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# Working Papers

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**WORKING PAPER NO. 99-5**

QUANTITATIVE ASSET PRICING IMPLICATIONS OF
ENDOGENOUS SOLVENCY CONSTRAINTS

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## ABSTRACT

We study the asset pricing implications of an economy where solvency constraints are determined to efficiently deter agents from defaulting. We present a simple example for which efficient allocations and all equilibrium elements are characterized analytically. The main model produces large equity premia and risk premia for long-term bonds with low risk aversion and a plausibly calibrated income process. We characterize the deviations from independence of aggregate and individual income uncertainty that produce equity and term premia.

## 1. Introduction

It is difficult to make a debtor pay: debt collection, litigation, and wage garnishment are costly, and the end result is uncertain because of the possibility of declaring bankruptcy. Motivated by these difficulties, we study a model where individuals with large financial obligations cannot contract further debts that would make them choose to default. In the model, individual income risks are only incompletely shared and become one of the determinants of asset prices.

In a companion paper, Alvarez and Jermann (1998a), we present a framework for studying the asset pricing implications when agents can default on their debts, built on earlier work by Kocherlakota (1996) and Kehoe and Levine (1993). In the model, agents default on their debts if this makes them better off. We assume that if agents default, their labor earnings cannot be seized, but they are excluded from asset markets forever. In our equilibrium concept, this lack of commitment results in state-specific and agent-specific borrowing constraints. These constraints ensure that agents will not default, since they will never owe so much as to make them choose to default. At the same time, the constraints ensure that there is as much risk-sharing as possible. In this paper, we focus on the quantitative effects of these endogenous solvency constraints for asset returns.

Our work is related to the asset pricing literature that studies the effects of portfolio restrictions, such as He and Modest (1995) and Luttmer (1996) among others. He and Modest (1995) conclude that "none of the market frictions alone—with the possible exception of solvency constraints—can explain the apparent rejection of the first-order equilibrium condition between consumption and asset returns." In our model we propose a coherent story of how solvency constraints and asset prices are endogenously determined.

We start by presenting a set of results about general properties of equilibrium allocations. Since the welfare theorems hold, we characterize equilibria by solving a planning problem. We prove that incomplete risk-sharing is possible only for low risk aversion, low time preference parameter, and for persistent but not too volatile individual shocks.

We present an example that we can fully characterize analytically. We show how to compute efficient allocations and all the elements of an equilibrium. The example characterizes the type of parameters that deliver different levels of risk-sharing, shows that interest rates are lower in economies with solvency constraints, and illustrates that poor agents face binding solvency constraints. We calibrate a simple income process and show that for low risk aversion the pricing kernel can be volatile enough to pass the Hansen-Jagannathan test. The example also illustrates how to design efficient algorithms to solve for optimal allocations in more general cases.

For a detailed quantitative analysis we specify an endowment process that allows varying degrees of dependence between aggregate and individual income uncertainty. Using moderate values for risk aversion and an income process calibrated to aggregate and household U.S.-data, the model generates large risk premia for equity and for long term bonds. With limited risk-sharing there is no "risk-free rate puzzle," as interest rates are lower than in the corresponding representative agent economy. In fact, to explain risk-free rates in the order of 1% per annum, the model requires a time-discount factor that is lower than the ones typically used in other studies. Finally, we characterize the deviations from independence of aggregate and individual risks that generate equity and term premia. The equity premium depends on the comovement between individual income risk and the contemporaneous aggregate income growth. This result is driven by solvency constraints that bind frequently, as opposed to the well known results by Mankiw (1986), Weil (1992), and Constantinides and Duffie (1996) that require convexity of the marginal utility. The term premium depends on the comovement between the forecast of future individual income risk and aggregate income growth.

Compared with several studies on incomplete markets economies, our study differs along some of the following dimensions.[1] First, we have complete markets where every claim can be traded and priced; most incomplete markets models consider only a very limited set of assets. Second, in our model the extent of risk-sharing depends on the cost and benefits of defaulting. In most incomplete market models, the extent of risk-sharing depends primarily on the assumptions about available assets. Third, in our model agent-specific and state-specific solvency constraints bind frequently; in most incomplete markets models portfolio constraints rarely bind. Finally, our model is very tractable since we solve a planner's problem and then compute the prices for the corresponding equilibrium—incomplete markets models are solved directly as a complicated fixed point over market prices. Given the tractability and the presence of complete markets we are able to study equity, one-period bonds, and the entire term structure of interest rates.

In section 2 we present the environment. Section 3 presents the equilibrium with solvency constraints. In section 4 we characterize the optimal allocations. In section 5 we solve and analyze a simple two-state case. Section 6 contains the calibration and the quantitative findings of a more general case. Section 7 concludes.

[1] For instance, Mankiw (1986), Telmer (1993), Constantinides and Duffie (1996), Heaton and Lucas (1996), and Zhang (1997).

## 2. Environment

We consider a pure exchange economy with two (types of) agents.[2] Agents' endowments follow a finite state Markov process; agents' preferences are identical and given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarchy at any time and for any history.

We use a Markov chain with generic elements $z \in Z$, a set with $N$ elements. To refer to particular elements of $Z$ we use $Z = \{\mathfrak{z}_1, \mathfrak{z}_2, ...., \mathfrak{z}_N\}$, and to refer to the time $t$ realization of the process we use $z_t$. We denote by $z^t = (z_0, z_1, z_2, ..., z_t)$ a length $t$ history of $z$. We use $\Pi$ for the matrix containing the transition probabilities, which generates conditional probabilities for histories that we denote as $\pi(z^t|z_0 = z)$. We use the notation $\{c_i\}$ and $\{e_i\}$ for the stochastic process of consumption and endowment of each agent, hence $\{c_i\} = \left\{c_{i,t}(z^t) : \forall\ t \geq 0,\ z^t \in Z^t\right\}$. We assume that aggregate endowment $e_t\left(z^t\right) \equiv e_{1,t}\left(z^t\right) + e_{2,t}\left(z^t\right)$ is constant and equal to $e$.[3]

Individual endowments are given by a function $\epsilon_i$ that depends only on $z_t$, so that $e_{i,t}\left(z^t\right) = \epsilon_i\left(z_t\right)$. We assume that $\epsilon_i\left(z\right) > 0$ for all $i$ and $z$. The utility for an agent corresponding to the consumption process $\{c\}$ starting at time $t$ at history $z^t$ is denoted by $U(c)(z^t)$ and is given by:

$$U(c)(z^t) \equiv \sum_{s=0}^{\infty} \sum_{z^{t+s} \in Z^{t+s}} \beta_{t,t+s}\left(z^{t+s-1}\right) u\left(c_{t+s}\left(z^{t+s}\right)\right) \pi(z^{t+s}|z_t)\,, \tag{2.1}$$

where $u$ is the period utility and $\beta_{t,t+s}$ is a time-discount factor. We assume that $u : R_+ \to R$ is strictly increasing, strictly concave, and $C^1$. The multi-period time-discount factor $\beta_{t,t+s+1}$ is defined recursively using the one-period state-contingent discount factor $\beta : Z \to [0,1]$. Specifically, $\beta_{t,t}\left(z^{t-1}\right) \equiv 1$ for all $z^{t-1}$, $\beta_{t,t+1}\left(z^t\right) = \beta\left(z_t\right)$ for all $z^t$, and for $s > 1$, $\beta_{t,t+s+1} : Z^{t+s} \to [0,1]$ satisfies

$$\beta_{t,t+s+1}\left(z^{t+s}\right) = \beta_{t,t+s}\left(z^{t+s-1}\right) \cdot \beta\left(z_{t+s}\right).$$

Note that the standard case with constant discount factor $\beta\left(z\right) = \bar{\beta}$ corresponds to $\beta_{t,t+s}\left(z^{t+s-1}\right) = \bar{\beta}^s$. As we will show below, letting the time-discount factor be state-contingent in this way allows us to introduce aggregate stochastic growth as a special case within the same notational framework.

[2] Our companion paper, Alvarez and Jermann (1998a), considers the more general case with $I$ agents, when $I \geq 2$.

[3] We will show below how to introduce aggregate uncertainty into this notational framework.

We assume that the matrix $\Pi$ is such that the process for $\{z_t\}$ has $Z$ as its unique ergodic distribution and has no cyclically moving subsets. Additionally we assume that the shocks are symmetric across agents in the following sense. Let us denote by $\widetilde{\boldsymbol{\epsilon}}$ and $\widetilde{\boldsymbol{\epsilon}}'$ two vectors of arbitrary values for the agents' endowments, and let us denote by $\tilde{\beta}$ and $\tilde{\beta}'$ two arbitrary values for the time-discount factor; then $\Pi, \epsilon_i(\cdot)$ and $\beta(\cdot)$ are assumed to satisfy

$$\begin{aligned} & \Pr\left(\left(\epsilon_1(z'), \epsilon_2(z')\right) = \widetilde{\boldsymbol{\epsilon}}', \beta(z') = \tilde{\beta}' \,|\, \left(\epsilon_1(z), \epsilon_2(z)\right) = \widetilde{\boldsymbol{\epsilon}}, \beta(z) = \tilde{\beta}\right) \\ = \ & \Pr\left(\left(\epsilon_2(z'), \epsilon_1(z')\right) = \widetilde{\boldsymbol{\epsilon}}', \beta(z') = \tilde{\beta}' \,|\, \left(\epsilon_2(z), \epsilon_1(z)\right) = \widetilde{\boldsymbol{\epsilon}}, \beta(z) = \tilde{\beta}\right). \end{aligned}$$

An allocation $\{c_i\}_{i=1,2}$ is resource feasible if:

$$c_{1,t}\left(z^t\right) + c_{2,t}\left(z^t\right) = e_t\left(z^t\right) \qquad \forall\, t \geq 0,\ z^t \in Z^t, \tag{2.2}$$

and it satisfies the participation constraints if:

$$U(c_i)(z^t) \geq U(e_i)(z^t) \equiv U^i\left(z_t\right) \qquad \forall\, t \geq 0,\ z^t \in Z^t. \tag{2.3}$$

where we use the notation $U^i(z_t)$ to refer to $U(e_i)(z^t)$ to emphasize that it only depends on $z_t$.

Except for the state-contingent time discount factor our environment is a special case of the one studied by Kehoe and Levine (1993). In particular, we consider the case with one good, two agents, and where the participation constraints have autarchy as the outside option. It is identical to the one studied by Kocherlakota (1996), except that we allow for a stochastic discount factor and non-i.i.d. process for the income shocks.

### 2.1. Aggregate uncertainty

Our specification of the time-discount factor allows us to consider stochastic aggregate income growth to be introduced as in Mehra-Prescott (1985) and in much of the quantitative consumption based asset pricing literature. An economy with stochastic growth, constant time-discount factor, and constant relative risk aversion can be expressed as an economy with constant aggregate endowment and state-contingent discount factor such as the one presented in section (2). Let

$$e_{t+1}\left(z^t, z_{t+1}\right) = \lambda(z_{t+1}) \cdot e_t\left(z^t\right) \qquad \text{and} \qquad e_{i,t}\left(z^t\right) = \epsilon_i(z_t) \cdot e_t\left(z^t\right) \text{ for } i = 1, 2$$

and define $\hat{e}_{i,t}\left(z^t\right) \equiv e_{i,t}\left(z^t\right)/e_t\left(z^t\right) = \epsilon_i(z_t)$ for all $i$ so that $\hat{e}_t\left(z^t\right) = 1$ all $z^t$.

Assuming a constant time-discount factor $\beta$ and a period utility function of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some positive $\gamma$ (for simplicity, $\gamma \neq 1$); defining $\hat{c}_{i,t}\left(z^t\right) \equiv c_{i,t}\left(z^t\right)/e_t\left(z^t\right)$, then $U(\cdot)$ satisfies

$$U\left(\hat{c}_i\right)\left(z^t\right) = \frac{\left(\hat{c}_{i,t}\left(z^t\right)\right)^{1-\gamma}}{1-\gamma} + \hat{\beta}\left(z_t\right) \sum_{z_{t+1}\in Z} U\left(\hat{c}_i\right)\left(z^t, z_{t+1}\right) \hat{\pi}\left(z_{t+1}|z_t\right),$$

with probabilities and discount factor

$$\hat{\pi}\left(z'|z\right) \equiv \frac{\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}}{\sum_{z'}\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}} \text{ and } \hat{\beta}\left(z\right) \equiv \beta\cdot\sum_{z'}\pi\left(z'|z\right)\cdot\lambda\left(z'\right)^{1-\gamma}.$$

Clearly, the resource and participation constraints are satisfied for an allocation $\{c_i\}_{i=1,2}$ in an economy with aggregate growth $\lambda(\cdot)$ and constant discount factor $\beta$ *if and only if* they are satisfied for the corresponding $\{\hat{c}_i\}_{i=1,2}$ allocation in the economy with constant aggregate endowment, discount factor $\hat{\beta}(\cdot)$, and probabilities $\hat{\pi}$. Moreover, the preference orderings are identical in the two corresponding economies. For later reference, when we want to distinguish between the two representations of the same economy, we refer to the one with stochastic discount factors and constant aggregate endowment as the *share* representation of the economy.

## 3. Equilibrium with endogenous solvency constraints

In this section, we define a competitive equilibrium with complete markets in Arrow securities and with endogenous solvency constraints. The solvency constraints prohibit agents from holding large amounts of contingent debt, hence preventing default. In general, these solvency constraints will be state-contingent, since the value of default (reverting to autarchy) varies with the state.

Let $q_t\left(z^t, z'\right)$ denote the period $t$, state $z^t$, price of one unit of the consumption good delivered at $t+1$, contingent on the realization of $z_{t+1} = z'$, in terms of period $t$ consumption goods. The holdings of agent $i$ at $t$ of this security are denoted by $a_{i,t+1}(z^t, z')$, and the lower limit on the holdings of agent $i$ is denoted by $B_{i,t+1}(z^t, z')$. Following our notational convention, we use $\{q\}$, $\{a_i\}$ and $\{B_i\}$ for the corresponding stochastic processes. For given $\{q\}$ and $\{B_i\}$ the problem for household $i$ is defined as

$$J_{i,t}\left(a, z^t\right) = \max_{\substack{c,\\ \{a_{z'}\}_{z'\in Z}}} \left\{ u(c) + \beta(z_t)\sum_{z'} J_{i,t+1}\left(a_{z'}, \left(z^t, z'\right)\right)\pi\left(z'|z_t\right)\right\} \quad (3.1)$$

$$
\begin{aligned}
e_{i,t}\left(z^t\right)+a &= \sum_{z'\in Z} a_{z'} q_t\left(z^t, z'\right)+c && (3.2)\\
a_{z'} &\geq B_{i,t+1}\left(z^t, z'\right) \quad \text{all } z' \in Z. && (3.3)
\end{aligned}
$$

**Definition 3.1.** *An equilibrium with Solvency Constraints $\{B_i\}$ for initial conditions $a_{i,0}$ has quantities $\{a_i\}$ and prices $\{q\}$ such that for $i = 1, 2$,*

a. $\left\{a_{i,t+1}\left(z^t, z'\right)\right\}_{z'\in Z}$ achieves the right hand side of (3.1) at $z^t$ given $a = a_{i,t}\left(z^t\right)$.

b. market clearing,

$$
a_{1,t}\left(z^t\right)+a_{2,t}\left(z^t\right)=0, \quad \text{all } t, \text{ all } z^t.
$$

From the first order condition of the agent problem, one concludes that if an agent's marginal rate of substitution is strictly lower than the corresponding Arrow price, this agent's solvency constraint must bind. Additionally, if an agent's solvency constraint does not bind, this agent's marginal rate of substitution must equal the corresponding Arrow price. Consequently, in an equilibrium with solvency constraints, Arrow prices are equal to the highest marginal rate of substitution, *i.e.*

$$
q_t\left(z^t, z_{t+1}\right)=\max_{i=1,2} \beta\left(z_t\right) \frac{u'\left(c_{i,t+1}\left(z^t, z_{t+1}\right)\right)}{u'\left(c_{i,t}\left(z^t\right)\right)} \pi\left(z_{t+1} | z_t\right). \tag{3.4}
$$

Furthermore, if the solvency constraints do not bind for either agent, the Arrow price (3.4) is equal to the one from the corresponding representative agent economy.

In problem (3.1) agents *never contemplate the option of default.* Now we move to the analysis of the decision of default; this consideration describes our theory of the solvency constraints presented in Alvarez and Jermann (1998a). The next condition makes the solvency constraints endogenous.

**Definition 3.2.** *An equilibrium with solvency constraints that are not too tight is such that the solvency constraints satisfy*

$$
J_{i,t+1}\left(B_{i,t+1}\left(z^{t+1}\right), z^{t+1}\right)=U\left(e_i\right)\left(z^{t+1}\right), \tag{3.5}
$$

*for all $t = 0, 1, ...$ and for all $z^{t+1} \in Z^{t+1}$ and for $i = 1, 2$.*

The left hand side of (3.5) is the utility of an agent that participates in the market, starting with financial wealth $B_{i,t+1}\left(z^{t+1}\right)$. The right hand side of (3.5) is the agent's utility if he defaults, given our assumption that default is punished by permanent exclusion from asset markets. This condition ensures that solvency constraints prevent default by prohibiting agents from accumulating more contingent debt than they will be willing to pay back. At the same time, it allows as much insurance as possible: if the solvency constraint binds and the continuation utility is strictly higher than the value of autarchy, the constraint could be relaxed, without inducing the agent to accumulate so much debt that he will prefer to default.

To simplify the notation, we state our equilibrium using Arrow prices. The budget set and the incentives to default are the same with other dynamically complete sets of securities, provided that the solvency constraints are stated in terms of the value of the portfolio at the beginning of the period. Thus, the value of any security—not just Arrow securities—is equal to the value of the sum (weighted by the payoffs) of the corresponding Arrow securities given by (3.4). In fact, the pricing kernel is the highest marginal rate of substitution, so that for the one period return $R_{t,t+1}$ of any asset, the following must hold

$$1 = E_t \left[ R_{t,t+1} \cdot \left( \max_{i=1,2} \beta_t \frac{u'\left(c_{i,t+1}\right)}{u'\left(c_{i,t}\right)} \right) \right].$$

We think *equilibria with solvency constraints that are not too tight* are interesting, since they restrict endogenously the amount of risk-sharing and make a direct connection between asset prices and constraints on borrowing. In Alvarez and Jermann (1998a) we show that the first and second welfare theorems hold for our equilibrium definition. The reason that *equilibria with solvency constraints that are not too tight* and constrained efficient allocations are equivalent is that the condition on the solvency constraints (3.5) serves the same purpose as the participation constraints (2.3).

Our interest is in asset prices, but analyzing equilibria directly is in general difficult. Given the equivalence between efficient allocations and *equilibria with solvency constraints that are not too tight*, we analyze efficient allocations as a way to characterize equilibrium prices.

## 4. Characterizing constrained efficient allocations

In this section we provide a recursive formulation of efficient allocations, which we use to establish the following properties of efficient allocations. First, we characterize the parameters that determine the extent of risk-sharing, which allows us to concentrate on the cases in which individual risk is important for asset prices. Second, we characterize the process for the highest marginal rate of substitution and its interaction with the participation constraints. This is important since the pricing kernel is equal to the highest marginal rate of substitution. Finally, we establish some properties of the dynamics of efficient allocations that are useful for the design of fast computational algorithms.

Constrained efficient allocations are defined as the processes $\{c_i\}$ that maximize period zero expected lifetime utility for agent 1, subject to resource balance and the participation constraints for both agents, given some initial (time zero) expected lifetime utility for agent 2. Optimal allocations solve the following maximization problem:

$$V^*(w, z) \equiv \max \{U(c_1)(z)\}$$

subject to (2.2), (2.3), $U(c_2)(z_0) \geq w$, and $z_0 = z$.

A pair $(w, z)$ is in the domain of $V^*$ if and only if $(V^*(w, z), w)$ is in the utility possibility set for $z_0 = z$. Indeed the function $V^*(w, z)$ describes the utility possibility frontier at time zero when $z_0 = z$.

Now we restate the previous problem recursively. We introduce a functional equation and relate its fixed points to the function $V^*$. The functional equation is not completely standard. The operator, $T$, maps a function $V$ defined in a given domain into another function $TV$. Using $TV$, a new domain, denoted by $D(z)$, is defined as specified below. Specifically, $V(\cdot, z) : D(z) \rightarrow R$ for each $z$, where $D(z) \subset [U^2(z), \infty)$, and the operator $T$ generates $TV$ as

$$TV(w, z) = \max_{c_{i=1,2}\ , w'(z'), z' \in Z} \left\{ u(c_1) + \beta(z) \sum_{z' \in Z} V(w'(z'), z')\ \pi(z'|z) \right\} \tag{4.1}$$

$$c_1 + c_2 \leq 1 \tag{4.2}$$

$$u(c_2) + \beta(z) \sum_{z' \in Z} w'(z')\ \pi(z'|z) \geq w \tag{4.3}$$

$$w'(z') \geq U^2(z') \quad \text{all } z' \in Z \tag{4.4}$$

$$V(w(z'), z') \geq U^1(z') \quad \text{all } z' \in Z. \tag{4.5}$$

For this operator to be well defined, $V$ has to be such that there are $c_1$, $c_2$, $\{w'(z')\}_{z' \in Z}$ for which the constraints (4.2), (4.3), (4.5), and (4.4) are satisfied. Consequently, the domain of $TV$ for each $z \in Z$, is defined as

$$D(z) \equiv \left\{ w : (TV)(w, z) \geq U^1(z) \text{ and } w \geq U^2(z) \right\}. \tag{4.6}$$

It is straightforward to show that the function $V^*$, defined previously, and its associated domain are a fixed point of $T$. However, the functional equation

(4.1) has more than one fixed point: hence it cannot be a contraction.[4] In particular, "autarchy" is always a fixed point, since it is immediate to verify that the trivial function $v$ defined on the domain given by singleton $\left\{U^2(z)\right\}$ and equal to $v\left(U^2(z), z\right) = U^1(z)$ is a fixed point of $T$. Nevertheless, for many parameter values there are other solutions, namely $V^*$.

Even though the $T$ operator is not a contraction, it is useful in computing $V^*$. Consider the operator $\tilde{T}$ defined exactly like $T$ in 4.1 except that the participations constraints for both agents 4.5 and 4.4 are removed, and denote its unique fixed point by $\tilde{V}$. The function $\tilde{V}(\cdot, z)$ is the full risk-sharing frontier when $z_t = z$.

**Proposition 4.1.** $\lim_{n\to\infty} T^n \tilde{V} = V^*$ *pointwise.*

**Proof.** The proof uses standard monotonicity arguments. It follows from a direct extension of Theorem 4.14 in Stokey and Lucas with Prescott (1989). Alternatively, it follows from the results of Abreu (1988) and Abreu, Pierce and Stacchetti (1990). ■

Because $\tilde{V}$ is easily computed, the previous proposition makes $T$ useful for computing the fixed point $V^*$ and its associated policies.

### 4.1. Risk-sharing regimes

This section describes the types of parameters for which this model has asset pricing implications different from the ones of the representative agent model. Depending on parameter values for preferences and endowments there are three possible "regimes" for the process for $\{w, z\}$. Independent of the initial condition $(w_0, z_0)$, one can show that:

1. Full risk-sharing forever is possible;
2. Only limited risk-sharing is possible;
3. Only autarchy is possible.

By full risk-sharing we mean that the allocation is Pareto efficient in the standard sense, ignoring the participation constraints, for some initial condition $(w_0, z_0)$. Parameter values that produce the first case are not interesting for us, since for the purposes of asset pricing, their implications are the same as for the representative agent economy.

[4] $T$ does not satisfy one of the Blackwell sufficient conditions for a contraction, namely *discounting*. $T(V + a)$ could be bigger than $TV + \beta a$ for a constant $a$, since the feasible set of choices for $(w(z'))_{z' \in Z}$ is bigger for $V + a$ than for $V$.

We discuss briefly how to verify whether full risk-sharing is possible. Kocherlakota (1996) presents sufficient conditions for each case when the shocks are i.i.d. and the discount factor is constant. We consider a slightly different case in the following proposition.

**Proposition 4.2.** *Full risk-sharing is possible if and only if*

$$u\left(1/2\right)\sum_{s=0}\sum_{z^t\in Z^t}\beta_{0,t}\left(z^t\right)\pi\left(z^t|z_0\right)\geq\max_{i=1,2}U^i(z_0) \tag{4.7}$$

*for all* $z_0\in Z$.

**Proof.** Full risk-sharing is characterized by resource feasibility and constancy of the ratio of the marginal utilities across agents. If condition (4.7) is satisfied, then the participation constraints are satisfied for the allocation $c_i=1/2$ . Conversely, if full risk-sharing is feasible, clearly condition (4.7) is satisfied. ■

For the case in which the time-discount factor is constant, equal to $\beta$, (for example if there is no aggregate uncertainty) the left hand side of equation (4.7) simplifies to $u(1/2)/\left(1-\beta\right)$. Figure 1 illustrates this case when full risk-sharing is possible for a range of $w$. When full risk-sharing is not possible, there are two cases: one case in which autarchy is the only feasible allocation that satisfies the participation constraints and the other in which some other allocations satisfy the participation constraints. This last case is the one that we are interested in, since it is not equivalent to a representative agent economy. Figure 2 illustrates the case when full risk-sharing is not possible.

Which case applies depends on how attractive autarchy is relative to some form of risk-sharing. This depends on the parameter values as explained in the following remark.

**Remark 1.** *Let,* $\Pi_\delta\equiv\delta I+\left(1-\delta\right)\Pi$ *for* $\delta\in\left(0,1\right)$ , *then full risk-sharing is not possible in any of the following cases:*

(a) The time preference parameter, $\max_z\beta(z)$ is sufficiently small;
(b) The persistence of $\Pi_\delta,\delta$, is sufficiently close to one;
(c) The variance of $\epsilon_i(z)$ is sufficiently close to zero;
(d) With CRRA utility, the relative risk aversion, $\gamma$, is sufficiently small.

The proof of this remark follows by taking the appropriate limit in each of the four cases and verifying that the inequality of the previous proposition (4.2) does not hold. In Alvarez and Jermann (1998a) we extend this result to show that not only is full risk-sharing not possible, but as the parameters approach the limit values mentioned in each of the four cases, autarchy is the only feasible allocation.

Mehra and Prescott (1985) and Weil (1992) emphasize that the representative agent model requires values of the risk aversion $\gamma$ and time preference $\beta$ that are too high to produce a high equity premium and a low interest rate. This model has asset pricing implications different from the representative agent model for low values of $\gamma$ and $\beta$, so our results can not rely upon high $\gamma$ and $\beta$.

### 4.2. Marginal rates of substitution with limited risk-sharing

This section analyzes the stochastic process for the marginal rate of substitution for each agent in a constrained efficient allocation. We are interested in these processes because in an equilibrium with solvency constraints, state prices are given by the highest marginal rate of substitution (see equation (3.4)).

Let $W_{z'}(w,z)$ and $C_i(w,z)$ denote the optimal decision rules of problem (4.1) given state $(w,z)$ for continuation utility $w'(z')$ and current consumption $c_i$ respectively.

**Proposition 4.3.** *$V$ is strictly decreasing, strictly concave and differentiable in its interior with respect to $w$. The optimal policy rules are single-valued and continuous.*

The proof follows immediately from the strict monotonicity, strict concavity and differentiability of the period utility function and convexity of the feasible set.

The next proposition says that if some risk-sharing is feasible, then at least one agent is unconstrained.

**Proposition 4.4.** *Assume that $\Pi$ is such that it has a unique invariant distribution with ergodic set $Z$, one of the following is true:*

$$V\left(U^2(z), z\right) = U^1(z) \quad \text{for all } z \in Z, \tag{4.8}$$

*which we refer to as saying that "autarchy is the only feasible allocation" or else*

$$V\left(U^2(z), z\right) > U^1(z) \quad \text{for all } z \in Z, \tag{4.9}$$

*which we refer to as saying that "some risk-sharing is feasible." In this case, at least one agent is unconstrained in each period.*

**Proof.** It follows by a straightforward adaptation of the arguments on page 600 in Kocherlakota (1996). ■

The next proposition shows that an unconstrained agent has the highest marginal rate of substitution: thus if both agents are unconstrained they equalize their marginal rates of substitution.

**Proposition 4.5.** *If some risk-sharing is feasible, for any* $(w, z)$ *and* $z'$*, if for agent* $i = 2$

$$W_{z'}(w, z) > U^i(z'), \tag{4.10}$$

*then*

$$\beta(z) \frac{u'(C_i(w(z'), z'))}{u'(C_i(w, z))} \pi(z'|z) = \max_{j=1,2} \beta(z) \frac{u'(C_j(w(z'), z'))}{u'(C_j(w, z))} \pi(z'|z)$$

*and analogously for agent* $i = 1$.

**Proof.** It follows from the first order conditions of (4.1) and from symmetry. ■

These propositions, together with the characterization for state prices (3.4) and the definition of the solvency constraints (3.5), imply the following: state prices are given by the marginal rate of substitution of the agent whose solvency constraint (3.3) does not bind; the solvency constraint (3.3) does not bind, if and only if the participation constraints (2.3) does not bind; and at any time and for any history there is at least one agent that is unconstrained.

### 4.3. Decision rules for the planning problem

In this subsection we present some further results about the properties of the decision rules for consumption $C_i(\cdot)$ and for Agent 2's continuation utility $W_{z'}(\cdot)$. We define $H(z)$ and $L(z)$ as the upper and lower bound of $D(z)$ so that by monotonicity of $V(\cdot, z)$,

$$L(z) \equiv U^2(z) \text{ and } H(z) \equiv V^{-1}(\cdot, z)(U^1(z)) \text{ for } z \in Z.$$

The time separability of the utility function implies that consumption is increasing in the current continuation utility.

**Proposition 4.6.** *Consumption is strictly monotone on* $z$*, i.e.,* $\forall z$*,* $C_2(w, z)$ *(*$C_1(w, z)$*, respectively) is strictly increasing (strictly decreasing) in* $w$*.*

**Proof.** By using Benveniste and Scheinkman, one shows that

$$\frac{\partial V(w, z)}{\partial w} = -\frac{u'(e - C_2(w, z))}{u'(C_2(w, z))} \tag{4.11}$$

which is strictly decreasing in $w$ since the value function is strictly concave. Thus $C_2(w,z)$ is increasing. ■

Next we analyze the decision rules for future continuation utility as a function of current continuation utility. We find that the decision rules are weakly increasing. Specifically, we consider two cases. First, if the shock $z_t$ is repeated, then consumption and continuation utility are the same. If the shock is different, then the decision rule is weakly increasing in $w$. Furthermore, the decision rule is flat only if the assigned continuation utility is such that either agent 1 or 2 is constrained in the next period.

**Proposition 4.7 (I).** *If $z' = z$, then we have $W_{z'}(w,z) = w$ (the "$45^0$–rule"). [II] If $z' \neq z$ and $(\tilde{w}, w, z, z')$ are such that (i) $w < \tilde{w}$ and (ii) both $W_{z'}(\tilde{w}, z)$ and $W_{z'}(w,z)$ are in the interior of the range of $W_{z'}(\cdot, z)$ i.e.,*

$$W_{z'}(\tilde{w}, z), W_{z'}(w,z) \in \left(L\left(z'\right), H\left(z'\right)\right),$$

*then $W_{z'}(\tilde{w}, z) > W_{z'}(w,z)$.*

**Proof.** The proof of [I] follows from examining the case where neither agent is constrained in the future, then by Proposition (4.5)

$$\frac{\partial V(w,z)}{\partial w} = -\frac{u'(e - C_2(w,z))}{u'(C_2(w,z))} = -\frac{u'(e - C_2(W_z(w,z), z))}{u'(C_2(W_z(w,z), z))} = \frac{\partial V(W_z(w,z), z)}{\partial w}.$$

Finally, using Proposition (4.6) [I] is obtained. [II] follows from a variation of the previous argument. ■

## 5. Analysis of the 2 shocks case

To illustrate how the model works and to get a rough idea about the quantitative potential of our model for explaining asset returns, we analyze a simple example. We completely characterize the optimal allocations and all the elements of an equilibrium with *solvency constraints that are not too tight,* and we compute some numerical cases. Among other things, this example illustrates the circumstances under which agents' solvency constraints bind. Moreover, our examination of the Hansen-Jagannathan bounds shows promise for cases with low risk aversion.

### 5.1. Efficient allocations

Consider the case with a constant discount factor $\beta$ and only two shocks $Z = \{\mathfrak{z}_1, \mathfrak{z}_2\}$, with $\epsilon_2(\mathfrak{z}_1) < \epsilon_2(\mathfrak{z}_2)$, which by symmetry imply $\epsilon_1(\mathfrak{z}_2) = \epsilon_2(\mathfrak{z}_1) <$

$\epsilon_2(\mathfrak{z}_2) = \epsilon_1(\mathfrak{z}_1)$. We first show that when full risk-sharing is not possible, the decision rules imply a unique ergodic set, where continuation utility $w$, and hence consumption, depend exclusively on the current value of $z$. Second, we characterize the values of consumption in the ergodic set by a single equation in one unknown.

The decision rules of problem (4.1) are completely described by analyzing two cases. If $z' = z$, then the optimal policy is the $45^o$ line as we have shown before. For the remaining case of $z' \neq z$, the following proposition, proven in the Appendix, shows that the policies rules are constant, a result that we refer to as saying that they are flat after reversal.

**Proposition 5.1.** *Decision rules that achieve $V$ are "flat" after a reversal of the shock, that is, for all $w \in D(z) = [L(z), H(z)]$*

$$W_{\mathfrak{z}_2}(w, \mathfrak{z}_1) = L(\mathfrak{z}_2) \equiv \bar{w}(\mathfrak{z}_2) \quad \text{and}$$
$$W_{\mathfrak{z}_1}(w, \mathfrak{z}_2) = H(\mathfrak{z}_1) \equiv \bar{w}(\mathfrak{z}_1)\ .$$

Figure 3 plots the decision rules $W_{z'}(w, z)$. For any initial $(w, z)$, after one reversal of the shock $z$, continuation utility and consumption for Agent 2 will attain the values $\overline{w}(z)$ and $\overline{c}(z)$, and depend only on the current state $z$. By inspection of these decision rules, if risk-sharing is not possible, the process for $\{w, z\}$ has a unique invariant distribution, with mass on $(\bar{w}(\mathfrak{z}_1), \mathfrak{z}_1)$ and $(\bar{w}(\mathfrak{z}_2), \mathfrak{z}_2)$. If full risk-sharing is possible, then the domains $D(\mathfrak{z}_1)$ and $D(\mathfrak{z}_2)$ have non-empty intersection, thus any constant value of $w$ in that intersection is optimal, and hence any distribution over $D(\mathfrak{z}_1) \cap D(\mathfrak{z}_2)$ is an invariant distribution.

We now solve for the values of consumption for Agent 2 in the ergodic set, which consists in solving one equation in one unknown.

Given our characterization, Agent 2's continuation utilities in the ergodic set, $(\bar{w}(\mathfrak{z}_1), \bar{w}(\mathfrak{z}_2))$, and the corresponding consumptions, $(\overline{c}(\mathfrak{z}_1), \overline{c}(\mathfrak{z}_2))$, have to satisfy the following system of four equations: two promise keeping conditions,

$$\bar{w}(\mathfrak{z}_1) = u(\overline{c}(\mathfrak{z}_1)) + \beta\bar{\pi}\bar{w}(\mathfrak{z}_1) + \beta(1-\bar{\pi})\bar{w}(\mathfrak{z}_2), \text{ and}$$
$$\bar{w}(\mathfrak{z}_2) = u(\overline{c}(\mathfrak{z}_2)) + \beta\bar{\pi}\bar{w}(\mathfrak{z}_2) + \beta(1-\bar{\pi})\bar{w}(\mathfrak{z}_1),$$

the boundary condition $\bar{w}(\mathfrak{z}_2) = U^2(\mathfrak{z}_2)$, and, due to the symmetry across agents and the resource constraint, $\overline{c}(\mathfrak{z}_1) = e - \overline{c}(\mathfrak{z}_2)$. By repeated substitution, this system reduces to one equation, $u_2 = h(u_2)$, in one unknown, $u_2 \equiv u(\overline{c}(\mathfrak{z}_2))$, where the equation is defined as:

$$\begin{aligned} u_2 &= h(u_2) \equiv \frac{(1-\beta)(1-2\beta\bar{\pi}+\beta)}{1-\beta\bar{\pi}} U^2(\mathfrak{z}_2) - \frac{\beta(1-\bar{\pi})}{1-\beta\bar{\pi}} f(u_2), \qquad (5.1) \\ f(u_2) &\equiv u\left(e - u^{-1}(u_2)\right). \end{aligned}$$

Since the previous characterization of the ergodic set of the efficient allocation requires that full risk-sharing is not possible, using Proposition (4.2), we only consider the case where $\beta, \bar{\pi}, \epsilon_2(\mathfrak{z}_2)/\epsilon_2(\mathfrak{z}_1)$, and $\gamma$ satisfy

$$U^2(\mathfrak{z}_2) > \frac{u(e/2)}{1-\beta} \equiv \frac{u^*}{1-\beta},$$

where $u^*$ is the period utility corresponding to having consumption in $\mathfrak{z}_1$ and $\mathfrak{z}_2$ equal to half the aggregate endowment, and hence it is the only solution to $u^* = f(u^*)$. Direct computation gives that,

$$U^2(\mathfrak{z}_2) = \frac{u(\epsilon_2(\mathfrak{z}_2))}{1-\beta}\left[\frac{(1-\beta\bar{\pi})}{(1-\beta\bar{\pi})+\beta(1-\bar{\pi})}\right] + \frac{u(\epsilon_2(\mathfrak{z}_1))}{1-\beta}\left[\frac{\beta(1-\bar{\pi})}{(1-\beta\bar{\pi})+\beta(1-\bar{\pi})}\right] \tag{5.2}$$

which, we use in the next remark to give sufficient conditions when the participation constraints do not bind.

**Remark 2.** *By direct computation, as the standard deviation of the endowment is large enough, i.e. as $\epsilon_2(\mathfrak{z}_2)/\epsilon_2(\mathfrak{z}_1) \uparrow \infty$, or as agents are patient enough, i.e. $\beta \uparrow 1$, or, for the CRRA case, as agents are risk averse enough, i.e. as $\gamma \uparrow \infty$, then full risk-sharing is feasible.*

Since $u$ is strictly increasing and strictly concave, the function $h$ is strictly increasing and strictly convex, and hence (5.1) has at most two solutions. Notice that autarchy, *i.e.*, $\bar{w}(\mathfrak{z}_i) = U^2(\mathfrak{z}_i)$, $\bar{c}(\mathfrak{z}_i) = \epsilon_2(\mathfrak{z}_i)$ for $i = 1, 2$ satisfies (4.3), resource feasibility, and the boundary condition, and hence (5.1) has at least one solution. We now describe which of the two solutions characterizes the efficient allocation, which we denote as $\bar{u}_2$.

**Proposition 5.2.** *Consider the two solutions of (5.1), $u_{2l}$ and $u_{2h}$, ordering them as $u_{2l} \leq u_{2h}$. The efficient allocation, $\bar{u}_2$, equals the smallest one, i.e., $\bar{u}_2 = u_{2l}$, because efficient allocations have a variability smaller or equal to the endowment, i.e., $\epsilon_2(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_1) \leq \bar{c}(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$. Finally, autarchy is efficient, and hence is the only feasible allocation, if and only if*

$$\frac{\beta(1-\bar{\pi})}{1-\beta\bar{\pi}}\frac{u'(\epsilon_2(\mathfrak{z}_1))}{u'(\epsilon_2(\mathfrak{z}_2))} \leq 1. \tag{5.3}$$

**Proof.** We start by showing that $u_{2l}$ is the efficient allocation. First, if $u_{2l}$ equals autarchy, then $u_{2h}$ corresponds to an allocation that is more volatile than the allocation at $u_{2l}$, with the same mean, thus by concavity $u_{2l}$ gives higher utility. Second, we consider the case where $u_{2h}$ equals autarchy. By assumption $U^2(\mathfrak{z}_2) > \frac{u^*}{1-\beta}$, which by direct computation implies that $h(u^*) > u^*$ and by convexity of $h$, $u^* < u_{2l}$ or $u_{2h} < u^*$. Because $u_{2h} < u^*$ is ruled if $u_{2h}$ equals autarchy, we have $u^* < u_{2l}$, that is: $\frac{e}{2} < \overline{c}_l(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$. Thus, $u_{2l}$ has higher utility than autarchy $u_{2h}$, and by symmetry $\epsilon_2(\mathfrak{z}_1) \leq \overline{c}(\mathfrak{z}_1) \leq \overline{c}(\mathfrak{z}_2) \leq \epsilon_2(\mathfrak{z}_2)$.

We now show the condition for autarchy being efficient. By convexity of $h$,

$$h'(u_{2l}) \leq 1 \leq h'(u_{2h}).$$

Thus, if $h'$ evaluated at autarchy is smaller than one, autarchy is efficient, and hence is the only feasible allocation. By direct computation of the derivative, $h'(u(\epsilon_2(\mathfrak{z}_2)))$, Equation (5.3) is obtained. ■

The inequality (5.3) implies, by direct computation, the following sufficient conditions under which autarchy is the only feasible allocation. As the standard deviation of endowment decreases, *i.e.* as $\frac{\epsilon_2(\mathfrak{z}_2)}{\epsilon_2(\mathfrak{z}_1)} \downarrow 1$, or as the autocorrelation of the endowment increases, *i.e.* $\bar{\pi}$ as $\uparrow 1$, or as the agents are more impatient, *i.e.*, $\beta \downarrow 0$, or, for the CRRA case, as the agents are less risk averse, *i.e.* $\gamma \downarrow 0$, then autarchy is efficient.

### 5.2. Equilibrium allocations

In this section we construct the elements of an *equilibrium with solvency constraints that are not too tight* corresponding to the efficient allocations found in the previous section. This illustrates the second welfare theorem for this environment and clarifies how the equilibrium works. In particular, we discuss the sense in which the "poor" agents are the ones that face binding solvency constraints, and we show that interest rates are lower and the pricing kernel is more variable than in the corresponding representative agent economy.

We analyze the equilibrium, once the allocation is in the ergodic set defined in the previous section. Arrow prices depend exclusively on whether the current state $z_t$ is the same as the state $z_{t+1}$ where they give the right to receive one unit of the consumption good,

$$\begin{aligned} q_t\left(z^t, z_{t+1}\right) &= \bar{q}_r \text{ if } z_{t+1} = z_t \text{ and} \\ q_t\left(z^t, z_{t+1}\right) &= \bar{q}_{nr} \text{ if } z_{t+1} \neq z_t,\ \forall z^t, \forall z_{t+1}. \end{aligned}$$

Using the relationship defining Arrow prices in equation (3.4), we obtain that

$$\begin{aligned} \bar{q}_r &= \beta\bar{\pi} \text{ for } z' = z, \\ \bar{q}_{nr} &= \beta\frac{u'(\bar{c}(\mathfrak{z}_1))}{u'(\bar{c}(\mathfrak{z}_2))}(1-\bar{\pi}) \text{ for } z' \neq z. \end{aligned} \tag{5.4}$$

Since one period bond prices are equal to the sum of the Arrow prices, the price of an uncontingent bond, and hence the interest rate $1+i$, is constant, and equal to:

$$\frac{1}{1+i} = \beta\left[\bar{\pi} + \left(\frac{u'\left(\bar{c}\left(\mathfrak{z}_1\right)\right)}{u'\left(\bar{c}\left(\mathfrak{z}_2\right)\right)}\right)(1-\bar{\pi})\right]. \tag{5.5}$$

By inspection of (5.5), the interest rate for the economy with solvency constraints is lower than the interest rate for the corresponding representative agent economy, which equals $\beta$. Also, by inspection of (5.4), the pricing kernel, given by $q_t\left(z_{t+1}, z^t\right)/\pi\left(z_{t+1}|z_t\right)$ is more volatile than the pricing kernel of the corresponding representative agent economy, which is constant in this case.

Agent 2's consumption, depends only on the current state $z_t$, *i.e.*,

$$c_{2,t}(z^t, z_{t+1}) = \bar{c}\left(z_{t+1}\right), \ \forall z^t, \forall z_{t+1}$$

for the $\bar{c}$ given by the efficient allocation, and Agent 2's purchases of Arrow securities depend only on the state in which they pay,

$$a_{2,t+1}\left(z^t, z_{t+1}\right) = \bar{a}\left(z_{t+1}\right), \forall z^t, \forall z_{t+1}.$$

Agent 1's consumption and Arrow securities holdings are given by:

$$\begin{aligned} c_{1,t+1}\left(z^{t+1}\right) &= e - c_{2,t+1}\left(z^{t+1}\right) \\ a_{1,t+1}(z^{t+1}) &= -a_{2,t+1}(z^{t+1}). \end{aligned}$$

We find the values of $\bar{a}\left(z\right)$ using the sequence budget constraints (3.2) for Agent 2 together with the resource constraint (2.2),

$$\begin{aligned} \bar{a}\left(\mathfrak{z}_2\right) &= \frac{\bar{c}\left(\mathfrak{z}_2\right) - \epsilon_2\left(\mathfrak{z}_2\right)}{1+\overline{q}_{nr} - \overline{q}_r} < 0 \\ \bar{a}\left(\mathfrak{z}_1\right) &= -\bar{a}\left(\mathfrak{z}_2\right) > 0. \end{aligned}$$

Notice that, very intuitively, $\bar{a}\left(\mathfrak{z}_1\right) > 0$ means that Agent 2 saves contingent on having his low income, and $\bar{a}\left(\mathfrak{z}_2\right) < 0$ means that Agent 2 borrows contingent on having his high income.

The solvency constraints for the high-income state can be found easily following our definition in equation (3.5). In an efficient allocation, the continuation utility corresponding to the high-income shock ($z_{t+1} = \mathfrak{z}_2$ for Agent 2) is given by $U^i\left(z_{t+1}\right)$, then $B_{i,t+1}\left(z^{t+1}\right) = a_{i,t+1}\left(z^{t+1}\right)$, which for Agent 2 gives $B_{2,t+1}\left(\mathfrak{z}_2, z^t\right) \equiv \overline{B}\left(\mathfrak{z}_2\right) = \bar{a}\left(\mathfrak{z}_2\right)$.

The solvency constraints corresponding to the low-income shock ($z_{t+1} = \mathfrak{z}_1$ for Agent 2) can not be determined directly from the optimal allocations, because they depend on the solutions to off-equilibrium consumption and portfolio choice problems.

**Proposition 5.3.** *$B_{2,t+1}\left(z^t, \mathfrak{z}_1\right) = \overline{B}\left(\mathfrak{z}_1\right) = \left[\frac{\overline{q}_{nr}}{1-\overline{q}_r}\right]\overline{B}\left(\mathfrak{z}_2\right) \leq 0$; and $B_{1,t+1}\left(z^t, \mathfrak{z}_2\right) = \overline{B}\left(\mathfrak{z}_1\right)$.*

**Proof.** For the given prices $\{\overline{q}\}$, we directly check the definition for solvency constraints that are not too tight for the postulated solvency constraints for Agent 2, $\{B_2\}$. For $z_{t+1} = \mathfrak{z}_2$, with initial wealth $\underline{a}_{t+1} = \overline{B}\left(\mathfrak{z}_2\right)$, the efficient choice for consumption and asset holdings from the equilibrium is still feasible with these constraints, and $J_{2,t+1}\left(\overline{B}\left(\mathfrak{z}_2\right), \left(z^t, \mathfrak{z}_2\right)\right) = U^2\left(\mathfrak{z}_2\right)$ as is required for solvency constraints that are not too tight. For $z_{t+1} = \mathfrak{z}_1$,with initial wealth $\underline{a}_{t+1} = \overline{B}\left(\mathfrak{z}_1\right)$, for any future path for which state $\mathfrak{z}_1$ continues to repeat itself uninterruptedly for the next $k$ periods, the Euler equation and the price $\overline{q}_r$ imply optimal individual choices such that $\underline{c}(z_{t+1} = \mathfrak{z}_1) = \underline{c}(z_{t+k} = \mathfrak{z}_1) = \underline{c}_0$, $\underline{a}(z_{t+k} = \mathfrak{z}_1) = \overline{B}\left(\mathfrak{z}_1\right)$ and $J_{2,t+k}\left(\overline{B}\left(\mathfrak{z}_1\right), \left(z^t, \mathfrak{z}_1\right)\right) = U^2\left(\mathfrak{z}_1\right)$. For $z_{t+1} = \mathfrak{z}_1$,with initial wealth $\underline{a}_{t+1} = \overline{B}\left(\mathfrak{z}_1\right)$, followed by $z_{t+2} = \mathfrak{z}_2$, the solvency constraint will bind at $\overline{B}\left(\mathfrak{z}_2\right)$, and from there on optimal consumption and portfolio choices will be identical to the one in the equilibrium, given that prices are unchanged, initial wealth identical and constraints such that these choices are feasible. With this characterization we have the following two equations for $\underline{c}_0$ and $\overline{B}(\mathfrak{z}_1)$, a budget constraint and the equation defining implicitly the solvency constraints:

$$\epsilon_2\left(\mathfrak{z}_1\right) + \overline{B}(\mathfrak{z}_1) = \overline{q}_r\overline{B}(\mathfrak{z}_1) + \overline{q}_{nr}\overline{B}(\mathfrak{z}_2) + \underline{c}_0, \text{ and}$$

$$J_{2,t+1}\left(\overline{B}\left(\mathfrak{z}_1\right), \left(z^t, \mathfrak{z}_1\right)\right) = U^2\left(\mathfrak{z}_1\right) = u(\underline{c}_0) + \beta\overline{\pi}U^2\left(\mathfrak{z}_1\right) + \beta(1-\overline{\pi})U^2\left(\mathfrak{z}_2\right),$$

which give that

$$\underline{c}_0 = \epsilon_2\left(\mathfrak{z}_1\right), \text{ and}$$

$$\overline{B}\left(\mathfrak{z}_1\right) = \left[\frac{\overline{q}_{nr}}{1-\overline{q}_r}\right]\overline{B}\left(\mathfrak{z}_2\right) \leq 0.$$

The same argument can be applied for any time period; for Agent 1, the solvency constraints are obtained by symmetry. ■

In this equilibrium, agents are constrained against borrowing against the future state $z_{t+1}$ where their income will be high, regardless of the current state $z_t$. But there is an important difference depending on the current state $z_t$. If the agent has his high-income shock (say, $z_t = \mathfrak{z}_2$ for Agent 2), the agent does not want to borrow more against either the bad or the good future state, since for both cases, his marginal rate of substitution is equal to the Arrow price. In this case, he is *at* the constraint but it is a"false corner." If the solvency constraint were relaxed a bit, he will not change the optimal choice of consumption and asset holdings. On the other hand, if the current state is the one where the agent has his low income ($z_t = \mathfrak{z}_1$ for Agent 2), then the agent wants to borrow against his good state, but he cannot. In this case, the solvency constraint binds and the marginal rate of substitution between consumption in $z_t = \mathfrak{z}_1$ and in $z_{t+1} = \mathfrak{z}_2$ for Agent 2 is strictly lower than the corresponding Arrow price. In this state, the solvency constraint cannot be relaxed without changing the choice of consumption and asset holdings. Hence, the solvency constraints 'bind' only in the case where the agent's current income shock is low, *i.e.* poor agents are constrained from borrowing.

### 5.3. Calibration: risk-sharing and the Hansen-Jagannathan bounds

We calibrate individual income following Heaton and Lucas (1996) based on a large sample from the PSID. In particular, the log of an agent's income, relative to the aggregate, that is $\ln \epsilon_{i,t}$, is stationary with a first order serial correlation of 0.5 and a standard deviation of 0.29 for annual data. Initially we set $\beta = 0.65$ and explore the effect of risk aversion for consumption and for asset pricing implications, we will explore the quantitative effects of $\beta$ below.[5]

Figure 4 in the top panel presents consumption of Agent 2 as a function of risk aversion. For values of risk aversion between about 2 and 4, the type of efficient allocations is sequentially autarchy, partial risk-sharing and full risk-sharing. In the bottom panel, solvency constraints and asset positions are presented. The picture also contains the solvency constraints for full risk-sharing allocations. In these cases, solvency constraints can be computed but they never bind. The graph suggests, quite intuitively, that the range of full risk-sharing allocations with different consumption levels across agents increases with risk aversion. The derivations of the constraints and asset positions for the full risk-sharing case are in the Appendix.

Hansen and Jagannathan's volatility bounds for stochastic discount factors provide a concise and widely used diagnostic device.[6] Candidate kernels obtained from theoretical model structures can be compared to the benchmark given by the volatility bound. Figure 5 presents test results for kernels generated by our model. The model is able to generate kernels that fall inside the HJ-bound for risk aversion coefficients around 2, whereas the representative agent economy fails this

[5]The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\begin{bmatrix} \epsilon_1(\mathfrak{z}_1) \\ \epsilon_1(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.641 \\ 0.359 \end{bmatrix}$, $\begin{bmatrix} \epsilon_2(\mathfrak{z}_1) \\ \epsilon_2(\mathfrak{z}_2) \end{bmatrix} = \begin{bmatrix} 0.359 \\ 0.641 \end{bmatrix}$.

[6]For a detailled survey of applications of this test see, for example, Cochrane and Hansen (1992).

test for such values of risk aversion.[7] This positive finding also confirms empirical results by He and Modest (1995) that show that solvency constraints can pass a modified type of volatility bound test for aggregate consumption data. A close look at this picture reveals the two-sided role of the risk aversion coefficient. In most of the asset pricing literature, increasing risk aversion increases the volatility of the pricing kernel, because for a given consumption process marginal utility is more volatile. In our framework however, the extent of risk-sharing and thus the volatility of the consumption process is endogenous. The highest volatility for the pricing kernel is achieved with modest values of risk aversion, which correspond to very limited risk-sharing.

## 6. Quantitative predictions about asset returns

We now consider a more general endowment process. We choose it to be general enough to display wide ranging degrees of dependence between individual and aggregate income uncertainty. The form and the extent of this dependence turns out to be very important for asset prices. For that reason, our objective in this section is not only to document to what extent a plausibly calibrated model can explain a set of asset pricing moments, but also to derive some qualitative properties about how specific forms of dependence between individual and aggregate income uncertainty generate specific asset return properties.

### 6.1. Specification of the endowment process

We specify the endowment process with four values for the share of income of each agent and two values for the aggregate growth rate, respecting symmetry across agents. The set $Z$ has four elements. With symmetry we end up with a total of 10 parameters to be calibrated, six for $\Pi$, two for $\epsilon_2(\cdot)$ and two for $\lambda(\cdot)$.[8]

The subscripts $r$ and $e$ index a recession and an expansion respectively. The subscripts $h$ and $l$ index a high- and a low-income share for Agent 2. This table summarizes the four states' characteristics:

$$\begin{array}{lll} \mathfrak{z}_1: & \lambda(\mathfrak{z}_1) = \lambda_r, & \epsilon_2(\mathfrak{z}_1) = \epsilon_{lr} \\ \mathfrak{z}_2: & \lambda(\mathfrak{z}_2) = \lambda_e, & \epsilon_2(\mathfrak{z}_2) = \epsilon_{le} \\ \mathfrak{z}_3: & \lambda(\mathfrak{z}_3) = \lambda_r, & \epsilon_2(\mathfrak{z}_3) = \epsilon_{hr} \\ \mathfrak{z}_4: & \lambda(\mathfrak{z}_4) = \lambda_e, & \epsilon_2(\mathfrak{z}_4) = \epsilon_{he}. \end{array}$$

[7] Introducing aggregate uncertainty would of course generally give further volatility to the pricing kernel and help it pass the test.

[8] There is no need to specify $\epsilon_1(\cdot)$ since $\epsilon_1(z) + \epsilon_2(z) = 1$.

where $\lambda_r < \lambda_e$ and

$$\epsilon_{lr} < \frac{1}{2} < \epsilon_{hr} \text{ and } \qquad \epsilon_{le} < \frac{1}{2} < \epsilon_{he}.^{9}$$

### 6.2. Calibration of the benchmark endowment process

We use 10 moments describing the aggregate and household income data to select the 10 free parameters of the endowment process. Our endowment process replicates four moments of U.S. aggregate output in the $20^{th}$ century, while the remaining six moments characterize household income risk. The preference parameters, risk aversion and pure time-discounting, are discussed in the next section.

The first two moments imply a 2 by 2 nonsymmetric matrix (each element of this matrix is a sum of different probabilities in the matrix defined in the previous subsection).

**M1**. $\rho(\lambda) = -0.14$, first order serial correlation, Mehra and Prescott (1985).

**M2**. $\Pr(expansion)/\Pr(recession) = 2.65$, NBER business cycle chronology for 1889-1991.

Given the previous matrix, the next two conditions determine the two values of $\lambda$.

**M3**. $E(\lambda) = 1.83\%$, Mehra and Prescott (1985).

**M4**. $Std(\lambda) = 3.57\%$, Mehra and Prescott (1985).

The remaining free parameters are determined jointly.[10] We use the studies by Heaton and Lucas (1996), henceforth HL, and Storesletten, Telmer and Yaron (1997), henceforth STY, to guide us in determining a benchmark calibration. After defining these moments and our benchmark values, we discuss further below how these choices are related to the two original data studies.

**M5**. $Std\left(\ln \epsilon\left(z\right)\right) = 0.296$.

**M6**. $\rho\left(\ln \epsilon\left(z\right)\right) = 0.53$, first order serial correlation.

[9] By symmetry, for each $z$, and each value of $\epsilon_2\left(z\right)$ there must be another $\tilde{z}$ such that $\epsilon_1\left(\tilde{z}\right) = \epsilon_2\left(z\right)$. This implies the following two equations:

$$\begin{aligned} \epsilon_{lr} + \epsilon_{hr} &= 1 \\ \epsilon_{le} + \epsilon_{he} &= 1. \end{aligned}$$

[10] The system is solved with a nonlinear equation solver. For part of the region in the moment space there are two sets of parameters that replicate the selected moments. In these cases we pick the solution that has less variations in the means conditional on whether the destination is a recession or an expansion. This is the process that is closer to the linear processes estimated by HL and STY for which conditional means do not depend on the destination state.

**M7**. $v_r/v_e = 1$, the cross sectional dispersion of the shares in recessions relative to expansions. In a given state this is defined as

$$v^2(z_t) \equiv \frac{1}{2}\sum_{i=1,2}\left[\epsilon_i(z_t) - \frac{1}{2}\right]^2$$

and by symmetry we have

$$v(\mathfrak{z}_1) = v(\mathfrak{z}_3) \text{ and } v(\mathfrak{z}_2) = v(\mathfrak{z}_3).$$

**M8**. Relative standard deviation of individual shares conditional on current and past realizations of the aggregate shock:

$$\frac{\sigma_{r'e}}{\sigma_{e'e}} \equiv \frac{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e\right)}{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_e, \lambda_t = \lambda_e\right)} = 1.5\,.$$ [11]

**M9**. Relative standard deviation of individual shares conditional on current and past realizations of the aggregate shock:

$$\frac{\sigma_{r'r}}{\sigma_{e'r}} \equiv \frac{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_r\right)}{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_e, \lambda_t = \lambda_r\right)} = 1.5.$$

**M10**. Relative standard deviation of individual shares conditional on past realization of the aggregate shock:

$$\frac{\sigma_r}{\sigma_e} \equiv \frac{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_t = \lambda_r\right)}{std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_t = \lambda_e\right)} = 0.95.$$

The following Table contains the moments implied by the HL and the STY estimation of the household income processes. For the HL calibration there are two values in each cell, the first for the entire sample of 860 households and the second for the subsample of 327 stockholders. Since HL and STY estimate the individual income process conditional on the aggregate income, we combine their estimates of the individual income process with our specification for aggregate income. Our calibration mainly follows the HL calibration, given that they are calibrating a model with the same infinite horizon, two-agent structure. Given that earlier work on asset pricing with incomplete markets has argued and shown that the moments 8 and 9 are important we choose a value slightly higher than HL, but still below STY's. In any case we will provide extensive sensitivity analysis.

[11] It can be shown that $std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e\right) = std\left(\ln\epsilon_i(z_{t+1})\,|\lambda_{t+1} = \lambda_r, \lambda_t = \lambda_e, \epsilon_{i,t}\right)$, that is, the conditional standard deviation of the share (and the log) does not depend on the current idiosynchratic shock, only on the aggregate growth state that is fully described by $\lambda_t$.

| | | HL | HL, our $\lambda_t$ | STY, our $\lambda_t$ | Benchmark |
|---|---|---|---|---|---|
| M5 | $Std(\ln \epsilon)$ | 0.3/0.4 | | 0.71 | .3 |
| M6 | $\rho\,(\ln \epsilon\,(z))$ | 0.53/.3 | | 0.87 | .53 |
| M7 | $v_r/v_e$ | 1.02/0.93 | 1.03/0.9 | $[.93 - 1.07]$ | 1 |
| M8 | $\sigma_{r'e}/\sigma_{e'e}$ | 0.99/1.19 | 0.99/1.27 | 1.88 | 1.5 |
| M9 | $\sigma_{r'r}/\sigma_{e'r}$ | 0.99/1.19 | 0.99/1.27 | 1.88 | 1.5 |
| M10 | $\sigma_r/\sigma_e$ | 1.37/1.01 | 0.45/0.86 | 0.90 | .95.[12] |

### 6.3. Solving the constrained efficient allocations

The model can be solved by iterating on the functional equation (4.1). However, this would be relatively costly in computing time and may introduce computational errors for the process of the highest marginal rate of substitution. Instead, we solve the model by extending the analysis of section (5) to the present four state case. In particular, we first solve 16 nonlinear equation systems describing subsets of all necessary conditions. We then select the appropriate one by checking the sufficient conditions described in Alvarez and Jermann (1998a) for a constrained efficient allocation: resource constraints, participation constraints, first order conditions, and the condition that the value of aggregate endowment is finite. We find constrained efficient allocations where consumption shares have at most 12 different values in the ergodic set. This approach is much faster than iterating on a functional equation and leaves virtually no room for numerical approximation errors.

### 6.4. Quantitative implications for the benchmark case

We document the implications for risk-sharing and asset pricing as a function of risk aversion, $\gamma$, and the pure time-discount factor, $\beta$, for the benchmark endowment process. Figure 6 presents consumption share volatility, the average risk-free rate, the equity premium, and the premium for long term bonds. We define equity as a claim to aggregate endowment and the long term bond as a real consol. We find that the extent of risk-sharing, as measured by the consumption volatility, is increasing in risk aversion and the time-discount factor. This is a quantitative illustration of the results in Remark (1). With limited risk-sharing there is only a small region, close to the one corresponding to autarchy, where the risk free rate attains reasonably low values. Interestingly, the equity premium is

[12] The two studies define their idiosynchratic income variable in a slightly different way. HL use $y = \ln(\epsilon)$, whereas, STY use $x = \ln \epsilon_i\,(z_t) - \frac{1}{I}\sum_{j=1,I} \ln \epsilon_j\,(z_t)$. Using a first order log-linear approximation the two measures differ only by a constant, so that the two can be considered, to a first approximation, as identical for the moments we consider here.

highest for values of risk aversion that are higher than what is required to be in autarchy. Finally, the shape of the premium for long term bonds, as function of $\gamma$ and $\beta$, is similar to the one for the equity premium, but the values are lower.

To gain intuition about what determines the level of the risk-free rate, we compare it to the one of the representative agent economy. In the later case

$$\frac{1}{R^f_{t,t+1}} = E_t \left[ \beta \left( \frac{e_{t+1}}{e_t} \right)^{-\gamma} \right],$$

and the risk-free rate, $R^f_{t,t+1}$, changes one for one with changes in the time-discount factor $\beta$. In the economy with solvency constraints the risk-free rate satisfies

$$\frac{1}{R^f_{t,t+1}} = E_t \left[ \beta \left( \frac{e_{t+1}}{e_t} \right)^{-\gamma} \max_{i=1,2} \left\{ \left( \frac{\hat{c}_{i,t+1}}{\hat{c}_{i,t}} \right)^{-\gamma} \right\} \right],$$

and thus $\beta$ changes the risk-free rates one for one for constant consumption allocation $\hat{c}_{i,t}, \hat{c}_{i,t+1}$. Because the equilibrium allocation depends also on $\beta$ there is a second effect on the level of the interest rate. For instance, if $\beta$ is lowered, there is less risk-sharing and individual consumption shares become more dissimilar and thus the 'max' increases, leading to a reduction in the interest rate. This is an illustration of the more general result in Alvarez and Jermann (1998a) that state prices in the solvency constraints economy are higher than the prices in the corresponding representative agent economy.

In Table 1, we set risk aversion $\gamma = 3$, and choose $\beta$ to match the historical average of the U.S. risk-free rate of 0.80% per annum. The implications from our benchmark case are very encouraging. We can generate a sizeable equity premia with low risk aversion.[13] Compared with U.S. data, the volatility of the risk-free rate and the premium for long-term bonds is close to their empirical counterparts. Table 1 illustrates that, compared with the representative agent economy, the restrictions on the portfolio choices implied by the endogenous solvency constraints improve the ability to generate realistic asset pricing implications substantially.

The implied value of $\beta$ is lower than the values used in other studies. We have followed the typical procedure used to identify $\beta$, *i.e.* we have chosen it to match the average risk-free rate. The last row of Table 1 presents an alternative "high beta" parametrization of the endowment process where individual shocks are more persistent (M6 = .9) and where recessions and expansions are more asymmetric ( M8 = M9 = 4 and M10 =.85). For this parametrization we set

[13] Note that the model economy does not have leverage, therefore we should compare it to the unlevered equity premium in the data, which is lower than the 6.18% of the reported, levered, equity premium.

$\gamma = 3.5$; then $\beta = .78$ is required to match the risk free rate. The other asset pricing implications are virtually unchanged.[14]

### 6.5. Determinants of the equity premium and the term structure

To understand the determinants of the equity premium and of the term structure, we analyze calibrations with different types of dependence between aggregate and individual income uncertainty. We start with a calibration where aggregate and individual income shocks are independent. Then we document the consequences for asset prices of the types of dependence described by the moments M8 to M10. We find that the equity premium depends on the comovement between individual income uncertainty and the contemporaneous aggregate income growth. The term premium depends on the comovement between the forecast of future individual income uncertainty and aggregate income growth.

We refer to the case where the aggregate income growth shock is *i.i.d.* and independent of the individual income shock as the calibration with *independent risks*. In particular, we specialize our framework to the case where $z$ can be decomposed as $z = (x, y) \in Z = X \times Y$, and where $\lambda$ and $\boldsymbol{\epsilon}_i$ are functions of $y$ and $x$, respectively. Specifically, $\lambda : Y \rightarrow R_+$ and $\boldsymbol{\epsilon}_i : X \rightarrow (0, 1)$.

**Definition 6.1.** *('Independent risks') We say that the aggregate shock is i.i.d. and independent of the individual income shock if there is a probability distribution $\phi$ and a stochastic matrix $\psi$ such that*

$$\pi\left(z'|z\right) \equiv \pi\left(\left(x', y'\right) | (x, y)\right) = \phi\left(y'\right) \cdot \psi\left(x'|x\right)$$

*for all $z, z'$.*

The first panel of Table 2 contains the case with independent risks, for which $M1 = 0$ and $M2, M7, M8, M9, M10 = 1$.[15] In all cases presented in Table 2, risk aversion $\gamma = 3$ and the time-discount factor $\beta = 0.5$, so that all the equilibria display some, but not complete risk-sharing as can be seen by the fact that $0 < Std(\ln \widehat{c}) < 0.296 = Std(\ln \epsilon)$. We summarize our findings in four results.

- **Result 1**: With independent risks, interest rates are constant and thus there is no risk premium for bonds (see panel 1 of Table 2).

[14] Interestingly, Ligon, Thomas and Worral (1997), using a version of this model and data from individual consumption and income from poor villages, have estimated values of $\beta$ in the neigbourhood of 1/2.

[15] The economy also displays symmetry between expansions and recessions, that is, $M2 = 1$. This is not necessary for the results presented here. Even with $M2 \neq 1$, the same qualitative properties hold.

As shown in the Appendix, the assumption of independent risks is not sufficient to generate this result in general. This result also depends on the particular number of states and agents chosen in this calibration. Nevertheless, we think this is a useful benchmark, because starting with a case that has constant interest rates will allow us to consider separately what determines term premia and premia for payout uncertainty.

- **Result 2**: With independent risks, the (multiplicative) equity premium in the solvency constraints economy is identical to the one in the corresponding representative agent economy.

As the reader can see in panel 1 of Table 2, the multiplicative risk premium for both economies is equal to 0.3689%.[16] It is in general true that with independent risks, the risk premium for a one-period risky claim with payout contingent on the aggregate endowment is the same in the solvency constraints economy as the one in the corresponding representative agent economy. As shown in the Appendix, this equivalence result extends here to the equity premium because interest rates are constant.

The following result illustrates a departure from independent risks that produces a different equity premium in the solvency constraints economy.

- **Result 3:** A negative covariance of the individual income variance with the contemporaneous aggregate income growth ($M8 = M9 > 1$) increases the equity premium and a positive covariance ($M8 = M9 < 1$) reduces the equity premium relative to the corresponding representative agent economy.

With $M8 = M9 > 1$ recessions are associated with higher individual income risk. Therefore, states where risk-sharing is very limited are more likely in recessions. Thus, in recessions the pricing kernel is higher, as is suggested by equation (3.4), making equity more risky. This is illustrated in the second panel of Table 2. Result 3 is related to a well known result by Mankiw (1985), who shows that the price of a risky strip will be lower in a static model with *exogenous incomplete* markets. In his environment, the convexity of the marginal utility is a necessary condition for a higher premium. Constantinides and Duffie (1995) use an assumption analogous to $M8 = M9 > 1$ in a model with *exogenously incomplete* markets and permanent individual income shocks to show that the pricing kernel is identical to the one in an economy with complete markets but for an agent

[16] We define the *multiplicative equity premium* as $E_t(R^e_{t,t+1})/E_t(R^f_{t,t+1}) - 1$. We define the *equity premium* in the standard way as $E(R^e_{t,t+1}) - E(R^f_{t,t+1})$. In the case considered for result 2, the multiplicative equity premium is constant across states.

with higher risk aversion. In both cases, the results follow from the assumption that agents have convex marginal utility and that there are no binding portfolio constraints: hence all agents have the same valuation of assets. Instead, in the economy with solvency constraints, Result 2 follows from the fact that marginal valuation across agents differ, and prices are equal to the highest marginal rate of substitution.

Note that interest rates remain constant even if we introduce dependence of the aggregate and individual risk such that $M8 = M9 \neq 1$, as can be seen in the second panel of Table 2. For this case, the dependence introduced between the aggregate and the individual shocks does not introduce any predictable changes in the pricing kernel, therefore interest rates remain constant.

The next result isolates a feature that explains volatility in interest rates and the existence of non-zero risk premia for bonds.

- **Result 4:** Introducing dependence between the aggregate income growth and next period's individual income risk makes the risk-free rate variable. A positive covariance of the aggregate income growth and next period's individual income risk, $M10 < 1$, creates positive term premia. A negative covariance of the aggregate income growth and next period's individual income risk, $M10 > 1$, creates negative term premia.

Predictable movements in the pricing kernel are required for nonconstant interest rates. In this case, the predictable changes are introduced through the conditional variance of the individual's income. For instance, if in recessions the variance for next period's consumption share is expected to be lower than in expansions, then the price of bonds is lower in recessions and higher in expansions, thus a positive term premium is required to compensate bond holders. In other words, with countercyclical interest rates, capital gains accrue to bondholders when they are valued relatively less—requiring positive term premia. Panel 3 of Table 2 illustrates this case.

## 7. Conclusions

The objective of this paper was to explore the quantitative asset pricing implications of a model with endogenously restricted risk-sharing. We show under which circumstances the endogenous solvency constraints will bind and we characterize the pricing kernel. We describe an algorithm to compute an equilibrium for given preference parameters and specification of the stochastic process for individual and aggregate income. This algorithm is based on the equivalence between equilibrium with endogenous solvency constraints and constrained efficient allocations. We found that for plausibly calibrated income processes and for low values of risk aversion the model produces solvency constraints that bind frequently, and as a consequence, individual consumption is volatile enough so that the resulting pricing kernel passes the Hansen and Jagannathan test. Additionally, given the calibrated correlations between the individual income risk and the aggregate income, the model produces sizable premia for equity and long-term bonds. We also provide characterizations of how the dependence between individual and aggregate risks determines both equity and bond premia.

We think that this model improves on the standard representative agent economy and on the—arbitrary—incomplete markets economies, since it makes the portfolio constraints endogenous and simultaneously obtains asset pricing implications that are closer to the data. Nevertheless, our assumption about punishment from defaulting is also arbitrary. We assume that default is punished by permanent exclusion of all asset markets but entails no garnishment of labor income. Instead, if the exclusion from asset markets is temporary, it will make default more attractive. On the other hand, if a fraction of the labor income is garnished, it will make default less attractive. Incorporating these more realistic alternatives may produce similar results, since they compensate each other. We leave the investigation of these alternatives, as well as the introduction of endogenous costly punishment, as a topic for future research.

In some related work (Alvarez and Jermann (1998b, 1999)), we have applied the model used in this paper to measure the cost of business cycle fluctuations and to analyze optimal international portfolio choices. We find that the cost of business cycles can be small in a model that generates an equity premium of several percentage points. For international portfolio diversification we find that countries display a high degree of home bias as we see it empirically.

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**Appendix**

**Proof of Proposition (5.1)**

In order to demonstrate this property we state the following two lemmas.

**Lemma 7.1.** *If*

$$\partial V\left(\bar{w}\left(\mathfrak{z}_1\right), \mathfrak{z}_1\right) / \partial w > \partial V\left(\bar{w}\left(\mathfrak{z}_2\right), \mathfrak{z}_2\right) / \partial w$$

*then the decision rules that achieve $TV$ are flat after reversal.*

**Proof.** The proof follows immediately from the first order conditions of the problem defined by the RHS of equation (4.1). ■

**Lemma 7.2.** *If the decision rules after reversal are flat, then*

$$\partial TV(\bar{w}(\mathfrak{z}_1), \mathfrak{z}_1)/\partial w > \partial TV(\bar{w}(\mathfrak{z}_2), \mathfrak{z}_2)/\partial w.$$

**Proof.** Using the "$45^0$ line" result from Proposition (4.7) and the assumption that the "decision rules are flat after reversal" in the two promise keeping equations (4.3) evaluated at $\bar{w}(\mathfrak{z}_1)$ and $\bar{w}(\mathfrak{z}_2)$ we obtain

$$\bar{w}(\mathfrak{z}_1) - \bar{w}(\mathfrak{z}_2) = \frac{u\left(e - C_2\left(\bar{w}(\mathfrak{z}_1), \mathfrak{z}_1\right)\right) - u\left(C_2\left(\bar{w}(\mathfrak{z}_2), \mathfrak{z}_2\right)\right)}{1 - \beta\bar{\pi} + \beta(1-\bar{\pi})}.$$

The desired result now follows by using the envelop condition. ■

If full risk-sharing is not possible, then $\partial\tilde{V}\left(\bar{w}\left(\mathfrak{z}_1\right), \mathfrak{z}_1\right)/\partial w > \partial\tilde{V}\left(\bar{w}\left(\mathfrak{z}_2\right), \mathfrak{z}_2\right)/\partial w$. The result that the optimal decision rules are flat after reversal for the fixed point $V^*$ follows by the combination of the previous two lemmas with the result that $\lim_{n\to\infty} T^n\tilde{V} = V^*$. At each iteration, the two lemmas are applied sequentially and then the domains are computed for $TV$. The fact that the described property is preserved for the limit follows directly from the fact that the limit is differentiable.

**Derivation of asset position and constraints with full risk-sharing**

Full risk-sharing allocations are characterized by constant consumption across the two states for each agent. Combining this fact with the sequential budget constraints, it is immediate to find that the asset positions for Agent 2 are given by:

$$\begin{aligned}
\overline{a}_2(\mathfrak{z}_2) &= \frac{\left(\frac{\overline{q}_{nr}}{1-\overline{q}_r}\right)\left(\overline{c}_2 - \epsilon_2(\mathfrak{z}_1)\right) + \left(\overline{c}_2 - \epsilon_2(\mathfrak{z}_2)\right)}{(1-\overline{q}_r) - \left(\frac{\overline{q}_{nr}}{1-\overline{q}_r}\right)\overline{q}_{nr}}, \text{ and} \\
\overline{a}_2(\mathfrak{z}_1) &= \frac{q_{nr}\overline{a}_2(\mathfrak{z}_2) + \overline{c}_2 - \epsilon_2(\mathfrak{z}_1)}{1-\overline{q}_r},
\end{aligned}$$

where $\overline{c}_2$ is the constant consumption level for Agent 2. For the solvency constraints, $\overline{B}_2(\mathfrak{z}_2)$ is determined by the same function as $\overline{a}_2(\mathfrak{z}_2)$ above except that $\overline{c}_2$ is replaced by $\underline{c}_0 = u^{-1}(U^2(\mathfrak{z}_2))$. $\overline{B}_2(\mathfrak{z}_1) = \left(\frac{\overline{q}_{nr}}{1-\overline{q}_r}\right) \overline{B}_2(\mathfrak{z}_2)$ as in the cases without full risk-sharing; for Agent 1, symmetry is applied.

**Proof of Result 1:** Consider the price of a one-period bond,

$$\beta \sum_{x_{t+1}} \sum_{y_{t+1}} \max_{i=1,2} \left[ \left( \frac{\widehat{c}_{i,t+1}}{\widehat{c}_{i,t}} \right)^{-\gamma} \right] \lambda(y_{t+1})^{-\gamma} \phi\left(y_{t+1}\right) \cdot \psi\left(x_{t+1}|x_t\right).$$

In Alvarez and Jermann (1998a) we show in proposition 4.18 that for an economy with independent risks the consumption shares $\hat{c}_i$ do not depend on the aggregate state $y$ (otherwise unnecessary volatility in consumption would be introduced). Thus, the price of a one-period bond can be written as

$$\beta \left\{ \sum_{y_{t+1}} \lambda(y_{t+1})^{-\gamma} \phi\left(y_{t+1}\right) \right\} \left\{ \sum_{x_{t+1}} \max_{i=1,2} \left[ \left( \frac{\widehat{c}_{i,t+1}}{\widehat{c}_{i,t}} \right)^{-\gamma} \right] \cdot \psi\left(x_{t+1}|x_t\right) \right\}.$$

Each of the two terms in curly brackets can be shown to be constant. For the first term, it is immediate since $\lambda_{t+1}$ is assumed to be *i.i.d.*. For the second term, this follows by recognizing that with independent risks the share representation of the economy has a constant discount factor and that two states, out of the four, are identical. The discount factor for the representation of the economy in terms of shares is $\widehat{\beta}(z) = \beta \sum_{y_{t+1}} \lambda(y_{t+1})^{-\gamma} \phi\left(y_{t+1}\right)$. Hence this economy is equivalent to one with only two values of the shocks $z$. In particular, $\epsilon_{lr} = \epsilon_{le} = \epsilon_l$ and $\epsilon_{hr} = \epsilon_{he} = \epsilon_{h.}$ In section (5.1) we show that for an economy with only two shocks, in the ergodic set, consumption is such that the highest marginal rate of substitution follows a simple two point Markov chain, see (5.4). Moreover its expected value is constant and hence by (3.4) interest rates are constant. At last, since the risk-free rate is constant, by arbitrage, the risk premium for bonds of any maturity is zero.

**Proof of Result 2:**

We start by rewriting the equity premium as a weighted average of risk premia to individual dividends. Let $D_{t+k}$ be the dividend of a stock at $t+k,$ and $\{D_{t+k}\}_{k=1}^{\infty}$ be the dividend process, and $V_t\left[\cdot\right]$ the value at $t$ of $D_{t+k}$. Define the value of the stock as

$$V_t[\{D_{t+k}\}_{k=1}^{\infty}] = \sum_{k=1}^{\infty} V_t\left[D_{t+k}\right],$$

so that the price of a stock can be seen to be the price of a portfolio containing claims to each dividend, that is, a portfolio of *dividend strips.* The one-period holding return of equity is:

$$R_{t,t+1}[\{D_{t+k}\}_{k=1}^{\infty}] = \sum_{k=1}^{\infty} \left( \frac{V_t[D_{t+k}]}{V_t[\{D_{t+k}\}_{k=1}^{\infty}]} \right) \cdot R_{t,t+1}\left[D_{t+k}\right],$$

where the one-period return of a dividend strip is:

$$R_{t,t+1}\left[D_{t+k}\right] = \frac{V_{t+1}\left[D_{t+k}\right]}{V_t[D_{t+k}]}.$$

Denoting by $1_{t,t+1}$ a constant dividend, the conditional multiplicative equity premium equals

$$\frac{E_t(R_{t,t+1}[\{D_{t+k}\}_{k=1}^{\infty}])}{R_{t,t+1}[1_{t,t+1}]} = \sum_{k=1}^{\infty} w_t[D_{t+k}] \cdot \frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]}, \tag{7.1}$$

where $w_t$ are non-negative weights

$$w_t[D_{t+k}] = \frac{V_t[D_{t+k}]}{V_t[\{D_{t+k}\}_{k=1}^{\infty}]}$$

and where

$$\frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]}$$

is the conditional multiplicative risk premium of a strip paying $D_{t+k}$ at $t+k$.

Equation (7.1) shows that the equity premium is equal to a weighted average of all the strip premia. We have shown in Alvarez and Jermann (1998a) in Proposition (4.18) that with independent risks the risk premium for a one-period dividend strip, that is $E_t(R_{t,t+1}[D_{t+1}])/R_{t,t+1}[1_{t,t+1}]$, is identical in the solvency constraint economy to the one in the corresponding representative agent economy. Thus, in order to show that the multiplicative equity premium is equal across the two economies it is sufficient to show that risk premia for strips with different maturity dates $k$ are equal for each economy separately.

We first show that for a representative agent economy risk premia for strips with different maturities are equal for all $k$. Given *i.i.d.* aggregate growth rates

$$V_t[D_{t+k}] = \left[\beta E(\lambda^{1-\gamma})\right]^k,$$

and we have

$$E_t(R_{t,t+1}[D_{t+k}]) = \frac{1}{\left[\beta E(\lambda^{1-\gamma})\right]},$$

and thus

$$\frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]} = \frac{E(\lambda^{-\gamma})}{E(\lambda^{1-\gamma})}.$$

We now show that for the solvency constraints economy considered here, risk premia for strips with different maturities are equal for all $k$. In Alvarez and Jermann (1998a) we show in proposition 4.18 that for an economy with independent risks the consumption shares $\hat{c}_i$ do not depend on the aggregate state $y$ (otherwise unnecessary volatility in consumption would be introduced). Thus, the pricing kernel can be written as

$$\beta\left\{\lambda(y_{t+1})^{-\gamma}\phi\left(y_{t+1}\right)\right\}\left\{m\left(x_{t+1}|x_t\right)\cdot\psi\left(x_{t+1}|x_t\right)\right\},$$

where the first part in curly brackets depends only on the aggregate shock and the second part in curly brackets depends only on the individual shock. Using this notation, we can write that

$$V_t[D_{t+1}] = \beta E(\lambda^{1-\gamma})E(m),$$

where we have used the fact that, aggregate growth rates are *i.i.d.*, that interest rates are constant so that $E_t\left(m\left(x_{t+1}|x_t\right)\right) = E(m)$, and that the probabilities are separable for $y$ and $x$ so that $cov_t\left(\lambda(y_{t+1})^{-\gamma}, m\left(x_{t+1}|x_t\right)\right) = 0$. More generally, we have that

$$V_t[D_{t+k}] = \left[\beta E(\lambda^{1-\gamma})E(m)\right]^k,$$

where we have used the additional result that $cov_t\left(m\left(x_{t+1}|x_t\right), m\left(x_{t+2}|x_{t+1}\right)\right) = 0$. This last result follows directly from the fact that with constant interest rates, term premia are all equal to zero. We then have

$$E_t(R_{t,t+1}[D_{t+k}]) = \frac{1}{\left[\beta E(\lambda^{1-\gamma})E(m)\right]},$$

and thus

$$\frac{E_t(R_{t,t+1}[D_{t+k}])}{R_{t,t+1}[1_{t,t+1}]} = \frac{E(\lambda^{-\gamma})}{E(\lambda^{1-\gamma})}.$$

Thus, as before, the term structure for strip premia is flat, thus it is at the same level as the one of the representative agent economy. ■

Figure 1

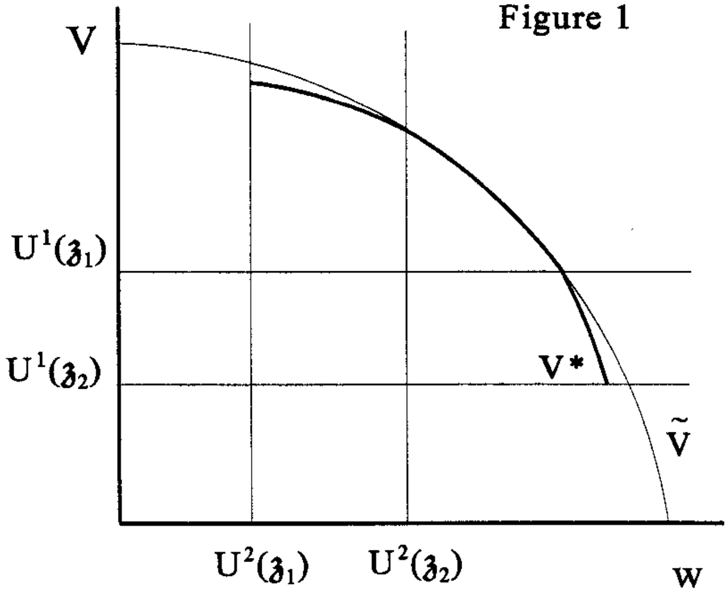



Figure 2

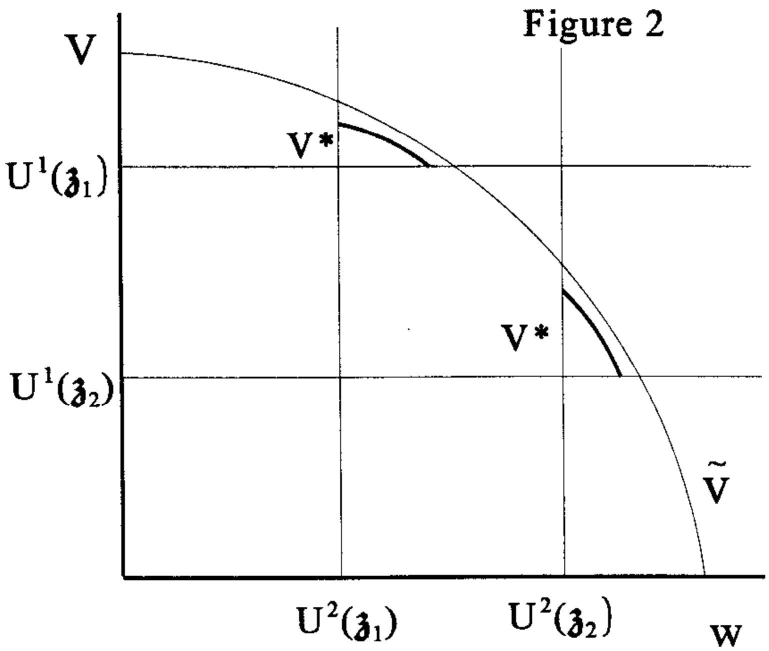


Figure 3

Decision rules with 2 shocks

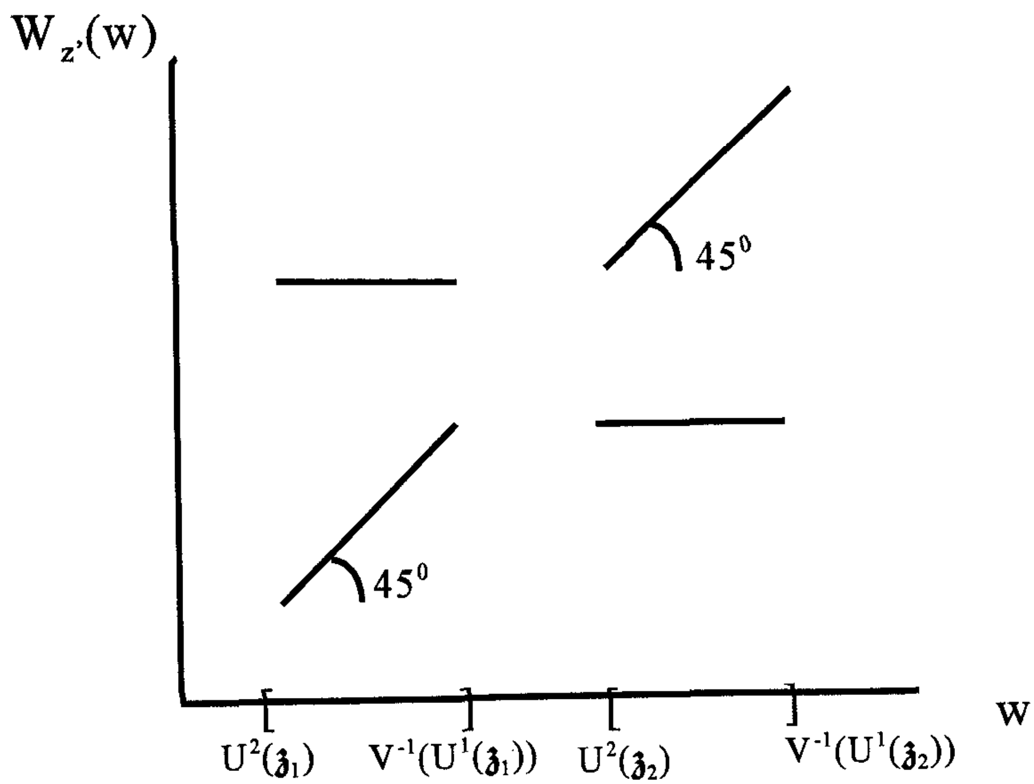


Figure 4: Consumption of Agent 2

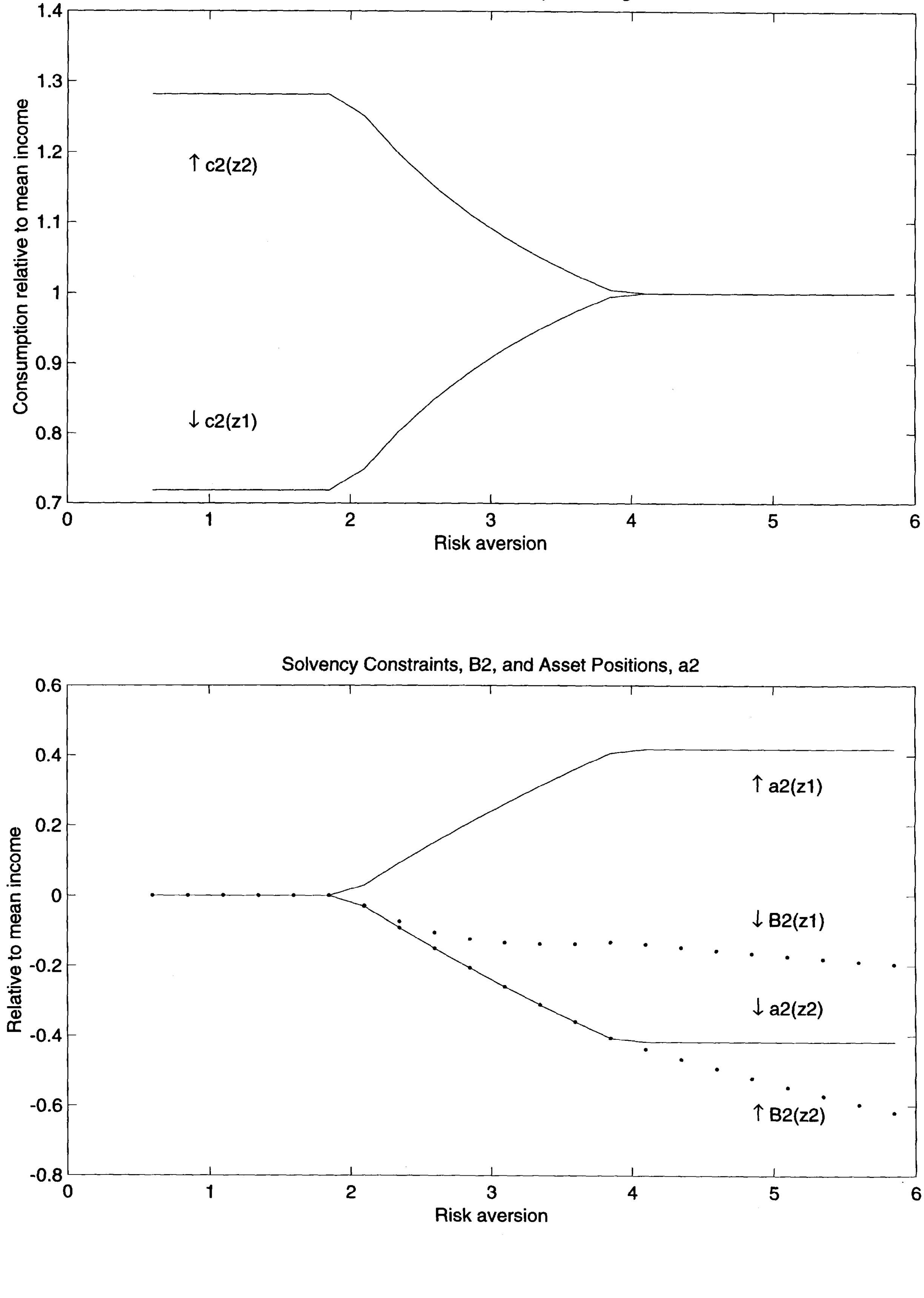
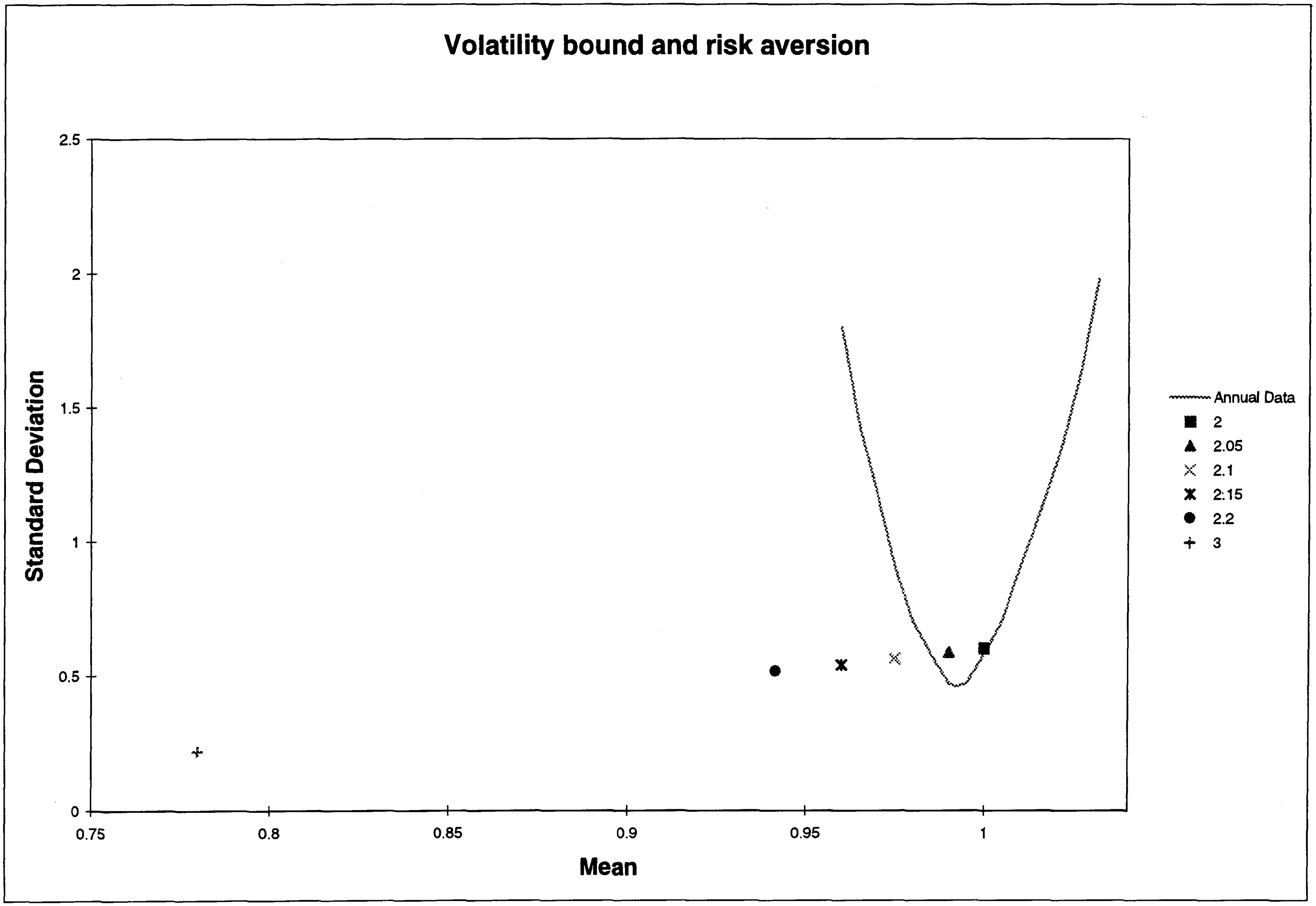
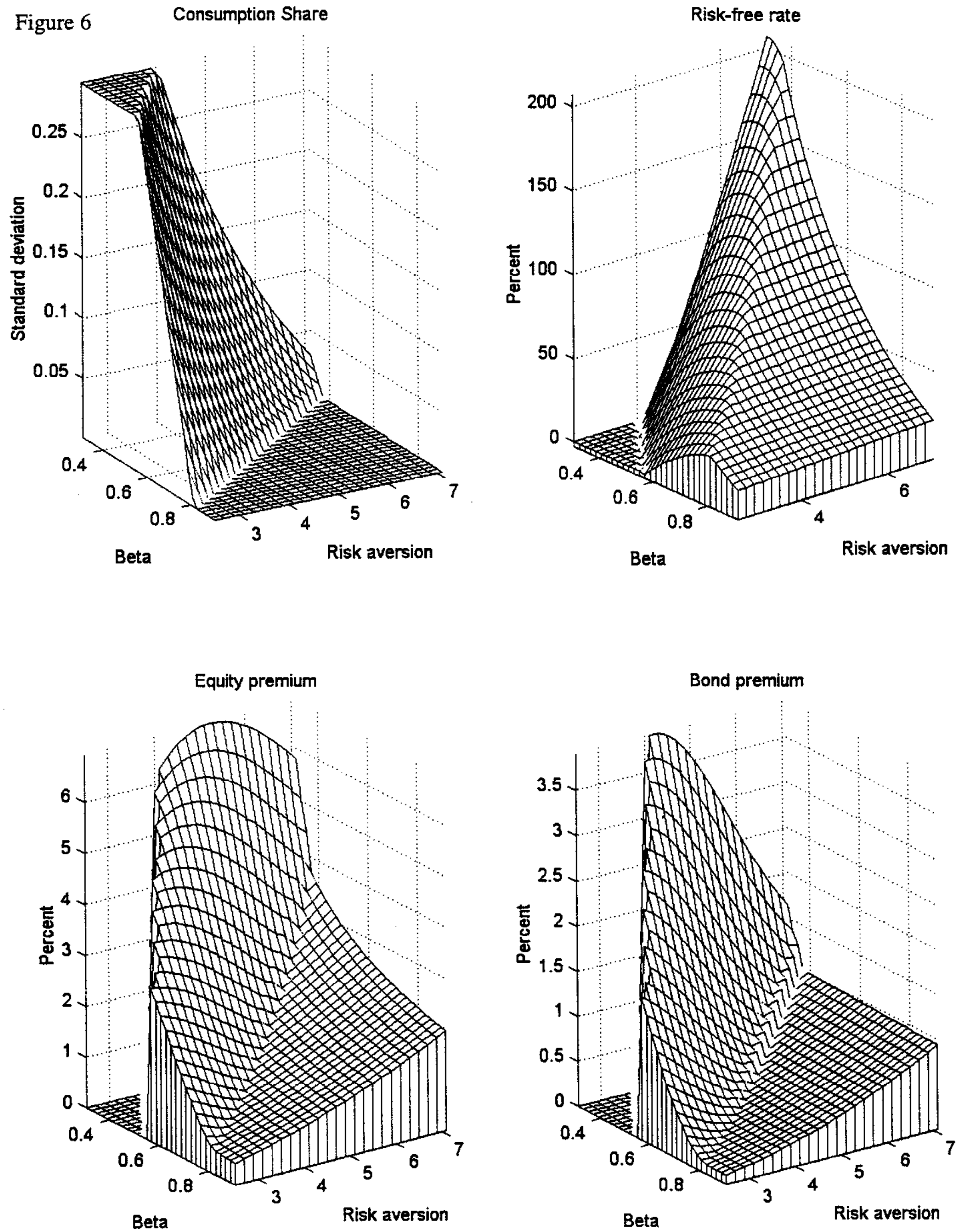


Figure 5




The annual data is from Cochrane and Hansen (1992). The individual income process has standard deviation of 29% and first order serial correlation of 0.5, this is consistent with estimates from Heaton and Lucas (1996).

Figure 6



Consumption Share
Standard deviation
0.25
0.2
0.15
0.1
0.05
0.4
0.6
0.8
Beta
3
4
5
6
7
Risk aversion
Risk-free rate
Percent
200
150
100
50
0
0.4
0.6
0.8
Beta
4
6
Risk aversion
Equity premium
Percent
6
5
4
3
2
1
0
0.4
0.6
0.8
Beta
3
4
5
6
7
Risk aversion
Bond premium
Percent
3.5
3
2.5
2
1.5
1
0.5
0
0.4
0.6
0.8
Beta
3
4
5
6
7
Risk aversion

Table 1

# Return premia and volatilities for the benchmark calibration

(Time-preference 'beta' chosen so as to match E(rf))

| | E(rf) | E(rb-rf) | E(re-rf) | Sharpe | Std(rf) | Std(rb) | Std(re) | Std(Inc) | rho1(Ine) | rho2(Ine) | beta |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Benchmark | 0.80 | 2.19 | 3.91 | 0.54 | 4.78 | 6.39 | 8.83 | 0.28 | 0.53 | 0.27 | 0.4887 |
| Represent. Agent | 4.00 | 0.16 | 0.53 | 0.12 | 1.68 | 2.16 | 4.81 | 0.00 | 0.00 | 0.00 | 1 |
| "High Beta" | 0.80 | 2.09 | 3.54 | 0.48 | 5.47 | 6.97 | 9.16 | 0.27 | 0.90 | 0.81 | 0.7801 |
| US-Data | 0.80 | 1.70 | 6.18 | 0.30 | 5.67 | | 16.54 | | | | |

*Returns are defined as follows: rf is the one-period risk-free rate; rb is the return of a long-term bond (a perpetual bond in the model); re is the return to equity. E(.) and Std(.) are unconditional means and standard deviations. Rho1 and rho2 are first- and second-order serial correlation coefficients. Finally, 'Inc' and 'Ine' stand for the log of the consumption shares and the log of the individual income share. Risk aversion = 3, Std(ln(e))=.296. For the 'High beta' case we use: M8,9=4, M10=.85, M6=.9, risk aversion=3.5. US-Data for equity and risk-free rate is from Mehra and Prescott (1985); long term government bond returns are from Ibbotson (1994). The model is annual.*

Table 2

# Determinants of the Equity premium and the Term structure

Deviations from 'Independent Risks'

| | E(rf) | E(rb-rf) | E(re-rf) | Sharpe | Std(rf) | Std(rb) | Std(re) | Std(Inc) | rho1(e) | rho2(e) | beta | (E(Re)/E(Rf)-1)*100 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | |
| Independent risks | | | | | | | | | | | | | |
| Solvency constraints | 5.62 | 0.00 | 0.39 | 0.1048 | 0.00 | 0.00 | 3.72 | 0.27 | 0.53 | 0.28 | 0.50 | 0.3689 | M8,9,10,2=1,M1=0 |
| Represent. Agent | 109.63 | 0.00 | 0.77 | 0.1048 | 0.00 | 0.00 | 7.38 | 0.00 | 0.00 | 0.00 | 0.50 | 0.3689 | +M5=0 |
| 2 | | | | | | | | | | | | | |
| Solvency constraints economy: | | | | | | | | | | | | | |
| Cov(dY',std(ei'))<0 | 6.63 | 0.00 | 1.29 | 0.34 | 0.00 | 0.00 | 3.78 | 0.26 | 0.53 | 0.28 | 0.50 | | M8=M9=1.5 |
| Cov(dY',std(ei'))>0 | 4.85 | 0.00 | -0.34 | -0.09 | 0.00 | 0.00 | 3.66 | 0.28 | 0.53 | 0.28 | 0.50 | | M8=M9=0.75 |
| 3 | | | | | | | | | | | | | |
| Solvency constraints economy: | | | | | | | | | | | | | |
| Cov(dY,std(ei'))>0 | 5.15 | 0.42 | 0.83 | 0.11 | 3.85 | 5.32 | 8.39 | 0.28 | 0.53 | 0.28 | 0.50 | | M10=.95 |
| Cov(dY,std(ei'))<0 | 6.22 | -0.31 | 0.06 | 0.26 | 3.65 | 5.01 | 3.67 | 0.27 | 0.53 | 0.28 | 0.50 | | M10=1.05 |

*Returns are defined as follows: rf is the one-period risk-free rate; rb is the return of a long-term bond (a perpetual bond in the model); re is the return to equity. E(.) and Std(.) are unconditional means and standard deviations. Rho1 and rho2 are first- and second-order serial correlation coefficients. Finally, 'Inc' and 'Ine' stand for the log of the consumption shares and the log of the individual income share. Risk aversion = 3, Std(ln(e))=.296.*