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Maturity, Indebtedness, and Default Risk [1]

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Abstract

In this paper, we advance the theory and computation of Eaton-Gersovitz style models of sovereign debt by incorporating long-term debt and proving the existence of an equilibrium price function with the property that the interest rate on debt is increasing in the amount borrowed and implementing a novel method of computing the equilibrium accurately. Using Argentina as a test case, we show that incorporating long-term debt allows the model to match the average external debt-to-output ratio, average spread on external debt, the standard deviation of spreads and simultaneously improve upon the model's ability to account for Argentina's other cyclical facts.

1 Introduction

Until recently, the existing literature on debt and default – both the consumer debt and the sovereign debt parts – has considered only one-period debt. In reality, both consumers and countries can and do borrow for more than one period. In this paper, we present a new approach to incorporating long-term debt in equilibrium models of unsecured debt and default in the style of Eaton and Gersovitz (1981).

We make four contributions. First, we show that there exists an equilibrium price function for unsecured long-term debt with the property that the supply curve for credit is rising in the interest rate. Thus, a key implication of the Eaton-Gersovitz framework is shown to carry over to the case of long-term debt.[1]

Second, we contribute to the quantitative-theoretic literature on emerging market business cycles by providing a more complete accounting of the facts. Specifically, we show that our model can easily account for the observed high debt-to-output ratio, the average spread, as well as the volatility of spreads in emerging markets without compromising the model's ability to account for emerging market business cycle facts.[2] We show that the role of long-term debt is critical in this accounting in that a model with one-period debt cannot match the three key first and second moments without generating excessive cyclical volatility in consumption and the trade balance and low correlations of these quantities with output.[3]

[1]The models of consumer bankrupcty analyzed in Chatterjee et al. (2007) and others bear a strong resemblance to the Eaton-Gersovitz model. The model and the solution procedure developed in this paper are thus relevant to the consumer bankruptcy literature as well.

[2]As documented in Neumeyer and Perri (2005), open emerging market economies display a high cyclical volatility of consumption and a countercyclical trade balance. Aguiar and Gopinath (2006) and Arellano (2008) showed that the Eaton-Gersovitz framework, with its rising supply curve for credit, can quantitatively account for these patterns.

[3]We suspect that any endowment model that (i) has only one-period (i.e., quarterly) debt, (ii) matches a high level of debt (70 percent of quarterly output on average, as in our case) and (iii) has a spread volatility as high as in the data will tend to imply a counterfactually high consumption volatility. The reason is that in such a model a large fraction of consumption must be financed by issuance of new debt and it would take counterfactually small shifts in the Eaton-Gersovitz supply curve for credit to generate consumption volatility of the magnitude we see in the data. To have a model consistent with realistic debt-to-output ratios and observed volatilities of spreads and consumption we must recognize that only a small portion of consumption is financed by new issuance of debt. This is precisely what a model with long-term debt permits.

Third, we investigate the optimal maturity length. Although long-term debt is a better hedge against low realizations of output, the fact that the sovereign cannot commit to limit its future borrowing makes long-term debt more expensive – the so-called "debt dilution problem." In our calibrated model, the cost imposed by the debt dilution problem turns out to be the dominant effect and welfare is decreasing in maturity. However, we show that this result is reversed when we modify our model to allow for a small probability of a self-fulfilling rollover crisis. A sovereign that issues long-term debt is less vulnerable to a rollover crisis than one that issues short-term debt.[4] Indeed, we show that this additional source of shock does not affect the properties of the baseline long-term debt model but affects the properties of the one-period debt model significantly. Specifically, if debt is short-term, the sovereign constrains its borrowing in order to reduce the likelihood of rollover crises. In our calibrated model, this scaling back of debt makes one-period debt inferior to long-term debt.

Finally, we present a novel approach to accurately computing models with unsecured long-term debt and default. Our approach relies on the presence of a low-variance i.i.d. output shock drawn from a continuous CDF. As we explain later in the paper, continuity of the CDF is key to avoiding a lack of convergence and the i.i.d nature of the shock is key to developing an algorithm that can solve for the equilibrium accurately.

The paper is organized as follows. Section 2 provides a brief literature review. Section 3 introduces the sovereign debt environment. Section 4 gives the main theoretical results and explains the computational challenge involved in solving this class of models and how this challenge can be met. Section 5 presents all the quantitative results when the model is calibrated to Argentina's experience in the 1990s. The three appendices contain results of a more technical nature, including the proofs of the main propositions, the logic and performance details of our computational algorithm, and robustness of results to pure computational assumptions.

[4] A self-fulfilling rollover crisis is default resulting from a coordination failure, wherein if all lenders continue to lend the sovereign will continue to repay but if each lender suspects that other lenders will not extend new loans and therefore refuse to lend in anticipation of a default then the sovereign defaults. Cole and Kehoe (2000) provide a theoretical demonstration that long-term debt can reduce the probability of rollover crisis relative to short-term debt.

2 Literature Review

There is a related literature on long-term sovereign debt. Hatchondo and Martinez (2009) introduce consols with geometrically declining coupon payments. They show that having long-duration debt considerably improves the cyclical volatility of spreads relative to a one-period debt model.[5] Nevertheless, the standard deviation of spreads in their model is very low, at most 0.33 percent, compared to the data, which they report to be 2.51 percent.

Bi (2008a) examines maturity choice in a model of one- and two-period debt with renegotiation on defaulted debt and shows that when default is likely in the near future, it is relatively attractive for the sovereign to borrow short-term. Thus, the maturity structure of debt shortens when approaching a crisis. Arellano and Ramanarayanan (2009) examine maturity choice using the long-duration debt model of Hatchondo and Martinez. The focus of their paper is on the cyclical properties of the term spread and the duration of debt (maturity).

There have been a number of recent additions to the quantitative sovereign debt literature that extend the Eaton-Gersovitz framework in important directions while maintaining the assumption that debt is one-period. Bi (2008a), D'Erasmo (2008), Benjamin and Wright (2009) and Yue (2009) explicitly model the debt renegotiation process that follows sovereign default. Cuadra and Sapriza (2008) examine the role of political uncertainty in affecting the level and volatility of sovereign spreads. Mendoza and Yue (2009) endogenize the costs of default by combining a production model featuring foreign working capital loans (as in Neumeyer and Perri (2005) and Uribe and Yue (2006)) with the Eaton-Gersovitz framework. Several of these extensions were motivated by the desire to generate debt-to-output ratios that come closer to the high levels observed for emerging markets. However, with one exception, these studies do not come close to generating the high debt-to-output ratios we

[5]They depart from the Eaton-Gersovitz framework in assuming that sovereigns do not lose access to international capital markets upon default but simply suffer a one-period proportional output loss (up to 50 percent in the worst case).

see in the data.[6]

3 Environment

3.1 Preferences and Endowments

Time is discrete and denoted $t \in \{0, 1, 2, ...\}$. The sovereign receives a strictly positive endowment x_t each period. The stochastic evolution of x_t is governed by the following process:

$$x_t = y_t + m_t. \tag{1}$$

Here $m_t \in M = [-\bar{m}, \bar{m}]$ is a transitory income shock drawn independently each period from a mean zero probability distribution with continuous cdf $G(m)$, and y_t is a persistent income shock that follows a finite-state Markov chain with state space $Y \subset \mathbb{R}_{++}$ and transition law $\Pr\{y_{t+1} = y' | y_t = y\} = F(y, y') > 0,\ y$ and $y' \in Y$. As noted in the introduction, the i.i.d shock m is included to make robust computation of the model possible. In the quantitative analysis to follow, the endowment process (1) is estimated assuming a very small variance for m.

The sovereign maximizes expected utility over consumption sequences, where the utility from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1 \tag{2}$$

The momentary utility function $u(\cdot) : [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing, strictly

[6] Arellano (2008) obtained a mean debt-to-output ratio of 6 percent, Bi (2008a) 21.2 percent, Aguiar and Gopinath (2006) 19 percent, Yue (2009) 10.1 percent, Cuadra and Sapriza (2008) 6.9 percent and Mendoza and Yue (2009) 23.1 percent. Benjamin and Wright (2009) are able to generate a debt-to-output ratio of 65 percent but they do not compare the level and the standard deviation of spreads in their model to that in the data.

concave, and bounded above by the quantity U.

3.2 Option to Default and the Market Arrangement

The sovereign can borrow in the international credit market and has the option to default on a loan. Default is costly in several ways. First, upon default, the sovereign loses access to the international credit market – cannot borrow or save in the period of default – and remains in financial autarchy for some random number of periods. Specifically, following the period of default, the sovereign is let back into the international credit market with probability $0 < \xi < 1$. Second, during its sojourn in financial autarchy, the sovereign loses some amount $\phi(y) > 0$ of the persistent component of output y. Third, the sovereign's transitory component of income drops to $-\bar{m}$ in the period of default.[7] We assume that $y - \phi(y) - \bar{m} > 0$ (which ensures that $y - \phi(y) + m > 0$ for all $(y, m) \in (Y \times M)$) and that $y - \phi(y)$ is increasing in y.[8]

We analyze long-term debt contracts that mature probabilistically. Specifically, each unit of outstanding debt matures next period with probability λ. If the unit does not mature, which happens with probability $1 - \lambda$, it gives out a coupon payment z. Note that, going forward, a unit bond of type (z, λ) issued $k \geq 1$ periods in the past has exactly the same payoff structure as another (z, λ) unit bond issued $k' > k$ periods in the past. This means that we need to keep track of the *total* number of (z, λ) bonds only. This cuts down on the number of state variables.[9] In what follows we assume that unit bonds are infinitesimally

[7] This technical assumption is made for the purpose of speeding up computation. It is not important that m take the lowest value possible. As we verify in the sensitivity analysis section, setting $m = 0$ (the mean value of the transitory shock) works just as well.

[8] In this paper, a function $f(x)$ is *increasing (decreasing) in* x if $x' > x$ implies $f(x') \geq (\leq) f(x)$ and *is strictly increasing (strictly decreasing)* in x if $f(x') > (<) f(x)$.

[9] If bonds mature deterministically in T periods, the sovereign's state vector will contain the vector $(b_0, b_1, b_2, \ldots, b_{T-1})$, where b_τ is the quantity of bonds due for repayment τ periods in the future. Thus one has to keep track of at least T state variables, each of which can take many values. Hatchondo and Martinez (2009) use a similar trick of rendering outstanding obligations "memoryless." In their setup, all bonds last forever (consols) but each pays a geometrically declining sequence of coupon payments. Thus, a bond issued in the current period promises to pay the sequence $\{1, \delta, \delta^2, \delta^3, \ldots\}$. This payoff structure is the same as that of a unit random maturity bond with $\lambda = 1 - \delta$ and $z = 1$. Our specification has the advantage that it allows separate targeting of maturity length and coupon payments.

small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer's coupon obligations next period will be $z \cdot (1 - \lambda)b$ with certainty and the issuer's payment-of-principal obligations will be λb with certainty.

We assume that there is a single type of bond (z, λ) available in this economy. We assume that lenders are risk-neutral and that the market for sovereign debt is competitive. The unit price of a bond of size b is given by $q(y, b)$. The price of a unit bond does not depend on the transitory shock m because knowledge of current period m does not help predict either m or y in the future and, therefore, does not inform the likelihood of future default. We assume that the sovereign can choose the size of its debt from a finite set $B = \{b_I, b_{I-1}, \ldots b_2, b_1, 0\}$, where $b_I < b_{I-1} < \ldots < b_2 < b_1 < 0$.[10] As is customary in this literature, we will view debt as negative assets.

3.3 Decision Problem

Consider the decision problem of a sovereign with $b \in B$ of type (z, λ) bonds outstanding and endowments $(y,\, m)$. Denote the sovereign's lifetime utility conditional on repayment by the function $V(y, m, b) : Y \times M \times B \to \mathbb{R}$, its lifetime utility conditional on being excluded from international credit markets by the function $X(y, m) : Y \times M \to \mathbb{R}$, and its unconditional (optimal) lifetime utility by the function $W(y, m, b) : Y \times M \times B \to \mathbb{R}$.

Then,

$$X(y, m) = u(c) + \beta\{[1 - \xi]E_{(y'\, m')|y}X(y', m') + \xi E_{(y'\, m')|y}W(y', m', 0)\} \tag{3}$$

s.t.

$$c = y - \phi(y) + m$$

The sovereign's lifetime utility under financial autarchy reflects the fact that it loses $\phi(y)$ of its output and can expect to be let back into the international credit market next period

[10] For simplicity, we do not allow the sovereign to accumulate assets. In our application, the no-accumulation constraint is never binding in the simulations.

with probability ξ.

And,

$$V(y,m,b) = \max_{b' \in B} u(c) + \beta E_{(y'\, m')|y} W(y',m',b') \tag{4}$$
$$\text{s.t.}$$
$$c = y + m + [\lambda + (1-\lambda)z]\, b - q(y,b')\, [b' - (1-\lambda)b]$$

The above assumes that the budget set under repayment is nonempty, meaning there is at least one choice of b' that leads to nonnegative consumption. But it is possible that (y,b,m) is such that all choices of b' lead to negative consumption. In this case, repayment is simply not an option, and the sovereign must default.

Finally,

$$W(y,m,b) = \max\{V(y,m,b), X(y,-\bar{m})\}. \tag{5}$$

Since W determines both X and V (via equations (3) and (4), respectively) equation (12) defines a Bellman equation in W.

We assume that if the sovereign is indifferent between repayment and default, it repays. Hence, the sovereign defaults if and only if $X(y,-\bar{m}) > V(y,m,b)$. This decision problem implies a default decision rule $d(y,m,b)$ (where $d=1$ is default and $d=0$ is repayment) and, in the region where repayment is feasible, a debt decision rule $a(y,m,b)$. We assume that if the sovereign is indifferent between two distinct b's, it chooses the larger one (i.e., chooses a lower debt level over a higher one).

3.4 Equilibrium

The world one-period risk-free rate r_f is taken as exogenous. Given a competitive market in sovereign debt, the unit price of a bond of size b, $q(y,b')$, must be consistent with zero profits adjusting for the probability of default. That is:

$$q(y,b') = E_{(y'\,m')|y}\left[[1-d(y',m',b')]\frac{\lambda+[1-\lambda][z+q(y',a(y',m',b'))]}{1+r_f}\right] \tag{6}$$

In the event of default, the creditors get nothing. In the event of repayment, the creditors get λ, which is the fraction of a unit bond that matures next period, and on the remaining fraction, $(1-\lambda)$, the creditors get the coupon payment z. In addition, the fraction that remains outstanding has some value that depends on the persistent component of the sovereign's endowment next period and on the sovereign's debt next period. Since the right-hand side of the equation depends directly and indirectly (through the decision rules d and a) on $q(y,b')$, equation (6) defines a functional equation in $q(y,b')$, namely, $q = H(q)$.

4 Theory and Computational Algorithm

4.1 Theory

This section states results regarding how default and borrowing decisions change with regard to variations in b. These results are then used to establish the main theoretical result of this paper, namely, that an equilibrium pricing function exists with the property that price is increasing in b' (proofs of all assertions are available either in Appendix A or the Web Appendix).

Proposition 1 (Characterization of the Default Decision Rule): $d(y,m,b)$ is decreasing in b.

For one-period debt, Proposition 1 implies that the equilibrium pricing function is increasing in b', or, equivalently, that more credit is supplied at higher interest rates. In the one-period case $z = 0$ and $\lambda = 1$ and the equilibrium pricing equation (6) reduces to $q(y,b') = E_{(y'\,m')|y}[1-d(y',m',b')]/[1+r_f]$. Since the right-hand side of this equality is increasing in b' by virtue of Proposition 1, $q(y,b')$ is increasing in b'. But if $\lambda < 1$ (average maturity is greater than 1 period) the price function also depends on $a(y,m,b)$. The behavior of $q(y,b')$ with respect to b' now depends on how $a(y,m,b)$ varies with b. But the behavior of $a(y,m,b)$ with respect to b depends, in turn, on how $q(y,b')$ varies with b'. Proposition 2 states that if the price function is increasing in b', the debt decision rule is increasing in b. Proposition 3 then establishes that there always exists a solution to the pricing equation (6) that is increasing in b'.

Proposition 2 (Characterization of Debt Decision Rule): If $q(y,b')$ is increasing in b' then $a(y,m,b)$ is increasing in b.

Proposition 3 (Existence and Characterization of the Equilibrium Price Function): There exists an equilibrium price function $q^*(y,b')$ that is increasing in b'.

The existence of an equilibrium price function requires the presence of the random variable m with continuous CDF. With this addition, the operator H is a continuous function of q. The reason why m is needed to make H continuous is given in the next section.

4.2 Computation of the Equilibrium Price Function

In this section, we explain why computing the equilibrium price function can be challenging and how this challenge is met in our paper. The solution procedure is to iterate on (6) until convergence. More precisely, let k denote the iteration number, let $Z^k(y,b') = E_{(y',m'|y)}W^k(y',m',b')$ be the expected lifetime utility conditional on y and $b'$and let $d(y,m,b;q^k,Z^k)$ and $a(y,m,b;q^k,Z^k)$ be the default and debt decision rules when the price function is q^k and expected lifetime utility is Z^k; with a slight abuse of notation, let $H[q^k,Z^k]$ denote the expectation on the r.h.s of (6) given $d(y,m,b;q^k,Z^k)$ and $a(y,m,b;q^k,Z^k)$ and let ζ be a "relaxation parameter." Then,

$$q^{k+1} = (1-\zeta)H[q^k,Z^k] + \zeta q^k, \tag{7}$$

$$Z^{k+1} = E_m \max\{X(y,-\bar{m};Z^k)), V(y,m,b;q^k,Z^k)\} \tag{8}$$

and the iteration is continued until $\max|H(q^k, Z^k) - q^k|$ and $\max|Z^{k+1} - Z^k|$ are both sufficiently close to zero.

For the iterations to converge, a solution to (6) must exist. But if both y and b are discrete and m is identically zero, there is no assurance that (6) will have a solution because $H(q)$ need not be continuous in q: For instance, for some q and $(y, b, 0)$, the sovereign will be indifferent between default and repayment and an infinitesimal change in q will cause a switch in behavior and, therefore, a discrete change in the expectation on the r.h.s. of (6). Similarly, even if the sovereign strictly prefers to repay, it may be indifferent between two different choices of debt. Once again, an infinitesimal change in q can result in a discrete change in behavior and a discrete change in the expectations term. Note that the scope for getting jumps due to indifference is much greater for long-term debt than short-term debt because of the fact that $a(y, m, 0)$ appears in the pricing equation.[11] This additional complication of long-term debt cannot be attenuated by making the grid on B fine because the points of indifference can be far apart on the grid. The budget set under repayment is typically not convex. Figure 1 shows a portion of $q(y, b')[b' - (1 - \lambda)b]$ function for the case in which $b = 0$ and $m = 0$ from our quantitative model presented later in the paper. Observe the kink and the ensuing depression in the upward sloping portion of the function. Figure 2 displays the variation in total lifetime utility from different choices of b' for $b = 0$ and $m = 0$. Observe the many nonconcave segments in this function. These nonconcavities imply that, given (y, b) and q, the sovereign may be indifferent between two widely separated values of b', which will make the r.h.s of (6) discontinuous in q.

In Appendix B we document that solving the model without the m shock leads to poor convergence outcomes for the equilibrium pricing function. The points marked A (the global maximum) and B (a local maximum) in Figure 2 illustrate what goes wrong. When the variance of m is set to 0, it often happens that a point like A is the optimal choice for some iteration k, but when that choice is incorporated in the price function, a point like B

[11] Exact indifference never happens in computations but changes in q from one iteration to the next are not infinitesimal either. The point is that if two options are near-indifferent, very small changes in q^k can cause a discrete change in behavior and, therefore, a discrete change in q^{k+1}.

becomes the optimum choice for some iteration $k' > k$; and when that choice is incorporated in the price function, the point A re-emerges as the new optimum for some iteration $k'' > k'$. Thus, the asset decision rule meanders back and forth and (7) fails to converge. We suspect that this lack of convergence occurs because there is, in fact, no solution to (6).[12]

We know from general equilibrium theory that nonexistence resulting from nonconvexities can be avoided by allowing agents to randomize over decisions. Introducing m is like introducing randomization: There is now a *probability* that an action d or b' is chosen given (y,b) and q and this probability changes continuously with q. The nonconvexity of the budget set and nonconcavity of the value function continue to imply that the decision rule $a(y, m, b; q)$ is a discontinuous function of q. But as long as the points of discontinuity are finite in number, infinitesimal changes in q will *not* cause jumps in the expectation term (which is now an integral over y' *and* the continuous variable m') since each jump point has probability zero. In this way, a continuous CDF for m ensures the continuity of the r.h.s of the functional equation (6) with respect to q and the existence of an equilibrium.[13]

However, the introduction of m brings its own computational issues. Since m is a *continuous* variable and non-convexities make $a(y, m, b; q^k, Z^k)$ potentially discontinuous in m, it is not obvious how this potentially discontinuous decision rule is to be computed. This is where the assumption that m is i.i.d. plays an important role – it allows us to establish that $d(y, m, b)$ and $a(y, m, b)$ are monotone with respect to m, which, in turn, allows us to devise an algorithm to recover the decision rules near-exactly. We have:

[12] We consider an algorithm to have failed to converge if the maximum relative error between two successive iterates of the price function does not fall below 10^{-5} within 3000 iterations. If the lack of convergence is due to non-existence of an equilibrium then no matter how long we allow the program to run, it will not converge. Given this, some stopping rule is needed and we chose 3000 (we have confirmed that the algorithm without the iid shock does not converge for higher bounds as well).

[13] An alternative strategy to prove existence is to work with decision correspondences (as opposed to decision rules). In this approach, if the decision-maker is indifferent between two (or more) actions then each action is taken with some probability that is determined in equilibrium. The proof of existence of an equilibrium relies on the Kakutani Fixed Point Theorem for compact and convex-valued correspondences. While this approach solves the existence issue, it does not appear to be computationally tractable. In particular, computing mixed strategies when the "support points" of the mixed strategy are not known in advance – and the choice set is very large – seems to be a challenging task.

Proposition 4: $a(y, m, b)$ is increasing in m and $d(y, m, b)$ is decreasing in m.

The task of computing the decision rules with respect to m thus boils down to (i) locating the value of m at which $d(y, m, b; q^k, Z^k)$ switches from 1 to 0 and (ii) the values of m at which the $a(y, m, b; q^k, Z^k)$ switches from one debt level to a lower debt level.[14] Note that it is not known in advance which lower value of debt the sovereign switches to as m increases because $a(y, m, b; q^k, Z^k)$ may be discontinuous in m and the lower debt level need *not* be the next lower debt level on the grid. However, an algorithm exists, described in Appendix B, that can check for these discontinuities and recover $a(y, m, b; q^k, Z^k)$ correctly.[15] Once behavior with respect to m is known, $\max\{X^k(y, -\bar{m}), V^k(y, m, b; q^k, Z^k)\} = W^k(y, m, b; q^k, Z^k)$ can be integrated with respect to m accurately using the integration method also described in Appendix B.

Another issue is that the iteration (7) may fail to converge if the variance of m is too small. As q changes, the thresholds for m change. If the variance of m is very small, any given change in thresholds will result in a large change in the choice probabilities. Setting ζ very close to 1 can counteract this sensitivity (by making the change $q^{k+1} - q^k$ itself very small) but at the expense of making the number of iterations needed to achieve convergence much larger than our upper bound of 3000. Thus, to achieve convergence σ_m must not be

[14] The behavior of $d(y, m, b)$ with respect to m is easy to characterize also because of the assumption that the act of default resets m to $-\bar{m}$. This assumption makes the payoff from default independent of m. If the level of transitory income were to remain unaffected by the act of default, the payoff from default, X, would also depend positively on m. As shown in Chatterjee et al. (2007), this would result in the default set being characterized by *two* thresholds, m^L and m^U, with default occurring when $m \in (m^U, m^L)$. Since the role of the transitory shock in this paper is to ensure convergence of (7), it is computationally efficient to eliminate the dependence of X on m so that only one default threshold needs to be computed.

[15] For each (y, b) and q, the algorithm recovers $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ such that b'^K is chosen for $m \in [-\bar{m}, m^{K-1})$, b'^{K-1} is chosen for $m \in [m^{K-1}, m^{K-2})$, $\ldots$, b'^1 is chosen for $m \in (m^1, \bar{m}]$ ($K = 1$ means the same debt level b'^1 is chosen for all $m \in M$). Note that $K - i$ need not be adjacent to $K - (i+1)$ on the grid. Note also that it is not possible to apply these methods to a continuous y because y is not i.i.d and, therefore, the debt decision rule and the default decision rule are nonmonotonic in y. For instance, if current y is above its mean, the price of debt is low, and the sovereign has an incentive to issue more debt. On the other hand, an above-mean y implies that the country will be poorer in the future, which gives the sovereign an incentive to borrow less. These two effects pull in opposite directions and result in nonmonotonic behavior of b' with respect to y. In the absence of monotonicity, it is unclear if an algorithm can be devised to locate the values of y at which there is a switch in debt levels or default decision.

too small.[16] More generally, there is a tradeoff between σ_m and ζ with regard to achieving convergence: The lower is σ_m, the higher must ζ be to achieve convergence (an example of this tradeoff is given in the Web Appendix).

The final computational point is whether there are advantages to solving long-duration debt models assuming that y and b are discrete as opposed to continuous. We believe there are two advantages. First, if either y or b (or both) is a continuous variable, $q(y, b')$ is infinite-dimensional and it is much harder to establish the existence of a solution to $q = H(q)$. If a solution is not guaranteed, and a computational algorithm fails to converge, it is not possible to tell if this failure results from a lack of a solution or from a defective algorithm. Second, with continuous y and/or b', any computation scheme must involve interpolating value functions and the price function. For the interpolations to be justified, the functions must be smooth (i.e. differentiable) (see, for instance, Theorems 6.7.3 and 6.9.1 in Judd (1998)). But for this class of models neither value functions nor the price function are smooth everywhere.[17]

[16] Although we cannot prove that there is a unique equilibrium, we have not found instances of multiple equilibria. We do know, theoretically, that given the price vector $q(y, b')$, and a tie-breaking rule in case of indifference, the decision rules $d(y, m, b)$ and $a(y, m, b)$ are unique (see the Web Appendix for a proof).

[17] How well interpolation techniques work in practice is an open research question. Hatchondo, Martinez and Sapriza (2010) show that for the one-period debt model, interpolation techniques can deliver accurate results in the sense that interpolation methods give the same answer as the discrete state space method with a very fine grid for the model described in Arellano (2008). They apply a variant of their method to their long-duration bond model in Hatchondo and Martinez (2009) but they do not compare how well their method performs in solving the pricing equation relative to the discrete state space method. Also, it is not known if interpolation methods work well for the empirically relevant parameter space. In Arellano (2008), Hatchondo, Martinez and Sapriza (2010) as well as in Hatchondo and Martinez (2009), the level and volatility of spreads, as well as the level of debt, are quite low relative to the data.

5 Maturity, Indebtedness, and Spreads: The Argentine Case

5.1 Calibration

We apply the framework developed in the previous sections to Argentina. The main contribution is to show that long-duration bonds, besides being a closer fit with reality, can help account for the average level of spreads, the volatility of spreads, and the average level of debt in Argentina without generating counterfactual implications regarding Argentina's business cycle facts. Thus, introducing long-duration bonds into the Eaton-Gersovitz model significantly improves its quantitative performance. We focus on the 8-year period between 1993:Q1 and 2001:Q4 during which Argentina was on a fixed exchange rate vis-a-vis the dollar and was borrowing in international credit markets via marketable bonds.[18]

For the quantitative work we make the following specific functional form or distributional assumptions.

- Endowment processes:

$$\ln y_t = \rho \ln y_{t-1} + \epsilon_t, \text{ where } 0 < \rho < 1 \text{ and } \epsilon_t \sim N\left(0, \sigma_\epsilon^2\right)$$
$$m_t \sim \text{ trunc } N\left(0, \sigma_m^2\right) \text{ with points of truncation } -\bar{m} \text{ and } \bar{m}$$

- Utility function: $u(c) = c^{1-\gamma}/(1-\gamma)$.

- The loss in the persistent component of output in the event of default or exclusion:

$$\phi(y) = \max\{0, d_0 y + d_1 y^2\}, d_1 \geq 0.$$

The specification for $\phi(y)$ allows for a variety of cost functions. If $d_0 > 0$ and $d_1 = 0$, the

[18] This is also the time period analyzed in Arellano (2008).

cost is proportional to output; if $d_0 = 0$ and $d_1 > 0$, the cost rises more than proportionately with output; if $d_0 < 0$ and $d_1 > 0$, the cost is 0 for $0 \leq y \leq -d_0/d_1$ and rises more than proportionately with output for $y > -d_0/d_1$. This last case resembles the cost function in Arellano (2008).[19] The reasons for choosing this flexible form are discussed in the findings section.[20] With these assumptions, the numerical specification of the model requires giving values to 11 parameters. These are (i) three endowment process parameters, σ_m, ρ and σ_ϵ^2; (ii) two preference parameters, β and γ; (iii) two parameters describing the bond, the maturity parameter λ, and the coupon payment z; (iv) two default output loss parameters, d_0 and d_1, (v) the probability of re-entry following default, ξ; and (vi) the risk-free rate r_f.

The parameters of the endowment process are estimated on linearly detrended quarterly real GDP data for the period 1980:1-2001:4.[21] As noted earlier, convergence of (7) requires that the standard deviation of the i.i.d. shock m be not too low. Experimentation shows that $\sigma_m = 0.003$ is a good lower bound for our purposes – meaning that convergence is achieved within 3000 iterations for a wide range of parameter values. Thus, the endowment process is estimated assuming that $\sigma_m = 0.003$. The estimated value of ρ and σ_ϵ are 0.948503 and 0.027092, respectively.[22] In the computations, we approximate the y process by a 200-state Markov chain and set $\bar{m} = 2\sigma_m = 0.006$.[23] Of the preference parameters, the value of γ is set equal to 2, which is the standard value used in this literature.

[19] In Arellano, $\phi(y) = \max\{0, y - \bar{y}\}$. Thus, cost is 0 for $0 \leq y \leq \bar{y}$ and rises linearly at rate 1 beyond $\bar{y}$. Thus, default costs as a proportion of y, namely, $(1- \bar{y}/y)$, increase strongly with y.

[20] With this specification, the cost can exceed y for large y This situation never arises in our application but could be formally ruled out by setting $\phi(y) = \min\{y, \max\{0, d_0 y + d_1 y^2\}\}$.

[21] The quarterly data series on real GDP, real aggregate consumer expenditure, real exports, real imports and the (nominal) interest rate on Argentine sovereign debt is taken from Neumeyer and Perri (2005). All the quantity variables were deseasonalized using the multiplicative X-12 routine in EViews.

[22] If the process is estimated without the transitory shock, the estimates of ρ and σ_ϵ are 0.930139 (0.038395) and 0.027209 (0.001577), respectively, where the values in parenthesis are standard errors. Note that the values of ρ and σ_ϵ used in the calibration are well within 1 standard deviation of these AR1 estimates and statistically indistinguishable from them. Note also that adding m to the AR1 equation is equivalent to assuming that log GDP is measured with some noise. Since the standard deviation of log GDP in the sample is 0.076107, setting $\sigma_m = 0.003$ implies that σ_m^2 is 0.16 percent of the variance of log GDP. This is small compared with the standard deviation of measurement errors assumed in estimation of DSGE models (see, for instance, Ireland (2004); see Del Negro and Schorfheide (2010, p. 53) for a discussion of this practice).

[23] The value of $\bar{m}$ is small enough that the requirement $y - \max\{0, d_0 y + d_1 y^2\} - \bar{m} > 0$ is satisfied for all values of y in the Markov chain and for all d_0 and d_1 used in the computations.

The parameters describing the bond were determined to match the maturity and coupon information for Argentina reported in Broner, Lorenzoni, and Schmukler (2007). The median maturity of Argentine bonds is 20 quarters so $\lambda = 1/20 = 0.05$. We set $z = 0.03$, corresponding to an annual coupon rate of 12 percent. In the data, the value-weighted average coupon rate is about 11 percent.[24]

We set $\xi = 0.0385$, which gives an average period of exclusion of 26 quarters, or 6.5 years. One measure of the exclusion period is the time it took to reach settlement on the defaulted debt. Beim and Calomiris (2000, Table A) report that for the 1982 Argentine default, settlement did not occur until 1993. For the 2001 default, Argentina reached settlement with a majority of its creditors in 2005. Benjamin and Wright (2009, Figure 15) also report Argentina as being in a state of default between 1982 and 1993 and between 2001 and 2005. By these measures, the average exclusion period for Argentina is 7.5 years. Gelos, Sahay and Sandleris (2008) measure exclusion as the years between default and the date of the next issuance of public and publicly guaranteed bonds or syndicated loans. By this measure, exclusion following the 1982 default lasted only 4 years (Table A7). They do not report the exclusion period for the 2001 default. In our calibration, we give somewhat more weight to the settlement-date measures of the exclusion period and set the average exclusion period to 6.5 years.

The risk-free rate, r_f, was set at 0.01, which is roughly the real rate of return on a 3-month (one quarter) U.S. Treasury bill.

The three remaining parameters β, d_0, and d_1 are selected to match (i) an average external debt-to-output ratio of 0.7, which is 70 percent of the average external debt-to-output ratio for Argentina over the period 1993Q1-2001:Q4; (ii) the average default spread over the same period of 0.0815; and (iii) the standard deviation of the spread of 0.0443.[25] We seek to

[24]We chose 12 percent because with an annual risk-free rate of 4 percent and an average spread of around 8 percent, a bond with coupon of 12 percent will trade roughly at par. So, whether the debt is recorded at face value (which is the accounting practice) or at market prices (which is economically more sensible) will not matter for the calibration of the model.

[25]Debt is total long-term public and publicly guaranteed external debt outstanding and disbursed and owed to private and official creditors at the end of each year, as reported in the World Bank's

match only a portion of debt because we do not model repayment. In reality, sovereign debt that goes into default eventually pays off something. In Argentina's case, the repayment on debt defaulted on in 2001 has been around 30 cents to the dollar. Thus, we treat only 70 cents out of each dollar of debt as the truly unsecured portion of the debt. But, as part of our sensitivity analysis, we also examine the case in which we fully match average external debt-to-output ratio.

Finally, we need to specify the model analogs of the external debt-to-output ratio and spreads. In the GDF database, the external commitments of a country are reported on a cash-accounting basis, which means that commitments are recorded at their face value, i.e., they are recorded as the undiscounted sum of future promised payments of principal.[26] The coupon payments agreed to do not figure directly in this accounting because they are not viewed as obligations until they are past due. Given this valuation principle, the model analog of debt as reported in the data is simply b, and the external debt-to-output ratio is simply b/y.[27] The default spread in the model is calculated as in the data. We compute an internal rate of return $r(y, b')$which makes the present discounted value of the promised sequence of future payments on a unit bond equal to the unit price, that is, $q(y, b') = [\lambda + (1-\lambda)z]/[\lambda + r(y, b')]$. The difference between $(1 + r(y, b'))^4 - 1$ and $(1 + r_f)^4 - 1$ is the annualized default spread in the model.[28]

The parameter selections are summarized in the following two tables. Table 1 lists the

Global Development Finance Database (series DT.DOD.DPPG.CD). The average debt-to-output ratio is the average ratio of debt to GNP measured at a quarterly rate. The spread was calculated as the difference between the interest rate data reported in Neumeyer and Perri (which is the same as the EMBI data) and the 3-month T-bill rate. The T-bill rate series used is the TB3MS series available at `http://research.stlouisfed.org/fred2/categories/116`. Both the interest rate data and the T-bill rate are reported in annualized terms.

[26] See "Coverage and Accounting Rules" in Section 3 of the World Bank Statistical Manual on External Debt (also available at http://go.worldbank.org/6FB4093970).

[27] The reason for this is that each bond can be viewed as a combination of unit bonds with varying maturities. For instance, a measure λ of unit bonds is due next period, a measure $(1-\lambda)\lambda$ is due in 2 periods, $\ldots$, a measure $(1-\lambda)^{j-1}\lambda$ is due in j periods, and so on. Since each of these obligations has a face value of 1, each would be recorded as a unit obligation. Thus, the total obligation is simply $\sum_{j=1} \lambda(1-\lambda)^{j-1} = 1$.

[28] If there is no possibility of default, the unit price would be a constant $\bar{q}$ such that $\bar{q} = [\lambda + (1-\lambda)(z + \bar{q})]/[1 + r_f]$,which implies $\bar{q} = [\lambda + (1-\lambda)z]/[\lambda + r_f]$. Since $q(y, b) \leq \bar{q}$, it follows that $r(y, b') \geq r_f$. Furthermore, the higher is the probability of default, the lower is $q(y, b')$ and higher is $r(y, b')$.

values of the parameters that are selected directly without solving for the equilibrium of the model. Table 2 lists the parameter values that are selected by solving the equilibrium of the model and choosing the parameters so as to make the model moments come as close as possible to the three data moments mentioned above.

Table 1: Parameters Selected Directly

Parameter	Description	Value
γ	risk aversion	2
$\bar{m}$	bound on m	0.006
σ_m	standard deviation of m	0.003
σ_ϵ	standard deviation of ϵ	0.027092
ρ	autocorrelation	0.948503
ξ	probability of reentry	0.0385
r_f	risk-free return	0.01
λ	reciprocal of avg. maturity	0.05
z	coupon payments	0.03

Table 2: Parameters Selected by Matching Moments

Parameter	Description	Value
β	discount factor	0.95402
d_0	default cost parameter	-0.18819
d_1	default cost parameter	0.24558

5.2 Findings

The results of the moment matching exercise are reported in Table 3. The first column of numbers shows the data for Argentina. The second column reports the moments in the model. All model moments are (sample) averages calculated by simulating the economy over many periods but always discarding the first 20 periods after re-entry following each default.[29] Evidently, the matching exercise is fully successful.[30] Figure 3 shows the path of the model-simulated spreads for 1993-2001 if the initial level of debt is chosen to exactly match the spread in 1993:Q1. The close correspondence between the model-implied spreads and the data is striking.

For comparison purposes, the last column reports the corresponding model statistics of Arellano's one-quarter debt model. Although Arellano did not target these statistics, the fact remains that there are significant deviations between her model and the data: The debt-to-output ratio is very low, the average spread is about 50 percent lower and the volatility of spreads is about 44 percent higher.

Table 3: Results and Comparison

	Data	Baseline	Arellano
		($\beta = 0.95, d_0 = -0.19, d_1 = 0.25$)	(2008, Table 4, p.706)
Avg. Spread	0.0815	0.0815	0.0358
Std Dev of Spread	0.0443	0.0443	0.0636
Debt-to-Y	1.00	0.70	0.06

Table 4 reports some key cyclical properties of Argentine data and corresponding model moments.[31] Since we did not target these moments, the results are informative about the performance of our model.

[29]We do this because the model economy re-enters capital markets without any debt, whereas Argentina emerged from each default/restructuring episode with debt. By ignoring the first five years following re-entry, we ignore years with counterfactually low debt in our model.

[30]For the record, the average debt-to-output ratio in the baseline model when debt is measured at its market value is 0.703. So, it is only slightly higher than its face value. The reason is that the average interest rate on debt, 0.0292 percent per quarter, is only slightly larger than the 0.0285 ($= (1-\lambda)z = 0.95 \times 0.03)$) coupon payment on each unit of debt.

[31]Second moments for consumption and output were computed using logged and linearly de-trended series. Since net exports (NX) can be negative, it was expressed as a proportion of output and then linearly de-trended. The spread series was also linearly detrended, although the trend component is negligible.

Table 4: Cyclical Properties, Data and Models

Variable	Data (93Q1-01Q4)	Baseline Model	1-period Debt Model	Arellano (2008)
$\sigma(c)/\sigma(y)^{*}$	1.09	1.11	1.59	1.10
$\sigma(NX/y)/\sigma(y)$	0.17	0.20	1.06	0.26
$corr(c,y)^{*}$	0.98	0.99	0.73	0.97
$corr(NX/y,y)$	-0.88	-0.44	-0.16	-0.25
$corr(r-r_f,y)$	-0.79	-0.65	-0.55	-0.29
Avg. Debt Service[#]	0.053	0.055	0.699	0.056
Default Freq**	0.125	0.068	0.073	0.030
*Sample period: 1980:Q1-2001:Q4; **Sample period:1975-2001 # Principal and interest payments as a fraction of output				

The first column of numbers is the data for Argentina. Several features of the data stand out. First, the relative volatility of consumption is about the same as output – in stark contrast to small, open, developed economies. Second, the trade balance is countercyclical, which is also in contrast to small, open, developed economies. Third, spreads on sovereign debt are countercyclical and Argentina displayed a high propensity to default during the 1975-2001 period.[32] The following column reports the same statistics for the model. The model gets the qualitative patterns of the data right: Model consumption and trade balance have about the right level of volatility relative to output, and the trade balance and spreads are countercyclical while consumption is highly procyclical. The forces in the model that lead to these patterns are the ones emphasized in Aguiar and Gopinath (2006) and Arellano (2008).[33] The average probability of default in the model is lower than the observed frequency

[32]The frequency of default is the number of default episodes as a fraction of the number of years Argentina was in good standing with international creditors in the 27 years between 1975-2001. Argentina defaulted in August 1982 and re-gained access in March 1993. We assume that Argentina was in default for 11 years. Thus it defaulted twice in a 16-year period of good standing. We chose 1975 as the start date because that is when Argentina began accumulating significant amounts of debt. If we start in 1946 and use the "years in restructuring" reported in Beim and Calomiris (2000, Table A), Argentina would show three defaults in a 35-year period of good standing. This would give a default frequency of 0.086. If we start in 1800 and use Beim and Calomiris again, the default frequency would drop to 0.03. But it is questionable if our model is the right framework to address such a long sweep of history.

[33]When output is below trend, the probability of default on new loans rises. If this rise is sharp enough,

of default; but default is a rare event and it is hard to estimate its frequency accurately from relatively short data series.

The second column reports results if debt is assumed to be one period and (β, d_0, d_1) are chosen to match the same three statistics as in the baseline model. The parameter vector that achieves this match is $(0.67, -0.46, 0.57)$. Aside from the implausibility of such a low value of β, the match generates serious anomalies with respect to business cycle statistics. The relative volatility of consumption and the trade balance is now much higher than in the data and the correlations of consumption, the trade balance and spreads with output are much lower. Incorporating long-term debt moves virtually every model moment in Table 4 closer to the data, a clear indicator of the superior performance of the long-term debt model.

Why does the one-period debt model imply such a high volatility of consumption? The reason is simple: If there are b dollars of debt outstanding, the debt service obligation is b and the sovereign must refinance *all* of b at the new price $q(y, b)$ to maintain its debt level. Thus, changes in $q(y, b)$ will tend to imply large changes in consumption and the trade balance because b is large relative to output. In contrast, with long-term debt, the debt service obligation is only $[\lambda + (1 - \lambda)z]b$ and the sovereign can maintain its debt level by refinancing the much smaller quantity λb at the new price $q(y, b)$. Thus, for a given volatility of spreads, the long-term debt model can match the average level of debt without creating a counterfactually high volatility of consumption and the trade balance.

The last column reports the results for the benchmark model in Arellano (2008, Table 4). Even ignoring the frequency of default, the model with long-term debt comes substantially closer to accounting for the cyclical moments. In Arellano's model, the (negative) correlations between net exports and output and spreads and output are, on average, about 33 percent as large as in the data. In contrast, in the model with long-term debt these correlations, respectively, are 50 percent and 82 percent as large as in the data.

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it is optimal for the sovereign to *reduce* debt rather than to increase it (which is what would be optimal holding interest rates constant). Thus, there is a tendency for consumption to decline more than the decline in output and for the trade balance to improve with a fall in output.

5.3 Model Mechanics

5.3.1 Role of the Default Cost Function

Since the calibrated values d_0 and d_1 are negative and positive, respectively, our specification shares the feature that Arellano introduced in her specification of default costs, namely, that the default cost as a proportion of output declines with output and becomes zero for low enough output levels.

It is now well-understood that this structure of default punishment is important in generating higher default rates, whether the default cost function is endogenous or exogenous (see, for instance, the discussion in Mendoza and Yue (2009)). The key is the asymmetry in default costs: The country is punished much more severely for default when income is high than when income is low. The severe punishment for default in high-income periods implies that investors do not expect the sovereign to default in the near future (given the persistence in output). This results in low spreads, and the (impatient) sovereign borrows aggressively. But when output declines, the punishment from default declines as well. This raises the likelihood of default and spreads rise. The high spreads make debt servicing more onerous, and, eventually, if income stays low, the sovereign defaults. Without the asymmetry, it is impossible to generate a significantly positive default frequency without making the sovereign very impatient.[34]

What appears not to have been appreciated in the literature is that the structure of default costs is also important for the *volatility* of spreads. For Arellano's specification, the default cost as a proportion of output is $1 - \bar{y} \cdot y^{-1}$ (where $\bar{y}$ is the level of output below which costs are zero), which is very sensitive to changes in y. Consequently, the probability of default is correspondingly sensitive to fluctuations in y and so is the spread (recall that Arellano's model predicted a higher volatility of spread than the data). With

[34] For instance, with a proportional default cost, the cost does not vary much with the level of output and spreads remain relatively high over a wide range of output and debt levels. Consequently, the sovereign rarely borrows enough to enter into regions where the probability of default is measurably positive (unless the sovereign is very impatient).

our specification, we can match both the level *and* the volatility of spreads. The larger d_1 is, the more volatile the spreads are likely to be. This intuition is verified in Table 5 which shows the results of varying d_1 while choosing d_0 and β to match the targets for average debt-to-output ratio and the average spreads. Notice that the volatility of spreads rises with d_1.

Table 5: Role of Default Cost Parameters

d_1	d_0	β	$\sigma(r - r_f)$	avg. $(r - r_f)$	avg. b/y
0.15	-0.098	0.93696	0.0264	0.0815	0.70
0.25 (baseline)	-0.188	0.95402	0.0443	0.0815	0.70
0.35	-0.288	0.96195	0.0577	0.0815	0.70

The accompanying changes in β and d_0 are informative about the economics of the model and are worth commenting on. The higher is d_1 the more sensitive spreads are to variation in output and the easier it is for the model to achieve a higher frequency of default. Since higher default frequency and high spreads are easier to achieve with a higher d_1, the sovereign needs to be *more* patient in order for it to willingly hold the level of debt that implies the observed probability of default. This explains why the value of β rises with d_1. We also see that d_0 falls as d_1 rises. This is because an increase in d_1 shifts up the default cost function which expands the maximum amount of debt the sovereign can carry without defaulting. As long as β is sufficiently less than $1/(1 + r_f)$, the sovereign will gravitate to this maximum and that will increase the average level of debt. To keep the average debt level constant, the overall default punishment should remain roughly constant. Thus, d_0 falls to counterbalance the increase in d_1.

5.3.2 The Role of Long-term Debt

In this section, we explain the role of long-term debt in our model. One way to understand its role is to compute the equilibrium of the baseline model with short-term debt, i.e., all parameter values are held fixed at their baseline values but the sovereign is permitted to issue only one-period debt. The results of this exercise are shown in the last column in Table 6. The equilibrium has stark differences. The average spread, the volatility of spreads, and the default frequency are minuscule compared with the long-term bond case.

Table 6: Role of Long-term Debt

Moment	Data	Baseline	Baseline Model w/ $\lambda = 1$
Avg. $(r - r_f)$	0.0815	0.0815	0.0026
$\sigma(r - r_f)$	0.0443	0.0443	0.0037
Avg. b/y	1	0.70	0.81
$\sigma(c)/\sigma(y)$	1.09	1.11	1.14
$\sigma(NX/y)/\sigma(y)$	0.17	0.20	0.34
$corr(c, y)$	0.98	0.99	0.95
$corr(NX/y, y)$	-0.88	-0.44	-0.24
$corr(r - r_f, y)$	-0.79	-0.65	-0.42
Debt Service	0.053	0.055	0.812
Def Freq	0.075	0.068	0.002

This raises the question as to why increasing the maturity length beyond one period induces the sovereign to willingly extends its borrowing into the region where the probability of default is significantly positive. The answer lies in the differing incentives to issue additional debt in the two cases. Treating b' as a continuous variable, the marginal gain from borrowing is given by:

$$\left(-q(y,b') - \frac{\partial q(y,b')}{\partial b'}\left[b' - (1-\lambda)b\right]\right) u'\left(y + m + [\lambda + (1-\lambda)z]b - q(y,b')\left[b' - (1-\lambda)b\right]\right) \tag{9}$$

When the sovereign issues an extra unit of debt, it gets revenue from that extra unit but faces a decrease in the price of the bond, which decreases the revenue on all bonds being currently issued. In the case of short-term debt, the decrease in price applies to the entire stock of debt b', whereas with long-term debt it applies to $[b' - (1-\lambda)b]$. Thus, the sovereign faces a much greater disincentive to borrow when default probabilities become positive in the short-term case.

In addition, $|\partial q(y, b')/\partial b'|$ is larger for short-term debt in the region where default probabilities are positive, as shown in Figures 4 and 5, which plot how spreads and the default probability vary with debt in the two cases. The reason why spreads rise faster with short-term debt is that servicing one-period debt becomes onerous more quickly than servicing long-term debt. Although debt levels are not perfectly comparable across the two cases (they involve different future obligations), the fact that spreads rise faster for short-term bonds is another reason the sovereign is less willing to extend borrowing into the region where default probability is positive when debt is short-term.[35]

5.4 The Welfare Cost of Debt Dilution

In this section, we examine the welfare effects in the baseline model of moving from one-period debt ($\lambda = 1$) to long-term debt ($\lambda = 0.05$). We assume that the b and m are both zero and compute $\Sigma_y V_\lambda(y, 0, 0)\Pi(y)$, where $\Pi(y)$ is the invariant distribution of the Markov chain for y. Rather than report utilities, we report the value of c that makes $c^{1-\sigma}/[(1-\beta)(1-\sigma)]$ equal to $\Sigma_y V_\lambda(y, 0, 0)\Pi(y)$ (the flow certainty equivalent consumption). The results are given in Table 7.

[35]It is worth noting that spreads on long-term debt start out positive and rise even though the probability of default next period may be zero. Even when the sovereign borrows a very small amount in the current period (so the default probability for next period is 0), lenders understand that the sovereign's optimal decision *next* period is to take on a significant amount of debt. And since $q(y, b')$ is decreasing in b', lenders rationally expect to suffer a capital loss on the nonmaturing portion of the debt. This depresses the current price of debt and leads to a positive spread from the start. And, initially, the spread rises with debt simply because the $a(y', m', b')$ is increasing in b' (Proposition 3), and the expected capital loss is increasing. This shows that it is not necessary to invoke risk-aversion on the part of lenders to account for gaps between spreads and default probabilities. With long-term debt, a gap can arise (and vary) because of the dynamics of debt accumulation. A gap can also arise if there is repayment on defaulted debt, which we have ruled out.

Table 7: Welfare Comparison Across Maturity Length

(Quarters, λ)	Cert. Eqv. Cons	Avg. Spread	Avg. b/y	Def Freq
(1, 1)	1.0175	0.0026	0.81	0.0024
(2, 0.5)	1.0174	0.0049	0.81	0.0047
(4, 0.25)	1.0169	0.0102	0.79	0.0096
(6, 0.167)	1.0161	0.0166	0.76	0.0156
(8, 0.125)	1.0150	0.0241	0.74	0.0224
(10, 0.1)	1.0139	0.0327	0.73	0.0298
(12, 0.083)	1.0129	0.0420	0.71	0.0375
(14, 0.071)	1.0118	0.0519	0.70	0.0455
(16, 0.063)	1.0108	0.0619	0.70	0.0534
(18, 0.056)	1.0099	0.0719	0.70	0.0608
(20, 0.05)	1.0092	0.0815	0.70	0.0675

Welfare is highest for short-term debt and declines monotonically as λ falls toward 0.05. Thus, the sovereign is best off issuing short-term (one-quarter) debt. The difference in consumption equivalent in going from 20-quarter maturity to 1-quarter maturity is 0.81 percent, which is significant by the standards of welfare comparisons.

Why is short-term debt better than long-term debt? If the sovereign can commit to not default, the (implicit) interest rate on short-term and long-term debt would be the risk-free rate and maturity length would make no difference to welfare or consumption. Evidently, the risk of default makes a difference. But the reason for the difference is subtle. It turns out that if lenders insist that the sovereign compensate them for declines in the market value of outstanding debt and, conversely, the sovereign insists that the lenders compensate the sovereign for improvements in the value of outstanding debt, long-term debt becomes equivalent to short-term debt *even in the presence of the default risk*. This equivalence result is formally demonstrated in Appendix C. Thus, freezing the value of future outstanding debt at its current market value makes long-term and short-term debt equivalent.

If the future value of outstanding debt is fixed at its value at issue, the sovereign cannot dilute the future value of outstanding debt by issuing more debt in the future. This arrangement, therefore, solves the debt dilution problem and reduces the interest rate on debt. On the other hand, the future market value of debt can also change due to changes in y and these *exogenous* fluctuations in the market value of outstanding debt lead to corresponding fluctuations in disposable income of the risk-averse sovereign.[36] For our calibration, the welfare-reducing effect of a more volatile disposable income is dominated by the welfare-enhancing effect of lower borrowing costs, making short-term debt better than long-term debt.[37]

5.5 Rollover Crises and the Superiority of Long-Term Debt

The results of the previous section lead to the awkward conclusion that even though long-term debt improves model performance, in the model itself the sovereign would prefer to issue one-period debt. In this section we extend the baseline model to allow for a small probability of a rollover crisis (self-fullfilling default) and show that this small additional source of shocks makes long-term debt better than short-term debt without affecting the superior performance of the long-term debt model emphasized earlier.

This extension is motivated by the following two observations. First, if the sovereign issues short-term debt in our model, it issues a large amount of it – on average 81 percent of mean output. Thus, the sovereign rolls over a very large fraction of current consumption each period, on average. Second, many observers have noted that a large volume of short-term debt exposes a borrower to the possibility of a "run equilibrium" wherein lenders' refusal to roll over maturing debt can force the borrower into default, thereby justifying the lenders'

[36]When the market value falls, the sovereign makes a payment to lenders and when the market value rises the sovereign receives a payment from lenders.

[37]In an earlier version of the paper, we showed that if we give the sovereign a choice between short-term and long-term debt each period, and we match the same three statistics as in the baseline model, the sovereign always chooses to issue short-term debt in every state. These results are available in the working paper version of this paper. Note, however, that to solve this model in reasonable time we employed a much coarser grid for Y and b than in the baseline model.

refusal to lend. Cole and Kehoe (2000) provide a theoretical foundation for this view in the context of sovereign borrowing. Importantly for our purposes, they also show that the "run equilibrium" is less likely if the sovereign issues long-term debt. Even with a large stock of long-term debt, the *maturing* portion of debt can be small, so lenders' refusal to roll over is of little consequence to the borrower. Knowing this, lenders do not run and runs fail to be an equilibrium outcome.

Cole and Kehoe assumed that the level of debt is given. In our model, the sovereign gets to choose the level of debt. As in Cole and Kehoe, it is also the case for us that the probability of rollover crisis is higher for short-term debt when the level of debt is high. But, given the higher probability of a crisis, the sovereign chooses to borrow less when debt is short term and, in equilibrium, the probability of rollover crises is low for short-term debt also. But the reduction in borrowing leads to a reduction in welfare relative to long-term debt.

To proceed, consider the following static coordination game played by the sovereign and a single new lender at the start of any period in which the sovereign has some outstanding debt and, conditional on meeting its current obligations, desires to issue new loans. The columns give the strategies of the sovereign and the rows give the strategies of the lender. If the lender makes the new loan (L) and the sovereign repays its existing debt (R), the sovereign receives the payoff from repaying the loan and borrowing, denoted $V^{+}(y, m, b)$, and the lender earns a net return of 0 (i.e, the lender earns the risk-free return – in expectation – which is also the opportunity cost of its funds). If the lender lends and the sovereign defaults (D), we assume that the new loan is returned to the lender without it earning any interest – hence the (discounted) loss of interest earnings $(r_f/(1 + r_f))\Delta$, where Δ is the amount of new lending.[38] If the lender does not lend (N) and the sovereign repays but cannot borrow, the sovereign receives $V^{-}(y, m, b) \leq V^{+}(y, m, b)$ and the lender earns 0. Finally, if the lender

[38]In Cole and Kehoe, the game is sequential with the sovereign deciding on default after lenders have made their lending decisions. If we had followed Cole and Kehoe, the payoff from default conditional on having received new loans would take into account the additional consumption afforded by the new loan. And, if lenders lend and the sovereign defaults lenders would lose not only the interest but the entire loan as well.

does not lend and the sovereign defaults, the payoffs are 0 and $X(y, -\bar{m})$ for the lender and sovereign, respectively.

	R	D
L	0, $V^{+}(y, m, b)$	$-(r_f/1 + r_f)\Delta$, $X(y, -\bar{m})$
N	0, $V^{-}(y, m, b)$	0, $X(y, -\bar{m})$

.

Under the tie-breaking rule that if the sovereign is indifferent between repaying and defaulting it always repays and if the lender is indifferent between lending and not lending it always lends, this game has the following set of Nash equilibria, depending on the value of $X(y, -\bar{m})$. When $X(y, -\bar{m}) \leq V^{-}(y, m, b) \leq V^{+}(y, m, b)$, the unique equilibrium is (L, R). Similarly, if $V^{-}(y, m, b) \leq V^{+}(y, m, b) < X(y, -\bar{m})$, the unique equilibrium is (N, D). But when $V^{-}(y, m, b) < X(y, -m) \leq V^{+}(y, m, b)$, both (L, R) and (N, D) are equilibria of the game. In this case, we assume that the equilibrium selected depends on the realization of a sunspot variable, denoted ω : If $\omega = 0$, the (L, R) equilibrium is selected and if $\omega = 1$, the (N, D) equilibrium is selected. The latter case corresponds to a self-fulfilling "rollover crisis": The lender refuses to lend because it believes that the sovereign will default and the sovereign defaults because it believes that the lender will refuse to lend.[39]

In what follows, we modify the model presented in earlier sections in light of this game. First, we need to be precise about the values $V^{+}(y, m, b)$ and $V^{-}(y, m, b)$ and $X(y, -\bar{m})$. Let $W(y, m, b, \omega)$ denote the lifetime utility of the sovereign, which now depends on the sunspot variable ω in addition to the other state variables. Then,

$$
\begin{aligned}
V^{+}(y, m, b) &= \max_{b' \in B} u(c) + \beta E_{(y' m')|y}[(1 - \pi)W(y', m', b', 0) + \pi W(y', m', b', 1)] \\
\text{s.t.} & \qquad\qquad (10) \\
c &= y + m + [\lambda + (1 - \lambda)z]\, b - q(y, b')\, [b' - (1 - \lambda)b],
\end{aligned}
$$

[39] In Cole and Kehoe, the game is played between the sovereign and many lenders acting independently. The multiplicity of lenders makes the coordination failure implicit in a rollover crisis more plausible. For simplicity, we assume that a coordination failure may occur between the sovereign and a single lender.

where we assume that ω is i.i.d and takes the value 1 with probability π. If there is no b' for which consumption is non-negative, then we set $V^{+}(y, m, b)$ to $-\infty$. And

$$
\begin{aligned}
& V^{-}(y, m, b) = \max_{b' \in B} u(c) + \beta E_{(y'\, m')|y}[(1-\pi)W(y', m', b', 0) + \pi W(y', m', b', 1)] \\
& \text{s.t.} \qquad (11) \\
& c = y + m + [\lambda + (1-\lambda)z]\, b - q(y, b')\, [b' - (1-\lambda)b] \\
& b' \geq (1-\lambda)b.
\end{aligned}
$$

Again, if there is no b' for which consumption is non-negative, we set $V^{-}(y, m, b)$ to $-\infty$. Evidently, $V^{-}(y, m, b) \leq V^{+}(y, m, b)$. If the sovereign desires to issue new loans (conditional on meeting its current obligations), $V^{-}(y, m, b) < V^{+}(y, m, b)$. The value under exclusion has the same structure as in the rest of the paper, specifically, $X(y, m)$ solves $X(y, m) = u(y - \phi(y) + m) + \beta\{[1-\xi]E_{(y'\, m'\omega')|y}X(y', m') + \xi E_{(y'\, m'\, \omega')|y}W(y', m', 0, \omega')\}$ which then pins down the value under default, $X(y, -\bar{m})$.

The functional equation that determines $W(y, m, b, \omega)$ is given by

$$
W(y, m, b, \omega) \qquad (12)
$$
$$
= \begin{cases}
V^{+}(y, m, b) & \text{if } X(y, -\bar{m}) \leq V^{-}(y, m, b) \text{ and } \omega \in \{0, 1\} \\
X(y, -\bar{m}) & \text{if } V^{+}(y, m, b) < X(y, -\bar{m}) \text{ and } \omega \in \{0, 1\} \\
V^{+}(y, m, b) & \text{if } V^{-}(y, m, b) < X(y, -\bar{m}) \leq V^{+}(y, m, b) \text{ and } \omega = 0 \\
X(y, -\bar{m}) & \text{if } V^{-}(y, m, b) < X(y, -\bar{m}) \leq V^{+}(y, m, b) \text{ and } \omega = 1
\end{cases}
$$

To see why (12) holds, observe that when $X(y, -\bar{m}) \leq V^{-}(y, m, b)$ the unique equilibrium of the game is (L, R). Therefore, regardless of the value of ω, the (equilibrium) lifetime utility of the sovereign is $V^{+}(y, m, b)$. Similarly, when $V^{+}(y, m, b) < X(y, -\bar{m})$, the unique equilibrium of the game is (N, D) and the lifetime utility of the sovereign is $X(y, -\bar{m})$, regardless of the value of ω. When $V^{-}(y, m, b) < X(y, -\bar{m}) \leq V^{+}(y, m, b)$, the equilibrium of the game depends on the value of ω. If $\omega = 0$, the equilibrium is (L, R) and lifetime utility is $V^+(y, m, b)$, but if $\omega = 1$ the equilibrium is (N, D) and lifetime utility is $X(y, -\bar{m})$. Correspondingly, the default decision rule, denoted $d(y, m, b, \omega)$, takes the value 1 in those cases where lifetime utility is $X(y, -\bar{m})$ and 0 otherwise; the asset decision rule conditional on repayment, denoted $a(y, m, b, \omega)$, is one that solves (10). Finally, the pricing equation now solves the functional equation

$$q(y, b') = E_{(y'\, m', \omega')|y} \left[[1 - d(y', m', b', \omega')] \frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b', \omega'))]}{1 + r_f} \right].$$

Before we turn to the quantitative results, we note that the model analyzed in the rest of the paper corresponds to the case where $\omega = 0$ with probability 1. In this case, the "rollover crisis" equilibrium is never played and the sovereign defaults only if $V^+(y, m, b) < X(y, -\bar{m})$. Note also that if $V^-(y, m, b) = V^+(y, m, b)$ (which happens if the sovereign does not wish to issue new loans or if the budget sets are empty) then the bottom two branches of $W(y, m, b, \omega)$ are not relevant and, once again, the sovereign defaults only if $V^+(y, m, b) < X(y, -\bar{m})$.

Table 8 reports the results for long-term and short-term debt for values of sunspot probabilities ranging from 0 to 10 percent. For long-term debt, the rising sunspot probability leaves the average spread, the average b/y and the default frequency essentially unchanged, although it does marginally lower welfare (the drops are in the order of 10^{-5} percent). The reason the sunspot probability has so little additional effect is that the gap between $V^+(y, m, b)$ and $V^-(y, m, b)$ is positive only when the sovereign wishes to increase its debt to more than $0.95b$. Generally, the gap will be positive if borrowing costs are low, which happens when output is high. During such times the default cost is high and $X(y, -\bar{m})$ is low. Thus, the conditions for a sunspot-driven default (which require that $V^-(y, m, b) < X(y, -\bar{m}) \leq V^+(y, m, b)$) rarely occur and the randomness introduced by ω is of little consequence. In contrast, if the sovereign is carrying a large amount of one-period debt, the gap between $V^+(y, m, b)$ and $V^-(y, m, b)$ will be very large since fully paying down a large amount of debt is likely to be very costly. Thus, it is much more likely that the conditions for a sunspot-driven default will be satisfied. Thus, the extraneous uncertainty matters more for short-term debt.

The effects can be seen in Panel B of Table 8. When the probability of a sunspot is half a percent, the sovereign reduces its average level of short-term debt to about 70 percent of output.[40] As the sunspot probability is raised to 1 percent, the sovereign responds by cutting debt back to 43 percent of output, on average. The reason for the strong response is twofold. Most importantly, with a lot of short-term debt outstanding the sovereign is vulnerable to a rollover crisis whenever $\omega = 1$. Furthermore, default triggered by $\omega = 1$ tends to be more costly than default triggered by poor fundamentals because the sunspot shock is independent of fundamentals and can happen when output is high (recall that in our model the cost of default as a proportion of output is increasing in output). In sum, even though $\omega = 1$ with low probability, its occurrence is sufficiently painful for the sovereign that it wants to avoid any crisis that might ensue. It does so by limiting its borrowing and thereby eliminating the bad "run" equilibrium. Importantly, the scaling back of debt reduces welfare, and for a sunspot probability of 1 percent or higher, the sovereign is better off issuing long-term debt than short-term debt.

[40]Even with this reduced level of debt, spreads rise to 2.40 percent and default frequency to 2.23 percent. The strong response of default frequency to a small probability of a sunspot occurs because $V^{+}(y, m, b) - X(y, -\bar{m})$ also declines with a rise in spreads so that the likelihood of default when $\omega = 0$ also goes up.

Table 8: Maturity Length and the Effects of Sunspot Probability

$\Pr[\omega = 1]$	Cert. Eqv. Cons	Avg. Spread	Avg. b/y	Def Prob
Panel A: Maturity Length 5 Years ($\lambda = 0.05$)				
0.00	1.0092	0.0815	0.70	0.0676
0.005	1.0092	0.0815	0.70	0.0675
0.01	1.0092	0.0815	0.70	0.0674
0.02	1.0092	0.0815	0.70	0.0675
0.05	1.0092	0.0815	0.70	0.0677
0.1	1.0091	0.0816	0.70	0.0676
Panel B: Maturity Length 1 Quarter ($\lambda = 1$)				
0.00	1.0175	0.0026	0.81	0.0024
0.005	1.0108	0.0240	0.71	0.0223
0.01	1.0079	0.0066	0.43	0.0062
0.02	1.0074	0.0029	0.40	0.0027
0.05	1.0071	0.0024	0.39	0.0021
0.1	1.0069	0.0022	0.38	0.0021

These results provide one possible explanation for why emerging market economies choose to issue longer-term debt despite the fact that the debt dilution problem makes longer-term debt expensive. The benefit of longer-term debt is that it allows the sovereign to credibly commit to service the debt in the event of a sudden stop in lending even when the level of outstanding debt is large relative to output. This commitment in turn reduces the likelihood of such crises. If the sovereign is impatient, it may prefer to issue longer-term debt and borrow a large amount rather than issue cheaper short-term debt but constrain its borrowing so as to limit the likelihood of rollover crises. It is worth noting that even when the probability of a crisis is 10 percent, the gain in welfare from issuing long-term debt is only 0.002 percent of consumption. This is a rather small gain but it is in line with the common finding that the welfare costs of fluctuations tend to be quite small (Lucas (1987).

6 Conclusion

In this paper, we developed a novel and computationally tractable model of long-term unsecured debt and default. We showed that the model shares the key insight of Eaton and Gersovitz's original contribution, namely, that the option to default implies that the sovereign faces a rising supply schedule for credit. We established the existence of an equilibrium pricing function with this property and developed a novel and very accurate computational approach to compute it. Using Argentina as a test case, we showed that the model with long-term debt can easily match key first and second moments related to sovereign debt and improves model performance with regard to cyclical facts. We also investigated the welfare properties of maturity length and showed that if the possibility of self-fulfilling rollover crises is taken into account, long-term debt is superior to short-term debt.

7 References

Aguiar M. and G. Gopinath (2006) "Defaultable Debt, Interest Rate and the Current Account," *Journal of International Economics*, 69, pp. 64-83.

Arellano C. (2008) "Default Risk and Income Fluctuations in Emerging Markets," *American Economic Review*, 98(3), pp. 690-712.

Arellano, C. and A. Ramanarayanan (2009) "Default and Maturity Structure in Sovereign Bonds," Manuscript, University of Minnesota and Federal Reserve Bank of Dallas, December.

Athreya, K. (2002) "Welfare Implications of the Bankruptcy Reform Act of 1999," *Journal of Monetary Economics*, 49, 1567-95.

Beim D., and C. Calomiris (2000). *Emerging Financial Markets,* McGraw-Hill/Irwin, New York, USA.

Benjamin, D. and M.L. Wright (2009) "Recovery Before Redemption: A Theory of Sovereign Debt Renegotiation," Manuscript, SUNY Buffalo and UCLA.

Bi, R. (2008a) “Debt Dilution and the Maturity Structure of Sovereign Debt,” Ph.D. Dissertation, Department of Economics, University of Maryland.

Bi, R. (2008b) “Beneficial Delays in Debt Restructuring Negotiations,” Ph.D. Dissertation, Department of Economics, University of Maryland.

Broner, F., G. Lorenzoni, and S. Schmukler (2007) “Why Do Emerging Economies Borrow Short Term?” Working paper, MIT.

Chatterjee, S., D. Corbae, M. Nakajima, and J. Rios-Rull (2007) “A Quantitative Theory of Unsecured Consumer Credit with Risk of Default,” *Econometrica*, 75(6), pp. 1525-1589.

Cole, H., and T. Kehoe (2000) “ Self-Fulfilling Debt Crises,” *Review of Economic Studies*, vol 67(1), no. 230, 91-116, 2000.

Cuadra, G., and H. Sapriza (2008) “Sovereign Default, Interest Rates and Political Uncertainty in Emerging Markets,” *Journal of International Economics,* 76, pp. 77-88.

Del Negro, M. and F. Schorfheide (2010) “Bayesian Macroeconometrics,” Manuscript (prepared for the *Handbook of Bayesian Econometrics*), Department of Economics, University of Pennsylvania.

D’Erasmo, P. (2008) “Government Reputation and Debt Repayment, ” Manuscript, Department of Economics, University of Maryland.

Eaton J. and M. Gersovitz (1981) “Debt with Potential Repudiation: Theoretical and Empirical Analysis,” *Review of Economic Studies*, 47, pp. 289-309.

Gelos, R. G., R. Sahay, and G. Sandleris (2008) “Sovereign Borrowing by Developing Countries: What Determines Market Access?” mimeo, Centro de Investigacion en Finanzas, Universidad Torcuato Di Tella.

Hatchondo, J. C. and L. Martinez (2009) “Long Duration Bonds and Sovereign Defaults,” *Journal of International Economics*, 79(1), pp. 117-125.

——— and H. Sapriza (2010), “Quantitative Properties of Sovereign Default Models: Solutions Methods Matter,” *Review of Economic Dynamics*, 13, pp. 919-933.

Hutson, V., and J.S. Pym (1980) *Applications of Functional Analysis and Operator Theory,* Academic Press, New York.

Ireland, Peter (2004)."A Method for Taking Models to the Data," *Journal of Economic Dynamics and Control*, 28, pp.1205-1226.

Judd, K.L. (1998) *Numerical Methods in Economics,* MIT Press, Cambridge MA.

Lucas, R.E. Jr., (1987) *Models of Business Cycles,* New York: Basil Blackwell.

Livshits, I., J. MacGee, and M. Tertilt (2007) "Consumer Bankruptcy: A Fresh Start," *American Economic Review*, 97(1), pp. 402-418.

Mendoza, E. and V. Yue (2009) "Solution to the Default-Risk and Business Cycle Disconnect," Manuscript, University of Maryland and New York University.

Neumeyer, P., and F. Perri (2005) "Business Cycles in Emerging Economies: The Role of Interest Rates," *Journal of Monetary Economics*, 52(2):345-380 [30]

Reinhart, C. M., K.S. Rogoff, M.A. Savastano (2003) "Debt Intolerance," in W. Brainard and G. Perry (Eds.), Brookings Papers on Economic Activity, Spring 2003, (1), pp. 1–74.

Uribe, M. and V. Yue (2006) "Country Spreads and Emerging Countries: Who Drives Whom?" *Journal of International Economics*, 69, pp. 6-36.

Yue, V. (2009) "Sovereign Default and Debt Renegotiation," Manuscript, New York University.

8 Appendix A: Proofs of Propositions 1-4

Proposition 0 (Existence, Continuity and Monotonocity of Value Functions): Given any $q(y,b') \geq 0$, there exists a unique, bounded function $W(y,m,b)$ continuous in m that solves the functional equation (12). Furthermore, $X(y,m)$ is strictly increasing and continuous in m; in the region where repayment is feasible, $V(y,m,b)$ is strictly increasing in b and m and continuous in m; and $Z(y,b') = E_{(y',m')|y)}W(y',m',b')$ is strictly increasing in b', provided there is positive probability of repayment for every debt level.

Proof: See Web Appendix

Proof of Proposition 1: Suppose, to get a contradiction, that for some (y,m) we have $d(y,m,b^0;q) < d(y,m,b^1;q)$. Then $d(y,m,b^0;q) = 0$ and $d(y,m,b^1;q) = 1$. The former implies that $V(y,m,b^0) \geq X(y,-\bar{m})$ and the latter implies $X(y,-\bar{m}) > V(y,m,b^1)$. But these inequalities imply $V(y,m,b^0) > V(y,m,b^1)$, which contradicts Proposition 0. Hence, $d(y,m,b^0) \geq d(y,m,b^1)$.

Proof of Proposition 2: Fix m and y. Denote $a(y,m,b^0)$ by b'^0 and the associated consumption level by c^0. Let $\hat{b}'$ be some other feasible choice greater than b'^0 and let $\hat{c}$ be the associated consumption level. Then, by optimality and the tie-breaking rule that if the sovereign is indifferent between two b's it always chooses the higher one, we have

$$u(c^0) + \beta Z(y,b'^0) > u(\hat{c}) + \beta Z(y,\hat{b}'). \tag{13}$$

Since $Z(y,\hat{b}') > Z(y,b'^0)$ (Proposition 0), (13) implies $c^0 > \hat{c}$. Let $\Delta(b^0) = c^0 - \hat{c} > 0$. Thus, $\Delta(b^0)$ is the loss in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period debt is b^0. From the budget constraint we have that $\Delta(b^0) + q(y,b'^0)b'^0 - q(y,\hat{b}')\hat{b}' = [1-\lambda](-b^0)[q(y,\hat{b}') - q(y,b'^0)]$. Holding fixed $\hat{b}'$ and b'^0, let $\Delta(b^1)$ be the value of Δ that solves $\Delta(b^1) + q(y,b'^0)b'^0 - q(y,\hat{b}')\hat{b}' = [1-\lambda](-b^1)[q(y,\hat{b}') - q(y,b'^0)]$. Then $\Delta(b^1)$ is the change in current consumption from choosing $\hat{b}'$ over b'^0 when the beginning-of-period debt is b^1. Since, by assumption, $[q(y,\hat{b}') - q(y,b'^0)] \geq 0$, $b^1 < b^0$ implies $\Delta(b^1) \geq \Delta(b^0)$. Thus the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ is at least as large when the beginning-of-period debt is b^1 compared with b^0. Next, note that since $[\lambda+(1-\lambda)z] > 0$ and $q(y, b') \geq 0$, $b^1 < b^0$ implies $[\lambda + (1-\lambda)z]b^1 + (1-\lambda)q(y, b'^0)b^1 < [\lambda + (1-\lambda)z]b^0 + (1-\lambda)q(y, b'^0)b^0$. Therefore, from the budget constraint it follows that if the beginning-of-period debt is b^1, choosing ${b'}^0$ implies consumption $\tilde{c}$ strictly less than c^0. To complete the proof, observe that the strict concavity of u implies $u(\tilde{c}) - u(\tilde{c} - \Delta(b^1)) > u(c^0) - u(c^0 - \Delta(b^0)) = u(c^0) - u(\hat{c})$. Therefore, (13) implies that $u(\tilde{c})+\beta Z(y, {b'}^0) > u(\tilde{c}-\Delta(b^1))+\beta Z(y, \hat{b}')$. Since $\hat{b}'$ is any feasible b' greater than ${b'}^0$, the optimal choice of b' under repayment when beginning-of-period debt is b^1 cannot be greater than ${b'}^0$. Therefore, $a(y, m, b^1) \leq a(y, m, b^0)$.

Proof of Proposition 3: Let $\bar{q} = [\lambda+[1-\lambda]z]/[\lambda+r_f]$. Then $\bar{q}$ is the present discounted value of a bond with coupon payment z and probability of maturity λ on which there is no risk of default. Let S be the set of all nonnegative functions $q(y, b')$ defined on $Y \times B$ and let $Q \subset S$ be the subset of functions that are increasing in b' and bounded above by $\bar{q}$.

Define the $H(q)(y, b') : Q \to S$ as

$$E_{(y'\,m')|y}\left[[1 - d(y', m', b'; q)]\frac{\lambda + [1-\lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f}\right],$$

where $d(y, m, b; q)$ and $a(y, m, b; q)$ are the default and debt decision rule, given q. Then H has the following properties:

(i) $H(q)(y, b') \in Q$. Nonnegativity is obvious. We will show that $H(q)(y, b') \leq \bar{q}$. Observe that $\bar{q}$ satisfies the equation $\bar{q} = [\lambda + (1-\lambda)[z+\bar{q}]]/(1+r_f)$. Then, since $1 - d(y', m', b') \leq 1$ and $q(y', a(y', m', b'; q)) \leq \bar{q}$ for every (y', m', b'), it follows that

$$\left[[1 - d(y', m', b'; q)]\frac{\lambda + [1-\lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f}\right] \leq \bar{q} \text{ for every } y', m', b'.$$

Hence $H(q)(y, b') \leq \bar{q}$. Next, we will show that $H(q)(y, b')$ is increasing in b'. Fix y' and m'. Since $q(y, b') \in Q$, $q(y, b')$ is increasing in b' and, by Proposition 2, $a(y', m', b'; q)$ is increasing in b'. Thus, $q(y', a(y', m', b'; q))$ is increasing in b'. And, by Proposition 2 again,

$[1 - d(y', m', b'; q)]$ is increasing in b'. Hence $H(q)(y, b')$ is increasing in b'.

(iii) $H(q)(y, b')$ is continuous in q (see the Web Appendix for proof).

To complete the proof, note that Q is a compact and convex set. Since $H(q)$ is continuous, by Brouwer's Fixed Point Theorem there exists $q^* \in Q$ such that $q^*(y, b') = H(q^*)(y, b')$. This establishes the existence of an equilibrium price function that is increasing in b'.

Proposition 4: $a(y, m, b)$ is increasing in m and $d(y, m, b)$ is decreasing in m

Proof: To prove $a(y, m, b)$ is increasing in m, fix y and b and let $m^1 > m^0$. Assume also that repayment is feasible for both m^1 and m^0. Denote $a(y, m^1, b)$ by b'^1 and the associated consumption by c^1. Let $\hat{b}' > b'^1$ be some other feasible choice of b' greater than $\hat{b}'^1$ and denote the associated consumption by $\hat{c}$. Then, by optimality $u(c^1) + \beta Z(y, b'^1) > u(\hat{c}) + \beta Z(y, \hat{b}')$. Since $Z(y, \hat{b}') > Z(y, b'^1)$ (Proposition 0), the above inequality implies $c^1 > \hat{c}$. Let $\Delta = c^1 - \hat{c}$ denote the loss in current consumption from choosing $\hat{b}'$ over b'^1 when the transitory shock takes the value m^1. Now observe that the loss in current consumption from choosing $\hat{b}'$ over b'^1 when the transitory shock takes the value m^0 is also Δ. However, the level of consumption when the transitory shock takes the value m^0 and the sovereign chooses b'^1, denoted $\tilde{c}$, is strictly less than c^1. From the strict concavity of u, it follows that $u(\tilde{c}) - u(\tilde{c} - \Delta) > u(c^1) - u(c^1 - \Delta)$. Therefore, $u(\tilde{c}) + \beta Z(y, b'^1) > u(\tilde{c} - \Delta) + \beta Z(y, \hat{b}')$. Since $\hat{b}'$ was any b' greater than b'^1, $a(y, m^0, b)$ cannot exceed b'^1. Thus, $a(y, m^0, b) \leq a(y, m^1, b)$.

The fact that $d(y, m, b)$ is decreasing in m follows from the fact that $V(y, m, b)$ is strictly increasing in m (Proposition 0) and the utility from default, $X(y, -\bar{m})$, is independent of m.

9 Appendix B: Logic, Performance of the Computation Algorithm and Sensitivity Analysis

9.1 Logic

In this section, we describe the logic of our solution algorithm. The first part gives the logic of the algorithm for calculating the optimal debt choice as a function of m. The second part, taking the algorithm in the first part as given, provides the logic for the solution algorithm.

9.1.1 Method For Recovering $a(y, m, b; q)$ Given (y, b) and q

Proposition 5 implies that given (y, b) and q there exists $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ such that b'^K is chosen for $m \in [-\bar{m}, m^{K-1})$, b'^{K-1} is chosen for $m \in [m^{K-1}, m^{K-2})$, $\ldots$, b'^1 is chosen for $m \in (m^1, \bar{m}]$ ($K = 1$ means that b'^1 is chosen for all $m \in M$).

Since b'^k need not be adjacent to b'^{k+1} on the grid, the algorithm has to find both $\{m^k\}$ *and* $\{b'^k\}$. The decision rule is constructed recursively. The choice problem is initially solved for a choice set containing only one b'. The choice set is then expanded in steps until the entire set B is available, with the solution from each step being used to construct the solution for the next step.

Suppose that we have located pairs $\{(m^{h-1}, b'^h), (m^{h-2}, b'^{h-1}), \ldots (\bar{m}, b'^1)\}$ such that if the sovereign is permitted to choose *only from the set* $b' \geq b'^h$, the sovereign would choose b'^h for $m \in [-\bar{m}, m^{h-1})$, b'^{h-1} for $m \in [m^{h-1}, m^{h-2})$, $\ldots$, $b'^1 \in (m^1, \bar{m}]$. The next step is to compare the utility from choosing b'^h with the utility from choosing the next lower b' (i.e., next higher debt level) on the grid, denoted b'^-. Two cases are possible.

1. $-q(y, b'^-)[b'^- - (1-\lambda)b] \leq -q(y, b'^h)[b'^h - (1-\lambda)b]$. Then, the lifetime utility from b'^h is at least as high as the lifetime utility from b'^- for all $m \in M$. So we drop b'^- from further consideration and move to comparing b'^h to the next lower b' on the grid.

2. $-q(y, b'^{-})[b'^{-} - (1-\lambda)b] > -q(y, b'^{h})[b'^{h} - (1-\lambda)b]$. Then $\Delta(m) = u(\dots m - q(y, b'^{-})[b'^{-} - (1-\lambda)b]\dots) - u(\dots m - q(y, b'^{h})[b'^{h} - (1-\lambda)b]\dots) > 0$ for all m, where $u(\dots m - q(y, b')[b' - (1-\lambda)b]\dots)$ is the current utility from choosing b' (we have suppressed terms that do not depend on m and b'). Furthermore, from the strict concavity of u, $\Delta(m)$ is decreasing in m. Three subcases are possible.

   (a) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} \leq 0$. Then b'^{h} is at least as good as b'^{-} for all m and we can drop b'^{-} from further consideration.

   (b) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} \leq 0$. Then there must exist a unique $\tilde{m} \in (-\bar{m}, \bar{m}]$ such that $\Delta(\tilde{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} = 0$. If $\tilde{m} < m^{h}$, we prepend $(\tilde{m}, b'^{-})$ to the list of pairs and proceed to compare the utility between b'^{-} with the next lower b' on the grid. If $\tilde{m} \geq m^{h}$, we drop b'^{h} from further consideration and proceed backwards to compare b'^{-} with b'^{h-1}. The reason is that $\tilde{m} \geq m^{h}$ implies that b'^{-} is preferred to b'^{h} for any $m < \tilde{m}$ and at the same time b'^{h-1} is preferred to b'^{h} for any $m \geq m^{h}$. Thus, b'^{h} is dominated by the choices of b'^{h-1} and b'^{-} and can be dropped from further consideration. When this is the case, b'^{-} needs to be compared to b'^{h-1}. The process is continued by finding a new $\widetilde{m}$ between the choices of b'^{-} and b'^{h-1}. If $\tilde{m} < m^{h-1}$, we add $(\tilde{m}, b'^{-})$ to the list of pairs $\{(m^{h-2}, b'^{h-1}), \dots (\bar{m}, b'^{1})\}$ and proceed to compare the utility between b'^{-} with the next lower level of assets. If $\tilde{m} \geq m^{h-1}$, we drop b'^{h-1} from further consideration and continue to go backwards through the list. This process will either end in finding m^{h-j} such that $\tilde{m} < m^{h-j}$ or in the exhaustion of all pairs in the list $\{m^{k}, b'^{k}\}$. If the latter, we conclude that b'^{-} dominates any $b' > b'^{-}$ for all m (i.e., the list becomes a singleton $\{(\bar{m}, b'^{-})\}$ and proceed to compare b'^{-} with the next lower b' on the grid.

   (c) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$. Then b'^{-} dominates b'^{h} for all m and we can drop b'^{h} from further consideration. We then move to compare b'^{-} with b'^{h-1}.

3. To implement this algorithm we start off with the choice set being $\{0\}$.The solution for this stage is the list $\{(\bar{m}, 0)\}$ (meaning that no borrowing is optimal for all m). We then proceed to compare 0 with the next lower b' on the grid. The algorithm is applied until every element of B has been compared.

9.1.2 Method for Computing the Solution

We discretize the state space into N_y grids for persistent output shock and N_b grids for bonds. We enter the k-th iteration with guesses for $q^k(y, b')$ and $Z^k(y, b')$, where $Z^k(y, b') = E_{(y',m'|y)}W^k(y', m', b')$. All calculations below are for some specific (y, b) and k.

1. Given these guesses, we find what the sovereign would do if it repayed. This entails finding the decision rule for debt. The algorithm to accomplish this was outlined above. At the end of this stage, we have $\left\{(m^{K-1}, b'^K), (m^{K-2}, b'^{K-1}), \ldots, (\bar{m}, b'^1)\right\}$.

2. In the second step, we find default thresholds. For each interval from step 1, we compare the lifetime utility from choosing the indicated quantity of debt with the lifetime utility derived from default. Suppose that for $m \in (m^i, m^{i-1}]$ the sovereign chooses b'^i. Define $\Delta(m) = u\left(y + m - q^k(y, b'^i)[b'^i - [1-\lambda]b]\right) + \beta Z^k(y, b'^i) - X(y, -\bar{m})$. Evidently, $\Delta(m)$ is increasing in m. If $\Delta(m^i) \cdot \Delta(m^{i-1}) < 0$, there exists an $\tilde{m}$ such that default is optimal for $(m^i, \tilde{m})$ and b'^i is optimal for $[\tilde{m}, m^{i-1}]$. If $\Delta(m^i) \cdot \Delta(m^{i-1}) \geq 0$, then either default is optimal over the entire interval or b'^i is optimal over the interval. At the end of this stage, we have a maximum of $2(N_b - 1)$ intervals. Within each interval we know whether default or repayment is chosen and if repayment is chosen, the corresponding debt choice. Although the maximum number of intervals can be very large, in practice the number of intervals is usually less than 20.

3. Finally, with these intervals in hand we compute the functions $Z^{new}(y, b')$ and $q^{new}(y, b')$. We check if $\left|Z^{new}(y, b') - Z^k(y, b')\right| < \varepsilon_1$ and $\left|q^{new}(y, b') - q^k(y, b')\right| < \varepsilon_2$ where ε_1 and ε_2 are very small numbers. If these conditions hold, we end the program. If one of

them does not hold, we update

$$q^{k+1}(y,b') = (1-\zeta)\cdot q^{new}(y,b') + \zeta \cdot q^k(y,b')$$
$$Z^{k+1}(y,b') = (1-\nu)\cdot Z^{new}(y,b') + \nu \cdot Z^k(y,b'),$$

where $\zeta, \nu \in [0,1)$ and continue with step 1 (ν can be set to 0 without any impairment in performance).

4. To compute $Z^{new}(y,b')$ and $q^{new}(y,b')$ we need to integrate W^k and q^k with respect to m. To integrate, we divide M into 11 equally spaced intervals and assume that within each interval m is uniformly distributed. Consider an interval (m_1, m_2) and suppose that it contains one threshold, say $\hat{m} \in [m_1, m_2]$, where the optimal decision changes from a debt of b' to a debt of $\hat{b}'$. Then,

$$\int_{m_1}^{m_2} W(y,m,b)dG(m) \simeq \left[\int_{m_1}^{m_2} dG(m)\right] \times$$
$$\left(\frac{\hat{m}-m_1}{m_2-m_1}\right)\cdot (u(y+m_{12}+[\lambda+z(1-\lambda)]b+q(y,b')[b'-(1-\lambda)b]) + \beta Z(y,b')) +$$
$$\left(\frac{m_2-\hat{m}}{m_2-m_1}\right)\cdot \left(u(y+m_{12}+[\lambda+z(1-\lambda)]b+q(y,\hat{b}')\hat{b}')[b'-(1-\lambda)b] + \beta Z(y,\hat{b}')\right).$$

In other words, over each interval, we replace m by the midpoint of the interval but recognize that the choice of debt may switch as m varies over the interval. The overall variation in m is small and, with 11 intervals, the variation within each interval is smaller still. Thus, the differences between m and m_{12} are of little consequence for the evaluation of utility, given the choice of debt. Having obtained $\int_m W(y,m,b)dG(m)$ in this way for each y and b, we obtain $Z(y,b')$ as $\sum_{y'}[\int_m W(y',m',b')dG(m)]F(y,y')$. The procedure for integrating the price function is similar:

$$\int_{m_1}^{m_2} q(y',a(y',m',b'))dG(m') \simeq$$
$$\int_{m_1}^{m_2} dG(m') \times \left[\left(\frac{\hat{m}-m_1}{m_2-m_1}\right)\cdot (q(y',b')) + \left(\frac{m_2-\hat{m}}{m_2-m_1}\right)\cdot (q(y,\hat{b}')\right].$$

9.2 Performance

As explained in the computation section, the reason for adding the m shock (and calculating thresholds to solve the decision problem) is to ensure that (6) has a solution and that the iteration (7) converges. In this appendix we show that alternative methods that do not use "randomization" have significantly worse convergence performance. We make these comparisons by fixing all parameter values at baseline values and iterate each solution method 3000 times and report the maximum absolute error in the final 100 iterations as well as the relative value of this maximum error. The error for iteration k is defined as the largest absolute change in the price matrix from iteration $k-1$ to k. For purely discrete models, we also report the maximum jump in asset choice (in terms of the maximum number of grid points skipped) from one iteration to the next, for the final 100 iterations. All computations were implemented via parallelized (MPI) Fortran 90/95 running on a 16-node cluster.

Omitting M and refining Y. The following table compares the baseline method (Method I) with three other methods. Method II is the model without M, method III is the model without M in which the Y grid is doubled, and method IV is the baseline model but the M is discretized and thresholds are not computed.

Table 9: Omitting M and Refining Y

	Baseline ($M = 11$)	II	III	IV ($M = 11$)
Grids	$Y = 200, B = 350$	$Y = 200, B = 350$	$Y = 400, B = 350$	$Y = 200, B = 350$
$\Delta\lvert q^k - q^{k-1}\rvert$	9.47×10^{-14}	2.33×10^{-2}	1.14×10^{-2}	3.34×10^{-3}
$\Delta\lvert (q^k - q^{k-1})/[0.001 + q^k]\rvert$	4.85×10^{-13}	2.11×10^{-1}	1.01×10^{-1}	2.61×10^{-2}
$\Delta\lvert V^k - V^{k-1}\rvert$	1.78×10^{-14}	2.65×10^{-4}	1.48×10^{-4}	2.40×10^{-5}
$\Delta\lvert (V^k - V^{k-1})/[0.001 + V^k]\rvert$	8.84×10^{-13}	1.31×10^{-5}	7.27×10^{-6}	1.04×10^{-6}
Max jump in b' between iterations	NA	15	14	14
Method II: No M				
Method III: No M but refined Y grid				
Method IV: Baseline but M discretized and thresholds are not computed				

With the baseline method, we get convergence for very tight convergence criteria. In contrast, for Method II, where we omit M, even after 3000 iterations the price matrix is far from convergence; the error can be as much as 20 percent. The maximum change in debt choice is 15 grid points; these jumps occur because nonconvexities lead to multiple local maxima and the solution meanders between these local maxima (as discussed in the text in relation to Figure 2). In Method III, we double the grid on Y to 400. There is not much improvement in the results. Finally, in the last column, M is discretized and thresholds are not used. Convergence is somewhat better but still nowhere close to Method I.[41] For models II- IV, the convergence performance of value functions is considerably better than the convergence performance of the price function. This is because the jumps occur between actions that give roughly the same utility and, therefore, do not affect value functions as much (the same is true of models V and VI discussed below).

Omitting M and Refining B. In the following table we establish that the poor performance of the baseline model without M (Model II above) cannot be rectified by refining the B, or asset, dimension.

Table 10: Omitting M and Refining B

	Method II	Method V	Method VI (continuous B)
Grids	$Y = 200, B = 350$	$Y = 200, B = 700$	$Y = 200, B = 350$
$\Delta\|q^t - q^{t-1}\|$	2.23×10^{-2}	2.47×10^{-2}	2.22×10^{-2}
$\Delta\|(q^t - q^{t-1})/(0.001 + q^t)\|$	$2.11 \times 10^{-}1$	2.13×10^{-1}	2.10×10^{-1}
$\Delta\|V^t - V^{t-1}\|$	2.65×10^{-4}	1.82×10^{-4}	1.84×10^{-4}
$\Delta\|(V^t - V^{t-1})/(0.001 + V^t)\|$	1.31×10^{-5}	9.05×10^{-5}	9.10×10^{-6}
Max jump in b' between iterations	15	31	-

[41] Also this method takes longer to run relative to the baseline method because the discounted utility of the country is calculated for *all* current states (m, y, b) and for all choices of b'. In the baseline method, given current states (y, b), we find the thresholds for which there is a switch between different choices of assets. As those switches do not happen very frequently, the utility level given the choice of b' is computed much less frequently.

The column labeled Method V shows the case where we omit the M shock and increase the number of grids for the asset level. Evidently, increasing the grids for B makes no difference to convergence. In Method VI, we continue to omit the M shock but treat B as a continuous variable. We discretize B as in the other methods but allow for asset choices off the grid. In particular, if income is y and beginning-of-period debt is b, then for a debt level b' between two adjacent grids b_j and b_{j-1}, $c = y + [\lambda + (1-\lambda)z]b + wq(b_j, y)[b_j - (1-\lambda)b] + (1-w)q(b_{j+1}, y)[b_{j+1} - (1-\lambda)b]$, and $E_{y'|y}V(y', b') = wE_{y'|y}V(y', b_j) + (1-w)E_{y'|y}V(y', b_{j+1})$ where w is $(b_{j+1} - b')/(b_{j+1} - b_j)$. Since there is more than one local maxima in our problem, we first find the b' that maximizes utility confining our choice to the initial discrete grids and then do a refined search to locate the best choice of b' around that grid (this is the procedure followed in Hatchondo, Martinez and Sapriza (2010)). Treating B continuous in this fashion also does not improve convergence. The lotteries between adjacent grid points do not help because the problematic cycles are between grids that are far apart.

9.3 Sensitivity Analysis

It is known that model statistics in the Eaton-Gersovitz model can be sensitive to the choice of grid sizes (Hatchondo, Martinez and Sapriza (2010)). To check for this, we doubled the grid sizes on Y and B, separately. Table 9 reports the results for the baseline model as well as for the one-period debt model. The statistics for the baseline model are virtually unaffected. For the one-period model, mean spreads decline somewhat with an increase in grid size but other statistics are unaffected.

Table 11: Sensitivity to Grid Sizes

Moment	Baseline	Model I	Model II	1-Period Debt	Model III	Model IV
Avg. $(r - r_f)$	0.0815	0.0814	0.0815	0.0815	0.0806	0.0809
$\sigma(r - r_f)$	0.0443	0.0443	0.0443	0.0443	0.0434	0.0438
Avg. b/y	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7
$\sigma(c)/\sigma(y)$	1.11	1.11	1.11	1.59	1.59	1.59
$\sigma(NX/y)/\sigma(y)$	0.20	0.21	0.20	1.06	1.05	1.06
corr (c, y)	0.99	0.99	0.99	0.73	0.73	0.73
corr (NX, y)	-0.44	-0.44	-0.44	-0.16	-0.16	-0.16
corr $(r - r_f, y)$	-0.65	-0.65	-0.65	-0.55	-0.55	-0.55
Debt Service	0.055	0.055	0.055	0.699	0.699	0.701
Def Freq	0.068	0.068	0.068	0.073	0.072	0.073
Baseline: $N_y = 200$, $N_b = 350$;1-Period Model: $N_y = 200$,$N_b = 450$						
Model I: $N_y = 400$, $N_b = 350$; Model II: $N_y = 200$, $N_b = 700$						
Model III: $N_y = 400$,$N_b = 450$; Model IV: $N_y = 200$,$N_b = 900$						

In addition, we investigated sensitivity to some other aspects of our model (details are available in the Web Appendix). We targeted the full average debt level of 1 (instead of 0.7). There is an increase in the volatility of consumption and of the trade balance and an increase in debt service, but these increases are what we would expect for a higher average debt burden. The correlation patterns remain the same. We also investigated whether results are sensitive to the assumption that the value of m resets to $-\bar{m}$ in the period of default. As an alternative, we assumed that m resets to 0 instead of $-\bar{m}$, which might be viewed as a more neutral assumption. The change makes no difference to model statistics. Finally, we re-estimated the endowment process for $\sigma_m = 0.002$, which is the lowest σ_m for which we get convergence for the baseline model. The implied estimate of ρ was somewhat smaller than in the baseline and the estimate of σ_ϵ somewhat higher. These differences had no effect on model statistics.

10 Appendix C: Equivalence of Longer-Term Debt to One-Period Debt When Future Value of Debt is Protected

For the purposes of this demonstration, it is not important to include the i.i.d. income shock, so we suppress it. The value of repayment is:

$$V(y_{-}, y, b) = \max_{b'} \left\{ \begin{array}{c} u\left(y + \left[\lambda + (1-\lambda)\left(z + q(y_{-}, b)\right)\right] b - q(y, b')b'\right) \\ +\beta E_{y'|y}\left[\max\left\{V(y, y', b'), X(y')\right\}\right] \end{array} \right\}$$

where y_{-} is the realization of income in the previous period. Observe that the portion of the bond that does not mature in the current period pays off the coupon z and its *last period* market value as opposed to its *current* market value. This is equivalent to the sovereign transferring $[(q(y_{-}, b) - q(y, b')](1-\lambda)b$ to the lenders each period (the sovereign pays if this quantity is negative and receives if it is positive). The value of default is: $X(y) = u(y - \phi(y)) + \beta E_{y'|y}\{(1-\xi)X(y') + \xi W(y, y', 0)\}$ and $W(y_{-}, y, b) = \max\{V(y_{-}, y, b), X(y)\}$. Denote the decision rules by $d(y_{-}, y, b)$ and $a(y_{-}, y, b)$. Then, the equilibrium price of a unit bond is given by:

$$q(y, b') = \left[\lambda + (1-\lambda)\left(z + q(y, b')\right)\right] \left(\frac{E_{y'|y}(1 - d(y, y', b'))}{1+r}\right) \tag{14}$$

We can now do a change of variables that will allow us to re-write the above problem in terms of only two state variables. The key insight is that the default decision should depend only on y and the *total* obligation of the sovereign at the start of each period, which is $\left[\lambda + (1-\lambda)\left(z + q(y_{-1}, b)\right)\right] b$. Let $A = \left[\lambda + (1-\lambda)\left(z + q(y_{-1}, b)\right)\right] b$ and suppose that the default decision rule can be expressed as a function $d(y, A)$. Next, notice that multiplying both sides of (14) by b' gives $q(y, b')b' = \left[\lambda + (1-\lambda)\left(z + q(y, b')\right)\right] b' \left(E_{y'|y}(1 - d(y, y', b'))/(1+r)\right)$. Or, $q(y, b')b' = \left(E_{y'|y}(1 - d(A', y'))/(1+r)\right) A' = \tilde{q}(y, A')A'$. Thus, we can re-write the value of repayment as $V(y,A) = \max_{A'}\{u(y + A - \tilde{q}(y,A')A') + \beta E_{y'|y} \max\{V(y',A'), X(y')\}$. This repayment value is exactly the same as the one in which the sovereign issues one-period debt. From the solution to this one-period debt problem (namely the decision rules $d(y,A)$ and $A'(y,A)$) we can recover both the decision rules and the price function of the original long-term debt problem. Observe that (i) $q(y,b')b' = A'\left(E_{y'|y}(1 - d(y',A'))/(1+r)\right)$ and (ii) $A' = [(1-\lambda) + \lambda\,(z + q(y,b'))]\,b'$. Using (i), we can solve for b' from (ii): $A' = [(1-\lambda) + \lambda z]b' + A'\left(E_{y'|y}(1 - d(y',A'))/(1+r)\right)$. This gives b' as a function of y, y_{-} and b, since A' is a function of y and A. Then, using this solution, we can solve for $q(y,b')$ from (i). Thus, with this market arrangement, long-term debt is isomorphic to one-period debt.[42]

[42]Strictly speaking it needs to be proven that the decision rule and price function recovered in this way actually solve the long-term debt problem. But it is fairly evident that it will, so we omit this proof.

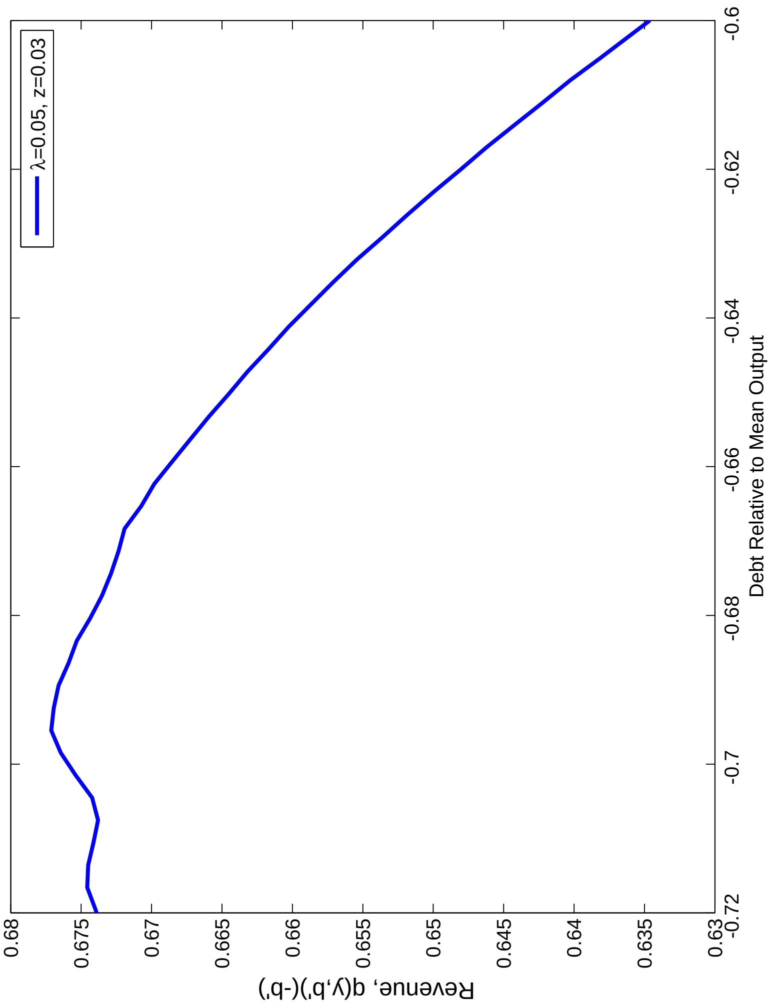


Figure 1: Revenue from Long-Term Debt (b=0)

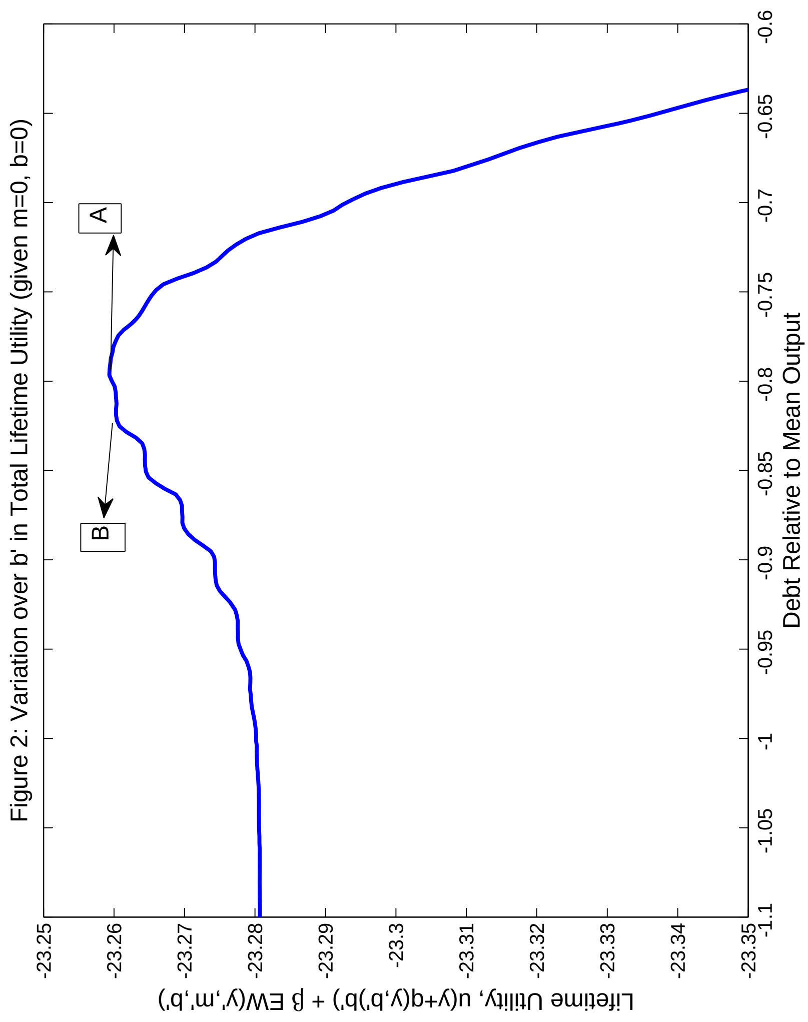


Figure 2: Variation over b' in Total Lifetime Utility (given m=0, b=0)
A
B
Lifetime Utility, u(y+q(y,b')b') + β EW(y',m',b')
Debt Relative to Mean Output
-23.25
-23.26
-23.27
-23.28
-23.29
-23.3
-23.31
-23.32
-23.33
-23.34
-23.35
-1.1
-1.05
-1
-0.95
-0.9
-0.85
-0.8
-0.75
-0.7
-0.65
-0.6

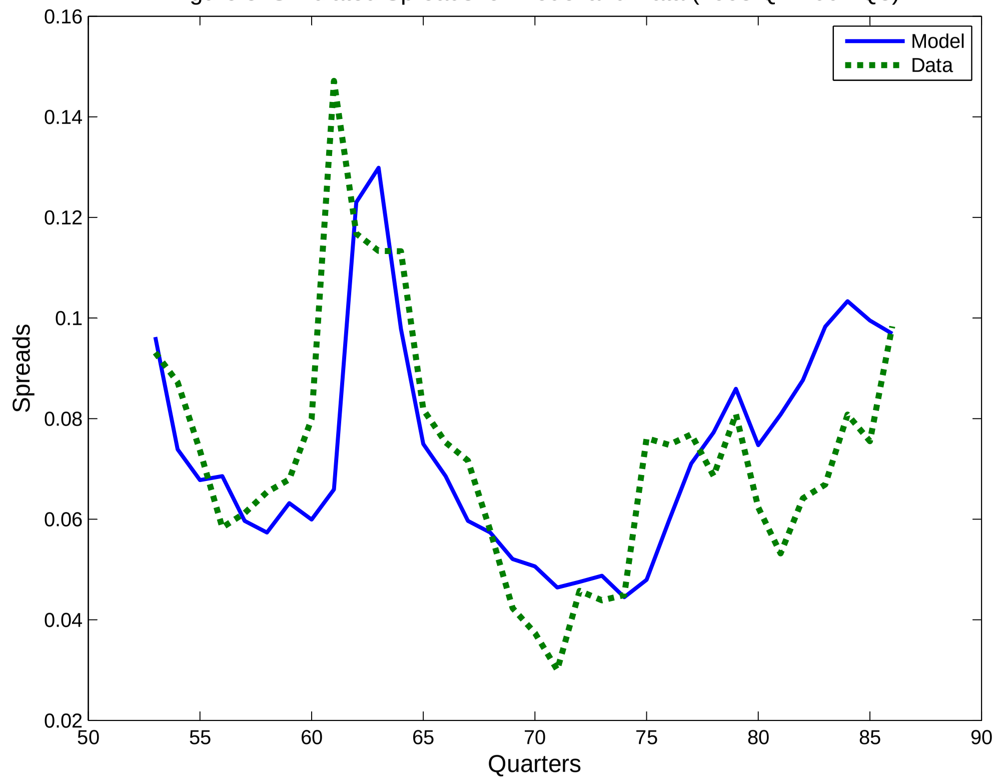


Figure 3: Simulated Spreads for Model and Data (1993:Q1-2001:Q3)

Figure 4: Current Spreads and Next-Period Default Probability for Long-term Bonds When y = Mean(y)

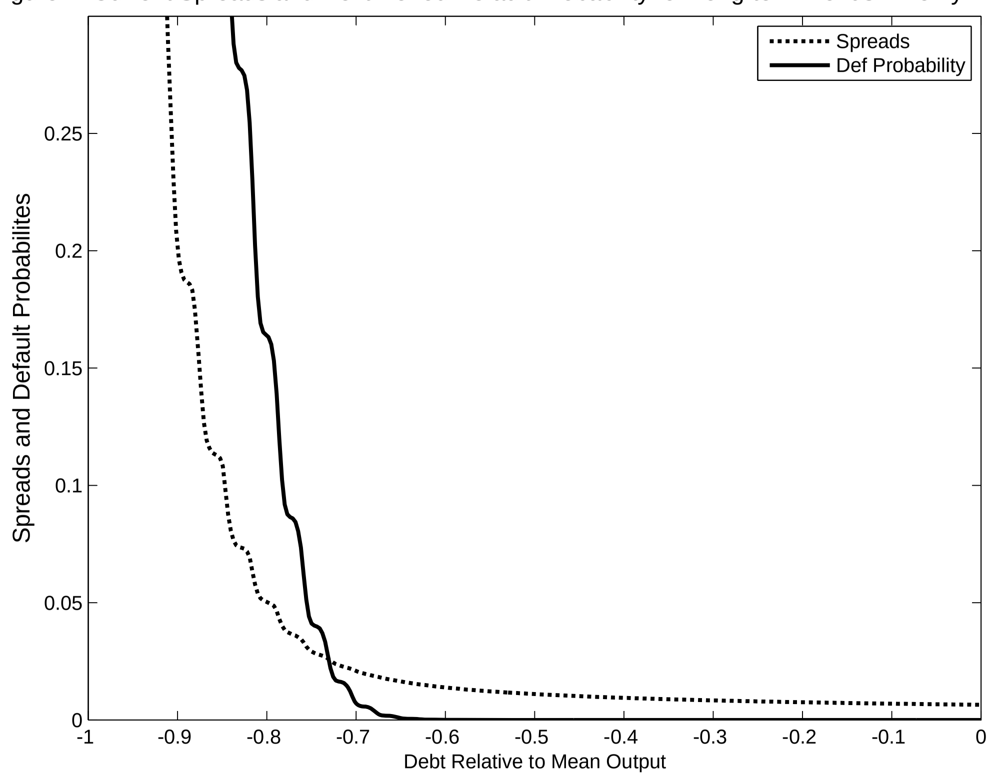
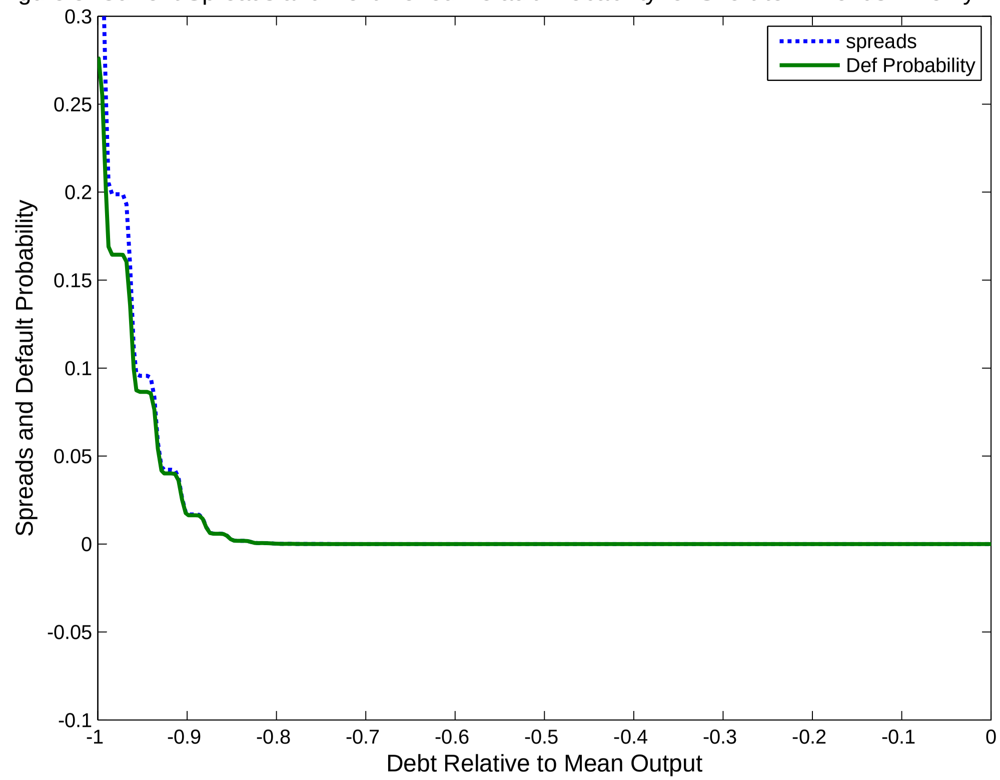


Figure 5: Current Spreads and Next-Period Default Probability for Short-term Bonds When y = Mean(y)



11 Web Appendix

11.1 Theory

Proof of Proposition 0 Let $\mathcal{W}$ be the set of all continuous (in m) functions on $Y \times M \times B$ that take values in the bounded interval $[u(0)/(1-\beta), U/(1-\beta)]$. Equip $\mathcal{W}$ with sup norm $||\cdot||_\infty$. Then $(\mathcal{W}, ||\cdot||_\infty)$ is a complete metric space.

For $W \in \mathcal{W}$, let $X(y, m; W)$ be the solution to (3). The solution exists because (3) defines a contraction mapping in X with modulus $\beta(1-\xi)$. By standard contraction mapping arguments, $X(y, m)$ is continuous in m because $c = y - \phi(y) + m$ is continuous in m and u is continuous in c.

For $W \in \mathcal{W}$, let $V(y, m, b; W, q)$ be the solution to (4). We index this solution by q because q appears as a parameter in (4). Here, however, we need to address the fact that V may not be well-defined because there may not be any feasible b' for some (y, m, b) and q. To extend the definition of V over the entire domain, we will assume that the utility from a choice of b' under repayment, denoted $V_{b'}(y, m, b; W, q)$, is given by $u(\max\{0, y + m + [\lambda + (1-\lambda)z]b + q(y, b')[b' - (1-\lambda)b]\}) + \beta E_{(y',m'|y)} W(y', m', b')$. Thus, for an infeasible choice of b', current consumption is set to 0. Then, $V(y, m, b; W, q) = \max_{b' \in B} V_{b'}(y, m, b; W, q)$. Since B is a finite set, $V(y, m, b; W, q)$ exists for all (y, m, b) and q. Also $V_{b'}$ is continuous in m for every b' since $\max\{0, y + m + [\lambda + (1-\lambda)z]b + q(y, b')[b' - (1-\lambda)b]\}$ is continuous in m and u is continuous in c. Therefore, $V(y, m, b; W, q)$ is continuous in m since the maximum of a finite set of continuous functions is also continuous. Furthermore, both $X(y, m; W)$ and $V(y, m, b; W, q) \in [u(0)/(1-\beta), U/(1-\beta)]$ for all y, m and b.

Next, define the operator

$$T(W)(y, m, b; q) = \max\{V(y, m, b; W, q), X(y, -\bar{m}; W)\} \tag{15}$$

on the space of functions $\mathcal{W}$. Then, (i) $T(\mathcal{W})(y, m, b; q) \subset \mathcal{W}$ which is obvious; (ii) If

$W \geq \hat{W}$ then $T(W) \geq T(\hat{W})$, which follows because $V(y, m, b, W, q)$ is clearly increasing in W and standard contraction mapping arguments can establish that $X(y, m; W)$ is increasing in W; and (iii) $T(W + k) \leq T(W) + \eta k$, where $\eta = \max\{\beta\xi/[(1-\beta) + \beta\xi], \beta\} < 1$. To see (iii), note that $V(y, m, b; W + k, q) = V(y, m, b; W, q) + \beta k$ and $X(y, m; W + k) = X(y, m; W) + (\beta\xi/[(1-\beta) + \beta\xi])k$. Therefore,

$$T(W+k)(y, m, b; q) = \max\left\{V(y, m, b; W, q) + \beta k,\ X(y, -\bar{m}; W) + \frac{\beta\xi}{[(1-\beta) + \beta\xi}k\right\}$$

and (iii) follows. Therefore, T is a contraction mapping with modulus η and the existence of a unique solution to (15) in $\mathcal{W}$, denoted $W_q^*(y, m, b)$, follows from the Contraction Mapping Theorem.

The strict monotonicity of $X(y, m)$ with respect to m follows from the endowment being strictly increasing in m, u being strictly increasing in c, and the fact that m does not affect the probability distribution of (y', m').

For the strict monotonicity of V with respect to m, observe that if $m^0 < m^1$ then every b' that is feasible under (y, m^0, b) is also feasible under (y, m^1, b) and yields strictly higher consumption. In the region where repayment is feasible, there must be at least one b' that is feasible. Then, since m does not affect the probability distribution of (y', m'), strict monotonicity of u implies $V(y, m^0, b) < V(y, m^1, b)$. For strict monotonicity with regard to b, observe that for $b^0 < b^1$ we have $[\lambda + (1-\lambda)z]b^0 + q(y, b')[1-\lambda]b^0 < [\lambda + (1-\lambda)z]b^1 + q(y, b')[1-\lambda]b^1$ for every feasible $b' \in B$ and every $y \in Y$. This follows because $[\lambda + (1-\lambda)z] > 0$ and $q(y, b') \geq 0$. Hence, every b' that is feasible under (y, m, b^0) is also feasible under (y, m, b^1) and affords strictly greater consumption. Again, in the region where repayment is feasible, there must be at least one feasible b'. Therefore, by the strict monotonicity of u, $V(y, m, b^0) < V(y, m, b^1)$.

From the strict monotonicity of V with respect to b, it follows that for $b'^1 > b'^0$, $W(y', m', b'^1) \geq W(y', m', b'^0)$. Hence, $Z(y, b'^1) \geq Z(y, b'^0)$. To show the inequality is strict, we will assume b_I (the smallest $b \in B$) is bounded below as

$$b_I > -[\phi(y_{\max}) + 2\bar{m}]/[\lambda + (1-\lambda)z], \tag{16}$$

where $y_{\max}$ is the largest $y \in Y$. Then, observe that (16) implies that $u(y_{\max} + [\lambda + (1-\lambda)z]b + \bar{m}) + \beta Z(y, 0) > u(y_{\max} - \phi(y_{\max}) - \bar{m}) + \beta Z(y, 0)$ for all $b \in B$. Also, observe that $Z(y, 0) > E_{(y',m')|y}[\xi W(y', m', 0) + (1-\xi)X(y', m')]$, since $X(y', m') < W(y', m', 0)$ for all $(y', m') \in Y \times M$. Thus, for every debt level there is a range of m values for which repayment without new borrowing is better than default if y is at its highest value. Therefore, for every $b' \in B$, there is a range of m' for which $V(y_{\max}, m', b') > X(y_{\max}, -\bar{m})$. By the strict monotonicity of V with respect to b, every m' for which $V(m', b'^0, y_{\max}) > X(y_{\max}, -\bar{m})$ it is also true that $V(m', b'^1, y_{\max}) > X(y_{\max}, -\bar{m})$. Thus, there is a range of m values for which $W(y_{\max}, m', b'^1) > W(y_{\max}, m', b'^0)$. Since $F(y, y_{\max}) > 0$ for all y, it follows that $Z(y', b'^1) > Z(y', b'^0)$. $\square$

Since we extended the domain of the definition of V to infeasible choices, we need to verify that this extension does not result in the sovereign actually choosing infeasible b'. In the following Lemma we establish that if $u(0)$ is set to a sufficiently low number, then it is never optimal to choose infeasible actions.

Lemma 0: If $u(0) + \beta U/(1-\beta) < u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$, where $y_{\min}$ is the smallest value in Y, then optimal consumption under repayment, $c(y, m, b)$, is uniformly bounded below by some strictly positive number $\bar{c}$.

Proof: By continuity of u there exists $\bar{c} > 0$ such that $u(\bar{c}) + \beta U/(1-\beta) < u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$. Since the sovereign can choose to consume its endowment each period, and it can always consume at least $y_{\min} - \phi(y_{\min}) - \bar{m}$ in every period, its lifetime utility in any period is bounded below by $u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$. On the other hand, the highest utility from selecting any action that leads to consumption $\bar{c}$ or less is $u(\bar{c}) + \beta U/(1-\beta)$. By assumption the former dominates the latter. Thus it can never be optimal to choose to consume $\bar{c}$ or less. In particular, it can never be optimal to choose an action that leads to 0 consumption. $\square$

The next three Lemmas are needed to establish the continuity of the operator $H(q)$ with respect to q.

Lemma 1: $W_q^*(y,m,b)$, $V(y,m,b;W_q^*,q)$, $X(y,m;W_q^*)$ and $Z_q^*(y,b')$ are all continuous in q.

Proof: To prove that $W^*(y,m,b;q)$ is continuous in q, it is sufficient to prove that the contraction operator $T(W)(y,m,b;q)$ is continuous in q (see Theorem 4.3.6 in Hutson and Pym (1980, pp. 117-118). In order to establish this, we need to prove only that $V(y,m,b;W,q)$ is continuous in q. Fix (y,m,b) and W. Observe that $V_{b'}(y,m,b;W,q)$ is continuous in q because $\max\{0, y+m+[\lambda+(1-\lambda)z]b+q(y,b')[b'-(1-\lambda)b]\}$ is continuous in q and u is continuous in c. Thus, $V(y,m,b;W,q)$, being the maximum of a finite set of continuous functions, is also continuous in q. Hence $W_q^*(y,m,b)$ is continuous in q. The continuity of $V(y,m,b;W_q^*,q)$ with respect to q follows from the same logic as before: $V_{b'}(y,m,b;W_q^*,q)$ is continuous in q for each $b'$and hence the maximum over b'must also be continuous in q; the continuity of $Z_q^*(y,b')$ with respect to q follows directly from its definition; and the continuity of $X(y,m;W_q^*)$ with respect q follows from noting that the contraction operator defining $X(y,m;W)$ depends on W via the quantity $Z(y,0)$ and that the operator is continuous in $Z(y,0)$. Since $Z_q^*(0,y')$ is continuous in q, it follows from another application of Theorem 4.3.6 of Hutson and Pym that $X(y,m;W_q^*)$ is continuous in q. $\square$

Lemma 2 establishes that the sovereign can be indifferent between default and repayment at exactly one value of m and it can be indifferent between any two borrowing levels at exactly one value of m. These results are needed for Lemma 3, which establishes almost sure convergence of decision rules with respect to prices q.

Lemma 2: (i) For any given b'^0, there can be at most one value of m for which choosing b'^0 gives the same lifetime utility as defaulting and (ii) for any given $b'^0 < b'^1$ there can be at most one value of m for which choosing the two debt levels gives the same lifetime utility.

Proof: (i) Fix y and b. (i) Suppose that there is an $\hat{m}$ such that $V_{b'^0}(y,\hat{m},b) = X(y,-\bar{m})$.

Since the l.h.s is strictly increasing in m, there cannot be any other $m \neq \hat{m}$ for which the same equality holds. (ii) Suppose there is an $\hat{m}$ for which $u(c^0(\hat{m})) + \beta Z(y, b'^0) = u(c^1(\hat{m})) + Z(y, b'^1)$, where $c^0(\hat{m})$ and $c^1(\hat{m})$ are the levels of consumption when b'^0 and b'^1 are chosen, respectively. Since $Z(y, b'^1) > Z(y, b'^0)$ (Proposition 0), it follows that $c^0(\hat{m}) > c^1(\hat{m})$. Suppose, to get a contradiction, there is another $\tilde{m} > \hat{m}$ such that $u(c^0(\tilde{m})) + \beta Z(y, b'^0) = u(c^1(\tilde{m})) + Z(y, b'^1)$. Then, $u(c^0(\hat{m})) - u(c^0(\tilde{m})) = u(c^1(\hat{m})) - u(c^1(\tilde{m}))$ and (from the budget constraint) $c^i(\tilde{m}) = c^i(\hat{m}) + [\tilde{m} - \hat{m}]$ for $i = 0, 1$. Thus, we must have $u(c^0(\hat{m})) - u(c^0(\hat{m}) + [\tilde{m} - \hat{m}]) = u(c^1(\hat{m})) - u(c^1(\hat{m}) + [\tilde{m} - \hat{m}])$. But, since $c^0(\hat{m}) > c^1(\hat{m})$, the preceding equality violates the strict concavity of u. Hence there can only be at most one m for which $u(c^0(m)) + \beta Z(y, b'^0) = u(c^1(m)) + \beta Z(y, b'^1)$. □

Corollary to Lemma 2: The thresholds $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and the corresponding debt choices $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ are unique.

Proof: Suppose, to get a contradiction, that there are two distinct pairs $\{m^{k-1}, b'^k\}$ and $\{\hat{m}^{k-1}, \hat{b}'^k\}$. Without loss of generality, assume that these lists deviate from each other for $k = 1$. That is, according to the first list the sovereign is indifferent between choosing 0 and b'^1 at m^1 and according to the second list it is indifferent between choosing 0 and $\hat{b}'^1$ at $\hat{m}^1$. Suppose also that $\hat{b}'^1 > b'^1$. If $\hat{m}^1 \neq m^1$, then there are two distinct values of m for which $\hat{b}'^1$ and b'^1 give the same utility. This contradicts Lemma 2(ii). And if $\hat{m}^1 = m^1$ then b'^1 is inconsistent with our assumption that, all else the same, two b' choices that give the same utility, the sovereign chooses the larger one. □

Lemma 3: Let $q^n(y, b')$ be a sequence converging to $\bar{q}(y, b')$. Let $d(y, m, b; q^n)$, $a(y, m, b; q^n)$ and $d(y, m, b; \bar{q})$, $a(y, m, b; \bar{q})$ be the corresponding optimal decision rules. Then, $d(y, m, b; q^n)$ converges pointwise to $d(y, m, b; \bar{q})$ and $a(y, m, b; q^n)$ converges pointwise to $a(y, m, b; \bar{q})$ except, possibly, at a finite number of points.

Proof: (Convergence of $a(y, m, b; q^n)$). Let $q^n \to \bar{q}$. Fix y and b. For a given m, let $b'^0 = a(y, m, b; \bar{q})$. Let $V_{b'}(y, m, b; W^*_{\bar{q}}, \bar{q})$ denote the lifetime utility if the sovereign chooses to borrow b' in the current period but follows the optimal plan in all future periods. Two cases are possible: (i) $V(y,m,b;W^*_{\bar{q}},\bar{q}) > V_{b'}(y,m,b;W^*_{\bar{q}},\bar{q})$ for all $b' \neq b'^0$ and (ii) $V(y,m,b;W^*_{\bar{q}},\bar{q}) = V_{b'}(y,m,b;W^*_{\bar{q}},\bar{q})$ for some $b' \neq b'^0$. Consider case (i). Let $V(y,m,b;W^*_{\bar{q}},\bar{q}) - V_{b'}(y,m,b;W^*_{\bar{q}},\bar{q}) = \Delta$. Since $V(y,m,b;W^*_q,q)$ is continuous in q there exists N_1 such that for all $n \geq N_1$ $V(y,m,b;W^*_{q^n},q^n) > V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta/2$. Next, note that

$$V_{b'}(y,m,b;W^*_{q^n},q^n) = u(y+m+[\lambda+(1-\lambda)z]b - q^n(y,b')[b'-(1-\lambda)b]) + \beta Z^*_{q^n}(y,b').$$

Since $Z^*_q(y,b')$ is continuous in q it follows that there exists N_2 such that for all $n \geq N_2$ $V_{b'}(y,m,b;W^*_{q^n},q^n) < V_{b'}(y,m,b;W^*_{\bar{q}},\bar{q})+\Delta/2$. Therefore $V(y,m,b;W^*_{q^n},q^n)-V_{b'}(y,m,b;W^*_{q^n},q^n)$ $> V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta/2 - V_{b'}(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta/2 = 0$ for all $n \geq \max\{N_1,N_2\}$. Hence $a(y,m,b;q^n) = b'^0$ for all $n > \max\{N_1,N_2\}$. Now consider case (ii). In this case, convergence may fail because $a(y,m,b;q^n)$ may converge to b' rather than b'^0. However, by Lemma 1 there can be only a finite number of m values for which case (ii) can hold. Therefore, $a(y,m,b;q^n)$ converges pointwise to $a(y,m,b;\bar{q})$ except, possibly, for a finite number of m.

(Convergence of $d(y,m,b;q^n)$). Let $q^n \to \bar{q}$. Fix y and b. Again, two cases are possible. (i) $X(y,-\bar{m};W^*_{\bar{q}}) \neq V(y,m,b;W^*_{\bar{q}},\bar{q})$ and (ii)$X(y,-\bar{m};W^*_{\bar{q}}) = V(y,m,b;W^*_{\bar{q}},\bar{q})$. Consider case (i). For concreteness, suppose that $X(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) = \Delta > 0$. Then, by continuity of $V(y,m,b;W^*_q,q)$ and $X(y,m;W^*_q)$ with respect to q there exists N_1 such that for all $n \geq N_1$, $V(y,m,b;W^*_{q^n},q^n) < V(y,m,b;W^*_{\bar{q}},\bar{q}) + \Delta/2$. And, by the continuity of $X(y,-\bar{m};W^*_q)$ with respect to q, there exists N_2, such that for $n \geq N_2$, $X(y,-\bar{m};W^*_{q^n}) > X(y,-\bar{m};W^*_{\bar{q}})-\Delta/2$. Then, for all $n \geq \max\{N_1,N_2\}$, $X(y,-\bar{m};W^*_{q^n})-V(y,m,b;W^*_{q^n},q^n) > X(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta = 0$. Hence $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 1$ for all $n \geq \max\{N_1,N_2\}$. If $\Delta < 0$, we can use a similar argument to show that there exists some N such that for all $n \geq N$, $V(y,m,b;W^*_{q^n},q^n) > X(y,-\bar{m};W^*_{q^n})$. Hence, for all such n, $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 0$. Now consider case (ii). Again, convergence may fail in this case because $d(y,m,b;q^n)$ may converge to 1 or 0 while $d(y,m,b;\bar{q})$ is 0 or 1. However, by Lemma 2(i), there can only be one value of m for which this can happen. Therefore,

$d(y, m, b; q^n)$ converges pointwise to $d(y, m, b; \bar{q})$ except, possibly, for one value of m. $\square$

Proof of Continuity of $H(q)$: Let $\{q^n\}$ be a sequence in Q converging to $\bar{q} \in Q$ and let $\{d(y, m, b; q^n),\ a(y, m, b; q^n)\}$ and $\{d(y, m, b; \bar{q}),\ a(y, m, b; \bar{q})\}$ be the corresponding default and debt decision rules. Then

$$H(q^n)(y, b') = E_{(y'\,m')|y}\left[[1 - d(y', m', b'; q^n)]\frac{\lambda + [1-\lambda][z + q^n(y', a(y', m', b'; q^n))]}{1 + r_f}\right].$$

Or,

$$H(q^n)(y, b') = \\ \sum_{y'}\left[\frac{\int_M [1 - d(y', m', b'; q^n)]\,[\lambda + [1-\lambda][z + q^n(y', a(y', m', b'; q^n))]]\,dG(m')}{1 + r_f}\right] F(y', y).$$

Fix y' and b'. By Lemma 3, $\lim_n [1 - d(y', m', b'; q^n)] = [1 - d(y', m', b'; \bar{q})]$ for all but a finite number of points (possibly) of m'. Since individual points of m have probability zero, $[1 - d(y', m', b'; q^n)]$ converges almost surely to $[1 - d(y', m', b'; \bar{q})]$ with respect to the measure induced by $G(m)$. Also, by Lemma 3, $\lim_n a(y', m', b'; q^n) = a(y', m', b'; \bar{q})$ for all but a finite number of points (possibly) of m'. If convergence holds then, since $a(\cdot; q^n)$ takes values in a finite set B, there must exist N such that for all $n > N$ $a(y', m', b'; q^n) = a(y', m', b'; \bar{q})$. Therefore, for $n > N$, $q^n(y', a(y', m', b'; q^n)) = q^n(y', a(y', m', b'; \bar{q}))$. Since $q^n \to \bar{q}$, it follows that $\lim_n q^n(y', a(y', m', b'; \bar{q})) = \bar{q}(y', a(y', m', b'; \bar{q}))$. Thus, viewed as a function of m', $q^n(y', a(y', m', b'; q^n))$ converges almost surely to $\bar{q}(y', a(y', m', b'; \bar{q}))$. Therefore, we have that

$$\lim_n [1 - d(y', m', b'; q^n)]\,[\lambda + [1-\lambda][z + q^n(y', a(y', m', b'; q^n))]] = \\ [1 - d(y', m', b'; \bar{q})]\,[\lambda + [1-\lambda][z + \bar{q}(y', a(y', m', b'; \bar{q}))]]$$

except, possibly, at a finite number of points.

Now observe that each function in the sequence is non-negative and bounded above by

$\lambda + (1-\lambda)[z+\bar{q}]$. Thus, by the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_n \int_M [1 - d(y', m', b'; q^n)] \left[\lambda + [1-\lambda][z + q^n(y', a(y', m', b'; q^n))]\right] dG(m') =$$
$$\int_M [1 - d(y', m', b'; \bar{q})] \left[\lambda + [1-\lambda][z + \bar{q}(y', a(y', m', b'; \bar{q}))]\right] dG(m').$$

Putting these results together, we get

$$\begin{aligned} & \lim_n H(q^n)(y, b') = \\ & \sum_{y'} \left[\frac{\int_M [1 - d(y', m', b'; q^n)] \left[\lambda + [1-\lambda][z + q^n(y', a(y', m', b'; q^n))]\right] dG(m')}{1 + r_f} \right] F(y', y). \\ = \; & H(\bar{q})(y, b'). \end{aligned}$$

Thus $H(q)(y, b')$ is continuous. □

11.2 Details of Sensitivity Analyses

Table 12 reports the results of three sensitivity analysis. In the first exercise, denoted Model I, the full average debt level of 1.0 is targeted. The model can successfully match all targets. There are some differences in the results. There are increases in the volatility of consumption and NX, and a measurable increase in the debt service. These increases are what we would expect for a higher average debt burden. The correlation patterns remain the same. Overall, model performance is somewhat inferior to the baseline model.

In Model II, we address one potential concern regarding the assumption about m in the period of default. Recall that we assumed that in the period of default, the value of m resets to $-\bar{m}$. This means that there is an additional source of punishment for default, and one may wish to know if this plays any role in the results. In Model II, we assume that in the period of default the value of m resets to 0 instead of $-\bar{m}$ – which might be viewed as a more neutral assumption. As is evident, there is virtually no difference in results between the baseline model and this one.

In the third sensitivity analysis, we examine if the results change with a lower standard deviation for m. We re-estimated the endowment process under the assumption that $\sigma_m = 0.002$, which is the lowest σ_m value for which we get convergence for our baseline model. The implied estimates of ρ and σ_ϵ are as reported in the table. As one would expect, ρ is somewhat lower, and σ_ϵ somewhat higher than in the baseline model. However, these changes in the endowment process have virtually no effect on model statistics.

Table 12: Additional Sensitivity Analyses

Moment	Baseline	Model I	Model II	Model III
Avg. $(r - r_f)$	0.0815	0.0815	0.0815	0.0815
$\sigma(r - r_f)$	0.0443	0.0443	0.0443	0.0444
Avg. b/y	-0.7	-0.9996	-0.7	-0.7
$\sigma(c)/\sigma(y)$	1.11	1.15	1.11	1.11
$\sigma(NX/y)/\sigma(y)$	0.20	0.28	0.21	0.20
$corr(c, y)$	0.99	0.96	0.99	0.99
$corr(NX/y, y)$	-0.44	-0.44	-0.44	-0.44
$corr(r - r_f, y)$	-0.65	-0.62	-0.65	-0.65
Debt Service	0.055	0.078	0.055	0.055
Def Freq	0.068	0.067	0.068	0.068
Model I: Average b/y target $= 1.0$				
Model II: Same targets as baseline but in period of default, m resets to 0				
Model III: Same targets as basline with $\rho = 0.948081$,$\sigma_\epsilon = 0.027203$,$\sigma_m = 0.002$				

11.3 The Trade-off Between σ_m and ζ in Achieving Convergence

This section of confirms that there is a trade-off between the variability of m and the relaxation parameter ζ with regard to convergence within $100,000$ iterations. We consider the model where all parameter values are as in the baseline model but the number of grids on

Y, M, and B are 25, 50 and 100, respectively. With fewer grids, the computations take less time and we can demonstrate that convergence can be achieved for *very* small values of σ_m, provided the value of ζ is increased correspondingly.

Table 13: (σ_m, ζ) Pairs For Which Convergence Is Achieved

σ_m	ζ
0.001	0.98
0.0005	0.98
0.0001	0.98
0.00005	0.995
0.00001	0.998
Grids	$Y = 25,\ M = 50,\ B = 100$