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Abstract

In this paper, we present a new approach to incorporating long-term debt into equilibrium models of unsecured debt and default. We make three sets of contributions. First, we advance the theory of sovereign debt begun in Eaton and Gersovitz (1981) by proving the existence of an equilibrium price function with the property that the interest rate on debt is increasing in the amount borrowed. Second, using Argentina as a test case, we show that unlike a one-period debt model, our model of long-term debt is capable of accounting for the average external debt-to-output ratio, average spread on external debt, and the standard deviation of spreads for the 1993-2001 period, without any deterioration in the model's ability to account for Argentina's other cyclical facts. Third, we propose a new and very accurate method for solving the model.

1 Introduction

Until recently, the existing literature on debt and default – both the consumer debt and the sovereign debt parts – has considered only one-period debt. In reality, both consumers and countries can and do borrow for more than one period. In this paper, we present a new approach to incorporating long-term debt into equilibrium models of unsecured debt and default.

We make three main contributions. First, we extend the seminal contribution of Eaton and Gersovitz (1981) to the case of long-term bonds. A key insight of their work is the connection between the size of a sovereign's borrowing and the interest rate at which it is obtained. They established that because a sovereign has the option to default, the interest rate at which a sovereign borrows increases with the amount borrowed. In their one-period debt model, the rising supply curve for credit followed directly from the rising likelihood of default. But with long-term debt, the connection between the likelihood of default and the price of credit is not straightforward because the price of credit also depends on how much the sovereign plans to borrow next period if it does not default. The first main contribution of our paper is to establish that there exists an equilibrium price function for unsecured long-term debt with the property that the supply curve for credit is rising in the interest rate.[1] Thus, a key implication of the Eaton-Gersovitz framework is shown to carry over to the case of long-term debt.

Second, we contribute to the burgeoning quantitative-theoretic literature on emerging market business cycles. As documented in Neumeyer and Perri (2005), open emerging market economies display a high cyclical volatility of consumption and a strongly countercyclical trade balance and interest rate, three facts that appear anomalous relative to the cyclical

[1]This extends the existence result for one-period debt in Chatterjee, Corbae, Nakajima, and Rios-Rull (2007) to the case of long-term debt with a constant risk-free rate. The model of unsecured consumer debt introduced in Athreya (2002) and extended and analyzed further in Chatterjee et al. (2007); Livshits, MacGee and Tertilt (2007); and others bears a strong resemblance to the Eaton-Gersovitz model of sovereign debt and default. Thus, the model and the solution procedure developed in this paper are relevant to the consumer debt and default literature as well.

experience of small (open) developed economies. Following up on Neumeyer and Perri's contribution, Aguiar and Gopinath (2006) and Arellano (2008) showed that the Eaton-Gersovitz framework can quantitatively account for these patterns. The key is the countercyclicality of the Eaton-Gersovitz supply curve for credit. Unlike developed economies, below-trend output in an emerging market economy shifts back the supply curve for credit because the probability of default goes up. The consequent rise in interest rates can cause the country to *reduce* its borrowing. This results in a countercyclical trade balance and interest rate and a higher sensitivity of consumption to output.

While these studies (and others mentioned below) make insightful contributions to the theory of aggregate fluctuations, they have left some questions unanswered. In particular, one would want theories along these lines to be consistent with what is observed regarding the average external debt-to-output ratio, the average default spreads on external debt, and the standard deviation of default spreads for the country in question. As we discuss in more detail below, there is not as yet a quantitative model of emerging market business cycles consistent with these basic facts. Using Argentina's experience during 1993-2001 as a test case, our second main contribution is to show that the long-term debt version of the Eaton-Gersovitz model can account for these three key first and second moments, without any deterioration in the model's ability to account for Argentina's other cylical facts.

Third, we present a novel approach to computing models with unsecured long-term debt and default. In the Eaton-Gersovitz framework, the budget set of the sovereign is typically nonconvex for any given (rising) supply curve for credit, and the future value function (which takes into account the option to default) is typically nonconcave in the amount borrowed. The lack of convexity implies that the optimal debt decision rule may fail to be continuous in state variables and prices. Unlike the one-period debt model, this (potentially discontinuous) debt decision rule appears in the equilibrium price equation for long-term debt. If a solution to the price equation is sought on a discretized state space (as in Aguiar and Gopinath (2006), Arellano (2008), Yue (2009) and others) any discontinuity in the debt decision rule with respect to prices implies a corresponding discontinuity in the price equation. This can lead to the lack of a solution to the price equation and, therefore, to a lack of convergence. The key to our computational approach is the introduction of an i.i.d. shock to output drawn from a continuous distribution with a very small variance. With this (slight) modification, a solution to the price equation is guaranteed to exist (and is the basis of the existence result mentioned earlier). We develop an algorithm that can almost exactly recover the default and debt decision rules with respect to the continuous i.i.d. shock, which then allows a very accurate solution to the equilibrium price equation. The accuracy of our solution procedure and the sensitivity of our findings to pure computational assumptions are reported.

There is a related literature on sovereign debt that goes beyond one-period debt. Hatchondo and Martinez (2008) introduce long-duration debt in a tractable way into the Eaton-Gersovitz framework.[2] They analyze consols with geometrically declining coupon payments and show that having long-duration debt considerably improves the cyclical volatility of spreads relative to the one-period debt version of their model. To give a flavor of their findings (Table 2, p.16), if the loss upon default is 50 percent of output and average duration of debt is 4.07 years, the average equilibrium debt-to-output ratio is 51 percent, the average spread is 2.73 percent and the standard deviation of spreads is 0.33 percent. These statistics decline to 44, 0.12 and 0.06 percent, respectively, if the duration of debt is one quarter. However, the average spread and standard deviation of spreads in the data (as reported by the authors) are 7.44 and 2.71 percent, respectively, so there still remains a large gap between the data and model statistics.

Bi (2008a) examines maturity choice in a model with one- and two-quarter debt. The focus of this study is not on business cycles *per se* but on explaining why the maturity structure of debt shortens when the sovereign is approaching default. The absence of a seniority structure on sovereign debt implies that new creditors understand that if there is default they can acquire a share of the debt recovery value at the expense of existing lenders. When default is likely in the near future, it is relatively attractive for the sovereign to borrow

[2]They depart, however, from the Eaton-Gersovitz framework in assuming that sovereign does not lose access to international capital markets upon default but simply suffers a one-period output loss.

short term. Thus, the maturity structure of debt shortens when approaching a crisis.

Arellano and Ramanarayanan (2009) examine maturity choice using the long-duration debt model of Hatchondo and Martinez. Their focus is on the cyclical behavior of the term spread and the duration of debt for Brazil. They document that both the term spread (the difference between interest rates on long-term and short-term debt) and the duration of debt are procyclical. They are able to replicate this procyclicality and emphasize the cyclical variation in the trade-off between the liquidity benefits of short-term debt and the insurance benefits of long-term debt (insurance against fluctuations in future default risk) as the key factor that helps them account for the facts.

Although maturity choice is not a focus of our paper, we examine this issue in order to better understand the welfare properties of short-term versus long-term debt in our model. First, we show that even though the model with long-term debt can better account for the facts, short-term debt is better than long-term debt in terms of welfare. This comes about because short-term debt is cheaper, since it does not suffer from the commitment problem that plagues long-term debt. We establish this by showing that if the long-term debt contract is modified so as to freeze the future value of outstanding debt at its current market value, long-term debt becomes equivalent to short-term debt even in the presence of default risk. Thus, the inferiority of long-term debt stems from the fact that the borrower cannot commit to not dilute the future value of outstanding debt by borrowing more in the future.

However, the commitment benefit of short-term debt comes at the expense of greater consumption volatility, since fluctuations in the market value of outstanding debt have to be absorbed by the sovereign. Our welfare analysis leaves open the question of whether the interest rate effect is always the dominant effect regardless of the sovereign's current state. To answer this question, we extend our model to the case in which the sovereign can choose to issue long-term or short-term debt each period. We match the same first and second moments as in the baseline long-term debt model and find that short-term debt is the preferred maturity through the business cycle. Although long-term debt reduces the risk of an adverse shift in the supply curve for credit, it is also more expensive and this fact is the dominant force in our simulations. This result is different from the above-mentioned studies by Bi and Arellano and Ramanarayanan. Differences in model structure (we do not allow repayment or risk-averse lenders) or calibration (our default cost function is different from Arellano and Ramanarayanan's) could account for the different results. It is the case, however, that the term spread is procyclical in our model as in Arellano and Ramanarayanan.

There have been a number of recent additions to the quantitative sovereign debt literature that extend the Eaton-Gersovitz framework in important directions while maintaining the assumption that debt is one-period. Bi (2008a), D'Erasmo (2008), Benjamin and Wright (2009) and Yue (2009) explicitly model the debt renegotiation process that follows sovereign default. Repayment makes debt more valuable to lenders and, all else remaining the same, tends to shift out the supply curve for credit and leads to an increase in the average debt-to-output ratio. Cuadra and Sapriza (2008) examine the role of political uncertainty in affecting the level and volatility of sovereign spreads. Mendoza and Yue (2009) endogenize the costs of default by combining a production model featuring foreign working capital loans (as in Neumeyer and Perri (2005) and Uribe and Yue (2006)) with imperfect substitutibility between foreign and domestic inputs. This extension, by endogenizing the default cost function, significantly increases the predicted equilibrium debt-to-output ratio.

Several of these extensions were motivated by the desire to generate debt-to-output ratio that comes closer to the high levels observed for emerging markets. With the exception of Benjamin and Wright, none of the above-mentioned studies generate debt-to-output ratios as high as those in the data.[3] Benjamin and Wright model the process of debt settlement following default and are able to generate a debt-to-output ratio of 65 percent in their baseline model. But it is not clear if their model delivers on the facts regarding spreads (they do not report the average spread or the standard deviation of spreads in their model).[4]

[3] Arellano (2008) obtained a mean debt-to-output ratio of 6 percent, Bi (2008) 21.2 percent, Aguiar and Gopinath (2006) 19 percent, Yue (2009) 10.1 percent, Cuadra and Sapriza (2008) 6.9 percent and Mendoza and Yue (2009) 23.1 percent.

[4] In Table 6, the authors report that the average default frequency in their model is 4.4 percent and the

Indeed, we suspect that any model that (i) has only one-period (i.e., quarterly) debt, (ii) matches a high level of debt (70 percent of quarterly output on average, as in our case) and (iii) has a spread volatility as high as that in the data will tend to imply a counterfactually high consumption volatility. The reason is that in such a model a large fraction of consumption must be financed by issuance of new debt and it would take very small shifts in the Eaton-Gersovitz supply curve for credit to generate consumption volatility of the magnitude we see in the data.[5] In reality, there is considerable volatility in spreads. To have a model consistent with realistic debt-to-output ratios and observed volatilities of spreads and consumption we must recognize that only a small portion of consumption is financed by new issuance of debt. This is precisely what a model with long-term debt permits.

The paper is organized as follows. In Section 2, we briefly discuss why incorporating long-term debt in the standard way into models of unsecured debt can lead to computationally intractable models and then describe our strategy for circumventing this problem. In Section 3, we introduce the sovereign debt environment we analyze. In Section 4, we establish the existence of an equilibrium pricing function that is decreasing in the amount borrowed. As mentioned above, grid-based algorithms for computing equilibrium models of default encounter convergence problems. These difficulties, and the way they are addressed in this paper, are explained in Section 5. Section 6 presents the results of incorporating long-term debt for Argentina and explains how incorporating long-term debt helps improve the ability of this class of models to explain the emerging market facts; sensitivity of findings to both

"haircut" in the event of default is 28 percent. Assuming a risk-free rate of 0, these values imply a default spread of 1.54 percent, which is much lower than in the data.

[5] A back-of-the-envelope calculation suggests why it must be true. Assume for simplicity that output is constant at 100 units and that short-term debt is 70 units. Assume also that the quarterly interest rate on debt is 3 percent. Therefore, about 2 units are paid out as interest, and consumption is 98 units. Of this total consumption, 30 units come from disposable income (income less debt obligations), and the remaining 68 units come from new issuances of short-term debt. Suppose there is a 5 percent fall in output so output declines to 95 units. This reduces disposable income to 25 units. Suppose also that the same issuance of debt as before now fetches *half-a-percent* less revenue than it did (which implies a very small increase in spreads). If the sovereign continues to issue the same face value of debt as before, consumption would fall to $25 + (68 \times 0.9950)$, or 92.66 units. The ratio of the percent decline in consumption to the percent decline in output is then 1.09, which is the relative standard deviation of consumption in the Argentine data.

substantive and purely computational assumptions are also discussed in this section. Section 7 discusses the optimal maturity length for Argentina and extends the model of Section 3 to incorporate maturity choice and shows that short-term debt is the preferred maturity choice. Section 8 summarizes the main findings of this paper and concludes. Finally, Appendix A contains proofs of the more technical results in the paper; Appendix B explains the logic of our computational algorithm; and Appendix C gives details regarding the solution accuracy of our method and compares it to other methods.

2 Random Maturity Bonds

A natural way to introduce long-term debt is to assume that debt issued in period t is due for repayment in period $t + T$. Since new debt can be issued each period, this means that the issuer's state vector contains the vector $(b_0, b_1, b_2, \ldots, b_{T-1})$, where b_τ is the quantity of bonds due for repayment τ periods in the future. The probability of default will, in general, depend on this vector. For even modest values of T (such as 3 or 4), computing default probabilities can become computationally demanding because one has to keep track of at least T state variables, each of which can take many values.

Our approach is to simplify the maturity structure of debt in a way that reduces the number of state variables relevant for computing default probabilities. We analyze long-term debt contracts that mature probabilistically. Specifically, each unit of outstanding debt matures next period with probability λ. If the unit does not mature, which happens with probability $1 - \lambda$, it gives out a coupon payment z. Note that, going forward, a unit bond of type (z, λ) issued $k \geq 1$ periods in the past has exactly the same payoff structure as another (z, λ) unit bond issued $k' > k$ periods in the past. This means that we need to keep track of the *total* number of (z, λ) bonds only. This cuts down on the number of state variables relevant to computing default probabilities.[6]

[6]Hatchondo and Martinez (2008) use a similar trick of rendering outstanding obligations "memoryless." In their setup, all bonds last forever (consols) but each pays a geometrically declining sequence of coupon

In what follows we assume that unit bonds are infinitesimally small – meaning that if b unit bonds of type (z, λ) are outstanding at the start of next period, the issuer's coupon obligations next period will be $z \cdot (1-\lambda)b$ with certainty and the issuer's payment-of-principal obligations will be λb with certainty. And if no new bonds are issued or no outstanding bonds redeemed next period, $(1 - \lambda)b$ unit bonds will be outstanding at the start of the following period.

3 Environment

3.1 Preferences and Endowments

Time is discrete and denoted $t \in \{0, 1, 2, ...\}$. The sovereign receives a strictly positive endowment x_t each period. The stochastic evolution of x_t is governed by the following process:

$$x_t = y_t + m_t. \tag{1}$$

Here $m_t \in M = [-\bar{m}, \bar{m}]$ is a transitory income shock drawn independently each period from a mean zero probability distribution with continuous cdf $G(m)$, and y_t is a persistent income shock that follows a finite-state Markov chain with state space $Y \subset \mathbb{R}_{++}$ and transition law $\Pr\{y_{t+1} = y' | y_t = y\} = F(y, y') > 0,\ y$ and $y' \in Y$. As noted in the introduction, the i.i.d shock m is included to make robust computation of the model possible. In the quantitative analysis to follow, the endowment process (1) is estimated assuming a very small variance for m.

The sovereign maximizes expected utility over consumption sequences, where the utility

payments. Thus, a bond issued in the current period promises to pay the sequence $\{1, \delta, \delta^2, \delta^3, \ldots\}$. This payoff structure is the same as that of a unit random maturity bond with $\lambda = 1 - \delta$ and $z = 1$. Our specification has the advantage that it can be used to separately target maturity length and size of coupon payments as we do later in the paper.

from any given sequence c_t is given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1 \tag{2}$$

The momentary utility function $u(\cdot) : [0, \infty) \to \mathbb{R}$ is continuous, strictly increasing, strictly concave, and bounded above by the quantity U.

3.2 Option to Default and the Market Arrangement

The sovereign can borrow in the international credit market and has the option to default on a loan. Default is costly in several ways. First, upon default, the sovereign loses access to the international credit market – cannot borrow or save in the period of default – and remains in financial autarchy for some random number of periods. Specifically, following the period of default, the sovereign is let back into the international credit market with probability $0 < \xi < 1$. Second, during its sojourn in financial autarchy, the sovereign loses some amount $\phi(y) > 0$ of the persistent component of output y. Third, the sovereign's transitory component of income drops to $-\bar{m}$ in the period of default.[7] We assume that $y - \phi(y) + m > 0$ for all $(y, m) \in (Y, M)$ and increasing in y.[8]

There is a single type of bond of type (z, λ) available in this economy. We assume that lenders are risk-neutral and that the market for sovereign debt is competitive. The unit price of a bond of size b is given by $q(y, b)$. The price of a unit bond does not depend on the transitory shock m because knowledge of current period m does not help predict either m or y in the future and, therefore, does not inform the likelihood of future default. We assume that the sovereign can choose the size of its debt from a finite set $B = \{b_I, b_{I-1}, \ldots b_2, b_1, 0\}$, where $b_I < b_{I-1} < \ldots < b_2 < b_1 < 0$.[9] As is customary in this literature, we will view debt

[7]This technical assumption is made for the purpose of speeding up computation. It is not important that m take the lowest value possible. As we verify in the sensitivity analysis section, setting $m = 0$ (the mean value of the transitory shock) works just as well.

[8]In this paper, a function $f(x)$ is *increasing (decreasing) in* x if $x' > x$ implies $f(x') \geq (\leq) f(x)$ and *is strictly increasing (strictly decreasing)* in x if $f(x') > (<) f(x)$.

[9]For simplicity, we do not allow the sovereign to accumulate assets. In our application, the no-

as negative assets.

3.3 Decision Problem

Consider the decision problem of a sovereign with $b \in B$ of type (z, λ) bonds outstanding and endowments $(y,\, m)$. Denote the sovereign's lifetime utility conditional on repayment by the function $V(y, m, b) : Y \times M \times B \to \mathbb{R}$, its lifetime utility conditional on being excluded from international credit markets by the function $X(y, m) : Y \times M \to \mathbb{R}$, and its unconditional (optimal) lifetime utility by the function $W(y, m, b) : Y \times M \times B \to \mathbb{R}$.whatw Then,

$$X(y,m) = u(c) + \beta\{[1-\xi]E_{(y'\,m')|y}X(y',m') + \xi E_{(y'\,m')|y}W(y',m',0)\} \tag{3}$$

s.t.

$$c = y - \phi(y) + m$$

The sovereign's lifetime utility under financial autarchy reflects the fact that it loses $\phi(y)$ of its output and can expect to be let back into the international credit market next period with probability ξ.

And,

$$V(y,m,b) = \max_{b' \in B}\, u\,(c) + \beta E_{(y'\,m')|y}W(y',m',b') \tag{4}$$

s.t.

$$c = y + m + [\lambda + (1-\lambda)z]\, b - q(y,b')\, [b' - (1-\lambda)b]$$

The above assumes that the budget set under repayment is nonempty, meaning there is at least one choice of b' that leads to nonnegative consumption. But it is possible that (y, b, m) is such that all choices of b' lead to negative consumption. In this case, repayment is simply not an option, and the sovereign must default.

accumulation constraint is never binding in the simulations.

Finally,

$$W(y,m,b) = \max\{V(y,m,b), X(y,-\bar{m})\}. \tag{5}$$

Since W determines both X and V (via equations (3) and (4), respectively) equation (5) defines a Bellman equation in W.

We assume that if the sovereign is indifferent between repayment and default, it repays. Hence, the sovereign defaults if and only if $X(y,-\bar{m}) > V(y,m,b)$. This decision problem implies a default decision rule $d(y,m,b)$ (where $d=1$ is default and $d=0$ is repayment) and, in the region where repayment is feasible, a debt decision rule $a(y,m,b)$. We assume that if the sovereign is indifferent between two distinct b's, it chooses the larger one (i.e., chooses a lower debt level over a higher one).

3.4 Equilibrium

The world one-period risk-free rate r_f is taken as exogenous. Given a competitive market in sovereign debt, the unit price of a bond of size b, $q(y,b')$, must be consistent with zero profits adjusting for the probability of default. That is:

$$q(y,b') = E_{(y'\,m')|y}\left[[1-d(y',m',b')]\frac{\lambda + [1-\lambda][z + q(y', a(y',m',b'))]}{1+r_f}\right] \tag{6}$$

In the event of default, the creditors get nothing. In the event of repayment, the creditors get λ, which is the fraction of a unit bond that matures next period, and on the remaining fraction, $(1-\lambda)$, the creditors get the coupon payment z. In addition, the fraction that remains outstanding has some value that depends on the persistent component of the sovereign's endowment next period and on the sovereign's debt next period. Since the right-hand side of the equation depends directly and indirectly (through the decision rules d and a) on $q(y,b')$, equation (6) defines a functional equation in $q(y,b')$.

4 Characterization and Existence of Equilibrium

In this section, we characterize the equilibrium pricing function and establish its existence. We begin with some basic continuity and monotonicity results concerning the value functions W, V, and X.

Proposition 1 : Given any $q(y, b') \geq 0$, there exists a unique, bounded function $W(y, m, b)$ continuous in m that solves the functional equation (5). Furthermore, $X(y, m)$ is strictly increasing and continuous in m; in the region where repayment is feasible, $V(y, m, b)$ is strictly increasing in b and m and continuous in m; and $Z(y, b') = E_{(y', m')|y)} W(y', m', b')$ is strictly increasing in b', provided there is positive probabililty of repayment for every debt level.

Proof : The proof existence of unique, bounded and continuous (in m) functions W, V, and X follows from standard contraction mapping arguments. The strict monotonicity properties of W, V and X follow from the strict monotonicity of u with respect to c and the fact that c is strictly increasing in m and b given b'; the strict monotonicity of Z with respect to b' follows from the strict monotonicity of V with respect to b and a lower bound on the set B (see Appendix A for details). □

The first characterization result establishes the standard result that the default decision rule is increasing in debt.

Proposition 2: If $b^0 < b^1$, $d(y, m, b^0) \geq d(y, m, b^1)$.

Proof: Suppose, to get a contradiction, that for some (y, m) we have $d(y, m, b^0; q) < d(y, m, b^1; q)$. Then $d(y, m, b^0; q) = 0$ and $d(y, m, b^1; q) = 1$. The former implies that $V(y, m, b^0) \geq X(y, -\bar{m})$ and the latter implies $X(y, -\bar{m}) > V(y, m, b^1)$. But these inequalities imply $V(y, m, b^0) > V(y, m, b^1)$, which contradicts Proposition 1. Hence, $d(y, m, b^0) \geq d(y, m, b^1)$. □

In a one-period debt model, Proposition 2 implies the important result that the equilibrium pricing function is increasing in b', or, equivalently, that more credit is supplied at higher interest rates. In the one-period debt case, $z = 0$ and $\lambda = 1$ and the equilibrium pricing equation (6) reduces to

$$q(y, b') = E_{(y' m')|y}[1 - d(y', m', b')]/[1 + r_f].$$

Since the right-hand side of this equality is increasing in b' by virtue of Proposition 2, it follows that $q(y, b')$ is increasing in b'. But when $\lambda < 1$, Proposition 2 no longer guarantees that $q(y, b')$ is increasing in b'. In particular, we must now take into account how the expected value of the outstanding portion of the unit bond is affected by the choice of b'. The dependence of the pricing function on the debt decision rule is one of the novel aspects of a model with long-term debt. Indeed, inspection of the right-hand side of (6) shows that it will not be possible to say much about how $q(y, b')$ varies with b' unless we know how the debt decision rule $a(y, m, b)$ varies with b. By the same token, it is unlikely we can say much about the behavior of the debt decision rule without knowing how the pricing function behaves with respect to debt. But the two propositions that follow together establish that there exists an equilibrium pricing function that is increasing in b'. Thus, one of the key properties of the one-period debt model continues to hold when bonds last more than one period.

The first (of two) proposition establishes that if the pricing function is increasing in b', the debt decision rule is increasing in b. The result depends on the strict concavity of the utility function. The second proposition establishes that there exists a solution to (6) in the space of functions that are increasing in b'.

Proposition 3: Let $b^1 < b^0$ be two debt levels for which repayment is feasible. If $q(y, b')$ is increasing in b' then $a(y, m, b^1) \leq a(y, m, b^0)$.

Proof: Fix m and y. Denote $a(y, m, b^0)$ by b'^0 and the associated consumption level by c^0. Let $\hat{b}'$ be some other feasible choice greater than b'^0 and let $\hat{c}$ be the associated consumption level. Then, by optimality we have

$$u(c^0) + \beta Z(y, {b'}^0) \geq u(\hat{c}) + \beta Z(y, \hat{b}'). \tag{7}$$

Since $Z(y, \hat{b}') > Z(y, {b'}^0)$ (Proposition 1), (7) implies $c^0 > \hat{c}$. Let $\Delta(b^0) = c^0 - \hat{c} > 0$. Thus, $\Delta(b^0)$ is the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ when the beginning-of-period debt is b^0. From the budget constraint we have that $\Delta(b^0) + q(y, {b'}^0){b'}^0 - q(y, \hat{b}')\hat{b}' = [1-\lambda](-b^0)[q(y, \hat{b}') - q(y, {b'}^0)]$. Holding fixed $\hat{b}'$ and ${b'}^0$, let $\Delta(b^1)$ be the value of Δ that solves $\Delta(b^1) + q(y, {b'}^0){b'}^0 - q(y, \hat{b}')\hat{b}' = [1-\lambda](-b^1)[q(y, \hat{b}') - q(y, {b'}^0)]$. Then $\Delta(b^1)$ is the change in current consumption from choosing $\hat{b}'$ over ${b'}^0$ when the beginning-of-period debt is b^1. Since, by assumption, $[q(y, \hat{b}') - q(y, {b'}^0)] \geq 0$, $b^1 < b^0$ implies $\Delta(b^1) \geq \Delta(b^0)$. Thus the loss in current consumption from choosing $\hat{b}'$ over ${b'}^0$ is at least as large when the beginning-of-period debt is b^1 as compared with b^0. Next, note that since $[\lambda + (1-\lambda)z] > 0$ and $q(y, b') \geq 0$, $b^1 < b^0$ implies $[\lambda + (1-\lambda)z]b^1 + (1-\lambda)q(y, b'^0)b^1 < [\lambda + (1-\lambda)z]b^0 + (1-\lambda)q(y, b'^0)b^0$. Therefore, from the budget constraint it follows that if the beginning-of-period debt is b^1, choosing ${b'}^0$ implies consumption $\tilde{c}$ strictly less than c^0.

To complete the proof, observe that the strict concavity of u implies $u(\tilde{c}) - u(\tilde{c} - \Delta(b^1)) > u(c^0) - u(c^0 - \Delta(b^0)) = u(c^0) - u(\hat{c})$. Therefore, (7) implies that $u(\tilde{c}) + \beta Z(y, {b'}^0) > u(\tilde{c} - \Delta(b^1)) + \beta Z(y, \hat{b}')$. Since $\hat{b}'$ is any feasible b' greater than ${b'}^0$, the optimal choice of b' under repayment when beginning-of-period debt is b^1 cannot be greater than ${b'}^0$. Therefore, $a(y, m, b^1) \leq a(y, m, b^0)$. □

Proposition 4: There exists an equilibrium price function $q^*(y, b')$ that is increasing in b'.

Proof: Let $\bar{q} = [\lambda + [1-\lambda]z]/[\lambda + r_f]$. Then $\bar{q}$ is the present discounted value of a bond with coupon payment z and probability of maturity λ on which there is no risk of default. Let S be the set of all nonnegative functions $q(y, b')$ defined on $Y \times B$ and let $Q \subset S$ be the subset of functions that are increasing in b' and bounded above by $\bar{q}$.

Define the $H(q)(y, b') : Q \to S$ as

$$E_{(y' m')|y} \left[[1 - d(y', m', b'; q)] \frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f} \right],$$

where $d(y, m, b; q)$ and $a(y, m, b; q)$ are the default and debt decision rule, given q. Then H has the following properties:

(i) $H(q)(y, b') \in Q$. Nonnegativity is obvious. We will show that $H(q)(y, b') \leq \bar{q}$. Observe that $\bar{q}$ satisfies the equation $\bar{q} = [\lambda + (1 - \lambda)[z + \bar{q}]]/(1 + r_f)$. Then, since $1 - d(y', m', b') \leq 1$ and $q(y', a(y', m', b'; q)) \leq \bar{q}$ for every (y', m', b'), it follows that

$$\left[[1 - d(y', m', b'; q)] \frac{\lambda + [1 - \lambda][z + q(y', a(y', m', b'; q))]}{1 + r_f} \right] \leq \bar{q} \text{ for every } y', m', b'.$$

Hence $H(q)(y, b') \leq \bar{q}$. Next, we will show that $H(q)(y, b')$ is increasing in b'. Fix y' and m'. Since $q(y, b') \in Q$, $q(y, b')$ is increasing in b' and, by Proposition 3, $a(y', m', b'; q)$ is increasing in b'. Thus, $q(y', a(y', m', b'; q))$ is increasing in b'. And, by Proposition 2, $[1 - d(y', m', b'; q)]$ is increasing in b'. Hence $H(q)(y, b')$ is increasing in b'.

(iii) $H(q)(y, b')$ is continuous in q (see the Appendix for proof).

To complete the proof, note that Q is a compact and convex set. Since $H(q)$ is continuous, by Brouwer's Fixed Point Theorem there exists $q^* \in Q$ such that $q^*(y, b') = H(q^*)(y, b')$. This establishes the existence of an equilibrium price function that is increasing in b'. □

The continuity of the function $H(q)$ depends on the presence of the *continuous* random variable m. The proof in the Appendix makes this point clear and some intuition for why this works is given in the next section. Another way of making H continuous in q is by making y a continuous random variable. But we do not pursue this strategy for two reasons. First, there is no assurance that a solution to the pricing equation exists. With y a continuous variable, $q(y, b')$ is infinite-dimensional object for each b'. Continuity of the operator H is no longer sufficient for existence; H must also be compact, which may not be true for our problem. A second set of issues has to do with computing the solution (assuming that one exists). With continuous y, any solution scheme must involve interpolating the value functions. In our application, consumption decision rules are not continuous, which means that value functions are not differentiable. This creates a problem because interpolation and integration techniques require that the functions being approximated, or integrated, be smooth, – meaning that at least the first derivative (and in some cases higher order ones as well) must exist. Thus, there is no assurance that interpolation techniques will solve our problem accurately.[10] In contrast, as we discuss in the next section, the method pursued in this paper avoids these approximation issues. Given the model environment, our method solves for (potentially discontinuous) decision rules and for equilibrium prices near-exactly.

5 Computation Issues

Computing the equilibrium price function for bonds with maturity greater than one period is challenging. In this section, we discuss the nature of the challenge and how this challenge is met in our paper. The solution procedure is to iterate on (6). That is,

$$q^{k+1} = (1-\zeta)E_{y'|y}\left[[1-d(y',m',b';q^k)]\frac{\lambda+[1-\lambda][z+q^k(y',a(y',m',b';q^k))]}{1+r_f}\right]+\zeta q^k, \quad (8)$$

where $\zeta \in (0,1)$ is the "relaxation" parameter and q^k is the k-th iterate of the price function.

To understand the new computational issues introduced by long-maturity bonds, it is useful to begin with the case of one-period bonds ($z = 0$ and $\lambda = 1$) and no m shocks. In

[10]Whether interpolation techniques work well in practice is an open research question. Hatchondo and Martinez (2009) show that for the one-period debt model, interpolation techniques can deliver accurate results for the model given in Arellano (2008) in that interpolation methods give the same answer as the discrete state space method with a fine grid size. They apply a variant of their method to their long-duration bond model in Hatchondo and Martinez (2008), but they do not compare how well their method performs in solving the pricing equation relative to discrete state space method. Also, it is not known if interpolation methods work well for the empirically relevant parameter space. In Hatchondo and Martinez (2008), the level and volatility of spreads, as well as the level of debt, is quite low relative to the data.

this case, the equilibrium pricing function reduces to

$$q(y, b') - E_{y'|y}[1 - d(y', b'; q)]\frac{1}{1 + r_f} = 0. \tag{9}$$

Since there are no preference shocks, the state variables in the decision rule are simply endowments and beginning-of-period debt. The solution procedure is to iterate on (9). That is

$$q^{k+1}(y, b') = (1 - \xi)E_{y'|y}[1 - d(y', b'; q^k)]\frac{1}{1 + r_f} + \xi q^k. \tag{10}$$

For this algorithm to converge, it is (obviously) necessary that a solution to (9) exists. However, if both y and b are discrete, there is no guarantee that (9) will have a solution because the l.h.s of (9) is typically not continuous in q: For some q and (y, b), the sovereign will be indifferent between default and repayment and an infinitesimal change in q will cause a switch in behavior and, therefore, a discrete change in the expectation on the l.h.s. of (9). If the zero line "intersects" the l.h.s. at one of these points of discontinuities, (9) will fail to have a solution. Of course, exact indifference never happens in the computations but neither are changes in q from one iteration to the next, infinitesimal. The point is that when two options are near-indifferent, very small changes in q^k can cause a discrete change in behavior and, therefore, a discrete change in q^{k+1}.

This problem becomes a more pervasive when debt is long-term. In this case, ignoring m shocks, we iterate on:

$$q^{k+1}(y, b') = E_{y'|y}\left[[1 - d(y', b'; q^k)]\frac{\lambda + [1 - \lambda][z + q^k(y', a(y', b'; q^k))]}{1 + r_f}\right]. \tag{11}$$

Now, the expectation term in (11) depends on the debt decision rule as well. Thus, even if the sovereign is not close to indifference between default and repayment for some (y, b) and would prefer to repay, it may be near- indifferent between two choices of debt. Once again, a small in q^k may cause a discrete change in q^{k+1}.

One might think that making the grid on B finer will solve this problem because, then, near-indifference will be between two very *similar* debt choices (unlike the case of default versus repayment). But this does not work. The budget set under repayment is typically not convex because $q(y, b')$ is a nonlinear function of b'. This means that the points of near-indifference can actually be far apart on the grid. Figure 1 shows a portion of $q(y, b')[b' - (1 - \lambda)b]$ function for the case in which $b = 0$ and $m = 0$ from our quantitative model presented later in the paper. Observe the kink and the ensuing depression in the upward sloping portion of the function. Futhermore, the future expected utility $Z(y, b')$, as a function of b', may also have nonconcave segments because of the possibility of default. Figure 2 displays an example of the variation in total lifetime utility from different choices of b', again for the case in which $b = 0$ and $m = 0$. Observe the many nonconcave segments in this function. These nonconcavities imply that, given (y, b) and q the sovereign may be indifferent between two widely separated values of b'. Thus, an infinitesimal change in q may cause big jumps in the decision $a(y, b; q)$.

In the computations, it is fairly common to get situations in which the sovereign is near-indifferent between two widely separated values of b'. The points marked A (the global maximum) and B (a local maximum) in Figure 2 illustrate this case. When the variance of m is set to 0, it often happens that a point like A is the optimal choice for some iteration k, but when that choice is incorporated in the new price iterate, a point like B becomes the optimum choice for iteration $k + 1$; and when that choice is incorporated in the next price iterate, point A re-emerges as the new optimum. Thus, both the price function and the optimal choice cycle back and forth. This cycling behavior causes convergence of iterative schemes (8) to fail.

Given these difficulties, one approach to solving the problem is to arrange matters so that the jumps in the expectation term on the l.h.s of (8) with respect to q are eliminated. It is for this purpose that we introduce the *continuous* i.i.d variable m. Now the decision rules are functions of y, m, and b. The nonconvexity of the budget set and nonconcavity of the value function continue to imply that the decision rule $a(y, m, b; q)$ is a discontinuous function of q. But as long as the points of discontinuity are finite in number, infinitesimal changes in q will *not* cause jumps in the expectation term (which is now an integral over y' *and* m') since each jump point has probability zero. In this way, a continuous m with a continuous CDF ensures the continuity of r.h.s of the functional equation (6) with respect to q and thereby ensures the existence of a solution.

There is another, perhaps more intuitive, way of viewing the role played by m. We know from general equilibrium theory that nonexistence of an equilibrium resulting from nonconvexities can often be solved by permitting agents to randomize over decisions (the same can be said for the use of mixed strategies in games with finite strategy sets). Based on this, one may surmise that the nonexistence problem in models of debt and default could be solved by allowing the debtor to randomize over repayment and default and over different choices of debt. From this perspective, introducing m is like introducing randomization: There is now a *probability* than an action d or b' is chosen given (y,b) and q.[11]

However, the introduction of m brings its own computational issues. One issue comes from the fact that nonconvexities make $a(y, m, b; q)$ potentially discontinuous in m, given (y, b, q). Since m is a *continuous* variable, it is not obvious how this potentially discontinuous decision rule is to be computed. Interpolation techniques – which assume that the decision rule being approximated is continuous and differentiable – are inapplicable. This is where the assumption that m is i.i.d. plays an important role. This assumption, and the fact that u is strictly increasing and concave, allows us to establish that $d(y, m, b)$ is decreasing in m and $a(y, m, b)$ is increasing in m. Thus the task of computing these decision rules boils down to (i) locating the value of m at which $d(y, m, b)$ switches from 1 to 0 and (ii) the values of

[11] An alternative strategy to prove existence (that does not involve introducing the continuous i.i.d. shock) is to work with decision correspondences (as opposed to decision rules). In this approach, if the decision-maker is indifferent between two (or more) actions then each action is taken with some probability that is determined in equilibrium. The proof of existence of an equilibrium relies on the Kakutani Fixed Point Theorem for compact and convex-valued correspondences. While this approach solves the existence issue, it does not appear to be computationally tractable. In particular, computing mixed strategies when the "support points" of the mixed strategy is not known in advance – and the choice set is very large – seems to be a challenging task.

m at which the $a(y, m, b)$ switches from one debt level to a lower debt level.[12] The decision rule $a(y, m, b)$ is somewhat more challenging to compute because it is not known in advance *which* lower value of debt the sovereign switches to as m increases (because $a(y, m, b)$ may be discontinuous in m, the lower debt level need *not* be the next lower debt level on the grid). However, an algorithm exists (described in Appendix B) that can check for these discontinuities and recover $a(y, m, b)$ correctly. That is, for each (y, b) and q, the algorithm recovers $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ such that b'^K is chosen for $m \in [-\bar{m}, m^{K-1})$, b'^{K-1} is chosen for $m \in [m^{K-1}, m^{K-2})$, $\ldots$, b'^1 is chosen for $m \in (m^1, \bar{m}]$ ($K = 1$ means the same debt level b'^1 is chosen for all $m \in M$). Note that K need not be adjacent to $K - 1$ on the grid.[13]

The second computational issue is that the iteration (8) may fail to converge if the variance of m is too small. As q changes, the thresholds for m change. If the variance of m is small, any given change in thresholds will result in a large change in the choice probabilities. Setting ζ very close to 1 can counteract this sensitivity (by making the change $q^{k+1} - q^k$ itself very small) but at the expense of increasing the time to convergence.[14] Thus, to achieve convergence in a reasonable amount of time, the iteration scheme (8) requires that σ_m not be too small. The lower bound on σ_m is must be determined through experimentation.

Although we cannot prove that there is a unique equilibrium, we have not found instances

[12]Behavior of $d(y, m, b)$ with respect to m is easy to characterize also because of the assumption that the act of default resets m to $-\bar{m}$. This assumption makes the payoff from default independent of m. If the level of transitory income were to remain unaffected by the act of default, the payoff from default, X, would also depend positively on m. As shown in Chatterjee et al (2007), this would result in the default set being characterized by *two* thresholds, m^L and m^U, with default occurring when $m \in (m^U, m^L)$. Since the role of the transitory shock in this paper is to ensure convergence of our numerical algorithm, it is computationally efficient to eliminate the dependence of X on m so that only one default threshold needs to be computed.

[13]It is not possible to apply these methods to a continuous y because y is not i.i.d and, therefore, the debt decision rule is nonmonotonic in y. If current y is above its mean, the price of debt is low, and the sovereign has an incentive to issue more debt. On the other hand, an above-mean y implies that the country will be poorer in the future, which gives the sovereign an incentive to borrow less. These two effects pull in opposite directions and results in nonmonotonic behavior of b' with respect to y. In the absence of monotonicity, it is unclear if an algorithm can be devised to recover (potentially) discontinuous decision rules with respect to a continuous state variable.

[14]We demonstrate the trade-off between the value of σ_m and the value of ζ in achieving convergence in Appendix C. We do this for the model with baseline parameterization but fewer grid points so that convergence can be checked for values of ζ very close to 1.

of multiple equilibria. If the algorithm converges, it converges to the same equilibrium regardless of starting values. We do know, theoretically, that given the price vector $q(y, b')$, and a tie-breaking rule in case of indifference, the decision rules $d(y, m, b)$ and $a(y, m, b)$ are unique (see, Appendix A for details).

We close this section with the statement of the monotonicity properties of decision rules with respect to the continous shock m that are key for our computational method. The proofs of these results are given in Appendix A.

Proposition 5: $a(y, m, b)$ is increasing in m and $d(y, m, b)$ is decreasing in m.

6 Maturity, Indebtedness, and Spreads: The Argentine Case

6.1 Calibration

We apply the framework developed in the previous sections to Argentina. The main contribution is to show that long-duration bonds, besides being a closer fit with reality, can help account for the average level of spreads, the volatility of spreads, and the average level of debt in Argentina without generating counterfactual implications regarding Argentina's business cycle facts. Thus, introducing long-duration bonds into the Eaton-Gersovitz model significantly improves its quantitative performance. We focus on the 8-year period between 1993:Q1 and 2001:Q4 during which Argentina was on a fixed exchange rate vis-a-vis the dollar and was borrowing in international credit markets via marketable bonds.[15]

For the quantitative work we make the following specific functional form or distributional assumptions.

[15]This is also the time period analyzed in Arellano (2008).

- Endowment processes:

$$\ln y_t = \rho \ln y_{t-1} + \epsilon_t, \text{ where } 0 < \rho < 1 \text{ and } \epsilon_t \text{ distributed } N\left(0, \sigma_\epsilon^2\right)$$
$$m_t \text{ distributed } N\left(0, \sigma_m^2\right)$$

- Utility function: $u(c) = c^{1-\gamma}/(1-\gamma)$.

- The loss in the persistent component of output in the event of default or exclusion:

$$\phi(y) = \max\{0, d_0 y + d_1 y^2\}, d_1 \geq 0.$$

The specification for $\phi(y)$ allows for a variety of cost functions. If $d_0 > 0$ and $d_1 = 0$, the cost is proportional to output; if $d_0 = 0$ and $d_1 > 0$, the cost rises more than proportionately with output; if $d_0 < 0$ and $d_1 > 0$, the cost is 0 for $0 \leq y \leq d_0/d_1$ and rises more than proportionately with output for $y > d_0/d_1$. This last case resembles the cost function in Arellano (2008).[16] The reasons for choosing this flexible form are discussed in the findings section.[17] With these assumptions, the numerical specification of the model requires giving values to 11 parameters. These are (i) three endowment process parameters, σ_m, ρ and σ_ϵ^2; (ii) two preference parameters, β and γ; (iii) two parameters describing the bond, the maturity parameter λ, and the coupon payment z; (iv) two default output loss parameters, d_0 and d_1, (v) the probability of re-entry following default, ξ; and (vi) the risk-free rate r_f.

The parameters of the endowment process are estimated on linearly detrended quarterly real GDP data for the period 1980:1-2001:4.[18] As noted earlier, convergence of (8) requires that the standard deviation of the i.i.d. shock m be not too low. Experimentation shows

[16] In Arellano, $\phi(y) = \max\{0, y - \bar{y}\}$. Thus, cost is 0 for $0 \leq y \leq \bar{y}$ and rises linearly at rate 1 beyond $\bar{y}$. Thus, default costs as a proportion of y, namely, $(1- \bar{y}/y)$, increase strongly with y.

[17] With this specification, the cost can exceed y for large y This situation never arises in our application but could be formally ruled out by setting $\phi(y) = \min\{y, \max\{0, d_0 y + d_1 y^2\}\}$.

[18] The quarterly data series on real GDP, real aggregate consumer expenditure, real exports, real imports and the (nominal) interest rate on Argentine sovereign debt is taken from Neumeyer and Perri (2005). All the quantity variables were deseasonalized using the multiplicative X-12 routine in EViews.

that $\sigma_m = 0.003$ is a good lower bound for our purposes – meaning that convergence is achieved within a reasonable time frame for a wide range of parameter values. Thus, the endowment process is estimated assuming that $\sigma_m = 0.003$.[19] The estimated value of ρ and σ_ϵ are 0.948503 and 0.027092, respectively.[20] In the computation, we truncate the support of the m distribution to $[-\bar{m}, \bar{m}]$,where $\bar{m} = 3\sigma_m = 0.009$. Of the preference parameters, the value of the γ is set equal to 2, which is the standard value used in this literature.

The parameters describing the bond were determined to match the maturity and coupon information for Argentina reported in Broner, Lorenzoni, and Schmukler (2007). The average coupon rate is about 12 percent per annum, or 0.03 per quarter, and the median maturity of Argentine bonds is 5 years or 20 quarters. Thus, $z = 0.03$ and $\lambda = 1/20 = 0.05$. In the post-1980 era, Argentina defaulted on its loans in 1982, and it took until 1993 to regain access to international borrowing. Following the 2001 default, access was regained in early 2005. Based on this experience, we set $\xi = 0.0385$, which gives an average period of exclusion of 26 quarters or 6.5 years. The risk-free rate, r_f, was set at 0.01, which is roughly the real rate of return on a 3-month (one quarter) U.S. Treasury bill.

The three remaining parameters β, d_0, and d_1 are selected to match (i) average external debt-to-output ratio of 0.7, which is 70 percent of the average external debt-to-output ratio for Argentina over the period 1993Q1-2001:Q4; (ii) the average default spread over the same period of 0.0815; and (iii) the standard deviation of the spread of 0.0443.[21] We seek to match

[19] Since m is transitory variation in the log of income, the level of output is $y \times \exp(m)$. In the numerical solution of the model, we approximate this by $y + m$ which is the first order Taylor expansion of $y \times \exp(m)$ around y=1 and m=0. By doing this we are adding slightly less i.i.d. variability to the level of income than implied by our process for log of income.

[20] If the process is estimated without the transitory shock, the estimates of ρ and σ_ϵ are 0.930139 (0.038395) and 0.027209 (0.001577), respectively, where the values in parenthesis are standard errors. Note that the values of ρ and σ_ϵ used in the calibration are well within 1 standard deviation of these AR1 estimates and statistically indistinguishable from them. Second, adding m to the AR1 equation is equivalent to assuming that log GDP is measured with some noise. Since the standard deviation of log GDP in the sample is 0.076107, setting $\sigma_m = 0.003$ implies that σ_m^2 is 0.16 percent of the variance of log GDP. This is small compared with standard deviation of measurement errors assumed in estimation of DSGE models (see, for instance, Ireland (2004); see Del Negro and Schorfheide (p. 53, 2010) for a discussion of this practice).

[21] Debt is total long-term public and publicly guaranteed external debt outstanding and disbursed and owed to private and official creditors at the end of each year, as reported in the World Bank's Global Development Finance Database (series DT.DOD.DPPG.CD). The average debt-to-output ratio is

only a portion of debt because we do not model repayment. In reality, sovereign debt that goes into default eventually pays off something. In Argentina's case, the repayment on debt defaulted on in 2001 has been around 30 cents to the dollar. Thus, we treat only 70 cents out of each dollar of debt as the truly unsecured portion of the debt. But, as part of our sensitivity analysis, we also report results for the case in which we match average external debt-to-output ratio fully.

Finally, we need to specify the model analogs of the external debt-to-output ratio and spreads. In the GDF database, the external commitments of a country are reported on a cash-accounting basis which means that commitments are recorded at their face value, i.e., they are recorded as the undiscounted sum of future promised payments of principle.[22] The coupon payments agreed to do not figure directly in this accounting because they are not viewed as obligations until they are past due. Given this valuation principle, the model analog of debt as reported in the data is simply b, and the external debt-to-output ratio is simply b/y.[23] The default spread in the model is calculated as in the data. Given the unit price $q(y,b)$ of the outstanding bonds, we compute an internal rate of return, $r(y,b')$, which makes the present discounted value of the promised sequence of future payments on a unit bond equal to the unit price, that is, $q(y,b') = [\lambda + (1-\lambda)z]/[\lambda + r(y,b')]$. The difference between $(1 + r(y,b'))^4 - 1$ and $(1 + r_f)^4 - 1$ is the annualized default spread in the model.

The parameter selections are summarized in the following two tables. Table 1 lists the values of the parameters that are selected directly without solving for the equilibrium of the model. Table 2 lists the parameter values that are selected by solving the equilibrium of

the average ratio of debt to GNP measured at a quarterly rate. The spread was calculated as the difference between the interest rate data reported in Neumeyer and Perri (which is the same as the EMBI data) and the 3-month T-bill rate. The T-bill rate series used is the TB3MS series available at `http://research.stlouisfed.org/fred2/categories/116`. Both the interest rate data and the T-bill rate are reported in annualized terms.

[22]See "Coverage and Accounting Rules" in Section 3 of World Bank Statistical Manual on External Debt (also available at http://go.worldbank.org/6FB4093970).

[23]The reason for this is that each bond can be viewed as combination of unit bonds with varying maturities. For instance, a measure λ of unit bonds is due next period, a measure $(1-\lambda)\lambda$ is due in 2 periods, ..., a measure $(1-\lambda)^{j-1}\lambda$ is due in j periods, and so on. Since each of these obligations has a face value of 1, each would be recorded as a unit obligation. Thus, the total obligation is simply $\sum_{j=1} \lambda(1-\lambda)^{j-1} = 1$.

the model and choosing the parameters so as to make the model moments come as close as possible to the three data moments mentioned above.

Table 1: Parameters Selected Directly

Parameter	Description	Value
γ	risk aversion	2
$\bar{m}$	upper bound on m	0.009
σ_m	standard deviation of m	0.003
σ_ϵ	standard deviation of ϵ	0.027092
ρ	autocorrelation	0.948503
ξ	probability of reentry	0.0385
r_f	risk-free return	0.01
λ	reciprocal of avg. maturity	0.05
z	coupon payments	0.03

Table 2: Parameters Selected by Matching Moments

Parameter	Description	Value
β	discount factor	0.95460
d_0	default cost parameter	-0.18845
d_1	default cost parameter	0.24559

6.2 Findings

The results of the moment matching exercise are reported in Table 3. The top row reports the data for Argentina. The second row reports the moments in the model. All model moments are (sample) averages calculated by simulating the economy over many periods but always discarding the first 20 periods after re-entry following each default.[24] Table 3 shows that the matching exercise is fully successful (the parameters values for which this matching is obtained are reported in the first column).[25] Figure 3 compares the simulated spreads in the model with the data. Although the match is not perfect, the two series track each other reasonably closely.

For comparison purposes, the third row of Table 3 reports the corresponding model statistics of Arellano's one-quarter debt model. It is important to point out that Arellano did not target these statistics, nevertheless, the fact remains that there are significant deviations between these statistics in her model and the data. In particular, the model's debt-to-output ratio is very low compared with the data and the average spread is about half.

Table 3: Results and Comparison

	Avg. Spread	SD Spread	Debt-to-Y
Data	0.0815	0.0443	1.00
Baseline $(\beta, d_0, d_1) = (0.95, 0.19, 0.25)$	0.0814	0.0444	0.70
Arellano (2008, Table 4, p.706)	0.0358	0.0636	0.06

Table 4 reports some key cyclical properties of Argentine data and corresponding model moments.[26] No attempt was made to target these data moments so the degree of correspondence between data and model is informative about model performance.

[24] We do this because the model economy re-enters capital markets without any debt, whereas Argentina emerged from each default/restructuring episode with debt. By ignoring the first five years following re-entry, we ignore years with counterfactually low debt in our model.

[25] For the record, the average debt-to-output ratio in the baseline model when debt is measured at its market value is 0.703. So, it is only slightly higher than its face value. The reason is that the average interest rate on debt, 0.0292 percent per quarter, is only slightly larger than the 0.0285 $(= (1-\lambda)z = 0.95 \times 0.03))$ coupon payment on each unit of debt.

[26] Second moments for consumption and output were computed using logged and linearly de-trended series. Since net exports (NX) can be negative, it was expressed as proportion of output and then linearly de-trended. The spread series was also linearly detrended, although the trend component is negligible.

Table 4: Cyclical Properties, Data and Models

Variable	Data (93Q1-01Q4)	Model	Arellano (2008)
$\sigma(c)/\sigma(y)^*$	1.09	1.11	1.10
$\sigma(NX/y)/\sigma(y)$	0.17	0.20	0.26
$corr(c,y)^*$	0.98	0.99	0.97
$corr(NX/y,y)$	-0.88	-0.45	-0.25
$corr(r-r_f,y)$	-0.79	-0.67	-0.29
Avg. Debt Service$^{\#}$	0.053	0.055	0.056
Default Freq**	0.125	0.066	0.030
*Sample period: 1980:Q1-2001:Q4; **Sample period:1975-2001			
$^{\#}$ Principal and interest payments as a fraction of output			

The first column of numbers are the data for Argentina. Several features of the data stand out. First, the relative volatility of consumption is about the same as output; second, the trade balance is strongly countercyclical – net exports (denoted NX) decline during periods of above-trend output and rise during periods of below-trend output; third, spreads on sovereign debt are countercyclical and, finally, Argentina displayed a high propensity to default during the 1975-2001 period.[27] The following column reports the same statistics for the model. The model gets the qualitative patterns of the data right: Model consumption and trade balance have about the right level of volatility relative to output, and the trade balance and spreads are countercyclical while consumption is highly procyclical. The forces in the model that lead to these patterns are the ones emphasized in Aguiar and Gopinath (2006) and Arellano (2008). When output is below trend, the probability of default on new

[27]The frequency of default is the number of default episodes as a fraction of the number of years Argentina was in good standing with international creditors in the 27 years between 1975-2001. Argentina defaulted in August 1982 and re-gained access in March 1993. We assume that Argentina was in default for 11 years. Thus it defaulted twice in a 16-year period of good standing. We chose 1975 as the start date because that is when Argentina began accumulating significant amounts of debt. If we start in 1946 and use the“years in restructuring” reported in Beim and Calomiris (2001, p. 32), Argentina would show three defaults in a 35-year period of good standing. This would give a default frequency of 0.086. If we start in 1800 and use Beim and Calomiris again, the default frequency would drop to 0.03. But it is questionable if a model with an unchanging output process and default cost function is right framework to address such a long sweep of history.

loans rises. If this rise is sharp enough, it is optimal for the sovereign to *reduce* debt rather than to increase it (which is what would be optimal holding interest rates constant). Thus, there is a tendency for consumption to decline more than the decline in output and for the trade balance to improve with a fall in output. The average probability of default in the model is lower than the observed frequency of default; but default is a rare event and it is hard to estimate its frequency accurately from relatively short data series. The final column reports the results for the benchmark model in Arellano (2008, Table 4). Even ignoring the frequency of default, the model with long-term debt clearly comes closer to accounting for the data moments. In particular, the model predicts a lower volatility of net exports and a stronger negative correlation between output and net exports and between output and spreads than the one-period debt model.

6.2.1 Role of the Default Cost Function

Recall that our specification for the default cost function is $\phi(y) = \max\{0, d_0 y + d_1 y^2\}$. Since the calibrated values d_0 and d_1 are negative and positive, respectively, our specification shares the feature that Arellano introduced in her specification of default costs, namely, that the default cost as a proportion of output declines with output and becomes zero for low enough output levels.

It is now well-understood that this structure of default punishment is important in generating higher default rates, whether the default cost function is endogenous or exogenous (see, for instance, the discussion in Mendoza and Yue (2009)). The key is the asymmetry in default costs: The country is punished much more severely for default when income is high than when income is low. The severe punishment for default in high-income periods implies that investors do not expect the sovereign to default in the near future (given the persistence in output). This results in low spreads, and the (impatient) sovereign borrows aggressively. But when output declines, the punishment from default declines as well. This raises the likelihood of default and spreads rise. The high spreads make debt servicing more onerous, and, eventually, if income stays low, the sovereign defaults. Without the asymmetry, it is impossible to generate a significantly positive default frequency without making the sovereign very impatient. For instance, with a proportional default cost, the cost does not vary much with the level of output and spreads remain relatively high over a wide range of output and debt levels. Consequently, the sovereign rarely borrows enough to enter into regions where the probability of default is measurably positive (unless the sovereign is very impatient).

What appears not to have been appreciated in the literature is that the structure of default costs is also important for the *volatility* of spreads. For Arellano's specification, the default cost as a proportion of output is $1 - \bar{y} \cdot y^{-1}$ (where $\bar{y}$ is the level of output below which costs are zero), which is very sensitive changes in y. Consequently, the probability of default is correspondingly sensitive to fluctuations in y and so is the spread. In contrast, with a proportional cost of default, there is very little sensitivity of the probability of default (and spreads) to variations in y. Given this, we adopted a "hybrid" default cost function that can potentially match both the level *and* the volatility of spreads. As noted earlier, $\phi(y) = 0$ for output levels below $(-d_0/d_1)$ and $\phi(y)/y = d_0 + d_1 y$ for y greater than $(-d_0/d_1)$. If $d_1 > 0$, default costs as a proportion of output rises with y, which makes the probability of default more sensitive to output than the proportional cost case and increases the volatility of spreads. Indeed, the larger d_1 is, the more volatile the spreads are likely to be. This intuition is verified in the table below, which shows the results of varying d_1 while choosing d_0 and β to match the targets for average debt-to-output ratio and the average spreads. Notice that the volatility of spreads rise with d_1. The baseline value of d_1 is such that the volatility of spreads matches that in the data.

Table 5: Role of Sensitivity of Default Cost to Output on Volatility of Spreads

d_1	d_0	β	$\sigma(r - r_f)$	avg. $(r - r_f)$	avg. b/y
0.15	-0.098	0.93753	0.0271	0.0816	0.70
0.25 (baseline)	-0.188	0.95460	0.0444	0.0815	0.70
0.35	-0.288	0.96217	0.0585	0.0815	0.70

To recap, the higher d_1 is, the more sensitive spreads are to variation in output and the easier it is for the model to achieve a higher frequency of default; the country borrows strongly in high output and low spread times and then gets caught with high debt and high spreads when output drops. Since higher default frequency and high spreads are easier to achieve with a higher d_1, the sovereign needs to be less *impatient* in order for it to willingly hold debt that implies positive probability of default. This explains why the value of β rises with d_1. We also see that d_0 falls (becomes more negative) as d_1 rises. This is because an increase in d_1 shifts up the default cost function, which expands the maximum amount of debt the sovereign can carry without defaulting. As long as β is sufficiently less than $1/(1 + r_f)$, the sovereign will gravitate to this maximum and that will increase the average level of debt. To keep the average debt level constant, the overall default punishment should remain roughly constant. Thus, d_0 falls to counterbalance the increase in d_1.

Overall, what we can say is that level and volatility of spreads are mostly determined by the interplay between β and how the cost of default is distributed between d_0 and d_1, while the the level of debt is determined by the overall costs of default.

6.2.2 Role of Long-Term Debt

In this section, we explain the contribution of long-term debt in our model. One way to understand this contribution is to compute the equilibrium of the baseline model with short-term debt. The results of this exercise are shown in the last column in Table 5. The equilibrium has stark differences. The average spread, the volatility of spreads, and the default frequency are miniscule compared with the long-term bond case.

Table 6: Role of Long-Term Debt

Moment	Data	Baseline	Recalibrate w/ $\lambda = 1$	Baseline w/ $\lambda = 1$
Avg. $(r - r_f)$	0.0815	0.0815	0.0814	0.0027
$\sigma(r - r_f)$	0.0443	0.0443	0.0443	0.0041
Avg. b/y	1	0.70	0.70	0.81
$\sigma(c)/\sigma(y)$	1.09	1.11	1.53	1.14
$\sigma(NX/y)/\sigma(y)$	0.17	0.20	0.99	0.93
$corr(c, y)$	0.98	0.99	0.75	0.95
$corr(NX/y, y)$	−0.88	−0.45	−0.15	−0.24
$corr(r - r_f, y)$	−0.79	−0.67	−0.48	−0.40
Debt Service	0.053	0.055	0.698	0.812
Def Freq	0.075	0.066	0.073	0.002

This raises the question as to why a sovereign borrowing long term willingly extends its borrowing into the region where the probability of default is significantly positive but chooses not to do so when borrowing short term. The answer lies in the differing incentives to issue additional debt in the two cases. Although this choice is discrete in the model, we may think of it as being continuous for the moment. Then, the marginal gain from borrowing is given by:

$$\left(-q(y,b') - \frac{\partial q(y,b')}{\partial b'}\left[b' - (1-\lambda)b\right]\right) u'\left(y + m + [\lambda + (1-\lambda)z]b - q(y,b')\left[b' - (1-\lambda)b\right]\right) \tag{12}$$

To understand the expression, it is easier to think of the sovereign as a "monopolistic" supplier of bonds. When the sovereign issues an extra unit of debt, it gets revenue from that extra unit but faces a decrease in the price of the bond, which decreases the revenue on all bonds being currently issued. In the case of short-term debt, the decrease in price applies to the entire stock of debt b', whereas with long-term debt it applies to $[b' - (1-\lambda)b]$. Thus, the sovereign faces a much greater disincentive to borrow when default probabilities become positive in the short-term case. Consequently, it rationally chooses to not extend borrowing into this region.

There is a second reason why the sovereign enters more readily into regions where default probability is positive. Figures 4 and 5 plot how spreads and the default probability vary with debt in the two cases. Observe that the spread is less sensitive to changes in long-term debt as compared with short-term debt. Spreads start out positive with long-term bonds and rise at a slower rate.[28] Although debt levels are not perfectly comparable across the two cases (they involve different future obligations), the spreads for short-term bonds rise at a much faster rate once the sovereign enters into positive default probability regions. Put differently, $|\partial q(y, b')/\partial b'|$ is larger for short-term debt in the region where default probabilities are positive. This creates a further disincentive to borrow short term. In sum, with short-term debt, the sovereign borrows as much as it can at the risk-free rate, but once it encounters the rising spreads portion of the pricing function, it chooses to borrow very little more. For comparison, Figure 5 shows the revenue function $q(y, b')(-b')$ for the long-term and the short-term debt. Even though the spread is higher for long-term debt, the revenue function for long-term debt is higher than for short-term debt because each unit of long-term debt fetches a higher price than each unit of short-term debt.

The above exercise might make one think that it would be impossible to match the spread and debt levels with short-term debt. We see, in fact, that is not the case. When

[28] The reason is that even if the sovereign borrows a very small amount in the current period (so the default probability for next period is 0), lenders understand that the sovereign's optimal decision *next* period is to take on a significant amount of debt. And since $q(y, b')$ is decreasing in b', lenders rationally expect to suffer a capital loss on the nonmaturing portion of the debt. This depresses the current price of debt and leads to a positive spread from the start. And, initially, the spread rises with debt simply because the $a(y', m', b')$ is increasing in b' (Proposition 3), and the expected capital loss is increasing. This shows that it is not necessary to invoke risk-aversion on the part of lenders to account for gaps between spreads and default probabilities. With long-term debt, a gap can arise (and vary) because of the dynamics of debt accumulation. A gap can also arise also if there is repayment on defaulted debt, which we have ruled out.

we permit (β, d_0, d_1) to adjust, we can match all three moments perfectly with short-term debt. The next-to-last last column in Table 5 shows the result of this matching exercise. But the matching creates some major anamolies. The parameter vector that achieves the match is $(\beta, d_0, d_1) = (0.67, -0.46, 0.57)$. The discount factor is very low – arguably outside the plausible range. But the reason why this vector succeeds in delivering a match is quite intuitive. We had seen that with $(\beta, d_0, d_1) = (0.96, -0.19, 0.25)$, the sovereign almost never enters the region of positive default probability. To get the sovereign to enter into this region, d_1 must be raised so that default cost is asymmetric in current output, the sovereign must be made more impatient, and, to keep the over all cost of default well contained, d_0 must rise as well.

Setting aside the implausible β, the more important anamolies are with respect to business cycle statistics. The relative volatilities of consumption and trade balance are now much higher than the baseline model (as well as the data). Some clue as to why this is happening can be gleaned from the previous experiment (the last column in Table 5) where, even with so little borrowing in the rising portion of the spread curve, the volatility of trade balance is almost as high as that of output, and the volatility of consumption is close to the baseline model as well as the data.

In different ways, these two experiments highlight the fundamental problem with a short-term debt model. Basically, short-term debt makes consumption and NX more volatile when interest rates on new loans can fluctuate with the state of the economy. If there are b dollars of debt outstanding, the debt service obligation in the case of long-term debt is only $[\lambda + (1 - \lambda)z]b$, and the sovereign can maintain its debt level at b by simply refinancing the quantity λb at the new price $q(y, b)$. In contrast, with short-term debt, the debt service obligation is b, and the sovereign must refinance b at the new price $q(y, b)$ to maintain its debt level. Thus, any change in $q(y, b)$ resulting from a change in y will mean a much bigger change in consumption and trade balance for short-term debt than for long-term debt. In reality, sovereigns do not have to refinance their entire debt each period. Once this basic fact is taken into account, the model becomes capable of accounting for both the levels and the volatility of key macroeconomic variables.

6.3 Sensitivity Analysis

Table 6 reports the results of three sensitivity analyses. In the first exercise, denoted Model I, the full average debt level of 1.0 is targeted. The model can successfully match all targets. There are some differences in the results. There are increases in the volatility of consumption and NX, and a measurable increase in the debt service. These increases are what we would expect for a higher average debt burden. The correlation patterns remain the same. Over all, model performance is somewhat inferior to the baseline model.

In Model II, we address one potential concern regarding the assumption about m in the period of default. Recall that we assumed that in the period of default, the value of m resets to $-\bar{m}$. This means that there is an additional source of punishment for default, and one may wish to know if this plays any role in the results. In model II, we assume that in the period of default the value of m resets to 0 instead of $-\bar{m}$ – which might be viewed as a more neutral assumption. As is evident, there is virtually no difference in results between the baseline model and this one.

In the third sensitivity analysis, we examine if the results change with a lower standard deviation for m. We re-estimated the endowment process under the assumption that $\sigma_m = 0.002$, which is the lowest σ_m value for which we get convergence for our baseline model. The implied estimates of ρ and σ_ϵ are as reported in the table. As one would expect, ρ is somewhat lower, and σ_ϵ somewhat higher than in the baseline model. However, these changes in the endowment process have virtually no effect on model statistics.

Table 7: Sensitivity Analyses I

Moment	Baseline	Model I	Model II	Model III
Avg. $(r - r_f)$	0.0815	0.0815	0.0815	0.0815
$\sigma(r - r_f)$	0.0443	0.0448	0.0448	0.0443
Avg. b/y	-0.7	-1.001	-0.7	-0.7
$\sigma(c)/\sigma(y)$	1.11	1.15	$1,11$	1.11
$\sigma(NX/y)/\sigma(y)$	0.20	0.27	0.20	0.20
$corr(c, y)$	0.99	0.98	0.99	0.99
$corr(NX/y, y)$	-0.45	-0.45	-0.45	-0.45
$corr(r - r_f, y)$	-0.67	-0.65	-0.67	-0.67
Debt Service	0.055	0.078	0.055	0.055
Def Freq	0.066	0.067	0.067	0.067
Model I: Average b/y target $= 1.0$				
Model II: Same targets as baseline but in period of default, m resets to 0				
Model III: Same targets as basline with $\rho = 0.948081$,$\sigma_\epsilon = 0.027203$,$\sigma_m = 0.002$				

Table 7 reports a second set of sensitivity analysis. It is well known that model statistics in the Eaton-Gersovitz model can be sensitive to the choice of grid sizes (see, for instance, Hatchondo and Martinez (2009)). We increase the grid sizes on Y and B, separately, by 50 percent relative to the baseline. The only statistic that changes somewhat is the average spread, which falls slightly when the grid on Y is increased. All other statistics are basically unaffected.

Table 8: Sensitivity Analysis II

Moment	Baseline	Model IV	Model V
Avg. $(r - r_f)$	0.0815	0.0800	0.0815
$\sigma(r - r_f)$	0.0443	0.0441	0.0444
Avg. b/y	-0.7	-0.7	-0.7
$\sigma(c)/\sigma(y)$	1.11	1.11	1.11
$\sigma(NX/y)/\sigma(y)$	0.20	0.20	0.20
corr (c, y)	0.99	0.99	0.99
corr (NX, y)	-0.45	-0.44	-0.45
corr $(r - r_f, y)$	-0.67	-0.66	-0.67
Debt Service	0.055	0.055?	0.055?
Def Freq	0.066	0.067	0.067
Baseline: $N_y = 50$, $N_b = 350$			
Model IV: $N_y = 75$, $N_b = 350$			
Model V: $N_y = 50$, $N_b = 525$			

7 Welfare and Maturity Choice

In this section, we examine the welfare effects of shortening maturity length. The results of this exercise motivate our exploration of a version of our model with a choice of two different maturity levels for new debt each period.

We compare the welfare effects in the baseline model of moving from one-period debt ($\lambda = 1$) to long-term debt ($\lambda = 0.05$). The comparison is done in two ways. In the first comparison, we compute $V_\lambda(1, 0, 0)$. In the second comparison, we compute $\Sigma_y V_\lambda(y, 0, 0)\Pi(y)$, where $\Pi(y)$ is the invariant distribution of the Markov chain for y. Thus, in both cases, we assume that the initial debt is 0 and the value of m is at its mean value of 0. Rather than report utilities, we report the perpetual constant flow of consumption that gives the same level of lifetime utility (certainty equivalent consumption) The results are reported in Table 9.

Table 9: Welfare Comparison Across Maturity Length

quarters to mat	λ	CE cons (1)	CE cons (2)	avg. spread	avg.b/y	def freq
1	1	1.020497253	1.016985118	0.0026	0.81	0.0024
2	0.5	1.020482885	1.016966606	0.0050	0.81	0.0047
4	0.25	1.019941300	1.016421132	0.0106	0.79	0.0100
6	0.167	1.019077219	1.015554751	0.0173	0.76	0.0161
8	0.125	1.018064597	1.014541280	0.0249	0.74	0.0228
10	0.1	1.016959172	1.013435222	0.0337	0.73	0.0303
12	0.083	1.015889466	1.012368509	0.0427	0.71	0.0379
14	0.071	1.014864264	1.011344013	0.0523	0.70	0.0456
16	0.063	1.013883872	1.010364402	0.0629	0.70	0.0538
18	0.056	1.013039754	1.009518194	0.0726	0.70	0.0610
20	0.05	1.012317490	1.008792999	0.0815	0.70	0.0675

For both comparisons, welfare is highest for short-term debt and declines monotonically as λ falls toward to 0.05. Thus, according to these measures, the sovereign is better off issuing short-term (one quarter) debt. For the first comparison, the difference in consumption equivalent in going from 20 quarter maturity to 1 quarter maturity is 0.81 percent, which is significant by the normal standards of welfare comparisons. The comparable gain for the second comparison is also 0.81 percent.

If the sovereign can commit to never default, the implicit one-period interest rate on short-term and long-term debt would be the risk-free rate. Then, the length of maturity of debt would make no difference to welfare or consumption. Evidently, the possibility of default makes a difference. But what, exactly, underlies this preference for short-term debt when there is a risk of default?

7.1 Role of Commitment

In this subsection we show that if lenders insist that the sovereign compensate them for declines in the market value of outstanding debt and, conversely, the sovereign insists that the lenders compensate it for improvements in the value of outstanding debt, long-term debt becomes equivalent to short-term debt *even in the presence of the default risk*. In what follows, we first derive this equivalence result and then discuss what lessons can be learnt from it.

For the purposes of this section, it is not important to include the iid income shock. Then, the value of default is:

$$X(y) = u(y - \phi(y)) + \beta E_{y'|y}\{(1-\xi)X(y') + \xi W(y', 0)\}.$$

And the value of repayment is:

$$V(y_-, y, b) = \max_{b'} \left\{ \begin{array}{c} Ru\left(y + \left[\lambda + (1-\lambda)\left(z + q(y_-, b)\right)\right] b - q(y, b')b'\right) \\ +\beta E_{y'|y}\left[\max\left\{V(y, y', b'), X(y')\right\}\right] \end{array} \right\}$$

where y_- is the realization of income in the previous period. Observe that the portion of the bond that does not mature in the current period pays off the coupon z and its *last period* market value as opposed to its *current* market value. This is equivalent to the sovereign transferring $[(q(y_-, b) - q(y, b')](1-\lambda)b$ to the lenders each period (the sovereign pays if this quantity is negative and receives if it is positive).

Denote the decision rules by $d(y_-, y, b)$ and $a(y_-, y, b)$. Then the equilibrium price of a unit bond is given by:

$$q(y, b') = \left[\lambda + (1-\lambda)\left(z + q(y, b')\right)\right] \left(\frac{E_{y'|y}(1 - d(y, y', b'))}{1+r}\right) \tag{13}$$

We can now do a change of variables that will allows us to re-write the above problem in terms of only two state variables. The key insight is that the default decision should depend only on y and the *total* obligation of the sovereign at the start of each period, which is $[\lambda + (1-\lambda)(z + q(y_{-1}, b))]\, b$. With this in mind, define this total obligation by $A = [\lambda + (1-\lambda)(z + q(y_{-1}, b))]\, b$ and suppose that the default decision rule can be expressed as a function $d(y, A)$. Next, notice that multiplying both sides of (13) by b' gives:

$$q(y, b')b' = [\lambda + (1-\lambda)(z + q(y, b'))]\, b' \left(\frac{E_{y'|y}(1 - d(y, y', b'))}{1+r} \right)$$

Or,

$$q(y, b')b' = \left(\frac{E_{y'|y}(1 - d(A', y'))}{1+r} \right) A' = \tilde{q}(y, A')A'.$$

Thus, we can re-write the value of repayment as follows:

$$V(y, A) = \max_{A'} \{u(y + A - \tilde{q}(y, A')A') + \beta E_{y'|y} \max\{V(y', A'), X(y')\}\,. \tag{14}$$

This repayment value is exactly the same as the one in which the sovereign issues one-period debt. From the solution to this one-period debt problem (namely the decision rules $d(y, A)$ and $A'(y, A)$) we can recover both the decision rules and the price function of the original long-term debt problem. Observe that (i) $q(y, b')b' = A' \left(E_{y'|y}(1 - d(y', A'))/(1+r)\right)$ and (ii) $A' = [(1-\lambda) + \lambda (z + q(y, b'))]\, b'$. Using (i), we can solve for b' from (ii): $A' = [(1-\lambda) + \lambda z]b' + A' \left(E_{y'|y}(1 - d(y', A'))/(1+r)\right)$. This gives b' as a function of y, y_{-} and b, since A' is a function of y and A. Then, using this solution, we can solve for $q(y, b')$ from (i). Thus, with this market arrangement, long-term debt is isomorphic to one-period debt.[29]

If freezing the value of future outstanding debt at its current market value makes long-term and short-term debt equivalent, the inferiority of long-term debt evident in Table 9 must stem from the fact that the market value of outstanding debt can change over time. There are two effects at work. First, with the future value of outstanding debt fixed at its value at issue, the sovereign cannot dilute the value of future outstanding debt by issuing more debt

[29]Strictly speaking it needs to be proven that the decision rule and price function recovered in this way actually solve the long-term debt problem. For the sake of brevity we omit this proof.

in the future. Thus, its inability to commit to not borrow more in the future no longer affects the market value debt in the current period. In other words, this arrangement solves the debt dilution problem. This reduces the interest rate on debt and is welfare improving. On the other hand, the future market value of debt can change due to a change in y. These exogenous fluctuations in the market value of outstanding debt lead to corresponding fluctuations in disposable income (when the market value falls, the sovereign must make lenders whole against the decline and when it rises the sovereign receives a payment from lenders) and, given that the sovereign is risk averse, is potentially welfare reducing. Evidently, for our calibration of the model, this welfare reducing effect is simply not as strong as the welfare increasing effect and long-term debt with value protection (which is equivalent to short-term debt) is the preferred arrangement.

7.2 Maturity Choice

These welfare comparisons leave open the question whether there are circumstances (in terms of initial debt and output) for which long-term debt might actually be the prefered maturity. To explore this possibility, we modified our baseline model to allow for a choice between debt of two different maturities each period. The modified environment is as follows. There are two types debt, denoted b_S and b_L, with maturity and coupon combinations of (z_j, λ_j), where $\lambda_S > \lambda_L$. Thus b_S is short-term debt while b_L is long-term debt. We assume that the sovereign can adjust the quantity of one of these bonds each period.[30] Let

$$\mathbf{b} = \begin{bmatrix} b_S \\ b_L \end{bmatrix} \text{ and } \mathbf{b}' = \begin{bmatrix} b_S' \\ b_L' \end{bmatrix} \text{ and } P = \begin{bmatrix} \lambda_S + (1-\lambda_S)z_S & 0 \\ 0 & \lambda_L + (1-\lambda_S)z_L \end{bmatrix}$$

[30]Simultaneous adjustment of the stock of both types of debt is ruled out only for computational reasons. Since we solve the model via grid search, permitting changes in both stocks simultaneously increases the dimesionality of the choice set greatly. Note also that restricting the issuance to only one type debt each model period is less binding if the model period is shortened. For example, if the model period is shortened to 6 weeks, the sovereign can issue different types of debt twice a quarter.

If the sovereign chooses to repay and adjust the quantity of maturity j bonds, the lifetime utility is $V_j(y, m, \mathbf{b})$. If the sovereign chooses to default, it defaults on all its debt, and the lifetime utility from default is $X(y, -\bar{m})$ and from exclusion $X(y, m)$. The unconditional lifetime utility of a sovereign in good standing is $W(y, m, \mathbf{b})$ The Bellman equations for this environment are as follows.

$$\begin{aligned} & V_j(y, m, b) = \max_{b'_j \in B} u\left(c\right) + \beta E_{(y'\, m')|y} W(y', m', \mathbf{b}') \\ & \text{s.t.} \\ & c = y + m + P \cdot \mathbf{b} - q_j(y, \mathbf{b}') \left[b'_j - [1 - \lambda_j] b_j\right] \\ & b'_{-j} = (1 - \lambda) b_{-j} \end{aligned} \tag{15}$$

$$\begin{aligned} & X(y, m) = u(c) + \beta\{[1 - \xi] E_{(y'\, m')|y} X(y', m') + \xi E_{(y'\, m')|y} W(y', m', \mathbf{0})\} \\ & \text{s.t.} \\ & c = y - \phi(y) + m \end{aligned} \tag{16}$$

and

$$W(y, m, \mathbf{b}) = \max\{V_S(y, m, \mathbf{b}), V_L(y, m, \mathbf{b}), X(y, -\bar{m})\} \tag{17}$$

The decision problem delivers decision rules $d(y, m, \mathbf{b})$ and $\mathbf{a}(y, m, \mathbf{b})$. Given these rules, the pricing relationships is given by the pair of equations

$$q_j(y, \mathbf{b}') = E_{(y'\, m')|y} \left[[1 - d(y', m', \mathbf{b}')] \frac{\lambda_j + [1 - \lambda_j][z_j + q_j(y', \mathbf{a}(y', m', \mathbf{b}'))]}{1 + r_f} \right] \text{ for } j \in \{S, L\} \tag{18}$$

The numerical specification of the model proceeds as follows. We assume that the two maturities available are three years ($\lambda_S = 0.083$) and seven years ($\lambda_L = 0.036$) and the coupon rate on both is 0.03 (3 percent at a quarterly rate). Given these choices, the parameters that are selected directly are set as in Table 1. The parameters (β, d_0, d_1) continue to be chosen to target the same three statistics as before. We take $(b_S + b_L)/y$ as the model analog of the debt-to-output ratio and the model analog of the spread to be the difference between the weighted sum $(b'_S r_S(\mathbf{b}', y) + b'_L r_L(\mathbf{b}', y))/(b'_S + b'_L)$ of interest rates and r_f, where r_S (r_L) is the implied annual internal rate of return on the short-term (long-term) debt. The results of the matching exercise is reported are Table 10.

Table 10: Moment Matching with Maturity Choice

	Avg. Spread	SD Spread	Debt-to-Y
Data	0.0815	0.0443	1.00
$(\beta, d_0, d_1) = (0.94, 0.21, 0.27)$	0.0816	0.0443	0.70

As is evident, the matching exercise is successful. We find that in equilibrium the sovereign always issues short-term debt. Thus, the logic underlying the benefits of short-term debt carries over to the empirically relevant portion of the state space.[31] In the following table we report the operating characteristics of the economy regarding the two spreads. Although long-term debt is never issued, we report statistics on the long-term spread based on the sale price of long-term debt if the sovereign issues a small amount of it.

[31] Arellano and Ramanarayanan (2009) find that the sovereign issues long-term debt when output is high and short-term debt when output is low. There are several differences between their model and ours which could account for the difference in findings. First, they do not attempt to match the average level of debt, spreads, or the volatility of spreads as we do; second, they allow for risk-aversion on the part of lenders while our lenders are risk-neutral; and third, they allow for repayment on defaulted debt. Bi (2008a) has noted the importance of repayment for maturity choice.

Table 11: Maturity and Spreads

Avg. $(r_S - r_f)$	0.0815
Avg. $(r_L - r_f)$	0.0801
$\sigma(r_S - r_f)$	0.0443
$\sigma(r_L - r_f)$	0.0277
$\sigma(r_L - r_S)$	0.0167
$corr(r_L - r_S, y)$	0.5973

An interesting finding is that the spread between long-term and short-term debt is procyclical. While both spreads rise when output falls, the spread on short-term debt rises more than the spread on long-term debt; and, as a result, the spread between long-term and short-term debt falls. The long-term spread is an average of spreads that puts relatively more weight on the distant future. When output is low, the sovereign has a high risk of default in the near future, which makes short spreads high. Long spreads take into account that spreads in the future will decline because output will rise (mean reversion in output). Thus, long-term spreads do not rise as much as short spreads when output is low and the gap falls with output.

For completeness, we report the statistics on model performance.

Table 12: Maturity Choice and Model Performance

Moment	Data	Baseline	Model w/ Maturity Choice
Avg. $(r - r_f)$	0.0815	0.0815	0.0814
$\sigma(r - r_f)$	0.0443	0.0443	0.0443
Avg. b/y	1	0.70	0.70
$\sigma(c)/\sigma(y)$	1.09	1.11	1.15
$\sigma(NX/y)/\sigma(y)$	0.17	0.20	0.29
$corr(c, y)$	0.98	0.99	0.97
$corr(NX/y, y)$	-0.88	-0.45	-0.38
$corr(r - r_f, y)$	-0.79	-0.67	-0.65
Debt Service	0.053	0.055	0.078
Def Freq	0.125	0.066	0.070

These findings raise an interesting issue. On the one hand, long-term nature of sovereign debt is an important factor in accounting for the level of debt and spreads and the cyclical volatility in key macroeconomic aggregates. On the other hand, the model environment implies that the sovereign would be better off issuing only short-term debt. The model seems to be missing some feature that makes long-term bonds more attractive than short-term bonds. In particular, there are no transaction costs of participating in the international credit market, and there are no coordination issues between lenders that might lead to a "run" on short-term debt; i.e., we do not model the possibility that lenders become unwilling to "roll over" short-term debt because they fear that future lenders may become reluctant to do the same. Adding these features would presumably change the result that short-term always dominates long-term debt.[32]

[32] Broner, Lorenzoni and Schmukler (2007) examine how the possibility of a roll-over crisis affects maturity choice. Cole and Kehoe (2000) provides an early discussion of roll over crisis in sovereign debt

8 Conclusion

In this paper, we developed a novel and computationally tractable model of long-term unsecured debt and default. The model extends the classic model of Eaton and Gersovitz. We showed that our long-term debt model shares the key insight of Eaton and Gersovitz's original contribution, namely, that the option to default implies that the sovereign faces a rising supply curve for credit. We established the existence of an equilibrium pricing function with this property for the case in which the arguments of the price function take discrete and a finite set of values. We also developed a novel computational approach to solving the model and demonstrated that our technique delivers very accurate solutions that are mostly insensitive to pure computational assumptions (such as the choice of grid size).

While the framework can be applied to both consumer and sovereign debt, we applied it to the recent quantitative literature on sovereign debt and emerging market business cycles. Using Argentina as a test case, we showed that our model with long-term debt can easily account for (or match) the average external debt-to-output ratio, the average spread on external debt, and the standard deviation of spreads on external debt. Existing quantitiative models have not matched all three facts. Furthermore, we show that accounting for these key first and second moment facts does not come at the expense of poor model performance along other business cycle dimensions. If anything, the model performance improves along these dimensions compared with existing one-period debt models. We also demonstrated that this improved model performance is due to the long-term nature of debt. Thus, incorporating long-term debt long the lines developed in this paper can lead to better models of emerging market business cycles.

9 References

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10 Appendix A

Proof of Proposition 1: Let $\mathcal{W}$ be the set of all continuous (in m) functions on $Y \times M \times B$ that take values in the bounded interval $[u(0)/(1-\beta), U/(1-\beta)]$. Equip $\mathcal{W}$ with sup norm $||\cdot||_\infty$. Then $(\mathcal{W}, ||\cdot||_\infty)$ is a complete metric space.

For $W \in \mathcal{W}$, let $X(y, m; W)$ be solution to the (3). The solution exists because (3) defines a contraction mapping in X with modulus $\beta\xi$. It is continuous in m because $c = y - \phi(y) + m$ is continuous in m and u is continuous in c.

For $W \in \mathcal{W}$, let $V(y, m, b; W, q)$ be the solution to (4). We index this solution by q because q appears as a parameter in (4). Here, however, we need to address the fact that V may not be well-defined because there may not be any feasible b' for some (y, m, b) and q. To extend the definition of V over the entire domain, we will assume that the utility from a choice of b' under repayment, denoted $V_{b'}(y, m, b; W, q)$, is given by $(\max\{0, y + m + [\lambda + (1-\lambda)z]b + q(y, b')[b' - (1-\lambda)b]\}) + \beta E_{(y',m'|y)} W(y', m', b')$. Thus, for an infeasible choice of b', current consumption is set to 0. Then,

$$V(y, m, b; W, q) = \max_{b' \in B} V_{b'}(y, m, b; W, q).$$

Since B is a finite set, $V(y, m, b; W, q)$ exists for all (y, m, b) and q. Also $V_{b'}$ is continuous in m for every b' since $\max\{0, y + m + [\lambda + (1-\lambda)z]b + q(y, b')[b' - (1-\lambda)b]\}$ is continuous in m and u is continuous in c. Therefore, $V(y, m, b; W, q)$ is continuous in m since the maximum of a finite set of continuous functions is also continuous. Furthermore, both $X(y, m; W)$ and $V(y, m, b; W, q) \in [u(0)/(1-\beta), U/(1-\beta)]$ for all y, m and b.

Next, define the operator

$$T(W)(y, m, b; q) = \max\{V(y, m, b; W, q), X(y, -\bar{m}; W)\} \tag{19}$$

on the space of functions $\mathcal{W}$. Then, (i) $T(\mathcal{W})(y, m, b; q) \subset \mathcal{W}$ which is obvious; (ii) If $W \geq \hat{W}$ then $T(W) \geq T(\hat{W})$, which follows because $V(y, m, b, W, q)$ is clearly increasing in

W and standard contraction mapping arguments can establish that $X(y,m;W)$ is increasing in W; and (iii) $T(W+k) \leq T(W) + \eta k$, where $\eta = \max\{\beta\xi/[(1-\beta)+\beta\xi], \beta\} < 1$. To see (iii), note that $V(y,m,b;W+k,q) = V(y,m,b;W,q) + \beta k$ and $X(y,m;W+k) = X(y,m;W) + (\beta\xi/[(1-\beta)+\beta\xi])k$. Therefore,

$$T(W+k)(y,m,b;q) = \max\left\{V(y,m,b;W,q) + \beta k,\ X(y,-\bar{m};W) + \frac{\beta\xi}{\beta\xi/[(1-\beta)+\beta\xi}k\right\}$$

and (iii) follows. Therefore, T is a contraction mapping with modulus η and the existence of a unique solution to (19) in $\mathcal{W}$, denoted $W_q^*(y,m,b)$, follows from the Contraction Mapping Theorem.

The strict monotonicity of $X(y,m)$ with respect to m follows from endowment being strictly increasing in m, u being strictly increasing in c, and the fact that m does not affect the probability distribution of (y',m').

For the strict monotonicity of V with respect to m, observe that if $m^0 < m^1$ then every b' that is feasible under (y,m^0,b) is also feasible under (y,m^1,b) and yields strictly higher consumption. In the region where repayment is feasible, there must be at least one b' that is feasible. Then, since m does not affect the probability distribution of (y',m'), strict monotonicity of u implies $V(y,m^0,b) < V(y,m^1,b)$. For strict monotonicity with regard to b, observe that for $b^0 < b^1$ we have $[\lambda+(1-\lambda)z]b^0 + q(y,b')[1-\lambda]b^0 < [\lambda+(1-\lambda)z]b^1 + q(y,b')[1-\lambda]b^1$ for every feasible $b' \in B$ and every $y \in Y$. This follows because $[\lambda+(1-\lambda)z] > 0$ and $q(y,b') \geq 0$. Hence, every b' that is feasible under (y,m,b^0) is also feasible under (y,m,b^1) and affords strictly greater consumption. Again, in the region where repayment is feasible, there must be at least one feasible b'. Therefore, by the strict monotonicity of u, $V(y,m,b^0) < V(y,m,b^1)$.

From the monotonicity of V with respect to b, it follows that for $b'^1 > b'^0$, $W(y',m',b'^1) \geq W(y',m',b'^0)$. Hence, $Z(y,b'^1) \geq Z(y,b'^0)$. To show the inequality is strict, we will assume

b_I (the smallest $b \in B$) is bounded below as

$$b_I > -[\phi(y_{\max}) + 2\bar{m}]/[\lambda + (1-\lambda)z], \tag{20}$$

where $y_{\max}$ is the largest $y \in Y$. Then, observe that (20) implies that $u(y_{\max} + [\lambda + (1-\lambda)z]b + \bar{m}) + \beta Z(y,0) > u(y_{\max} - \phi(y_{\max}) - \bar{m}) + \beta Z(y,0)$ for all $b \in B$. Also, observe that $Z(y,0) > E_{(y',m')|y}[(1-\xi)W(y',m',0) + \xi X(y',m')]$, since $X(y',m') < W(y',m',0)$ for all $(y',m') \in Y \times M$. Thus, for every debt level there is a range of m values for which repayment without new borrowing is better than default if y is at its highest value. Therefore, for every $b' \in B$, there is a range of m' for which $V(y_{\max}, m', b') > X(y_{\max}, -\bar{m})$. By the strict monotonicity of V with respect to b, every m' for which $V(m', b'^0, y_{\max}) > X(y_{\max}, -\bar{m})$ it is also true that $V(m', b'^1, y_{\max}) > X(y_{\max}, -\bar{m})$. Thus, there is a range of m values for which $W(y_{\max}, m', b'^1) > W(y_{\max}, m', b'^0)$. Since $F(y, y_{\max}) > 0$ for all y, it follows that $Z(y', b'^1) > Z(y', b'^0)$. □

Since we extended the domain of definition of V to infeasible choices, we need to verify that this extension does not result in the sovereign actually choosing infeasible b'. In the following Lemma we establish that if $u(0)$ is set to a sufficiently low number then it is never optimal to choose infeasible actions.

Lemma A1: If $u(0) + \beta U/(1-\beta) < u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$, where $y_{\min}$ is the smallest value in Y, then optimal consumption under repayment, $c(y,m,b)$, is uniformly bounded below by some strictly positive number $\bar{c}$.

Proof: By continuity of u there exists $\bar{c} > 0$ such that $u(\bar{c}) + \beta U/(1-\beta) < u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$. Since the sovereign can choose to consume its endowment each period, and it can always consume at least $y_{\min} - \phi(y_{\min}) - \bar{m}$ in every period, its lifetime utility in any period is bounded below by $u(y_{\min} - \phi(y_{\min}) - \bar{m})/(1-\beta)$. On the other hand, the highest utility from selecting any action that leads to consumption $\bar{c}$ or less is $u(\bar{c}) + \beta U/(1-\beta)$. By assumption the former dominates the latter. Thus it can never be optimal to choose to consume $\bar{c}$ or less. In particular, it can never be optimal to choose an action that leads to 0 consumption.

Proposition 5 : $a(y, m, b)$ is increasing in m and $d(y, m, b)$ is decreasing in m

Proof : To prove $a(y, m, b)$ is increasing in m, fix y and b and let $m^1 > m^0$. Assume also that repayment is feasible for both m^1 and m^0. Denote $a(y, m^1, b)$ by b'^1 and the associated consumption by c^1. Let $\hat{b}' > b'^1$ be some other feasible choice of b' greater than $\hat{b}'^1$ and denote the associated consumption by $\hat{c}$. Then, by optimality $u(c^1)) + \beta Z(y, b'^1) > u(\hat{c})) + \beta Z(y, \hat{b}')$. Since $Z(y, \hat{b}') > Z(y, b'^1)$ (Proposition 1), the above inequality implies $c^1 > \hat{c}$. Let $\Delta = c^1 - \hat{c}$ denote the loss in current consumption from choosing $\hat{b}'$ over b'^1 when transitory shock takes the value m^1. Now observe that the loss in current consumption from choosing $\hat{b}'$ over b'^1 when transitory shock takes the value m^0 is also Δ. However, the level of consumption when the transitory shock takes the value m^0 and the sovereign chooses b'^1, denoted $\tilde{c}$, is strictly less than c^1. From the strict concavity of u, it follows that $u(\tilde{c}) - u(\tilde{c} - \Delta) > u(c^1) - u(c^1 - \Delta)$. Therefore, $u(\tilde{c})) + \beta Z(y, b'^1) > u(\tilde{c} - \Delta)) + \beta Z(y, \hat{b}')$. Since $\hat{b}'$ was any b' greater than b'^1, $a(y, m^0, b)$ cannot exceed b'^1. Thus, $a(y, m^0, b) \leq a(y, m^1, b)$.

The fact that $d(y, m, b)$ is decreasing in m follows from the fact that $V(y, m, b)$ is strictly increasing in m (Proposition 1) and the utility from default, $X(y, -\bar{m})$, is independent of m. $\square$

The next Lemma establishes that the sovereign can be indifferent between default and repayment at exactly one value of m and it can be indifferent between any two borrowing levels at exactly one value of m.

Lemma A2: (i) For any given b'^0, there can be at most one value of m for which choosing b'^0 gives the same lifetime utility as defaulting and (ii) for any given $b'^0 < b'^1$ there can be at most one value of m for which choosing the two debt levels give the same lifetime utility.

Proof : (i) Fix y and b. (i) Suppose that there is an $\hat{m}$ such that $V_{b'^0}(y, \hat{m}, b) = X(y, -\bar{m})$. Since the l.h.s is strictly increasing in m, there cannot be any other $m \neq \hat{m}$ for which the same equality holds. (ii) Suppose there is an $\hat{m}$ for which $u(c^0(\hat{m})) + \beta Z(y, b'^0) =$

$u(c^1(\hat{m})) + Z(y, b'^1)$, where $c^0(\hat{m})$ and $c^1(\hat{m})$ are the levels of consumption when b'^0 and b'^1 are chosen, respectively. Since $Z(y, b'^1) > Z(y, b'^0)$ (Proposition 1), it follows that $c^0(\hat{m}) > c^1(\hat{m})$. Suppose, to get a contradiction, there is another $\tilde{m} > \hat{m}$ such that $u(c^0(\tilde{m})) + \beta Z(y, b'^0) = u(c^1(\tilde{m})) + Z(y, b'^1)$. Then, $u(c^0(\hat{m})) - u(c^0(\tilde{m})) = u(c^1(\hat{m})) - u(c^1(\tilde{m}))$ and (from the budget constraint) $c^i(\tilde{m}) = c^i(\hat{m}) + [\tilde{m} - \hat{m}]$ for $i = 0, 1$. Thus, we must have $u(c^0(\hat{m})) - u(c^0(\hat{m}) + [\tilde{m} - \hat{m}]) = u(c^1(\hat{m})) - u(c^1(\hat{m}) + [\tilde{m} - \hat{m}])$. But, since $c^0(\hat{m}) > c^1(\hat{m})$, the preceding equality violates the strict concavity of u. Hence there can only be at most one m for which $u(c^0(m)) + \beta Z(y, b'^0) = u(c^1(m)) + \beta Z(y, b'^1)$. □

Corollary to Lemma A2: The thresholds $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and the corresponding debt choices $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ are unique.

Proof: Suppose, to get a contradiction, that there are two distinct pairs $\{m^{k-1}, b'^k\}$ and $\{\hat{m}^{k-1}, \hat{b}'^k\}$. Without loss of generality, assume that these lists deviate from each other for $k = 1$. That is, according the first list the sovereign is indifferent between choosing 0 and b'^1 at m^1 and according to the second list it is indifferent between choosing 0 and $\hat{b}'^1$ at $\hat{m}^1$. Suppose also that $\hat{b}'^1 > b'^1$. If $\hat{m}^1 \neq m^1$, then there are two distinct values of m for which $\hat{b}'^1$ and b'^1 give the same utility. This contradicts Lemma A2. And if $\hat{m}^1 = m^1$ then b'^1 is inconsistent with our assumption that, all else the same, among two b' choices give the same utility, the sovereign chooses the larger one. □

Lemma A3: $W^*_q(y, m, b)$, $V(y, m, b; W^*_q, q)$, $X(y, m; W^*_q)$ and $Z^*_q(y, b')$ are continuous in q.

Proof: To prove that $W^*(y, m, b; q)$ is continuous in q, it is sufficient to prove that the contraction operator $T(W)(y, m, b; q)$ is continuous in q (see Theorem 4.3.6 in Hutson and Pym (1980, pp. 117-118). In order to establish this, we need to prove only that $V(y, m, b; W, q)$ is continuous in q. Fix (y, m, b) and W. Observe that $V_{b'}(y, m, b; W, q)$ is continuous in q because $\max\{0, y + m + [\lambda + (1 - \lambda)z]b + q(y, b')[b' - (1 - \lambda)B]\}$ is continuous in q and u is continuous in c. Thus, $V(y, m, b; W, q)$, being the maximum of a finite set of continuous functions, is also continuous in q. Hence $W^*_q(y, m, b)$ is continuous in q. The continuity of $V(y,m,b;W_q^*,q)$ with respect to q follows from the same logic as before: $V_{b'}(y,m,b;W_q^*,q)$ is continuous in q for each $b'$and hence the maximum over b'must also be continuous in q; the continuity of $Z_q^*(y,b')$ with respect q follows directly from its definition; and the continuity of $X(y,m;W_q^*)$ with respect q follows from noting that the contraction operator defining $X(y,m;W)$ depends on W via the quantity $Z(y,0)$ and that the operator is continuous in $Z(y,0)$. Since $Z_q^*(0,y')$ is continuous in q, it follows from another application of Theorem 4.3.6 of Hutson and Pym that $X(y,m;W_q^*)$ is continuous in q.

Lemma A4 : Let $q^n(y,b')$ be a sequence converging to $\hat{q}(y,b')$. Let $d(y,m,b;q^n)$, $a(y,m,b;q^n)$ and $d(y,m,b;\hat{q})$, $a(y,m,b;\hat{q})$ be the corresponding optimal decision rules. Then, $d(y,m,b;q^n)$ converges point []wise to $d(y,m,b;\hat{q})$ and $a(y,m,b;q^n)$ converges pointwise to $a(y,m,b;\hat{q})$ except, possibly, at a finite number of points.

Proof: (Convergence of $a(y,m,b;q^n)$). Let $q^n \to \bar{q}$. Fix y and b. For a given m, let $b'^0 = a(y,m,b;\bar{q})$. Let $V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q})$ denote the lifetime utility if the sovereign chooses to borrow b' in the current period but follows the optimal plan in all future periods. Two cases are possible: (i) $V(y,m,b;W_{\bar{q}}^*,\bar{q}) > V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q})$ for all $b' \neq b'^0$ and (ii) $V(y,m,b;W_{\bar{q}}^*,\bar{q}) = V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q})$ for some $b' \neq b'^0$. Consider case (i). Let $V(y,m,b;W_{\bar{q}}^*,\bar{q}) - V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q}) = \Delta$. Since $V(y,m[],b;W_q^*,q)$ is continuous in q there exists N_1 such that for all $n \geq N_1$ $V(y,m,b;W_{q^n}^*,q^n) > V(y,m,b;W_{\bar{q}}^*,\bar{q}) - \Delta/2$. Next, note that

$$V_{b'}(y,m,b;W_{q^n}^*,q^n) = u(y+m+[\lambda+(1-\lambda)z]b - q^n(y,b')[b'-(1-\lambda)b]) + \beta Z_{q^n}^*(y,b').$$

Since $Z_q^*(y,b')$ is continuous in q it follows that there exists N_2 such that for all $n \geq N_2$ $V_{b'}(y,m,b;W_{q^n}^*,q^n) < V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q}) + \Delta/2$. Therefore $V(y,m,b;W_{q^n}^*,q^n) - V_{b'}(y,m,b;W_{q^n}^*,q^n)$ $> V(y,m,b;W_{\bar{q}}^*,\bar{q}) - \Delta/2 - V_{b'}(y,m,b;W_{\bar{q}}^*,\bar{q}) - \Delta/2 = 0$ for all $n \geq \max\{N_1,N_2\}$. Hence $a(y,m,b;q^n) = b'^0$ for all $n > \max\{N_1,N_2\}$. Now consi []der case (ii). In this case, convergence may fail because $a(y,m,b;q^n)$ may converge to b' rather than b'^0. However, by Proposition 6 there can be only a finite number of m values for which case (ii) can hold.

Therefore, $a(y,m,b;q^n)$ converges pointwise to $a(y,m,b;\bar{q})$ except, possibly, for a finite number of m.

(Convergence of $d(y,m,b;q^n)$). Let $q^n \to \hat{q}$. Fix y and b. Again, two cases are possible. (i) $X(y,-\bar{m};W^*_{\bar{q}}) \neq V(y,m,b;W^*_{\bar{q}},\bar{q})$ and (ii)$X(y,-\bar{m};W^*_{\bar{q}}) = V(y,m,b;W^*_{\bar{q}},\bar{q})$. Consider case (i). For concreteness, suppose that $W(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) = \Delta > 0$. Then, by continuity of $V(y,m,b;W^*_q,q)$ and $X(y,m;W^*_q)$ with respect to q there exists N such that for all $n \geq N$, $V(y,m,b;W^*_{q^n},q^n) < V(y,m,b;W^*_{\bar{q}},\bar{q}) + \Delta$. For all such n, $W(y,-\bar{m};W^*_{q^n}) - V(y,m,b;W^*_{q^n},q^n) > W(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta = 0$. Hence $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 1$ for all $n \geq N$. If $\Delta < 0$ then there exists N such that for all $n \geq N$, $V(y,m,b;W^*_{q^n},q^n) > V(y,m,b;W^*_{\bar{q}},\bar{q}) + \Delta$. For all such n, $W(y,-\bar{m};W^*_{q^n}) - V(y,m,b;W^*_{q^n},q^n) < W(y,-\bar{m};W^*_{\bar{q}}) - V(y,m,b;W^*_{\bar{q}},\bar{q}) - \Delta = 0$. Hence $d(y,m,b;q^n) = d(y,m,b;\bar{q}) = 0$ for all $n \geq N$. Now consider case (ii). Again, convergence may fail in this case because $d(y,m,b;q^n)$ may converge to 1 or 0 while $d(y,m,b;\bar{q})$ is 0 or 1. However, by Proposition 7,, there can only be one value of m for which this can happen. Therefore, $d(y,m,b;q^n)$ converge pointwise to $d(y,m,b;\bar{q})$ except, possibly, for one value of m. $\square$

Proof of Continuity of $H(q)$: Let $\{q^n\}$ be a sequence in Q converging to $\hat{q} \in Q$ and let $\{d(y,m,b;q^n),\ a(y,m,b;q^n)\}$ and $\{d(y,m,b;\hat{q}),\ a(y,m,b;\hat{q})\}$ be the corresponding default and debt decision rules. Then

$$H(q^n)(y,b') = E_{(y'\,m')|y}\left[[1-d(y',m',b';q^n)]\frac{\lambda + [1-\lambda][z + q^n(y',a(y',m',b';q^n))]}{1+r_f}\right].$$

Or,

$$H(q^n)(y,b') = \\ \sum_{y'}\left[\frac{\int_M [1-d(y',m',b';q^n)]\,[\lambda+[1-\lambda][z+q^n(y',a(y',m',b';q^n))]]\,dG(m')}{1+r_f}\right]F(y',y).$$

Fix y' and b'. By Lemma A3, $\lim_n[1-d(y',m',b';q^n)] = [1-d(y',m',b';\hat{q})]$ for all but a finite number of points (pos []sibly) of m'. Since individual points of m have probability zero, $[1-d(y',m',b';q^n)]$ converge almost surely to $[1-d(y',m',b';\hat{q})]$ with respect to the measure induced by $G(m)$. Also, by Lemma A3, $\lim_n a(y',m',b';q^n) = a(y',m',b';\hat{q})$ for all but a finite number of points (possibly) of m'. If convergence holds then, since $a(\cdot;q^n)$ takes values in a finite set B, there must exist N such that for all $n > N$ $a(y',m',b';q^n)) = a(y',m',b';\hat{q})$. Therefore, for $n > N$, $q^n(y',a(y',m',b';q^n)) = q^n(y',a(y',m',b';\hat{q}))$. Since $q^n \to \hat{q}$, it follows that $\lim_n q^n(y',a(y',m',b';\hat{q})) = \hat{q}(y',a(y',m',b';\hat{q}))$. Thus, viewed as a function of m', $q^n(y',a(y',m',b';q^n))$ converges almost surely to $\hat{q}(y',a(y',m',b';\hat{q}))$. Therefore, we have that

$$\lim_n [1-d(y',m',b';q^n)]\,[\lambda + [1-\lambda][z + q^n(y',a(y',m',b';q^n))]] =$$
$$[1-d(y',m',b';\hat{q})]\,[\lambda + [1-\lambda][z + \hat{q}(y',a(y',m',b';\hat{q}))]]$$

except, possibly, at a finite number of points.

Now observe that each function in the sequence is non-negative and bounded above by $\lambda + (1-\lambda)[z+\bar{q}]$. Thus, by the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_n \int_M [1-d(y',m',b';q^n)]\,[\lambda + [1-\lambda][z + q^n(y',a(y',m',b';q^n))]]\,dG(m') =$$
$$\int_M [1-d(y',m',b';\hat{q})]\,[\lambda + [1-\lambda][z + \hat{q}(y',a(y',m',b';\hat{q}))]]\,dG(m').$$

Putting these result together, we get

$$\begin{aligned} & \lim_n H(q^n)(y,b') = \\ & \sum_{y'} \left[\frac{\int_M [1-d(y',m',b';q^n)]\,[\lambda + [1-\lambda][z + q^n(y',a(y',m',b';q^n))]]\,dG(m')}{1+r_f} \right] F(y',y). \\ = \; & H(\hat{q})(y,b'). \end{aligned}$$

Thus $H(q)(y,b')$ is continuous.

11 Appendix B

In this section, we give the logic of our solution algorithm. The first part gives the logic of the algorithm for calculating the optimal debt choice as a function of m. The second part, taking the algorithm in the first part as given, provides the logic for the solution algorithm for the model with maturity choice (the model with a single maturity is a special case).

11.1 Method For Recovering $a(y, m, b; q)$ Given (y, b) and q

Proposition 5 and Lemma A2 imply that given (y, b) and q there exists $\{-\bar{m} < m^{K-1} < m^{K-2} < \ldots < m^1 < \bar{m}\}$ and $\{b'^K < b'^{K-1} < \ldots < b'^1\}$ such that b'^K is chosen for $m \in [-\bar{m}, m^{K-1})$, b'^{K-1} is chosen for $m \in [m^{K-1}, m^{K-2})$, ..., b'^1 is chosen for $m \in (m^1, \bar{m}]$. $K = 1$ means the same debt level b'^1 is chosen for all $m \in M$.

Since b'^k need not be adjacent to b'^{k+1} the algorithm has to find both $\{m^k\}$ *and* $\{b'^k\}$. The decision rule is built up recursively. The problem is initially solved for a choice set containing only one choice of b' . The choice set is then expanded in steps until the entire set B is available. At each step, the solution from the previous step is used to determine the solution for the current step.

Suppose that we have located pairs $\{(m^{h-1}, b'^h), (m^{h-2}, b'^{h-1}), \ldots (\bar{m}, b'^1)\}$ such that if the sovereign is permitted to choose *only from the set* $b' \geq b'^h$, the sovereign would choose b'^h for $m \in [-\bar{m}, m^{h-1})$, b'^{h-1} for $m \in [m^{h-1}, m^{h-2})$, ..., $b'^1 \in (m^1, \bar{m}]$. The next step is to compare the utility from choosing b'^h with the utility from choosing the next lower b' (i.e., next higher debt level) on the grid, denoted b'^-. Two cases are possible.

1. $-q(y, b'^-)[b'^- - (1-\lambda)b] \leq -q(y, b'^h)[b'^h - (1-\lambda)b]$. Then, the life-time utility from b'^h is at least as high as life-time utility from b'^- for all $m \in M$. So we drop b'^- from further consideration and move to comparing b'^h to the next lower b' on the grid.

2. $-q(y, b'^{-})[b'^{-} - (1-\lambda)b] > -q(y, b'^{h})[b'^{h} - (1-\lambda)b]$. Then

$$\Delta(m) =$$
$$u(\ldots m - q(y, b'^{-})[b'^{-} - (1-\lambda)b]\ldots) - u(\ldots m - q(y, b'^{h})[b'^{h} - (1-\lambda)b]\ldots) > 0$$

for all m, where $u(\ldots m - q(y, b')[b' - (1-\lambda)b]\ldots)$ is the current utility from choosing b' (we have suppressed terms that do not depend on m and b'). Furthermore, from the strict concavity of u, $\Delta(m)$ is decreasing in m. Three subcases are possible

(a) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} \leq 0$. Then b'^{h} is at least as good as b'^{-} for all m and we can drop b'^{-} from further consideration.

(b) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} \leq 0$. Then there must exist a unique $\tilde{m} \in (-\bar{m}, \bar{m}]$ such that $\Delta(\tilde{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} = 0$.

i. If $\tilde{m} < m^{h}$, we prepend $(\tilde{m}, b'^{-})$ to the list of pairs and proceed to compare the utility between b'^{-} with the next lower b' on the grid.

ii. If $\tilde{m} \geq m^{h}$, we drop b'^{h} from further consideration and proceed backwards to compare b'^{-} with b'^{h-1}. The reason is that $\tilde{m} \geq m^{h}$ implies that b'^{-} is preferred to b'^{h} for any $m < \tilde{m}$ and at the same time b'^{h-1} is preferred to b'^{h} for any $m \geq m^{h}$. Thus, b'^{h} is dominated by the choices of b'^{h-1} and b'^{-} and can be dropped from further consideration. When this is the case, b'^{-} needs to be compared to b'^{h-1}. The process is continued by finding a new $\widetilde{m}$ between the choices of b'^{-} and b'^{h-1}. If $\tilde{m} < m^{h-1}$, we add $(\tilde{m}, b'^{-})$ to the list of pairs $\{(m^{h-2}, b'^{h-1}), \ldots (\bar{m}, b'^{1})\}$ and proceed to compare the utility between b'^{-} with the next lower level of assets. If $\tilde{m} \geq m^{h-1}$, we drop b'^{h-1} from further consideration and continue to go backwards through the list. This process will either end in finding m^{h-j} such that $\tilde{m} < m^{h-j}$ or in the exhaustion of all pairs in the list $\{m^{k}, b'^{k}\}$. If the latter, we conclude that b'^{-} dominates any $b' > b'^{-}$ for all m (i.e., the list becomes a singleton $\{(\bar{m}, b'^{-})\}$

and proceed to compare b'^{-} with the next lower b'on the grid.

(c) $\Delta(-\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ and $\Delta(\bar{m}) + \beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$. Then b'^{-} dominates b'^{h} for all m and we can drop b'^{h} from further consideration. We then move to compare b'^{-} with b'^{h-1} and proceed as in (ii) above.

3. To implement this algorithm we start off with the choice set being $\{0\}$.The solution for this stage is the list $\{(\bar{m}, 0)\}$ (meaning that no borrowing is optimal for all m). We then proceed to compare 0 with the next lower b' on the grid. The algorithm is applied until every element of B has been compared.

Remark 1: While we have described the algorithm logic for the single maturity (baseline) model, it applies equally well for the two maturity model. Propostion 5 can be easily adapted to establish that b'_j is increasing in m, given $y, \mathbf{b}$ and $b'_{\text{-}j}$. Thus, the decision rule for any one maturity type can be recovered holding y, $\mathbf{b}$ and the choice of the *other* maturity constant at some value (in our maturity choice model, this value is $(1 - \lambda)b_{\text{-}j}$ but for the purposes of this method it can be any value).

Remark 2: The fact that m is iid is used in step (2). For instance, the statement in 2(a) follows because $\beta\{Z(y, b'^{-}) - Z(y, b'^{h})\} > 0$ is independent of m.

11.2 Method for Computing the Model with Maturity Choice

We explain the method for the model with maturity choice. In the baseline model where there is only one maturity, step (2) does not apply. We discretize the state space into N_y grids for persistent output shock, N_S grids for short term bonds, and N_L grids for long term bonds. We enter the k-th iteration with guesses of price functions $q_S^k(y, b'_S, b'_L)$, $q_L^k(y, b'_S, b'_L)$ and a guess for $Z^k(y, b'_S, b'_L)$, where $Z^k(y, b'_S, b'_L) = E_{(y',m'|y)} W^k(y', m', b'_S, b'_L)$. All calculations below are for some specific current values of the *discrete* state variables (y, b_S, b_L) and for the k-th iteration. The iteration index and the discrete state variables are omitted from the notation unless there is an ambiguity.

1. Given these guesses, we find what the sovereign would do if it adjusted its short-term debt only and if it adusted its long-term debt only. This entails finding the decision rules for debt choices for each maturity type and for each current state variables (y, b_S, b_L). The algorithm to accomplish this was outlined above. At the end of this stage, we have the lists

$$\{(m_S^{K_s-1}, b_S'^{K_s}), (m_S^{K_s-2}, b_S'^{K_s-1}), ..., (\bar{m}, b_S'^1)\}$$

and

$$\left\{(m_L^{K_l-1}, b_L'^{K_l}), (m_L^{K_l-2}, b_L'^{K_l-1}), ..., (\bar{m}, b_L'^1)\right\}.$$

Next, we combine the two sets of thresholds to create one single ordering with respect to the m thresholds. For example, if one portion of the ordering is

$$\{...(m_S^{i-1}, b_S'^i), (m_L^{j-1}, b_L'^j), (m_S^{i-2}, b_S'^{i-1}), (m_L^{j-2}, b_L'^{j-1})...\}$$

this would imply $m_S^{i-1} \leq m_L^{j-1} \leq m_S^{i-2} \leq m_L^{j-2}$ and we would know, for example, that for $m \in (m_S^{i-1}, m_L^{j-1}]$ the country would choose $b_S'^{i-1}$ if it chose short-term debt and it would choose $b_L'^j$ if it chose long-term debt. Or, between $m \in (m_L^{j-1}, m_S^{i-2}]$ the country would choose $b_S'^{i-1}$ if it chose short-term debt and $b_L'^{j-1}$ if it chose long-term debt. Thus, with this combined ordering, we know the sovereign's choice of short and long-term debt for each interval. In this ordering there are a maximum of $K_S + K_L - 1$ intervals.

2. In the second step, we determine the maturity choice that gives higher utility for each of the intervals from step 1. It is possible that for a given interval there is be a point of indifference such that one maturity is better in part of the interval and the other maturity in the other part. Continuing with the above example, given that for

$(m_L^{j-1}, m_S^{i-2}]$ the country chooses $b_S'^{i-1}$ for short-term debt, and $b_L'^{j-1}$, let

$$
\begin{aligned}
&\Delta(m) = \\
&u\left(y + A + m - q_S^k(y, b_S'^{i-1}, (1-\lambda_L)b_L)\left[b_S'^{i-1} - (1-\lambda_S)b_S\right]\right) + \beta Z^k\left(y, b_S'^{i-1}, [1-\lambda_L]b_L\right) \\
&-u\Big(y + A + m - q_L^k(y, (1-\lambda_S)b_S, b_L'^{j-1})[b_L'^{j-1} - (1-\lambda_L)b_L]\Big) - \beta Z^k\left(y, [1-\lambda_S]b_S, b_L'^{j-1}\right)
\end{aligned}
$$

where $A = y + [\lambda_S + [1-\lambda_S]z_S]b_S + [\lambda_L + [1-\lambda_L]z_L]b_L$. By strict concavity of u, $\Delta(m)$ will be monotone in m (increasing or decreasing, depending on whether $q_S(y, b_S'^{i-1}, [1-\lambda_L]b_L)\left[b_S'^{i-1} - [1-\lambda_S]b_S\right]$ is smaller or larger than $q_L(y, [1-\lambda_S]b_S, b_L'^{j-1})[b_L'^{j-1} - [1-\lambda_L]b_L\ )$. If $\Delta(m_L^{j-1}) \cdot \Delta(m_S^{i-2}) < 0$ then there must exist $\tilde{m}$ where $\Delta(\tilde{m}) = 0$ and either the short-term debt is preferred over $(m_L^{j-1}\tilde{m}]$ and long-term over $(\tilde{m}, m_S^{i-2}]$ or vice versa. If $\Delta(m_L^{j-1}) \cdot \Delta(m_S^{i-2}) \geq 0$, either the short-term or the long-term debt is best choice for all $m \in (m_L^{j-1}, m_S^{i-2}]$. If $\Delta(m)$ is identically zero for all m, then either choice give the same utility and we assume that the sovereign chooses the short-term debt. Since each interval inherited from step 1 may potentially have an indifference point in it, at the end of step 2 we can have a maximum of $2(K_S + K_L - 1)$ intervals. Within each of these intervals we know which maturity type is chosen in what quantity.

3. In third step, we find default thresholds. For each interval from step 2, we compare the life-time utility from choosing the indicated maturity type and quantity with the lifetime utility derived from default. Suppose that for $m \in (m_L^{i-1}, m_S^{i-2}]$ the sovereign chooses $b_S'^i$. Define

$$
\begin{aligned}
&\Delta(m) = u(y + A + m - q^k(y, b_S'^i, [1-\lambda_L]b_L)\left[b_S'^i - [1-\lambda_S]b_S\right]) + \\
&\beta Z^k\left(y, b_S'^i, [1-\lambda_L]b_L\right) - X(y, -\overline{m}).
\end{aligned}
$$

Evidently, $\Delta(m)$ is increasing in m. If $\Delta(m_L^{i-1}) \cdot \Delta(m_S^{i-2}) < 0$, there exists an $\tilde{m}$ such that default is optimal for $(m_L^{i-1}, \tilde{m})$ and $b_S'^i$ is optimal $[\tilde{m}, m_S^{i-2}]$. If $\Delta(m_L^{i-1}) \cdot \Delta(m_S^{i-2}) \geq 0$, then either default is optimal over the interval or $b_S'^i$ is optimal over the interval. At the end of this stage, we have a maximum of $2\left(2(N_s + N_l - 1)\right)$ intervals. Within each interval we know whether default or repayment is chosen and, if repayment is chosen, which maturity type is chosen in what quantity. Although the maximum number of intervals can be very large, in practice the number of intervals usually less than 20.

4. Finally, with these intervals in hand we calculate (using the integration method described in Appendix C) the functions $Z^{new}\left(y, b'_S, b'_L\right)$, $q_S^{new}(y, b'_S, b'_L)$ and $q_L^{new}(y, b'_S, b'_L)$. We check if $\left|Z^{nw}\left(y, b'_S, b'_L\right) - Z^k\left(y, b'_S, b'_L\right)\right| < \varepsilon_1$, $\left|q_S^{new}(y, b'_S, b'_L) - q_S^k(y, b'_S, b'_L)\right| < \varepsilon_2$ and $\left|q_L^{new}(y, b'_S, b'_L) - q_L^k(y, b'_S, b'_L)\right| < \varepsilon_2$, where ε_1 and ε_2 are very small numbers. If these conditions hold, we end the program. If one of them does not hold, we update

$$
\begin{aligned}
q_S^{k+1}(y, b'_S, b'_L) &= (1-\zeta)\cdot q_S^{new}(y, b'_S, b'_L) + \zeta \cdot q_S^k(y, b'_S, b'_L) \\
q_L^{k+1}(y, b'_S, b'_L) &= (1-\zeta)\cdot q_L^{new}(y, b'_S, b'_L) + \zeta \cdot q_L^k(y, b'_S, b'_L) \\
Z^{k+1}\left(y, b'_S, b'_L\right) &= (1-\zeta)\cdot Z^{new}\left(y, b'_S, b'_L\right) + \zeta \cdot Z^k\left(y, b'_S, b'_L\right),
\end{aligned}
$$

where $\zeta \in [0, 1)$ and continue with step 1.

12 Appendix C: Comparison of Alternative Computation Methods

As explained in the computation section, the reason for adding the m shock (and calculating thresholds to solve the decision problem) is to ensure that (6) has a solution and that the iteration (8) converges. In this appendix we (i) show that alternative methods that do not use "randomization" have significantly worse convergence performance and (ii) establish that there is indeed a tradeoff between the how small σ_m can be and close must ζ be to 1 to obtain convergence.

One clarification regarding the baseline method: The grid for M is relevant only for integration of value functions and the price function. To integrate, we divide M into 50 equally spaced intervals and assume that within each interval m is uniformly distributed. As an example of how value functions are integrated, fix y and b and consider an interval (m_1, m_2) and suppose that it contains one threshold, say $\hat{m} \in [m_1, m_2]$, where the optimal decision changes from a debt of b' to a debt of $\hat{b}'$. Then,

$$\int_{m_1}^{m_2} W(y, m, b) dG(m) \simeq \int_{m_1}^{m_2} dG(m) \times$$
$$\left(\frac{\hat{m} - m_1}{m_2 - m_1}\right) \cdot (u(y + m_{12} + [\lambda + z(1-\lambda)]b + q(y, b')[b' - (1-\lambda)b]) + \beta Z(y, b')) +$$
$$\left(\frac{m_2 - \hat{m}}{m_2 - m_1}\right) \cdot \Big(u(y + m_{12} + [\lambda + z(1-\lambda)]b + q(y, \hat{b}')\hat{b}')[b' - (1-\lambda)b] + \beta Z(y, \hat{b}')\Big) .$$

In other words, over each interval, we replace m by the midpoint of the interval but recognize that the choice of debt may switch as m varies over the interval. The overall variation in m is small and, with 50 intervals, the variation within each interval is smaller still. Thus, the differences between m and m_{12} are of little consequence for the evaluation of utility, given the choice of debt. The important variation comes from the *switch* in debt choice and this is captured by taking account of the exact location of the switch within the interval. Thus, the integration is done in segments determined by the thresholds. With each segment, the function being integrated is continuous and differentiable so standard integration techniques can be applied. Since the variation in m within each interval (and, therefore, within each segment) is small, the simplest method (the Midpoint Rule) suffices. Having obtained $\int_m W(y, m, b) dG(m)$ in this way for each y and b, we obtain $Z(y, b')$ as $\sum_{y'} [\int_m W(y', m', b') dG(m)] F(y, y')$.

The procedure for integrating the price function is similar:

$$\int_{m_1}^{m_2} q(y', a(y', m', b')) dG(m') \simeq$$
$$\int_{m_1}^{m_2} dG(m') \times \left[\left(\frac{\hat{m} - m_1}{m_2 - m_1}\right) \cdot (q(y', b')) + \left(\frac{m_2 - \hat{m}}{m_2 - m_1}\right) \cdot (q(y, \hat{b}')\right] .$$

All computations were implemented via parallelized (MPI) Fortran 90/95 running on a

16-node cluster.

12.1 Omitting M and Refining Y

Turning first to comparison with alternative methods, we make these comparisons by fixing all parameter values at baseline values and iterate each solution method 3000 times and report the maximum absolute error in the final 100 iterations as well as the relative value of this maximum error. The error for iteration k is defined as the largest absolute change in the price matrix from iteration $k-1$ to k. For purely discrete models, we report the maximum jump in asset choice, in terms of the maximum number of grid points skipped, from one iteration to the next, for the final 100 iteration.

The following table compares the baseline method (Method I) with three other methods. Method II is the model without M. Method III is the model without M in which the Y grid is increased until each iteration takes roughly the same time as each iteration in the baseline model. Method IV is the baseline model but in which M is discretized and thresholds are not computed.

Table 13: Omitting M and Refining Y

	Baseline ($M = 50$)	II	III	IV ($M = 50$)
Grids	$Y = 50, B = 350$	$Y = 50, B = 350$	$Y = 400, B = 350$	$Y = 50, B = 350$
$\Delta \lvert q^k - q^{k-1} \rvert$	4.73×10^{-13}	9.49×10^{-2}	1.07×10^{-2}	5.60×10^{-4}
$\Delta \lvert (q^k - q^{k-1})/q^k \rvert$	4.14×10^{-12}	8.05	3.23×10^{-1}	5.35×10^{-3}
Max jump in b' between iterations	NA	19	14	15
Method II: No M				
Method III: No M but refined Y grid				
Method IV: Baseline but M discretized and thresholds are not computed				

With the baseline method, we get convergence for very tight convergence criteria. In contrast, for Method II, where we omit M, even after 3000 iterations the price matrix is far from convergence; the error can be as much as 8 times the price (if q^k is zero for some node, we replace it by a very small number). Notice also that the maximum change in debt choice is 19 grid points. These jumps occur because nonconvexities lead to multiple local maxima and the solution cycles between these local maxima from one iteration to the next (as discussed in the text in relation to Figure 2). When debt choice jumps, the change causes a jump in price which causes a jump in debt in the next iteration. This (mis)behavior suggests that interpolating between adjacent points (making the asset dimension continuous) is unlikely to help since the jumps are not necessarily between adjacent grid points but between grid points that are far apart.

It is possible, of course, to simply increase the number of grid points in Y so as to attenuate the impact of jumps in decisions on the expectation term (more grid points mean each grid point has a lower probability associated with it). In Method III, we increase the grid on Y to 400. 400 is chosen because with these many grid points each iteration takes roughly the same amount of time as in our baseline method (Method I). Convergence occurs for a tigher criteria relative to Method II but it is still nowhere close to Method I. And there are still large jumps in asset choice. But these results do suggest that if we could make $Y \to \infty$ we could get better results with respect to convergence even without the m shock.

Finally, in the last column, we see that if the M shock is discretized and thresholds are not used, convergence is better than the Methods II and III but still not as good as Method I. We see the same problem in terms of jumps in asset choice. But the results do suggest that performance will improve if $M \to \infty$ (which is equivalent to the baseline method). But one thing to be note is that this program takes longer to run relative to our baseline method. This is because when M is discretized, the discounted utility of the country is calculated for *all* current states $(m, y,\ b)$ and for all choices of b'. In the baseline method, given current states (y, b), we find the thresholds of M for which there is a switch between different choices of assets. As those switches do not happen very frequently, the utility level given the choice of b' is computed much less frequently.

12.2 The Trade-off Between σ_m and ζ in Achieving Convergence

Turning next to the role of σ_m and ζ in achieving convergence, we proceed as follows. We consider the model where all parameter values are as in the baseline model but the number of grids on Y, M, and B are 25, 50 and 100, respectively. With fewer grids, the computations take less time and we can demonstrate that convergence can be achieved for the same level of precision as in the baseline model for *very* small values of σ_m, provided the value of ζ is increased correspondingly.

Table 14: (σ_m, ζ) Pairs For Which Convergence is Achieved

σ_m	ζ
0.001	0.98
0.0005	0.98
0.0001	0.98
0.00005	0.995
0.00001	0.998
Grids	$Y = 25$, $M = 50$, $B = 100$

12.3 Omitting M and Refining B

In the following table we establish that the poor performance of the baseline model without M (Model II above) cannot be rectified by refining the B, or asset, dimension.

Table 15: Omitting M and Refining B

	Method II	Method V	Method VI (continuous B)
Grids	$Y = 50, B = 350$	$Y = 50, B = 1600$	$Y = 50, B = 350$
$\Delta\lvert q^t - q^{t-1}\rvert$	9.49×10^{-2}	1.69×10^{-1}	9.46×10^{-2}
$\Delta\lvert (q^t - q^{t-1})/q^t\rvert$	8.05	9.80	8.01
Max jump in b' between iterations	19	93	-

The column labeled Method V shows the case where we omit the M shock and increase the number of grids for the asset level. Evidently, increasing the grids for B makes convergence an even bigger problem. The cycles in prices are even larger than the case where we had lower number of grids for B.

In the next column (labeled Method VI), we continue to omit the M shock but treat B as continuous variable. We discretize B as in the other methods but allow for asset choices off the grid. In particular, if income is y and beginning of period debt is b,then for a debt level b' between two adjacent grids b_j and b_{j-1},

$$c = y + [\lambda + (1-\lambda)z]b + wq(b_j, y)[b_j - (1-\lambda)b] + (1-w)q(b_{j+1}, y)[b_{j+1} - (1-\lambda)b],$$

and

$$E_{y'|y}V(y', b') = wE_{y'|y}V(y', b_j) + (1-w)E_{y'|y}V(y', b_{j+1}),$$

where w is $(b_{j+1} - b')/(b_{j+1} - b_j)$. As there are more than one local maxima in our problem, we first find the b' that maximizes utility confining our choice to the initial discrete grids and then do a refined search to locate the best choice of b' around that grid (this is the procedure followed in Hatchondo and Martinez (2009)). Treating B continuous in this fashion also does not improve convergence. The lotteries between adjacent grid points do not help because the the problematic cycles are between grids that are far apart.

Figure 1: Revenue from Long-Term Debt (b=0)

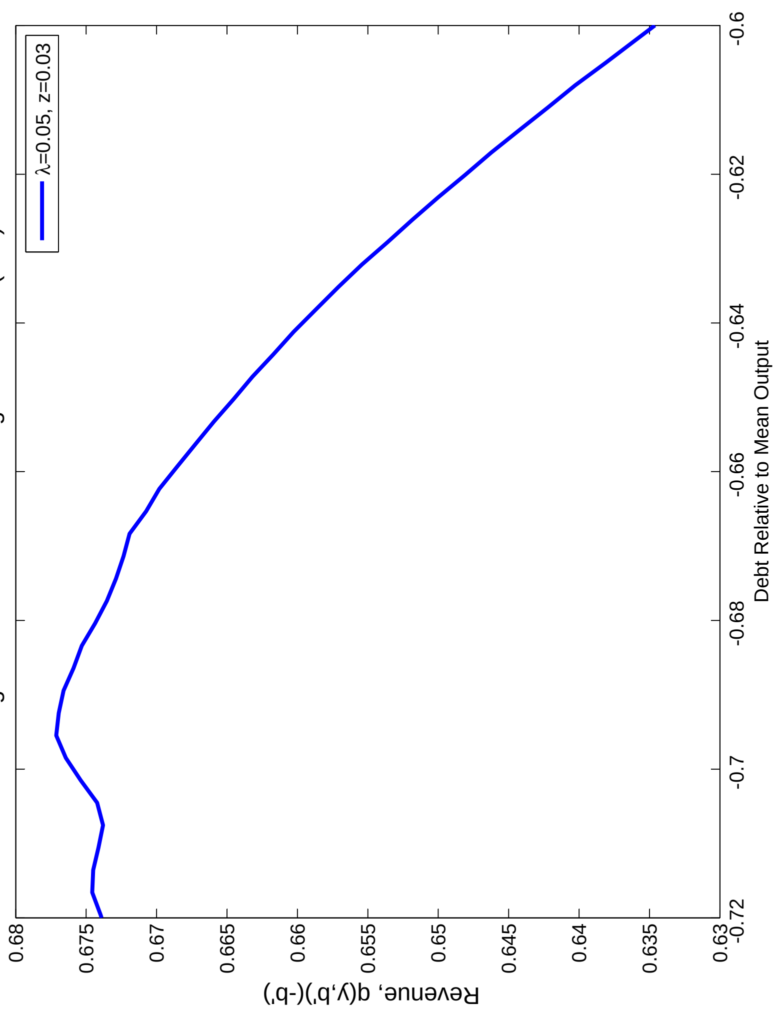

Figure 2: Variation over b' in Total Lifetime Utility (given m=0, b=0)

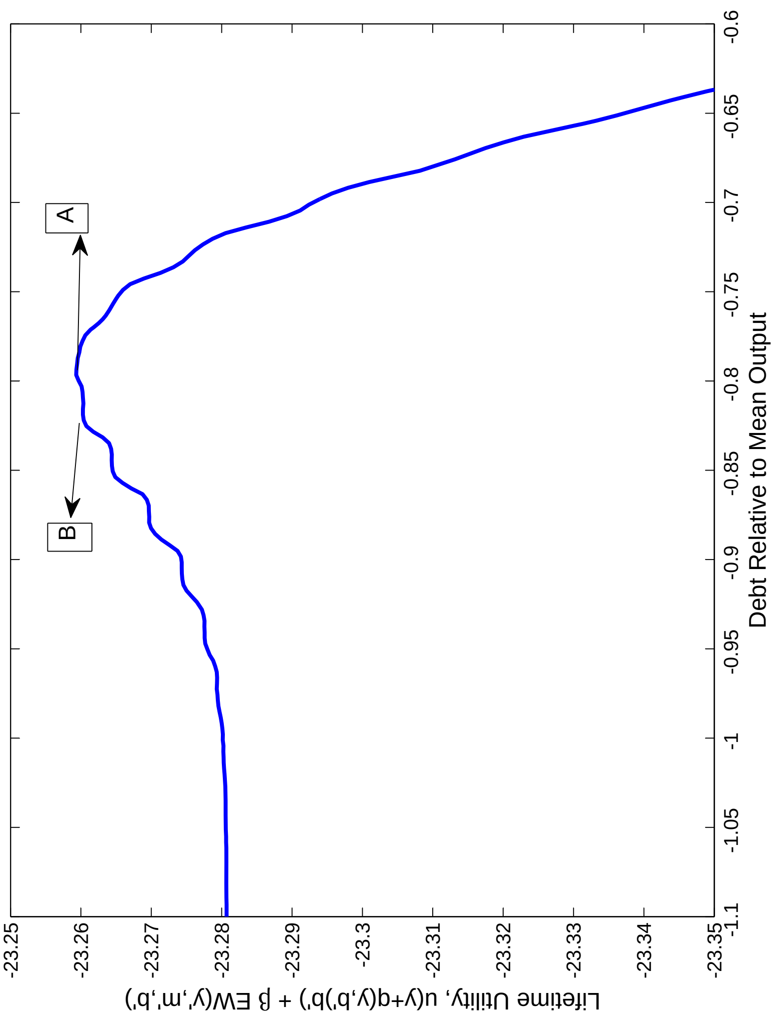

Figure 3: Simulated Spreads for Model and Data (1993:Q1-2001:Q3)

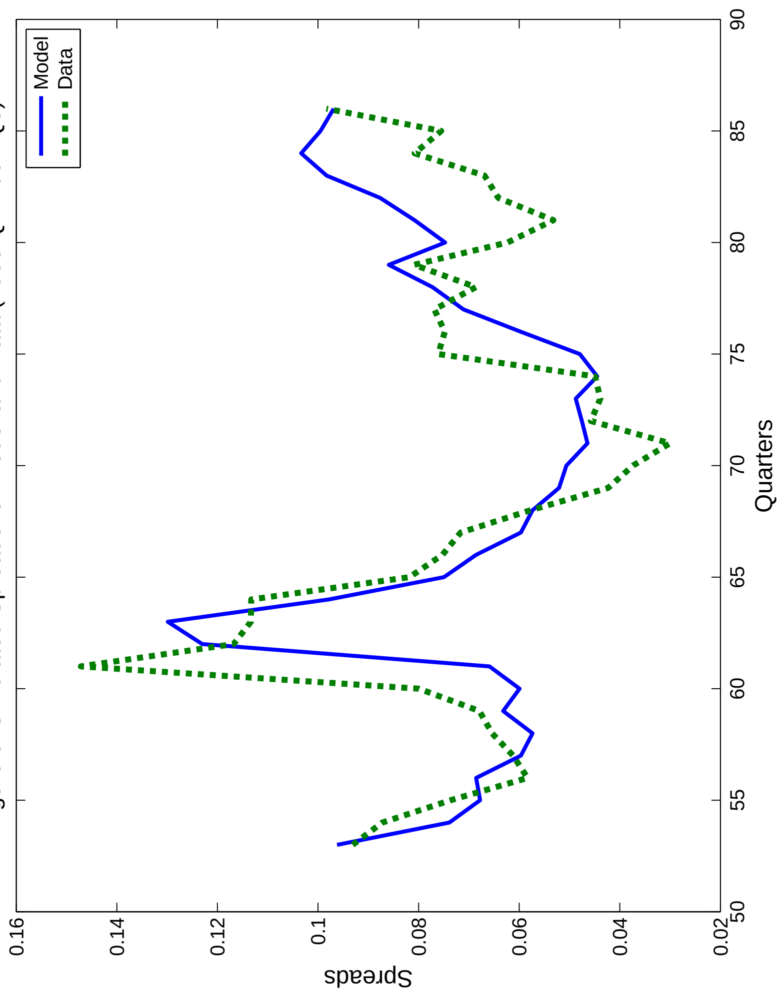

Figure 4: Current Spreads and Next-Period Default Probability for Long-term Bonds When y = Mean(y)

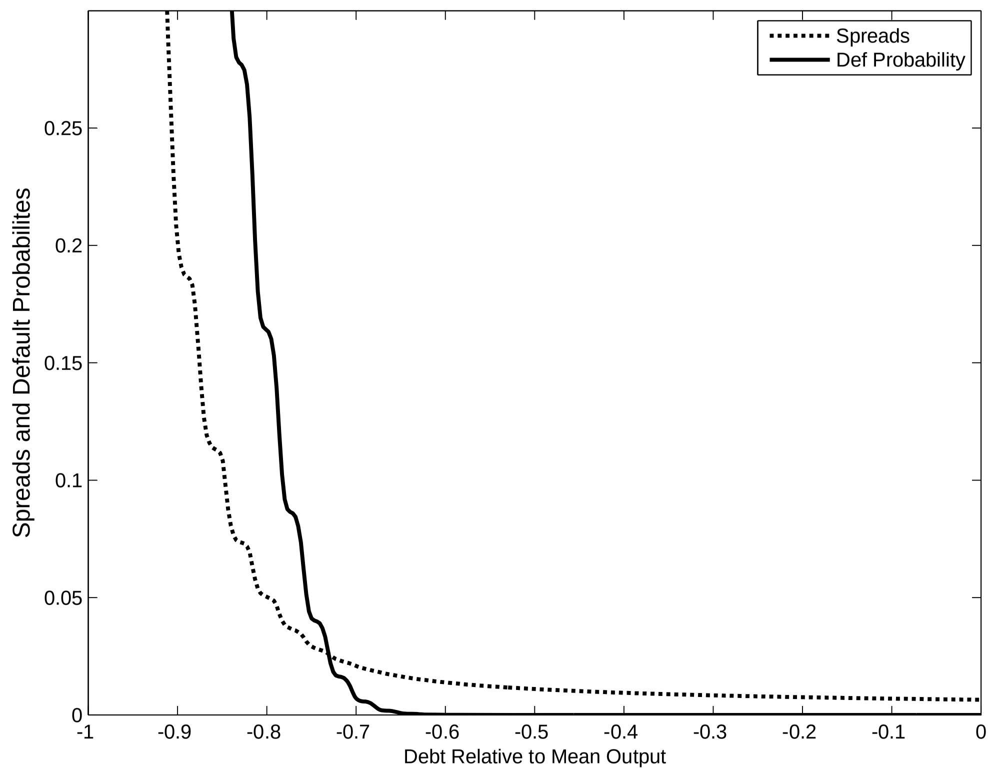

Figure 5: Current Spreads and Next-Period Default Probability for Short-term Bonds When y = Mean(y)