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Kevin X. D. Huang
Vanderbilt University
and
Qinglai Meng
Chinese University of Hong Kong

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Capital and macroeconomic instability in a discrete-time model with forward-looking interest rate rules*

Kevin X.D. Huang[a,b,†] , **Qinglai Meng**[c]

[a]*Department of Economics, Vanderbilt University, Nashville, TN 37235, USA*

[b]*Research Department, Federal Reserve Bank of Philadelphia, Philadelphia, PA, USA*

[c]*Department of Economics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong*

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†Corresponding author. Tel.: +1 615 936 7271; fax: +1 615 343 8495.

E-mail address: kevin.huang@vanderbilt.edu (K.X.D. Huang).

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Abstract

We establish the necessary and sufficient conditions for local real determinacy in a discrete-time production economy with monopolistic competition and a quadratic price adjustment cost under forward-looking policy rules, for the case where capital is in exogenously fixed supply and the case with endogenous capital accumulation. Using these conditions, we show that (i) indeterminacy is more likely to occur with a greater share of payment to capital in value-added production cost; (ii) indeterminacy can be more or less likely to occur with constant capital than with variable capital; (iii) indeterminacy is more likely to occur when prices are modelled as jump variables than as predetermined variables; (iv) indeterminacy is less likely to occur with a greater degree of steady-state monopolistic distortions; and (v) indeterminacy is less likely to occur with a greater degree of price stickiness or with a higher steady-state inflation rate. In contrast to some existing research, our analysis indicates that capital tends to lead to macroeconomic instability by affecting firms' pricing behavior in product markets rather than households' arbitrage activity in asset markets even under forward-looking policy rules.

JEL classification: E12, E31, E52

Keywords: Capital; Indeterminacy; Forward-looking interest rate rules; Jump prices; Predetermined prices

1. Introduction

As central banks around the world have become more independent and transparent in the past two decades or so, more systematic conduct of monetary policy has become increasingly popular in the policymaking circle. Most practices of systematic monetary policy have taken the form of interest rate feedback rules that set a short-term nominal interest rate as an increasing function of expected future inflation. This trend, which began in the industrial and middle-income countries in the late 1980s, and spread to the transition and emerging-market economies in the late 1990s, has become a practical phenomenon worldwide.

Those past years have also witnessed a large body of academic research on whether interest rate rules may lead to real indeterminacy of equilibrium, which would open the door to welfare-reducing fluctuations unrelated to economic fundamentals.[1] This literature, beginning with the seminal work of Sargent and Wallace (1975) and McCallum (1981), has accumulated some insightful results, which have greatly enhanced our understanding of the issue.[2] While most of the studies focus on the implications of how the rate of interest affects consumption-savings decisions, the channel by which the rate of interest affects investment decisions has also begun to draw attention recently.

Dupor (2001) analyzes the issue of local real determinacy in a continuous-time model with a quadratic nominal price adjustment cost and endogenous capital accumulation, where the monetary authority sets a nominal interest rate as a function of the instantaneous rate of inflation. As we know, the instantaneous rate of inflation in a continuous-time setting is the right-derivative of the logged price level and, thus, the discrete-time counterpart of a continuous-time policy rule that sets the interest rate in response to the instantaneous rate

[1] In a recent analysis, King and Wolman (2004) demonstrate how discretionary monetary policy can lead to multiple equilibria and sunspot fluctuations by generating dynamic complementarity between forward-looking private agents and a discretionary monetary authority.

[2] Important contributions include Leeper (1991), Taylor (1993), Kerr and King (1996), Bernanke and Woodford (1997), Rotemberg and Woodford (1997, 1999), Christiano and Gust (1999), Clarida, Galí, and Gertler (2000), Carlstrom and Fuerst (2001), and Benhabib, Schmitt-Grohé, and Uribe (2001a, b), among many others. See, for example, the references cited in Taylor (1999) and Dupor (2001). See, also, John and Wolman (1999, 2004).

of inflation is characterized by forward-looking policy that responds to expected future inflation.[3] Carlstrom and Fuerst (2005) obtain a necessary condition for local real determinacy under such a forward-looking rule in a discrete-time model with partial nominal price adjustment and endogenous capital accumulation. While both papers emphasize the endogenous nature of capital in their modelling and, in particular, an implied no-arbitrage condition that links (expected) real return on bonds to (expected) real return on capital, prices are modelled as predetermined variables in the former but as jump variables in the latter.

In this paper, we obtain the necessary and sufficient conditions for local real determinacy in a discrete-time production economy with monopolistic competition and a quadratic cost of nominal price adjustment[4] under forward-looking interest rate rules, for the cases with constant and variable capital, and with predetermined and jump prices. To our knowledge, in the literature on discrete-time sticky-price models this paper is the first to establish a necessary and sufficient condition for real determinacy with investment under forward-looking interest rate rules. In addition to their own interests, these necessary and sufficient conditions allow us to gain important insights into the determinacy issue that have not been explored in the existing literature. Using these conditions we find that,

(i) indeterminacy is more likely to occur with a greater share of payment to capital in value-added production cost, regardless of whether capital supply is constant or variable;

(ii) indeterminacy can be either more or less likely to occur with constant capital than with variable capital, depending on the values of parameters, in general, and the cost share of capital, in particular;

(iii) indeterminacy is more likely to occur when prices are modelled as jump variables than as predetermined variables;

[3]See, for example, Obstfeld and Rogoff (1995, p. 522, Footnote 10) for an enlightening discussion of this point.

[4]This approach of modelling nominal price adjustment cost has been used in both the continuous-time and the discrete-time literature. For examples in the continuous-time setting, see Benhabib et al. (2001a, b), in addition to Dupor (2001) and others. For examples in the discrete-time setting, see Rotemberg (1982), Hairault and Portier (1993), Ireland (2000), and Kim (2000), among many others. Our choice to use this approach here allows us to deal with two price-adjustment timings often used in the literature in a unified framework in a transparent way.

(iv) indeterminacy is less likely to occur with a greater steady-state monopolistic markup of price over marginal cost;

(v) indeterminacy is less likely to occur with a greater degree of price stickiness or with a higher steady-state inflation rate.

In the process of deriving the above results, we gain insights into how capital tends to induce macroeconomic instability under forward-looking interest rate rules. The lesson is that capital matters for macroeconomic instability through affecting the pricing behavior of firms in product markets (and thus the New Phillips curve), regardless of whether it is in fixed or variable supply. This provides an alternative perspective from the intuition offered by Carlstrom and Fuerst (2005). They attribute the tendency of variable capital to inducing indeterminacy to the fact that the investment activity of households in financial markets brings a zero eigenvalue into the dynamic equilibrium system via a no-arbitrage condition between investing in capital and holding bonds, which, they argue, is what makes forward-looking interest rate rules more prone to indeterminacy than if there is no (or a fixed stock of) capital.[5]

To help make our perspective transparent, we first show that local determinacy analysis in the constant-capital case is an analysis of a system of two linear difference equations (a New Phillips curve and a consumption Euler equation) in two jump variables, so that whether there is determinacy depends on whether the two eigenvalues of this linear system both have larger than unit modulus. Capital here matters for indeterminacy through how its share in value-added production cost affects firms' pricing behavior in product markets and thus the New Phillips curve.

We then illustrate for the variable-capital case that the arbitrage activity of households in asset markets between investing in capital and bonds brings with it two additional equations,

[5]Carlstrom and Fuerst's (2005) intuition for why capital matters for macroeconomic stability under forward-looking interest rate rules contrasts with those offered under current-looking policy rules by Benhabib and Eusepi (2005) and Sveen and Weinke (2005) that emphasize how capital affects firms' pricing behavior through a cost channel. What we show here is that how capital affects firms' pricing behavior is still the key player for indeterminacy even under forward-looking policy rules.

a no-arbitrage condition and a capital accumulation equation. We show that although this introduces a zero eigenvalue (corresponding to the no-arbitrage condition) and a greater than unit eigenvalue (corresponding to the capital accumulation equation) under forward-looking interest rate rules, it also brings in two additional variables, a jump variable and a predetermined variable. We further show how these additional two equations and two variables can form a self-closed subsystem, and thus can be dropped out of the determinacy analysis entirely. We show that, as a result, local determinacy analysis here becomes an analysis of the consumption Euler equation and the New Phillips curve, just as in the constant-capital case. Just as there, whether there is determinacy here depends on whether the two eigenvalues of this two-equation system both have greater than unit modulus, and capital here matters for determinacy also through how its share in value-added production cost affects firms' pricing behavior in product markets and thus the New Phillips curve.[6]

It is worth emphasizing that the necessary and sufficient conditions for real determinacy that we derive in this paper are essential for obtaining the aforementioned results (i)-(v). We view the comparison between the cases with constant and variable capital based on (i) and (ii) an important contribution to the literature. The comparison between the cases with jump and predetermined prices based on (iii) is useful for understanding in part why indeterminacy is more likely to occur in Carlstrom and Fuerst (2005) where prices are jump variables than in Dupor (2001) where prices are predetermined variables.[7] The role of steady-state monopolistic distortions for macroeconomic stability under forward-looking policy rules as reported in (iv) is also a valuable contribution. Our findings (v) under forward-looking policy rules contrast the results obtained under current-looking rules under which indeterminacy is

[6]Given that the variable-capital case and the constant-capital case can be viewed as the case with zero and with infinite capital adjustment cost, respectively, one should not be too surprised by our finding that the zero eigenvalue brought about by the investment activity of households in the variable-capital model is not the key story for why capital can make forward-looking interest rate rules susceptible to macroeconomic instability: when one goes from the case with zero capital adjustment cost, as in our variable-capital model here, into the case with an infinitesimal capital adjustment cost, the zero eigenvalue would disappear, but the (in)determinacy region would remain almost unchanged.

[7]The choice of discrete time versus continuous time in modelling per se can also make a difference in implications for determinacy that is beyond the difference between jump prices and predetermined prices. We do not attempt to address this issue here.

more likely to occur with a greater degree of price stickiness or with a higher steady-state inflation rate.[8]

The rest of the paper proceeds as follows. Section 2 sets up the model, which incorporates the case with jump prices and the case with predetermined prices in a unified framework, and which allows for both the case with constant capital and the case with variable capital.

Section 3 derives equilibrium conditions for the constant-capital case and shows that determinacy analysis here is an analysis of a two-equation system, and that capital matters for indeterminacy through how its share in production affects the New Phillips curve.

Sections 4 and 5 derive equilibrium conditions for the variable-capital case and show how the (in)determinacy of this four-equation system is equivalent to the (in)determinacy of a two-equation system similar to the two-equation system for the constant-capital case. It is demonstrated that the basic mechanisms within the constant-capital and the variable-capital models that ensure determinacy are similar, and that endogenous capital matters for indeterminacy also through how its share in production affects the New Phillips curve.

Section 6 presents our main results: the necessary and sufficient conditions for local real determinacy under forward-looking interest rate rules, for both the constant-capital model and the variable-capital model, and with both jump prices and predetermined prices. It then applies these conditions to establish the results (i)-(v) reported above and provides some underlying intuitions. Section 7 concludes. Most proofs are relegated to the appendix.

[8]Under current-looking interest rate rules Carlstrom and Fuerst (2005) and Sveen and Weinke (2005) find that a higher degree of price stickiness tends to give rise to a greater indeterminacy region. They use models with partial nominal price adjustment, and their analysis is based on log-linearization around a zero steady-state inflation rate and thus is not able to explore the implications of steady-state inflation rate for determinacy. Using a model featuring Taylor-type staggered price setting Hornstein and Wolman (2005) find that a greater steady-state inflation rate makes indeterminacy more likely to occur under current-looking interest rate rules. Our results obtained in a separate study under current-looking interest rate rules in a model with a quadratic price adjustment cost are consistent with these authors' findings.

2. A Model with Price Adjustment Cost and Capital

Time is discrete and indexed by $t = 0, 1, \ldots$ The economy is populated by a large number of household-firm units, each producing a differentiated good and having a lifetime utility,

$$\sum_{t=0}^{\infty} \rho^t \left[\log c_t + \psi \log \frac{M_{t+1}}{\bar{P}_t} - \upsilon n_t - \frac{\gamma}{2} \left(\frac{P_{t+\mathbf{J}}}{P_{t-1+\mathbf{J}}} - \pi^* \right)^2 \right], \quad \text{for } \ \psi > 0, \ \upsilon > 0, \ \gamma > 0, \tag{1}$$

where P_t denotes the nominal price that a unit's firm charges at t for the good it produces in the period, and $\mathbf{J}$ is either 0 or 1, corresponding to two price-adjustment timings often used in discrete-time models of monopolistic competition: The specification is such that, at date t, the firm chooses $P_{t+\mathbf{J}}$, taking $P_{t-1+\mathbf{J}}$ as given; thus, prices effective in the current period are set in the current period if $\mathbf{J} = 0$, but were set in the previous period if $\mathbf{J} = 1$. As such, individual prices and, by symmetry, the price level are jump variables if $\mathbf{J} = 0$, but they are predetermined variables if $\mathbf{J} = 1$.[9] The other notations in (1) are standard: $\rho \in (0, 1)$ is a discount factor, c_t is the unit's household consumption in period t, which is a composite of goods produced by all firms, n_t is the household's labor supply in period t, $\bar{P}_t$ is the economy-wide price level, π^* is the steady-state value of the gross rate of change in the price level, and M_{t+1} is the household's nominal money balances at the end of period t.[10] Note that the linearity of the period utility function in labor hours is a consequence of

[9] This distinction between jump prices and predetermined prices is valid regardless of whether there is a price adjustment cost ($\gamma > 0$) or there is no price adjustment cost ($\gamma = 0$). In fact, the idea of modelling prices as preset in models of monopolistic competition in discrete time goes back at least to Svensson's (1986) work, which does not feature any price adjustment cost. In the continuous-time literature, this idea can be traced back to an even earlier date, to Dornbush's (1976) model with no price adjustment cost. Interestingly, subsequent studies in the continuous-time setup that do model price adjustment costs almost entirely follow this convention about the timing of price setting. See Benhabib et al. (2001a, b), Kimball (1995), in addition to Dupor (2001) and others. In contrast, subsequent studies in the discrete-time setup that do model price adjustment costs mostly choose to use the alternative timing convention and model prices as jump variables. See Rotemberg (1982), Hairault and Portier (1993), Ireland (2000), and Kim (2000), among others. Although the discrete-time model of Dupor (2003) assumes preset prices, it does not feature any price adjustment cost. To our knowledge, the present paper provides the first discrete-time model with a price adjustment cost that features prices as predetermined variables. It, in fact, models the two timing conventions in a unified framework.

[10] We adopt here the convention of end-of-period real money balances in the utility function. Were we to assume beginning-of-period real money balances in the utility function, the result obtained herein for the case with $\mathbf{J} = 1$ would hold in its exact form, but it would be quite easy to get real indeterminacy for the case with $\mathbf{J} = 0$ under the assumption of a Ricardian fiscal policy, since in this case the initial-period real

aggregation when labor is assumed to be indivisible and such a utility function is consistent with any labor supply elasticity at the individual level (e.g., Hansen, 1985; Rogerson, 1988).

At each date t, the firm inputs labor and capital services, $\tilde{n}_t$ and $\tilde{k}_t$, to produce its differentiated good y_t according to

$$y_t = \tilde{k}_t^{\alpha} \tilde{n}_t^{\beta}, \quad \text{where } \alpha \in [0, 1), \ \alpha + \beta = 1. \tag{2}$$

The firm is an input-price taker, but a monopolistic competitor in its product market. With markup pricing, α and β determine respectively the share of payments to capital and labor in value-added production cost rather than in gross output (i.e., the cost share as opposed to the revenue share), as will be made more transparent below. It is assumed that, given the price P_t that the firm charges for its product, it must produce enough to meet the demand for its good given by the right-hand side of the following equation,

$$y_t = Y_t^d \left(\frac{P_t}{\bar{P}_t} \right)^{\phi}, \tag{3}$$

where Y_t^d denotes aggregate output, which, as the consumption good, is a composite of individually differentiated goods produced via a Dixit-Stiglitz technology, and ϕ is the elasticity of substitution between the differentiated goods that is equal to the price elasticity of this demand function faced by the individual firm. Note that $\phi < -1$ is a necessary assumption for a well-defined model of monopolistic competition. As we will show below, the ratio $\phi/(\phi + 1)$ determines the steady-state monopolistic distortion measured by the markup of price over marginal cost.

The objective of the household-firm unit is to maximize (1), subject to (2), (3), and a

money balances might not be determinate even when the entire paths for consumption and the expected inflation rate were.

flow budget constraint,

$$\frac{M_{t+1}+B_t}{\bar{P}_t} = \frac{M_t + B_{t-1}R_{t-1}}{\bar{P}_t} + \frac{P_t}{\bar{P}_t}y_t - r_t\tilde{k}_t - w_t\tilde{n}_t + r_t k_t + w_t n_t - \tilde{y}_t - \tau_t, \tag{4}$$

where $\tilde{y}_t$ denotes the household's demand for the composite good in period t, k_t is its capital supply at the beginning of period t, which, as the consumption good and aggregate output, is measured in units of the composite of the individually differentiated goods, B_{t-1} is its bond-holdings acquired in period $t-1$, R_{t-1} is the gross nominal rate of return on holding the bond from $t-1$ to t, w_t and r_t are real wage and real capital rental rate, respectively, and τ_t is a real lump-sum tax (or subsidy).

The specification of the flow budget constraint (4) implies that a nominal bond carried from period $t-1$, B_{t-1}, matures at the beginning of period t, so that $B_{t-1}R_{t-1}$ is available in period t. An alternative specification is: $(M_{t+1} + B_{t+1}/R_t)/\bar{P}_t = (M_t + B_t)/\bar{P}_t + \cdots$ (e.g., Ljungqvist and Sargent, 2000). These two specifications lead to identical first-order conditions for bonds.

This setup allows for both the case where capital is in exogenously fixed supply and the case with endogenous capital accumulation. In the case with exogenous capital supply (at either aggregate or individual household level), the fixed amount of capital available for firms to hire never depreciates and new capital good is never produced, although the demand for capital by firms can still be endogenously determined.[11] In this case, we have

$$k_t = k_0, \quad \text{and} \quad \tilde{y}_t \equiv c_t, \quad \text{for all } t \geq 0, \tag{5}$$

where k_0 denotes the household's initial holding of capital, which is treated as given. In the

[11]It could be assumed alternatively that firms directly own capital in a fixed stock. Since there is no relative price distortion in a symmetric equilibrium in our present setting, it does not matter whether it is assumed that each firm has the same amount of capital never being relocated, or it is assumed that capital in a fixed stock on aggregate can be relocated among firms. Either way, k_t, $\tilde{k}_t$, and r_t would drop out of the budget constraint (4) altogether. The result to be presented below for the case with exogenous capital supply would hold in its exact form under these alternative assumptions.

case with endogenous capital accumulation, both the supply of and the demand for capital are determined endogenously. In this case, (5) is replaced with a capital accumulation equation,

$$k_{t+1} = i_t + (1-\delta)k_t, \quad \text{and} \quad \tilde{y}_t \equiv c_t + i_t, \tag{6}$$

where δ is the capital depreciation rate, and i_t denotes the household's investment in units of the composite good during period t.

3. Equilibria with Exogenous Capital Supply

At each date t, a household-firm unit chooses c_t, M_{t+1}, B_t, n_t, $\tilde{n}_t$, $\tilde{k}_t$, and $P_{t+\mathbf{J}}$ to maximize (1) subject to (2)–(5), taking as given the initial conditions M_0, B_{-1}, k_0, and $P_{\mathbf{J}-1}$, as well as the time paths for τ_t, R_t, Y_t^d, w_t, r_t, and $\bar{P}_t$. The Lagrangian is given by

$$\sum_{t=0}^{\infty}\rho^t\{[\log c_t + \psi\log\frac{M_{t+1}}{\bar{P}_t} - \upsilon n_t - \frac{\gamma}{2}(\frac{P_{t+\mathbf{J}}}{P_{t-1+\mathbf{J}}} - \pi^*)^2] + \mu_t[Y_t^d(\frac{P_t}{\bar{P}_t})^{\phi} - \tilde{k}_t^{\alpha}\tilde{n}_t^{\beta}]$$
$$+\lambda_t[\frac{M_t + B_{t-1}R_{t-1}}{\bar{P}_t} + \frac{P_t}{\bar{P}_t}\tilde{k}_t^{\alpha}\tilde{n}_t^{\beta} - r_t\tilde{k}_t - w_t\tilde{n}_t + r_t k_0 + w_t n_t - c_t - \tau_t - \frac{M_{t+1}+B_t}{\bar{P}_t}]\}.$$

The resulting first-order conditions, when coupled with the market-clearing conditions for factor inputs and equilibrium symmetry (i.e., $\tilde{k}_t = k_0$, $\tilde{n}_t = n_t$, $P_t = \bar{P}_t$), imply that,

$$\frac{1}{c_t} = \lambda_t, \tag{7}$$

$$\frac{\psi}{M_{t+1}} = \frac{\lambda_t}{\bar{P}_t} - \rho\frac{\lambda_{t+1}}{\bar{P}_{t+1}}, \tag{8}$$

$$\frac{R_t}{\pi_t} = \frac{\lambda_t}{\rho\lambda_{t+1}}, \tag{9}$$

$$\frac{v}{w_t} = \lambda_t, \tag{10}$$

$$\frac{w_t \tilde{n}_t}{y_t} = \beta \left(1 - \frac{\mu_t}{\lambda_t}\right), \tag{11}$$

$$\frac{r_t \tilde{k}_t}{y_t} = \alpha \left(1 - \frac{\mu_t}{\lambda_t}\right), \tag{12}$$

$$\frac{\gamma \pi_{t+\mathbf{J}-1}(\pi_{t+\mathbf{J}-1} - \pi^*) - \rho\gamma\pi_{t+\mathbf{J}}(\pi_{t+\mathbf{J}} - \pi^*)}{\rho^{\mathbf{J}} y_{t+\mathbf{J}}} = \phi\mu_{t+\mathbf{J}} + \lambda_{t+\mathbf{J}}, \tag{13}$$

and the market-clearing condition for the composite good is simply $c_t = y_t$ $(= \tilde{k}_t^{\alpha}\tilde{n}_t^{\beta})$. Here we have used π_t to denote the expected inflation rate $\bar{P}_{t+1}/\bar{P}_t$ so our notion is consistent with the one used in the continuous-time setup where the instantaneous rate of inflation is given by the right-hand derivative of the log of the price level. This also implies that the discrete-time counterpart of a standard continuous-time nominal interest rate rule that responds to the instantaneous rate of inflation is characterized by a forward-looking rule under which the monetary authority sets the nominal interest rate as a function of the expected future inflation rate. For the purpose of our local analysis, it is without loss of generality to consider a linear rule

$$R_t = R^* + q(\pi_t - \pi^*), \tag{14}$$

where $q \geq 0$ measures the degree of activeness (or passiveness) of the policy around the steady state and R^* denotes the steady-state value of the nominal interest rate.

Combining the steady-state versions of (11), (12), and (13), one can show that the ratio $\phi/(\phi+1)$, given by the inverse of $(1-\mu/\lambda)$, is equal to the steady-state markup of price over unit cost, which is also marginal cost under the assumed constant-return-to-scale production technology. Note also that, with markup pricing, the share of payment to capital (labor) input in value-added production cost (i.e., the cost share) equals the product of the share of capital (labor) input in gross output (i.e., the revenue share) and the steady-state markup of price over marginal cost. It then follows from (12) and (11) that α and β measure the share of payments to capital and labor, respectively, in value-added production cost.

Before proceeding further, it is useful to simplify the equilibrium conditions (7)-(13) into a system of equations that are key to our determinacy analysis.[12] One such equation is derived by substituting (7), (10), and (11) into (13), which yields

$$\gamma\pi_{t+\mathbf{J}-1}(\pi_{t+\mathbf{J}-1}-\pi^*)-\rho\gamma\pi_{t+\mathbf{J}}(\pi_{t+\mathbf{J}}-\pi^*)=\rho^{\mathbf{J}}(1+\phi)\frac{y_{t+\mathbf{J}}}{c_{t+\mathbf{J}}}-\rho^{\mathbf{J}}\frac{\phi}{\beta}\upsilon n_{t+\mathbf{J}}. \tag{15}$$

Another such equation is obtained by combining (7) with (9), which gives rise to

$$\frac{R_t}{\pi_t}=\frac{c_{t+1}}{\rho c_t}. \tag{16}$$

Determinacy analysis in this case requires (15) and (16), along with the policy rule (14). Here, capital matters through how its share in value-added production cost ($\alpha=1-\beta$) affects the New Phillips curve (15), and policy matters through its interaction with households' intertemporal consumption decisions, as prescribed by (16).

In deriving (15) and (16), we have used neither (8) nor (12). Equation (8) is not used, since under an interest rate policy rule, money supply is endogenously determined, and real money balances $M_{t+1}/\bar{P}_t$ is solved as a residual from (8), once the paths for consumption and the expected inflation rate are pinned down. Yet, (8) does imply that, for real determinacy, not only real quantities but the expected inflation rate have to be determinate in equilibrium. Equation (12) is not used since in the present case with exogenous capital supply, this capital demand equation plays only a residual role to pin down equilibrium capital rental rate.

To proceed, denote by x^* the steady-state value of a variable x_t, and $\hat{x}_t \equiv \log x_t - \log x^*$

[12] Given the assumption of a Ricardian fiscal policy, neither the government's public budget constraint nor the household's private budget constraint is of any use for our equilibrium determinacy analysis.

the percentage deviation of the variable from its steady-state value. It follows that

$$x_t = x^* \mathrm{e}^{\hat{x}_t} = x^*(1 + \hat{x}_t) + \mathcal{O}(\|\hat{x}_t\|^2),$$

where $\mathcal{O}(\|\hat{x}_t\|^2)$ summarizes terms of the second or higher orders. Using this expression to rewrite (14), (15), and (16), and dropping all terms higher than the first order, we have

$$\rho\gamma\pi^{*2}\hat{\pi}_{t+\mathbf{J}} = \gamma\pi^{*2}\hat{\pi}_{t+\mathbf{J}-1} + \rho^{\mathbf{J}}\frac{\phi v n^*}{\beta^2}\hat{c}_{t+\mathbf{J}}, \tag{17}$$

$$\hat{c}_{t+1} = (\rho q - 1)\hat{\pi}_t + \hat{c}_t. \tag{18}$$

Local determinacy analysis in this case with exogenous capital supply involves an analysis of the log-linearized New Phillips curve, (17), and consumption Euler equation, (18), as linear difference equations in two jump variables, π and c, where the log-linearized policy rule is embedded. Thus, whether there is determinacy depends on whether the two eigenvalues of this linear system both have larger than unit modulus. Capital here matters for determinacy through how its share in value-added production cost ($\alpha = 1 - \beta$) affects firms' pricing behavior in product markets and thus the New Phillips curve.

4. Equilibria with Endogenous Capital Accumulation

At each date t, in addition to those choice variables specified in the previous section, a household also chooses capital to be supplied in the next date, k_{t+1}, to maximize (1) subject to (2), (3), (4), and (6), taking as given the initial conditions and the time paths for aggregate variables, as before. The Lagrangian is given by

$$\sum_{t=0}^{\infty}\rho^t\{[\log c_t+\psi\log\frac{M_{t+1}}{\bar{P}_t}-vn_t-\frac{\gamma}{2}(\frac{P_{t+\mathbf{J}}}{P_{t-1+\mathbf{J}}}-\pi^*)^2]$$
$$+\mu_t[Y_t^d(\frac{P_t}{\bar{P}_t})^\phi-\tilde{k}_t^\alpha\tilde{n}_t^\beta]+\lambda_t[\frac{M_t+B_{t-1}R_{t-1}}{\bar{P}_t}$$
$$+\frac{P_t}{\bar{P}_t}\tilde{k}_t^\alpha\tilde{n}_t^\beta-r_t\tilde{k}_t-w_t\tilde{n}_t+r_tk_t+w_tn_t-k_{t+1}+(1-\delta)k_t-c_t-\tau_t-\frac{M_{t+1}+B_t}{\bar{P}_t}]\}.$$

The resulting equilibrium conditions contain (7)-(13), as in their exact forms, plus the first-order condition for the household's endogenous capital supply,

$$r_{t+1}+1-\delta=\frac{\lambda_t}{\rho\lambda_{t+1}}. \tag{19}$$

This capital supply equation, when coupled with the bond-holding condition (9), gives rise to a no-arbitrage condition,

$$\frac{R_t}{\pi_t}=r_{t+1}+1-\delta, \tag{20}$$

which links the expected real return on bonds to the expected real return on capital when there is no arbitrage opportunity between investing in these two types of assets. In addition, there is now the capital accumulation equation which, when combined with the market-clearing condition for the composite good, gives rise to a difference equation,

$$k_{t+1}=k_t^\alpha n_t^\beta+(1-\delta)k_t-c_t. \tag{21}$$

Here, capital matters through how its share in value-added production cost affects (21), in addition to (15), and policy matters through interacting with household investment decisions, as prescribed by (20), in addition to consumption decisions prescribed by (16).

The first-order approximation to these four equations and the policy rule gives rise to the following log-linearized system of equilibrium conditions:

$$\rho\gamma\pi^{*2}\hat{\pi}_{t+\mathbf{J}} = \gamma\pi^{*2}\hat{\pi}_{t+\mathbf{J}-1} + \rho^{\mathbf{J}}\frac{\phi v n^*}{\beta}[\beta\hat{c}_{t+\mathbf{J}} + \alpha\hat{r}_{t+\mathbf{J}}], \tag{22}$$

$$\hat{c}_{t+1} = (\rho q - 1)\hat{\pi}_t + \hat{c}_t, \tag{23}$$

$$\hat{r}_{t+1} = \frac{\rho q - 1}{\rho r^*}\hat{\pi}_t, \tag{24}$$

$$\hat{k}_{t+1} = -[\beta\delta + (\beta+1)\frac{c^*}{k^*}]\hat{c}_t + \beta(\delta + \frac{c^*}{k^*})\hat{r}_t + (1 + \frac{c^*}{k^*})\hat{k}_t. \tag{25}$$

In deriving (22), we have used a equilibrium condition, $\beta r_t k_t = \alpha v c_t n_t$, which is implied by (7), (10), (11), as well as the capital demand equation (12). We have also used this condition in writing (25).

We argue that local determinacy analysis in this case with endogenous capital accumulation is essentially an analysis of the log-linearized New Phillips curve (22) and consumption Euler equation (23), just as in the case where capital is in fixed supply. What the endogenous nature of capital supply essentially does is to introduce two additional equations, (24) and (25), an additional jump variable, r, along with a predetermined variable, k, into the system for determinacy analysis. However, as we will show below, by doing so it introduces at the same time a zero eigenvalue, corresponding to (24), and a greater than unit eigenvalue, corresponding to (25), into the system of the four linear difference equations (22)-(25). As a result, whether there is determinacy hinges entirely on whether the two eigenvalues of the system of the consumption Euler equation and the New Phillips curve both have greater than unit modulus. Capital here matters for determinacy through how its share in value-added production cost affects firms' pricing behavior in product markets and thus the New Phillips curve, just as in the case with exogenous capital supply. Although the specific details of this effect are different across the two models, the basic mechanisms by which capital tends to lead to macroeconomic instability turn out similar.

5. A Further Comparison

To further illustrate the comparison laid out in the previous sections, consider $\mathbf{J} = 1$. The comparison under $\mathbf{J} = 0$, although a little more complicated, is similar.

By embedding how policy affects households' intertemporal consumption decisions (18) into the New Phillips curve (17), the system of the two equations (17) and (18) for the case with exogenous capital supply can be written as

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} + \frac{(1+\phi)(\rho q-1)}{\gamma\pi^{*2}\beta} & \frac{1+\phi}{\gamma\pi^{*2}\beta} \\ \rho q - 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{c}_t \end{bmatrix}. \tag{26}$$

Since both π_t and c_t are jump variables, this two-equation system is determinate if and only if the two eigenvalues of the system both have larger than unit modulus.

By embedding how policy affects households' consumption and investment decisions (23) and (24) into the New Phillips curve (22), the four-equation system (22)-(25) for the case with endogenous capital accumulation can be written as

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{c}_{t+1} \\ \hat{r}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} + \frac{(\gamma\pi^{*2})^{-1}\phi(1+\phi)(\rho q-1)[1-\rho(1-\delta)\beta]}{\phi[1-\rho(1-\delta)]-(1+\phi)\rho\delta(1-\beta)} & \frac{(\gamma\pi^{*2})^{-1}\phi(1+\phi)[1-\rho(1-\delta)]\beta}{\phi[1-\rho(1-\delta)]-(1+\phi)\rho\delta(1-\beta)} & 0 & 0 \\ \rho q - 1 & 1 & 0 & 0 \\ \frac{\rho q-1}{1-\rho+\rho\delta} & 0 & 0 & 0 \\ 0 & -[\beta\delta + (\beta+1)\frac{c^*}{k^*}] & \beta(\delta + \frac{c^*}{k^*}) & 1 + \frac{c^*}{k^*} \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{r}_t \\ \hat{k}_t \end{bmatrix}$$

This linear system always has a zero eigenvalue, corresponding to the third line, and a greater than unit eigenvalue, $1 + c^*/k^*$, corresponding to the fourth line. It is easy to show that this four-equation system is determinate if and only if the two eigenvalues of the following two-equation system,

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} + \frac{\phi(1+\phi)(\rho q-1)[1-\rho(1-\delta)\beta]}{\gamma\pi^{*2}\{\phi[1-\rho(1-\delta)]-(1+\phi)\rho\delta(1-\beta)\}} & \frac{\phi(1+\phi)[1-\rho(1-\delta)]\beta}{\gamma\pi^{*2}\{\phi[1-\rho(1-\delta)]-(1+\phi)\rho\delta(1-\beta)\}} \\ \rho q - 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{c}_t \end{bmatrix}, \qquad (27)$$

both have larger than unit modulus, just as in the case of (26). Notice that the two-equation system (27) results from removing the last two lines in the above four-equation system. This essentially removes a jump variable, r_t, and a predetermined variable, k_t, along with the two eigenvalues, 0 and $1 + c^*/k^*$, from the original system. Therefore, (in)determinacy of the original four-equation system is equivalent to (in)determinacy of the reduced two-equation system (27). Note the similarity between (26) and (27): the second lines (the consumption Euler equation) are identical, and β affects only the first lines (the New Phillips curve). The difference between the two first lines is that the first line in (26) embeds how policy affects households' consumption decisions, while the first line in (27) embeds how policy affects households' consumption as well as investment decisions. Thus, the basic mechanisms within exogenous-capital and endogenous-capital models by which capital tends to lead to macroeconomic instability are similar, while the specific details of this effect are different across the two cases.

As we will show in the following section, because of the similarity in the basic mechanisms, a greater cost share of capital (a larger α and thus a smaller β), regardless of whether capital supply is modelled as exogenous or as endogenous, always leads to a smaller determinacy region.

As we will illustrate, also in the subsequent section, because of the difference in the specific details of this effect, the determinacy region with exogenous capital supply can be smaller or larger than the determinacy region with endogenous capital accumulation, depending on the values of parameters, in general, and the cost share of capital, in particular.

6. Main Results

6.1. Necessary and sufficient conditions for determinacy

In this section, we present our main results: the necessary and sufficient condition for local real determinacy for the endogenous-capital model as well as for the exogenous-capital model. To our knowledge, in the literature on discrete-time sticky-price models this paper is the first to derive a necessary and sufficient condition for real determinacy with endogenous capital accumulation under forward-looking interest rate policy rules.

To drive home the points illustrated in the previous sections, we show first the necessary and sufficient condition for local real determinacy for the exogenous-capital model.

Proposition 1. *With exogenous capital supply, there is local real determinacy under the forward-looking policy rule* (14) *if and only if*

$$0 < \rho q - 1 < \frac{2\gamma\pi^{*2}\beta(1+\rho)}{-\rho^{\mathbf{J}}(1+\phi)} \equiv \bar{B}_1. \tag{28}$$

Otherwise, there is a continuum of equilibria.

We next present the necessary and sufficient condition for local real determinacy for the endogenous-capital model.

Proposition 2. *With endogenous capital accumulation, there is local real determinacy under the forward-looking policy rule* (14) *if and only if*

$$0 < \rho q - 1 < \frac{\gamma\pi^{*2}(1-\rho)[\rho\delta - \phi(1-\rho) - (1+\phi)\beta\rho\delta]}{(1-\beta)\rho^{\mathbf{J}}\phi(1+\phi)} \equiv \bar{B}_2, \quad \text{for} \quad \beta < \beta^*, \tag{29}$$

$$0 < \rho q - 1 < \frac{2\gamma\pi^{*2}(1+\rho)[\rho\delta - \phi(1-\rho) - (1+\phi)\beta\rho\delta]}{[2-\beta(1+\rho-\rho\delta)]\rho^{\mathbf{J}}\phi(1+\phi)} \equiv \bar{\bar{B}}_2, \quad \text{for} \quad \beta \geq \beta^*, \tag{30}$$

where

$$\beta^* \equiv \frac{4\rho}{(1+\rho)^2 + \rho(1-\rho)\delta} \in (0,1).$$

Otherwise, there is a continuum of equilibria.

Propositions 1 and 2 say that there is real determinacy of equilibrium if and only if policy is active, in the spirit of the Taylor principle, but not more active than what is specified by the upper bound in (28) for the constant-capital model or in (29)-(30) for the variable-capital model.[13] Clearly, these upper bounds are proportional to γ, and thus the greater (smaller) this price adjustment cost coefficient is, the larger (smaller) is the determinacy region, with either exogenous capital or endogenous capital, and under both timing conventions for price-setting.

Besides their own interests, these necessary and sufficient conditions make it feasible to compare the determinacy region for exogenous capital supply and the determinacy region for endogenous capital accumulation under forward-looking interest rate rules. They also allow us to gain important insights into the determinacy issue under forward-looking policy rules that have not attracted much attention before.

6.2. Cost share of capital

The necessary and sufficient conditions established in Propositions 1 and 2 for the case where capital is in exogenously fixed supply and for the case with endogenous capital accumulation reveal an essential implication of capital for indeterminacy.

It can be verified that the upper bound in (28) for the fixed-capital model and the upper bound in (29)-(30) for the variable-capital model are each increasing in β and thus decreasing in α (note that $\bar{B}_2$ and $\bar{\bar{B}}_2$ meet only at $\beta = \beta^*$). Therefore, regardless of whether capital supply is modelled as exogenous or as endogenous, a larger share of capital in value-added

[13] Under an alternative, non-linear policy rule, such as $R_t = R^*(\pi_t/\pi^*)^q$, Propositions 1 and 2 continue to hold with the only modification being $\rho q - 1$ replaced with $q - 1$. The linear specification adopted in this paper is in the spirit of Leeper (1991), who considers linear backward-looking interest rate policy in a discrete-time flexible price model.

production cost (a larger α and thus a smaller β) always results in a smaller determinacy region. This is true under both conventions about the timing of price-setting.

These results arise because the basic mechanisms within the exogenous-capital and the endogenous-capital models by which capital tends to lead to macroeconomic instability are similar: it matters for real indeterminacy through affecting firms' pricing behavior in product markets rather than households' arbitrage activity in financial markets. Nevertheless, the specific details of this effect are different across the two models and, as a consequence, indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the values of parameters, in general, and the cost share of capital, in particular, as we turn to illustrate now.

6.3. Constant capital versus variable capital

Examining the upper bound in (28) for the constant-capital model and the upper bound in (29)-(30) for the variable-capital model also reveals a theoretical possibility not explored before: the reduction in the determinacy region due to the use of capital in production can be either greater or smaller when capital supply is modelled as exogenous than as endogenous. In other words, from the theoretical point of view, indeterminacy can be either less likely or more likely to occur in the constant-capital model than in the variable-capital model.

For instance, it can be shown that, if the cost share of capital is very large, then the determinacy region with exogenous capital is smaller than that with endogenous capital. This can be illustrated by considering a close-to-one α, and thus a close-to-zero β. Then, $\bar{B}_1$ is close to zero and the determinacy region defined by (28) is close to an empty set, while $\bar{B}_2$ and $\bar{\bar{B}}_2$ are each bounded from below by strictly positive numbers ($\bar{B}_2$ is what really matters here) and the determinacy region defined by (29)-(30) stays as a non-empty interval (what really matters here is (29)). To understand this contrast, compare the first line in (26) with the first line in (27). As α gets larger and thus β gets smaller, the response of inflation expectations to even small variations in current inflation and output becomes unboundedly sensitive in the case with exogenous capital supply (the first line in (26)), but stays bounded in the case with endogenous capital supply (the first line in (27)). Since production uses mostly capital, the effects of changes in aggregate economic conditions are drastic changes in inflation expectations if capital is in fixed supply, but they are more evenly split between changes in inflation expectations and changes in expected quantity variables if capital is in varying supply.

In the case that the share of (variable) labor in value-added production inputs is moderate or large, the effects of changes in aggregate demand conditions are always split somewhat evenly between changes in inflation expectations and changes in expected quantity variables, regardless of whether capital is in varying or fixed supply, and thus the sensitivity of inflation expectations to variations in current inflation and output is bounded in both the case with endogenous capital supply and the case with exogenous capital supply. In fact, the determinacy region with exogenous capital is larger than the determinacy region with endogenous capital if the share of labor (capital) in value-added production inputs is very large (small), or, more specifically, if

$$\max\{\beta^*, \beta^{**}\} < \beta < 1, \tag{31}$$

where

$$\beta^{**} \equiv \frac{1 - \rho - \frac{\rho\delta}{\phi}}{1 + \rho - \rho\delta} \in (0, 1),$$

and β^* is as defined in Proposition 2. In particular, one can show that $\bar{B}_1$ is larger than $\bar{\bar{B}}_2$ if and only if the quadratic concave function $f(\beta) \equiv \phi(1 + \rho - \rho\delta)\beta^2 + [(1 + \phi)\rho\delta - 2\phi]\beta + [\phi(1 - \rho) - \rho\delta]$ is strictly positive. It can be verified that the equation $f(\beta) = 0$ has two distinct real roots, equal to 1 and β^{**}, respectively. Again, the analysis conducted in this section holds regardless of which convention about the timing of price-setting is used.

6.4. Timing of price adjustment

Recall that our theoretical approach allows us to model the two conventions about the timing of price adjustment often used in models of monopolistic competition in a unified framework. This unified approach leads to the following useful point.

The necessary and sufficient conditions (28) and (29)-(30) presented in Propositions 1 and 2 reveal that the choice of one timing convention versus the other has an implication for local real determinacy. In particular, these conditions show that determinacy is more likely to occur when prices are modelled as predetermined variables (i.e., when $\mathbf{J} = 1$) than as jump variables (i.e., when $\mathbf{J} = 0$). With a smaller discount factor (say, in a lower-frequency model), the increase in the likelihood of determinacy owing to price being preset can be more dramatic.[14] This conclusion holds regardless of whether capital is modelled as exogenous or as endogenous.

6.5. Steady-state monopolistic distortions

An interesting implication of Propositions 1 and 2, as can be seen from examining the upper bounds in (28) and (29)-(30), is that a greater ϕ always leads to a larger region for local real determinacy. It follows that indeterminacy is less likely to occur, the greater is the degree of steady-state monopolistic distortions, that is, the steady-state markup of price over marginal cost. This observation holds regardless of whether capital supply is modelled as exogenous or as endogenous, or which of the two conventions about the timing of price adjustment is used.

[14]Note that, with $\mathbf{J} = 1$, prices are preset by only one period. In the literature, the case that prices are set multi-periods in advance has also been considered. We conjecture that Propositions 1 and 2 would continue to hold for $\mathbf{J} > 1$ and determinacy could be even more likely to occur if prices are set more than one period in advance than if prices are set just one period in advance.

6.6. Price stickiness and steady-state inflation

It is clear that both the upper bound in (28) for the constant-capital model and the upper bound in (29)-(30) for the variable-capital model are proportional to $\gamma\pi^{*2}$ and, thus, a greater degree of price stickiness or a greater steady-state inflation rate implies a larger region for local real determinacy under forward-looking interest rate rules. These observations hold regardless of whether capital is in fixed or variable supply, or which of the two timings of price adjustment is used.

These implications of price stickiness and steady-state inflation rate for indeterminacy under forward-looking interest rate rules are exactly the opposite of those obtained under current-looking interest rate rules with endogenous capital accumulation: Under current-looking policy rules indeterminacy is more likely to occur with a greater degree of price stickiness (e.g., Carlstrom and Fuerst, 2005; Sveen and Weinke, 2005) or with a higher steady-state inflation rate (e.g., Hornstein and Wolman, 2005). In a separate study under current-looking interest rate rules we obtain results similar to those obtained by these authors. Because of the similarity we do not report our results under current-looking interest rate rules, but they are available upon request.

7. Conclusion

As monetary policymakers around the world have moved into the framework of setting a nominal interest rate in response to changes in expected future inflation for the conduct of monetary policy, it is of paramount interest to know whether such policy practice may lead to real indeterminacy in a modern production economy where capital is an important factor input.

The present paper addresses this issue by deriving the necessary and sufficient conditions for local real determinacy in a discrete-time economy with monopolistic competition and a quadratic nominal price adjustment cost under forward-looking interest rate feedback rules, for the case where capital is in exogenously fixed supply, as well as the case with endogenous capital accumulation.

Applying these conditions, we find that indeterminacy is more likely to occur with a greater share of capital in value-added production cost, regardless of whether capital supply is modelled as exogenous or as endogenous. We show that these results arise because, under forward-looking interest rate rules, the basic mechanisms by which capital tends to lead to macroeconomic instability are similar within the constant-capital model and the variable-capital model: capital matters for real indeterminacy by affecting firms' pricing behavior in product markets. Thus our perspective for the endogenous capital model differs from the one offered by Carlstrom and Fuerst (2005) who argue that with forward-looking interest rate rules capital matters for indeterminacy by affecting households' arbitrage activity in financial markets, but is consistent with the view of Benhabib and Eusepi (2005) and Sveen and Weinke (2005) which is obtained under current-looking policy rules. We also show that the specific details of the effect of capital on firms' pricing behavior under forward-looking interest rate rules are different across the constant-capital model and the variable-capital model and, as so, indeterminacy can be either more likely or less likely to occur with exogenous capital than with endogenous capital, depending on the values of parameters, in general, and the cost share of capital, in particular.

The necessary and sufficient conditions also lead us to another interesting observation. While two different conventions on the timing of price adjustment have been used in the literature, we find that indeterminacy is more likely to occur when prices are modelled as jump variables than as predetermined variables, and this contrast is sharper in lower-frequency models, regardless of whether capital supply is modelled as an exogenous or as an endogenous variable. These conditions also suggest that indeterminacy is more likely to occur with a smaller degree of steady-state monopolistic distortions.

Finally, these necessary and sufficient conditions allow us to reach two conclusions under forward-looking interest rate rules which are exactly the opposite of those under current-looking interest rate rules with endogenous capital accumulation: A greater degree of price stickiness or a higher steady-state inflation rate makes indeterminacy less likely to occur under forward-looking interest rate rules while in contrast either makes indeterminacy more likely to occur under current-looking interest rate rules.

Our present analysis raises an important question as to how to ensure macroeconomic stability in a modern production economy with endogenous capital accumulation in which a monetary authority systematically adjusts a nominal interest rate to changes in expected future inflation. Existing studies reveal that interest rate policies that respond to current inflation may render a greater determinacy region if they also respond to current output activity. We have started exploring this issue for interest rate policies that respond to expected future inflation and our preliminary findings suggest that allowing the nominal interest rate to also respond to movements in current output activity can help a great deal in terms of achieving real determinacy, in both the model where capital is in fixed supply and the model with endogenous capital accumulation. This issue is so important that it, in our view, deserves the full attention of a separate paper.

Appendix

To help exposition, we shall prove Proposition 2 first.

Proof of Proposition 2: We introduce some auxiliary notations to facilitate the proof:

$$\epsilon \equiv \rho q - 1, \; a \equiv \tfrac{\phi v n^*}{\gamma \pi^{*2}}, \; b \equiv -a\left(1 + \frac{\alpha}{\rho \beta r^*}\right).$$

We first prove the theorem for the case with $\mathbf{J} = 1$. The system (22)-(25) can be written as

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{c}_{t+1} \\ \hat{r}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} - b\epsilon & a & 0 & 0 \\ \epsilon & 1 & 0 & 0 \\ \frac{\epsilon}{\rho r^*} & 0 & 0 & 0 \\ 0 & -[\beta\delta + (\beta+1)\frac{c^*}{k^*}] & \beta(\delta + \frac{c^*}{k^*}) & 1 + \frac{c^*}{k^*} \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{r}_t \\ \hat{k}_t \end{bmatrix}.$$

The above 4×4 matrix has four eigenvalues, two of which are independent of the policy rule: a zero eigenvalue and a larger than unit eigenvalue given by $1 + \frac{c^*}{k^*}$. The other two are policy dependent and given by the two eigenvalues of the upper left 2×2 sub-matrix, which can be obtained by solving for the two roots of the following quadratic equation in λ

$$D(\lambda) = \lambda^2 - \left(1 + \frac{1}{\rho} - b\epsilon\right)\lambda + \left(\frac{1}{\rho} - b\epsilon - a\epsilon\right) = 0. \tag{32}$$

It follows from $a < 0$, $b > 0$, and $b + a > 0$ that, for any $\epsilon < 0$,

$$D(0) = \frac{1}{\rho} - (b+a)\epsilon > 0, \; D(1) = -a\epsilon < 0, \; D(+\infty) > 0,$$

implying that there are two real roots of (32), one is strictly between 0 and 1 and the other is larger than 1. Hence a necessary condition for determinacy is for $\epsilon \geq 0$. This can be verified by computing the two quadratic roots explicitly as

$$\lambda_1(\epsilon) = \frac{(1+\frac{1}{\rho}-b\epsilon)+\sqrt{(1-\frac{1}{\rho}+b\epsilon)^2+4a\epsilon}}{2},$$

$$\lambda_2(\epsilon) = \frac{(1+\frac{1}{\rho}-b\epsilon)-\sqrt{(1-\frac{1}{\rho}+b\epsilon)^2+4a\epsilon}}{2}.$$

Clearly, $\lambda_1(0) = \frac{1}{\rho} > 1$ and $\lambda_2(0) = 1$. It can also be verified that, for any $\epsilon < 0$,

$$\lambda_1(\epsilon) > \lambda_1(0) > 1,\ 0 < \lambda_2(\epsilon) < \lambda_2(0) = 1.$$

This validates our claim. On the other side, it is straightforward to show that a policy that is marginally more active than the benchmark case with $\epsilon = 0$ suffices to ensure determinacy. To see this, compute the derivatives of the two quadratic roots with respect to ϵ and evaluate them at $\epsilon = 0$. We have

$$\left.\frac{\partial\lambda_1(\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} = -\frac{b(\frac{1}{\rho}-1)-a}{\frac{1}{\rho}-1} < 0,$$
$$\left.\frac{\partial\lambda_2(\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} = -\frac{a}{\frac{1}{\rho}-1} > 0.$$

Of course, only the signs of the right derivatives obtained above are of value, given the global results for the case with $\epsilon < 0$. The signs of the right derivatives imply that when policy becomes a little bit more active than the benchmark case with $\epsilon = 0$, the larger-than-unit root decreases from its value of $1/\rho$ (at $\epsilon = 0$), but remains larger than 1 as long as policy is not too much more active, while the unit root (at $\epsilon = 0$) rises above 1, as the roots are continuous functions of ϵ. Thus, such a policy would lead to determinacy.

To generalize this local result to the global one characterized by the theorem, we first note that $D(1) = -a\epsilon \geq 0$ for any $\epsilon \geq 0$. This combined with the above observation implies that (32) can never have two real roots with one larger than 1 and the other smaller than

-1. Therefore, determinacy obtains if and only if (32) has two real roots with both larger than 1, or two real roots with both smaller than -1, or a pair of complex roots the module of which is larger than 1. We proceed next to characterize the range for ϵ under which one of the three mutually exclusive possibilities is realized.

We first note that, for either of the first two possibilities, it is necessary that $D(-1) > 0$, which is true if and only if

$$\epsilon < \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}. \tag{33}$$

We next note that $\lambda^* = (1 + 1/\rho - b\epsilon)/2$ is the value of λ that minimizes $D(\lambda)$. Clearly, $\lambda^* > 1$ if and only if

$$\epsilon < \frac{\frac{1}{\rho} - 1}{b}, \tag{34}$$

which is a necessary condition for both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be real and larger than 1, while $\lambda^* < -1$ if and only if

$$\epsilon > \frac{\frac{1}{\rho} + 3}{b}, \tag{35}$$

which is a necessary condition for both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be real and smaller than -1.

Third, we note that the term under the square-root operator in the expressions for $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ can be viewed as a quadratic function of ϵ:

$$\Delta(\epsilon) = b^2\epsilon^2 - \left[2\left(\frac{1}{\rho} - 1\right)b - 4a\right]\epsilon + \left(\frac{1}{\rho} - 1\right)^2, \tag{36}$$

and that $\Delta(\epsilon) = 0$ always has two distinguished and strictly positive real roots,

$$\epsilon_1 = \left[\frac{\sqrt{(\frac{1}{\rho} - 1)b - a} - \sqrt{-a}}{b}\right]^2,$$

$$\epsilon_2 = \left[\frac{\sqrt{(\frac{1}{\rho} - 1)b - a} + \sqrt{-a}}{b}\right]^2.$$

Thus, in order for $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be two real numbers with absolute values larger than 1, it is necessary that

$$0 < \epsilon \leq \epsilon_1 \qquad \text{or} \qquad \epsilon \geq \epsilon_2, \tag{37}$$

while under (37), both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ must be real. It is easy to verify that

$$\epsilon_1 < \frac{\frac{1}{\rho} - 1}{b} < \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}. \tag{38}$$

This implies that, if $0 < \epsilon \leq \epsilon_1$, then $\Delta(\epsilon) \geq 0$, $\lambda^* > 1$, and $D(1) > 0$, and thus both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ must be real and larger than 1. On the other hand, it is easy to show that

$$\epsilon_2 > \frac{\frac{1}{\rho} - 1}{b}. \tag{39}$$

This implies that, if $\epsilon \geq \epsilon_2$, then $\Delta(\epsilon) \geq 0$, $\lambda^* < 1$, and $D(1) > 0$, and thus both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ must be real but smaller than 1. We have therefore established that both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ are real and larger than 1 if and only if $0 < \epsilon \leq \epsilon_1$.

This also implies that, in order for $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ to be both real and smaller than -1, it is necessary that $\epsilon \geq \epsilon_2$, besides that (33) and (35) have to hold. On the other side, if $\epsilon \geq \epsilon_2$, and (33) and (35) also hold, then $\Delta(\epsilon) \geq 0$, $\lambda^* < -1$, and $D(-1) > 0$, and thus both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ must be real and smaller than -1. We have therefore established that both $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ are real and smaller than -1 if and only if $\epsilon \geq \epsilon_2$, and (33) and (35) hold.

Finally, it is clear that $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ are a pair of complex conjugates, the module of which is larger than 1 if and only if

$$\epsilon_1 < \epsilon < \epsilon_2, \tag{40}$$

and

$$(1 + \frac{1}{\rho} - b\epsilon)^2 - (1 - \frac{1}{\rho} + b\epsilon)^2 - 4a\epsilon > 4. \tag{41}$$

It can be shown that (41) holds if and only if

$$\epsilon < \frac{\frac{1}{\rho} - 1}{b + a}. \tag{42}$$

Taking the above three cases together, we have established the necessary and sufficient condition for determinacy as

$$\epsilon \in \left(0, \min\left\{\epsilon_2, \frac{\frac{1}{\rho} - 1}{b + a}\right\}\right) \cup \left(\max\left\{\epsilon_2, \frac{\frac{1}{\rho} + 3}{b}\right\}, \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}\right), \tag{43}$$

with the understanding that if the first element under the "max" operator is greater than the second one, then the second interval is left-closed. Using the steady-state real rate of return on capital $r^* = 1/\rho - 1 + \delta$, and going through some algebra, we show that

$$\beta < \frac{4\rho}{(1+\rho)^2 + \rho(1-\rho)\delta} \quad \longrightarrow\!\!\longrightarrow\!\!\longrightarrow \quad \frac{\frac{1}{\rho} - 1}{b + a} < \epsilon_2 < \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}} < \frac{\frac{1}{\rho} + 3}{b}, \tag{44}$$

$$\beta = \frac{4\rho}{(1+\rho)^2 + \rho(1-\rho)\delta} \quad \longrightarrow\!\!\longrightarrow\!\!\longrightarrow \quad \frac{\frac{1}{\rho} - 1}{b + a} = \epsilon_2 = \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}} = \frac{\frac{1}{\rho} + 3}{b}, \tag{45}$$

$$\beta > \frac{4\rho}{(1+\rho)^2 + \rho(1-\rho)\delta} \quad \longrightarrow\!\!\longrightarrow\!\!\longrightarrow \quad \frac{\frac{1}{\rho} + 3}{b} < \epsilon_2 < \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}} < \frac{\frac{1}{\rho} - 1}{b + a}. \tag{46}$$

Under (44), the second interval in (43) is an empty set while the first interval reduces to

$$\left(0, \frac{\frac{1}{\rho} - 1}{b + a}\right).$$

Under (45) or (46), the union of the two intervals in (43) collapses into a single interval

$$\left(0, \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}\right).$$

These, together with the steady-state relation,

$$vn^* = \frac{\beta(\frac{1}{\rho} - 1 + \delta)}{\frac{\phi}{1+\phi}(\frac{1}{\rho} - 1 + \delta) - \alpha\delta},$$

completes the proof of the theorem for the case with $\mathbf{J} = 1$.

We turn now to proving the theorem for the case with $\mathbf{J} = 0$. The system (22)-(25) can be written as

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{c}_{t+1} \\ \hat{r}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} & \frac{a}{\rho} & \frac{\alpha a}{\beta\rho} & 0 \\ \frac{\epsilon}{\rho} & \frac{a\epsilon}{\rho} + 1 & \frac{\alpha a\epsilon}{\beta\rho} & 0 \\ \frac{\epsilon}{\rho^2 r^*} & \frac{a\epsilon}{\rho^2 r^*} & \frac{\alpha a\epsilon}{\beta\rho^2 r^*} & 0 \\ 0 & -[\beta\delta + (\beta+1)\frac{c^*}{k^*}] & \beta(\delta + \frac{c^*}{k^*}) & 1 + \frac{c^*}{k^*} \end{bmatrix} \begin{bmatrix} \hat{\pi}_{t-1} \\ \hat{c}_t \\ \hat{r}_t \\ \hat{k}_t \end{bmatrix}.$$

Through some algebra, it can be shown that the four eigenvalues of the above 4×4 matrix can be obtained by solving for the four roots of the following fourth-order polynomial equation in λ

$$\lambda \left(\lambda - 1 - \frac{c^*}{k^*} \right) F(\lambda) = 0,$$

where $F(\lambda)$ is a quadratic equation in λ given by

$$F(\lambda) = \lambda^2 - \left(1 + \frac{1}{\rho} - \frac{b}{\rho}\epsilon \right) \lambda + \left(\frac{1}{\rho} - \frac{b}{\rho}\epsilon - \frac{a}{\rho}\epsilon \right). \tag{47}$$

Thus, there are two policy-independent eigenvalues, a zero eigenvalue and a larger than unit eigenvalue given by $1 + \frac{c^*}{k^*}$, which are the same as in the case with $\mathbf{J} = 1$. The other two eigenvalues are policy dependent and can be obtained by solving for the two roots of the quadratic equation $F(\lambda) = 0$. Notice the similarity between the two functions $F(\cdot)$ and $D(\cdot)$: if we correspond b/ρ and a/ρ in $F(\cdot)$ to b and a in $D(\cdot)$, respectively, then the two functions become identical. The rest of the proof is similar. This completes the proof of the proposition. **Q.E.D.**

Proof of Proposition 1: We again introduce some auxiliary notations to facilitate the proof:

$$\epsilon \equiv \rho q - 1, \;\; b \equiv \frac{-\phi v n^*}{\gamma \pi^{*2} \beta^2}, \;\; a \equiv -b,$$

where the steady-state employment level in this case satisfies the following relation

$$vn^* = \frac{\beta(1+\phi)}{\phi}.$$

For the case with $\mathbf{J} = 1$, equations (17) and (18) can be written as

$$\begin{bmatrix} \hat{\pi}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} - b\epsilon & a \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_t \\ \hat{c}_t \end{bmatrix},$$

while for the case with $\mathbf{J} = 0$, they can be written as

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho} & \frac{a}{\rho} \\ \frac{\epsilon}{\rho} & \frac{a\epsilon}{\rho} + 1 \end{bmatrix} \begin{bmatrix} \hat{\pi}_{t-1} \\ \hat{c}_t \end{bmatrix}.$$

The corresponding characteristic functions of the two systems are identical in form to the functions $D(\cdot)$ and $F(\cdot)$ presented in the proof of Theorem 1, with an identical definition for ϵ, modified definitions for b and a, and an additional relationship $b + a = 0$ holding here. Following the same procedure as set out above, it is straightforward to show that determinacy obtains if and only if

$$\rho q - 1 \in \left(0, \frac{\frac{1}{\rho} + 1}{b + \frac{a}{2}}\right).$$

Substituting in b, a, and the steady-state relation between vn^* and β and ϕ at the beginning of this proof gives rise to the result in the proposition. **Q.E.D.**

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