of Proposition 3.

Let $\mathcal{V}$ be the set of all real valued functions defined on $Y \times B$. For $v, v' \in \mathcal{V}$, define the (uniform) metric $\| v - v' \| = \max_{y,b} |v(y,b) - v'(y,b)|$. Then, $(\mathcal{V}, \| \cdot \|)$ is a complete metric space. Let $\mathcal{Q}$ be $\{q(y,b',s) : q(y,b',s) \in [0,L]$ and $q(y,b',s)$ is Lebesgue integrable in $s \geq b'\}$, where $L = (\lambda + [1-\lambda]z)/(r+\lambda)$.

To begin, observe that substituting (14) into objective function in (13) leads to exactly the same dynamic program as in (11). In what follows, we will need to make explicit the dependence of decision rules and value functions on the price function. Thus $V_q(y,b)$, $a_q(y,b)$ and $d_q(y,b)$ are the value under repayment and the asset and default/renegotiate decision rules corresponding to the pricing function $q(y,b',s)$. Let $b_D(y;\chi,q)$ be the renegotiated debt in the event of a renegotiation.

Lemma 3 *Let $q \in \mathcal{Q}$. Then $V_q(y,b) : Y \times B \times \mathcal{Q} \to R$ exists and is monotonically increasing and continuous in q.*

Proof. Fix $q \in \mathcal{Q}$. Define the operator $T_q : \mathcal{V} \to \mathcal{V}$ as follows:

$$
\begin{aligned}
&(T_qV)(y,b) = \max_{b' \in B \text{ and } b' \leq (1-\lambda)b} u(c) + \beta E_{y'|y} \max\{V(y',b'), X(y')\} \\
&\text{s.t.} \\
&c \leq y + [\lambda + (1-\lambda)z]b + \int_{b'}^{(1-\lambda)b} q(y,b',s)ds.
\end{aligned}
$$

if at least one feasible b' exists. If there are no feasible choices of b', then

$$(T_qV)(y,b) = -U/(1-\beta).$$

It is easy to establish that T_q satisfies monontonicity and discounting and is, therefore, a contraction map. The existence of $V_q(y,b)$ follows from the Contraction Mapping Theorem.

To show monotonicity of V_q with respect to q, let $q^1 \leq q^0$, both in Q. Observe that $(T_{q^1}V_{q^0})(y,b) \leq V_{q^0}(y,b)$. From the monotonicity of T_{q^1} it follows that $(T^k_{q^1}V_{q^0})(y,b) \leq V_{q^0}(y,b)$ for all $k \geq 1$. Since $V_{q^1}(y,b) = \lim_{k\to\infty}(T^k_{q^1}V_{q^0})(y,b)$, $V_{q^1}(y,b) \leq V_{q^0}(y,b)$. To establish the continuity of V with respect to q, it is sufficient to establish that T_q is continuous in q (Hutson & Pym (1980), Proposition 4.3.6, p. 117) and for this it is sufficient to establish that the r.h.s of the budget inequality is continuous in q. Let q_k be a sequence of functions in Q converging to $\bar{q}$. Since each member of the sequence is a nonnegative integrable function bounded above by L, by the Lebesgue Dominated Convergence Theorem,

$$\lim_k \int_{b'}^{(1-\lambda)b} q_k(y, b', s)ds = \int_{b'}^{(1-\lambda)b} \lim_k q(y, b', s)ds = \int_{b'}^{(1-\lambda)b} \bar{q}(y, b', s)ds.$$

This establishes that the operator T_q is continuous in q and, hence, $V_q(y, b)$ is continuous in q. ■

Corollary 1 *$d_q(y, b)$ and $b_D(y; \chi, q)$ are monotonically increasing in q.*

Proof. Follows from the monotonicity of $V_q(y, b)$ and the fact that $X(y)$ is independent of q. ■

Lemma 4 *For each $q(y, b', s) \in \mathcal{Q}$, (i) $d_q(y, b)$ is decreasing in b and (ii) if for each s, $q(y, b', s)$ is increasing in $b' \leq s$ then $a_q(y, b)$ is increasing in b.*

Proof. (i) Fix y. If $d_q(y, b) = 0$ for some $\hat{b}$, then it must be the case that $V_q(y, \hat{b}) \geq X(y)$. Since for any q, $V_q(y, b)$ is increasing in b (all repayment choices that are feasible under b^0 are also feasible under $b^1 > b^0$ and afford strictly greater consumption), it follows that $V_q(y, b) \geq X(y)$ for all $b > \hat{b}$. Hence, $d_q(y, b) = 0$ for all $b > \hat{b}$. It follows that for any $q(y, b', s)$, $d_q(y, b)$ must be decreasing in b.

(ii) Fix y and let $b^1 < b^0$. If $(1 - \lambda)b^1 \leq a_q(y, b^0)$ then by the "no buyback" restriction, we have $a_q(y, b^1) \leq (1 - \lambda)b^1 \leq a_q(y, b^0)$ and we are done. Suppose, then, that $a_q(y, b^0) < (1 - \lambda)b^1$. Denote $a_q(y, b^0)$ by b'^0 and the associated consumption level by c^0. Let $\hat{b}' \in B$ be such that $b'^0 < \hat{b}'^1$. Then, $\hat{b}'$ is feasible under b^0. Let $\hat{c}$ be the consumption level when $\hat{b}'$ is chosen under b^0. We claim that $c^0 > \hat{c}$. To see this, let $Z_q(y, b') = E_{y'|y} \max\{V_q(y', b'), X(y')\}$. Then, by optimality and the tie-breaking rule that if the sovereign is indifferent between two b's it always chooses less debt, we have

$$u(c^0) + \beta Z(y, b'^0) > u(\hat{c}) + \beta Z(y, \hat{b}'). \tag{17}$$

Since $Z(y, \hat{b}') \geq Z(y, b'^0)$, the inequality implies $c^0 > \hat{c}$.

Next, define $\Delta(b^0) = c^0 - \hat{c} > 0$. Thus, $\Delta(b^0)$ is the loss in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period debt is b^0. From the budget constraint, $\Delta(b^0) =$

$R(y, b'^0, b^0) - R(y, \hat{b}'^0)$. Holding fixed $\hat{b}'$ and b'^0, let $\Delta(b^1)$ be the value of Δ that solves $\Delta(b^1) = R(y, b'^0, b^1) - R(y, \hat{b}'^1)$. Then $\Delta(b^1)$ is the loss in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period debt is b^1. We claim that $\Delta(b^1) \geq \Delta(b^0)$. To see this, use the definition of $R(y, b', b)$ to get

$$\Delta(b^0) = \Delta(b^1) + \left[\int_{(1-\lambda)b^1}^{(1-\lambda)b^0} q(y, b'^0, s)ds - \int_{(1-\lambda)b^1}^{(1-\lambda)b^0} q(y, \hat{b}', s)ds \right].$$

Now, by assumption, $q(y, b', s)$ is increasing in b', so the term in square brackets is nonpositive. Therefore, $\Delta(b^1) \geq \Delta(b^0)$.

Next, let $\tilde{c}$ be the level of consumption when $\hat{b}'$ is chosen under b^1. We claim that $\tilde{c} < c^0$. To see this, note that $R(y, b'^0, b^1) \leq R(y, b'^0, b^0)$ (since $q(y, b', s) \geq 0$ and the first integration is over a smaller set of s) and, hence, $b^1 < b^0$ implies $[\lambda + (1-\lambda)z]b^1 + R(y, b'^0), b^1) < [\lambda + (1-\lambda)z]b^0 + R(y, b'^0, b^0)$. Therefore, $\tilde{c}$ is strictly less than c^0.

Finally, to complete the proof, observe that the strict concavity of u implies $u(\tilde{c}) - u(\tilde{c} - \Delta(b^1)) > u(c^0) - u(c^0 - \Delta(b^0)) = u(c^0) - u(\hat{c})$. Therefore, (17) implies that $u(\tilde{c}) + \beta Z(y, b'^0) > u(\tilde{c} - \Delta(b^1)) + \beta Z(y, \hat{b}')$. Since $\hat{b}'$ is any feasible b' greater than b'^0, the optimal choice of b' under repayment when the beginning-of-period debt is b^1 cannot be greater than b'^0. Therefore, $a_q(y, b^1) \leq a_q(y, b^0)$. ∎

Definition 1 *Let $Q \subset \mathcal{Q}$ be the set of $q(y, b', s)$ such that (i) for each $b' \in B$, $q(y, b', s)$ is increasing in $s \geq b'$, and (ii) for each $s \in (B_I, 0]$, $q(y, b', s)$ is constant for $b' \leq s$.*

Lemma 5 *Q is a closed under pointwise convergence.*

Proof. Let $q_k(y, b', s)$ be a sequence in Q converging pointwise to $\bar{q}(y, b', s)$. Since every element of the sequence is bounded above by L and satisfies property (i) and (ii), it is evident that the pointwise limit will also be bounded above by L and satisfy property (i) and (ii). To establish the result, all we need to confirm is that the pointwise limit of a sequence of integrable functions is also integrable. But this follows from an application of the Lebesgue Dominated Convergence Theorem (see, for instance, Stokey & Lucas (1989) Theorem 7.10, p. 192). ∎

Lemma 6 *Let* $(Hq)(y, b', s) : Q \to \mathcal{Q}$ *be the operator defined by*

$$
\begin{aligned}
&(Hq)(y, b', s) = \\
&E_{y'|y}\left[[1 - d(y', b'; q)]\frac{\lambda + [1-\lambda][z + q(y', a(y', b'; q), [1-\lambda]s)]}{1+r} + \right. \\
&\qquad \left. d(y', b'; q)1_{\{s \geq b_D(y';q)\}}\frac{\lambda + [1-\lambda][z + q(y', a(y', b_D(y'; q); q), [1-\lambda]s)]}{1+r}\right].
\end{aligned}
$$

Then (a) $(Hq)(y, b', s) \in Q$ *for* $q \in Q$ *and (b)* H *is monotone: If* $q^1 \leq q^0$*, both in* Q*, then* $(Hq^1)(y, b'^0)(y, b', s)$*.*

Proof. We will first prove that if $q \in Q$, then $(Hq) \in \mathcal{Q}$. First, observe that for any $q \in \mathcal{Q}$, $0 \leq (Hq)(y, b', s) \leq [\lambda + (1-\lambda)(z + L)]/(1+r)$. Substituting in for the value of L shows that $0 \leq (Hq)(y, b', s) \leq L$. Second, since linear combinations of Lebesgue integrable functions are also Lebesgue integrable, $(Hq)(y, b's)$ is integrable in s. This establishes that $(Hq) \in \mathcal{Q}$. To establish that (Hq) belongs in Q, observe that property (i) follows from inspection and property (ii) follows from Lemma 3 and Proposition 2.

To prove H is monotone, fix y' and b'. There are three cases to consider.

First, suppose that $d_{q^1}(y', b') = d_{q^0}(y', b') = 0$. For all $s \geq b'$ and $j = 0, 1$, the payoff from holding the unit bond is $\lambda + (1-\lambda)(z + q^j(y', a_{q^j}(y', b'), (1-\lambda)s))$. Since $q^1 \leq q^0$, it follows that $q^1(y', a_{q^0}(y', b'^0(y', a_{q^1}(y', b'^0(y', a_{q^0}(y', b'), (1-\lambda)s)$, where the last equality follows from the constraint $a_q(y', b') \leq (1-\lambda)b'$ and property *(ii)* of Q. Thus, the payoff from holding the unit bond when the price function is q^1 is at most the payoff when the price function is q^0.

Second, suppose that $d_{q^1}(y', b') = 1$ and $d_{q^0}(y', b') = 0$. Then, the payoff from holding the unit bond when the price function is q^1 is $1_{\{s \geq b_D(y'^1)\}} \times [\lambda + (1-\lambda)(z + q^1(y', a_{q^1}(y', b_D(y'^1), (1-\lambda)s))]$, while the payoff from holding the unit bond when the price function is q^0 is $\lambda + (1-\lambda)(z + q^0(y', a_{q^0}(y', a_{q^0}(y', b')), (1-\lambda)s))$. If $s < b_D(y'^1)$ the payoff in the former case is 0, while it is positive in the latter case. If $s \geq b_D(y'^1)$, then $q^1(y', b_D(y'^1), (1-\lambda)s) \leq q^0(y', b_D(y'^1), (1-\lambda)s) = q^0(y', a_{q^0}(y', a_{q^0}(y', b'), (1-\lambda)s)$, where the first inequality follows by assumption and the second inequality follows (again) from the constraint $a_q(y', b') \leq (1-\lambda)b'$ and property *(ii)* of Q. Therefore, it is the case again that the payoff when the price function is q^1 is no larger than the payoff when the price function is q^0.

Third, consider the case in which $d_{q^1}(y', b') = d_{q^0}(y', b') = 1$. Now suppose that $s \geq b_D(y; q^1)$. Then, since $b_D(y; q^1) \geq b_D(y; q^0)$, it follows that the payoff from holding the bond when the pricing function is q^j is $\lambda + (1-\lambda)(z + q^j(y', b_D(y'^j), (1-\lambda)s))$. Now observe that $q^1(y', b_D(y'^1), (1-\lambda)s)) \leq q^0(y', b_D(y'^1), (1-\lambda)s)) = q^0(y', b_D(y'^0), (1-\lambda)s))$. Hence, the payoff from holding the unit bond when the price function is q^1 is at most the payoff when the pricing function is q^1. Next, consider the case where $s < b_D(y, q^1)$. Then, if $b_D(y'^0) \leq s < b_D(y'^1)$, the payoff from holding the bond when the price function is q^0 is strictly positive whereas it is zero when the price function is q^1. If $s < b_D(y'^0) \leq b_D(y'^1)$, then the payoff is zero for either price function.

Thus, it follows that the payoff from holding the unit bond when the price function is q^1 is at most equal to the payoff when the price function is q^0. Since y' and b' were arbitrary, it follows that $(Hq^1)(y, b'^0)(y, b', s)$. This establishes the monotonicity of H. ∎

Proof of Proposition 3. Observe first that if $\chi = 0$, then $b_D(y; 0, q)$ is simply $b_D(y; q)$ defined in (12). Hence, the equilibrium price function q^* must satisfy $q^* = (Hq^*)$, where H is the operator defined in Lemma 6.

We will construct a convergent sequence of price functions whose limit will be the equilibrium price function, provided that H is continuous at the limit point. Let $q^0(y, b', s) = L$. Then by Lemma 6, $q^1 = (Hq^0)(y, b'^0(y, b', s)$. By monotonicity of H and induction, we have $(H^{k+1}q^0)(y, b'^k q^0)(y, b', s)$ for $k \geq 1$. Now fix (y, b', s). Then, $(H^k q^0)(y, b', s))$ is a decreasing sequence of nonnegative real numbers that must converge to some nonnegative number $q^*(y, b', s)$. By Lemma 5, the function $q^*(y, b', s)$ is in Q.

We now claim that q^* is an equilibrium pricing function provided H is continuous at q^*. Observe that $q^* = \lim_k(H^k q^0) = H(\lim_k(H^{k-1}q^0)) = (Hq^*)$, where the first equality is by definition, the second is just relabeling, and the third follows from the assumed continuity of H at q^*. ∎

Remark: A sufficient condition for H to be continuous at q^* is that the decision rules $d(y, b; q^*)$ and $a(y, b; q^*)$ be strictly optimal, meaning that for each (y, b) any feasible action other than the optimal one yields strictly lower utility. To see this, let $\{q_k(y, b)\}$ be the monotonically decreasing sequence converging to q^*. Since $V_q(y, b)$ and $R(y, b', b; q)$ are both continuous in q (Lemma 3) and the set of states (y, b) is finite, we can be assured that there exists K such that $d(y, b; q^*)$ and $a(y, b; q^*)$ are the optimal decision rules for all $q_k(y, b)$, $k > K$. This means that the optimal decision rules do not change in the "tail" of the sequence and, therefore, $\lim_{k>K}(Hq_k)(y,b'^*)(y,b',s)$.