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Speculative Investor Behavior and Learning

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Abstract

As traders learn about the true distribution of some asset's dividends, a *speculative premium* occurs as each trader anticipates the possibility of re-selling the asset to another trader before complete learning has occurred. Small differences in prior beliefs lead to large speculative premiums during the learning process. This phenomenon helps explain a paradox concerning the pricing of initial public offerings. The result casts light on the significance of the common prior assumption in economic models.

I Introduction

In their 1978 paper "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," Harrison and Kreps characterize the price of a risky asset in a world where traders are risk neutral, have heterogeneous expectations about the asset's value, and cannot sell the asset short. In equilibrium, the asset must - after every history - be held by the trader who values it the most after that history, and the price must equal her short term valuation of holding the asset (if the price was less than her valuation, she would demand an infinite quantity). On the other hand, the price will typically exceed the short-term valuation of holding the asset to other traders (since they cannot sell it short). How will the price compare with the traders' "fundamental valuations" - that is, the value to them of holding the asset forever? Clearly the price must be at least as great as the fundamental valuation of the trader holding the asset. Harrison and Kreps observed that - given the heterogeneous expectations - the price will typically be *strictly* greater, since the trader currently holding the asset will anticipate contingencies in the future where other traders will value the asset more, and she will be able to re-sell for strictly more than her fundamental valuation.

Harrison and Kreps interpreted this result as a formalization of the notion of speculation in Keynes [1936, Chapter 12]: speculation occurs when an asset is bought for its short term expected gain, at a price higher than the expected discounted value of dividends. This result has apparently been largely ignored, presumably because of the assumption of (unmodelled) heterogeneity of expectations. In this paper, I consider a special case of the model of Harrison and Kreps, where traders *initially* have heterogeneous beliefs about the asset's fundamental value, but - because dividends are assumed to be i.i.d. - their beliefs converge over time. What becomes of the speculative premium then? In particular, how does the speculative premium depend on the initial heterogeneity of prior beliefs?

The key property of prior beliefs is the following. Say that a trader is an *optimist* if, after *every* history, she has the highest valuation of the asset. I provide a necessary and sufficient condition on priors for the existence of an optimist and show that the price always equals the optimist's fundamental valuation. On the other hand, if no optimist exists, then the price always *strictly* exceeds *every* trader's fundamental valuation, but through time, the price and traders' valuations all converge to an objective valuation.

This leaves open the question of how large speculative premiums are before learning occurs. I show that apparently small differences in priors can lead to large speculative premiums. I investigate the case where an asset either pays a dividend or not in any one period. Traders are uncertain of the true binomial parameter generating the dividend process. One trader has a uniform prior on the binomial parameter, and thus after observing s dividends in t periods, believes that the probability of a dividend in the next period is $(s+1) / (t+2)$. Another trader puts more weight on the empirical frequency and has posterior $(s+1/2)/(t+1)$. Then the price of the asset (before any returns are realized) is eight percent higher than *either* trader's valuation of the returns from the asset. This eight percent premium is generated by the option to re-sell the asset at some future date.

This phenomenon is consistent with the "hot issue" anomaly in the pricing of initial public offerings. The opening market price of initial public offerings appears to be too high relative to long-run values. Miller [1977] has suggested that this may be because the price tends to reflect the valuation of the most optimistic investor. My model formalizes Miller's conjecture and makes clear that it is enough that traders have different initial beliefs about the distribution of the parameters of the data-generating process. It is not required that they have different ex ante expected valuations of the stream of dividends from the asset.

By showing how fully rational learning is consistent with the model of Harrison and Kreps, I hope to show how heterogeneous prior beliefs can and should be used (selectively) as an assumption in understanding financial and other markets.

The paper is organized as follows. In section II, I examine the evolution of fundamental valuations of traders with initially heterogeneous beliefs. In section III, I present the model and main result relating prices to traders' valuations of the asset. In section IV, I present a numerical example that illustrates how reasonable differences in priors lead to significant speculative premiums in asset prices. In section V, I discuss the relationship to the empirical literature on initial public offerings. In section VI, I argue that the model and examples in this paper suggest a (limited) re-appraisal of economists' suspicion of arguments that appeal to differences in prior beliefs. Section VII concludes.

II Fundamental Values with Learning

A group of risk neutral traders are learning about the underlying value of a risky asset. In each period, the asset either pays a dividend of \$1, or not. The probability of a dividend in each period is θ and the dividend process is i.i.d. All this is common knowledge among the group of traders. Traders do not (initially) know the true value of θ, but have possibly heterogeneous prior beliefs about θ.

This simple dividend structure enables us to identify a trader's valuation of the asset after any given history with her point estimate $\widehat{\theta}$ of the value of θ, given that history. To see why, note that $\widehat{\theta}$ is also the expected dividend of the asset in the next period. Given the i.i.d. assumption, it is also the expected dividend of the asset in any future period. Thus if the interest rate from a safe asset were r, the (risk neutral) trader would value the asset at $(1/(1+r))\widehat{\theta} + (1/(1+r))^2\widehat{\theta} + .. = \widehat{\theta}/r$. Thus (ignoring the constant $1/r$), I will refer to a trader's point estimate of θ as her *fundamental valuation* of the asset. Note that this is the expected value to the trader of holding the asset forever. In the next section, I will relate these fundamental valuations to market prices. In this section, I explore how these fundamental valuations evolve as traders learn the true value of θ. Key questions for the analysis of competitive markets in later sections are the following. When is it the case that one trader remains the most optimistic about the asset (i.e. has the highest fundamental valuation) after every history? Conversely, when is it the case that after every history, there is a positive probability that the trader with the highest current fundamental valuation will not have the highest valuation in the future?

To state these questions formally, I introduce notation for the set of traders. There is a finite collection of risk neutral traders, $\mathcal{I} = \{1, ..., I\}$. An alternative interpretation is that $\mathcal{I}$ is a finite set of *types* of traders and that there are an infinite number of each type. Trader i's prior beliefs are represented by a density π_i over possible values of θ in $[0, 1]$. Assume that each π_i is twice differentiable and uniformly bounded below (there exists ε such that $\pi_i(\theta) \geq \varepsilon$ for all $i \in \mathcal{I}$ and $\theta \in [0, 1]$).

Suppose trader i observes a history of t periods in which a total of s dividends are paid. Then his posterior density over θ is given by

(1) $$\xi_i(\theta|s,t) = \frac{\theta^s(1-\theta)^{t-s}\pi_i(\theta)}{\int\limits_{\zeta=0}^{1} \zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta},$$

and the probability he attaches to a dividend being paid in the next period is

(2) $$\mu_i(s,t) = \int\limits_{\theta=0}^{1} \theta\xi_i(\theta|s,t)d\theta = \frac{\int\limits_{\theta=0}^{1} \theta^{s+1}(1-\theta)^{t-s}\pi_i(\theta)d\theta}{\int\limits_{\theta=0}^{1} \theta^s(1-\theta)^{t-s}\pi_i(\theta)d\theta}.$$

For the reasons noted above, I will refer to $\mu_i(s,t)$ as trader i's *fundamental valuation* of the asset after history (s,t). I will be examining the properties of the following example throughout the paper.

Example 1 *(Ignorance Priors) Imagine a situation where the risky asset is being traded for the first time, so that traders do not have a history of past dividends on the basis of which to form beliefs about θ. They must form "ignorance priors" about θ. I want to consider traders who are conservative in the light of their ignorance, but this will not pin down the exact prior they use. Suppose that trader 1 has a uniform prior over the parameter space, i.e. $\pi_1(\theta) = 1$ for all $\theta \in [0,1]$; integration by parts and elementary algebra shows that his valuation of the asset after history (s,t) is $\mu_1(s,t) = (s+1)/(t+2)$. On the other hand, trader 2 has read Jeffreys' (1946) proposal for dealing with ignorance and chooses a prior that minimizes entropy. Thus $\pi_2(\theta) \sim 1\big/\sqrt{\theta(1-\theta)}$, so that her valuation of the asset after history (s,t) is $\mu_2(s,t) = (s+1/2)/(t+1)$. Note that her point estimate of the dividend is thus shifted toward the observed empirical frequency s/t (relative to trader 1's posterior). Both these posteriors are reasonable. Despite many attempts, there is no philosophical (or other) agreement on how to assign priors in the face of ignorance.*

Given the simple learning environment, fundamental valuations will converge to the "objective" value, i.e., the observed frequency of dividends.

Lemma 1 *For all $\theta_0 \in [0,1]$ and $i \in \mathcal{I}$, $\mu_i(\theta_0 t, t) \to \theta_0$ as $t \to \infty$.*

In the ignorance priors example, as $t \to \infty$, $\mu_1(\theta_0 t, t) = (\theta_0 t + 1)/(t+2) \to \theta_0$ and $\mu_2(\theta_0 t, t) = (\theta_0 t + 1/2)/(t+1) \to \theta_0$. Thus all valuations converge to the observed frequency or common objective value. They will, however, converge at different rates. For any given history (s,t), some traders will be more optimistic, some less so. But when is there a trader who values the asset the most after *every* history?

Definition 1 *Trader k is a global optimist if $\mu_k(s,t) \geq \mu_i(s,t)$ for all $i \in \mathcal{I}$ and all histories (s,t).*

In the ignorance priors example, there is no global optimist, since $\mu_1(s,t) = (s+1)/(t+2) > (s+1/2)/(t+1) = \mu_2(s,t)$ if $s < t/2$, while $\mu_1(s,t) < \mu_2(s,t)$ if $s > t/2$. Intuitively, trader 2 puts more weight on the data, so after "good histories" $(s > t/2)$ she values the asset the most; trader 1, whose uniform prior puts less weight on the data, is thus more optimistic after "bad histories" $(s < t/2)$.

Even if trader k is not a *global* optimist, we might be interested in a situation where at least after *some* history, trader k is and *remains* the most optimistic. Thus say that history (s',t') *follows* (s,t) if $t' \geq t$ and $t' - t + s \geq s' \geq s$.

Definition 2 *Trader k is a local optimist if there exists a history (s,t) such that for all histories (s',t') following (s,t), $\mu_k(s',t') \geq \mu_i(s',t')$ for all $i \in \mathcal{I}$.*

Is it the case, in the ignorance priors example, that if the history is sufficiently bad, trader 1 not only values the asset the most, but will value it the most after every continuation history? The answer is no: after *any* history (s,t), suppose that $2t$ periods follow in which a dividend is always paid. Then we will be at history $(s+2t, 3t)$. In this case, since $s + 2t > 3t/2$, $\mu_1(s+2t, 3t) < \mu_2(s+2t, 3t)$. Thus 1 is not a local optimist. An analogous argument shows that 2 is not a local optimist. No matter how much they have learned about θ, they always attach positive probability to switching from a good history to a bad history, or vice versa.

The existence of global and local optimists is critical to our analysis since it determines when there is a trader who ends up holding the asset forever with no retrading. An alternative way of expressing the non-existence of a local optimist is the following.

Definition 3 *Beliefs* $\{\pi_i\}_{i\in\mathcal{I}}$ *satisfy perpetual switching if, for every* $i \in \mathcal{I}$ *and history* (s,t)*, there exists* $j \neq i$ *and a history* (s',t') *following* (s,t) *such that* $\mu_j(s',t') > \mu_i(s',t')$.

Beliefs satisfy perpetual switching if and only if there is no local optimist (this follows from the definitions). The ignorance priors example showed that beliefs satisfy perpetual switching under reasonable circumstances. I will report a necessary and sufficient condition soon. But let us first record the canonical circumstance in which there is *not* perpetual switching.

Example 2 *(Common Priors)* $\pi_i = \pi_1$ *for each* $i \in \mathcal{I}$.

Then $\mu_i(s,t) = \mu_j(s,t)$ for all $i,j \in \mathcal{I}$ and history (s,t), and *each* trader is a local and global optimist and thus beliefs do not satisfy perpetual switching. These examples suggest the intuition that perpetual switching will typically hold when there is sufficient heterogeneity of prior beliefs. But how different must beliefs be to allow perpetual switching? It is possible to give a precise characterization in terms of prior beliefs.

Definition 4 *Trader k is rate dominant if* $\frac{d}{d\theta}(\ln[\pi_k(\theta)]) \geq \frac{d}{d\theta}(\ln[\pi_i(\theta)])$ *for all* $i \in \mathcal{I}$ *and* $\theta \in [0,1]$.

This condition ensures that there is a single trader whose density is always increasing at the fastest rate. To see why this is important, consider trader i's expected value of θ contingent on knowing that θ is in the interval $[\theta_0 - \varepsilon, \theta_0 + \varepsilon]$. For small ε, this will be approximately θ_0 plus a term of order ε that is linear in $\frac{d}{d\theta}(\ln[\pi_i(\theta_0)])$. In fact, this local property can be used to give a tight characterization of the properties that we are interested in.

Theorem 1 *The following are equivalent claims: (1) trader k is rate dominant; (2) trader k is a global optimist; (3) trader k is a local optimist.*

The proof is in the appendix. Since I have shown the equivalence of being a global optimist and being a local optimist, I will henceforth use the term "optimist" to mean either. So how likely is perpetual switching? Equivalently, how close to the common prior assumption is the requirement that there exists a rate dominant trader? I will argue that it is very close. For example, it might be conjectured that if some trader's prior (first order) stochastically dominated all others, then he must be an optimist. This is false, as the following example shows.

Example 3 *(Stochastic Dominance) Suppose there are two traders* 1 *and* 2. *Trader* 1 *believes that* $\theta = 1/5$ *with probability* $1/2$ *and* $\theta = 4/5$ *with probability* $1/2$, *while trader* 2 *believes that* $\theta = 1/5$ *with probability* $1/2$ *and* $\theta = 3/5$ *with probability* $1/2$. *Trader* 1*'s prior stochastically dominates trader* 2*'s. However, suppose that they observe history* $(s,t) = (3,6)$, *i.e. they observe* 3 *dividends in* 6 *periods. Trader* 1*'s posterior probability that* $\theta = 4/5$ *is*

$$\frac{(1/2)^3\,(1/2)^3}{(1/2)^3\,(1/2)^3 + (1/2)^3\,(1/2)^3} = 1/2;$$

thus his fundamental valuation is $\mu_1(3,6) = (1/2)(4/5) + (1/2)(1/5) = 1/2$. *Trader* 2*'s posterior probability that* $\theta = 3/5$ *is*

$$\frac{(3/5)^3\,(2/5)^3}{(3/5)^3\,(2/5)^3 + (1/5)^3\,(4/5)^3} = \frac{3^3 2^3}{3^3 2^3 + 4^3} = \frac{27}{27+8} = \frac{27}{35};$$

thus his fundamental valuation is $\mu_2(3,6) = (27/35)(3/5) + (8/35)(1/5) = 89/175 > 1/2$.

With its discrete probability distributions, this example does not satisfy the assumptions of this paper. But it would be straightforward to approximate the priors in the example with smooth densities uniformly bounded below, with the same result.

Some intuition of what lies behind the above results comes from considering a parameterized class of prior probability distributions.

Example 4 *(Beta Distributions) Each trader has a prior in the set of beta distributions, i.e., for each* $i \in \mathcal{I}$,

$$\pi_i(\theta) = \frac{\theta^{\alpha_i - 1}(1-\theta)^{\beta_i - 1}}{\int\limits_{\zeta=0}^{1} \zeta^{\alpha_i - 1}(1-\zeta)^{\beta_i - 1}\,d\zeta}, \text{ for some } \alpha_i > 0 \text{ and } \beta_i > 0.$$

In this example, trader k is an optimist if and only if

(3) $$\alpha_k \geq \alpha_i \text{ and } \beta_k \leq \beta_i \text{ for all } i \in \mathcal{I}.$$

There are two ways to show this. First, it can be proved directly from the following implied fundamental valuations [Hartigan 1983, p. 76-78]:

$$\mu_i(s,t) = \frac{s+\alpha_i}{t+\alpha_i+\beta_i}. \tag{4}$$

Second, theorem 1 can be used to check this from the rate dominance condition. Note that

$$\begin{aligned}
\frac{d}{d\theta}\left(\ln\left[\pi_i(\theta)\right]\right) &= \pi_i'(\theta)/\pi_i(\theta) \\
&= \left[\begin{array}{c} (\alpha_i-1)\,\theta^{\alpha_i-2}(1-\theta)^{\beta_i-1} \\ -(\beta_i-1)\,\theta^{\alpha_i-1}(1-\theta)^{\beta_i-2} \end{array}\right] \Big/ \left[\theta^{\alpha_i-1}(1-\theta)^{\beta_i-1}\right] \\
&= (\alpha_i-1)/\theta - (\beta_i-1)/(1-\theta),
\end{aligned}$$

so

$$\frac{d}{d\theta}\left(\ln\left[\pi_k(\theta)\right]\right) - \frac{d}{d\theta}\left(\ln\left[\pi_i(\theta)\right]\right) = (\alpha_k-\alpha_i)/\theta + (\beta_i-\beta_k)/(1-\theta).$$

Thus (3) implies that trader k is rate dominant. For the converse, note that as $\theta \to 0$, $\theta\frac{d}{d\theta}(\ln[\pi_i(\theta)]) \to \alpha_i - 1$ while as $\theta \to 1$, $(1-\theta)\frac{d}{d\theta}(\ln[\pi_i(\theta)]) \to \beta_i - 1$. Thus if $\alpha_i > \alpha_k$, $\frac{d}{d\theta}(\ln[\pi_i(\theta)]) > \frac{d}{d\theta}(\ln[\pi_k(\theta)])$ for θ sufficiently close to 0, while if $\beta_i < \beta_k$, $\frac{d}{d\theta}(\ln[\pi_i(\theta)]) > \frac{d}{d\theta}(\ln[\pi_k(\theta)])$ for θ sufficiently close to 1. Thus if trader k is rate dominant, (3) holds.

So within this two-dimensional family of prior distributions, a two-dimensional restriction (i.e., equation 3) must hold to ensure the existence of an optimist. Note that the ignorance priors case (example 1) falls in the class of beta distributions: trader 1's prior has $\alpha_1 = \beta_1 = 1$, while trader 2's prior has $\alpha_2 = \beta_2 = \frac{1}{2}$. Thus we can check directly from condition (3) that there is no optimist.

III Market Prices with Learning

In the previous section, traders' "fundamental valuations" of the risky asset were characterized. Fundamental valuations reflect the (expected) value to each trader of holding the asset forever. In a market setting, however, where the asset can be re-sold, traders will want to take into account the possibility that they can sell the asset at a price higher than their fundamental valuations in some future contingency.

Consider the following economic environment for the group of traders $\mathcal{I}$ studied in the previous section. There are two infinitely lived assets:

the risky asset of the previous section and a riskless asset with interest rate $r > 0$. Traders buy and sell the risky asset in a competitive market in each time period $t = 0, 1, ...$ after any dividend is paid. They cannot sell the risky asset short, but can sell the riskless asset short (i.e., borrow at a riskless rate).

I will characterize an equilibrium pricing scheme of this economy. I write $P(s,t,r)$ for the price of the risky asset (in terms of current dollars) after history (s,t) if the interest rate is r. Write $\mu_*(s,t)$ for the most optimistic fundamental valuation of the asset of any trader after history (s,t), i.e.

$$\mu_*(s,t) = \max_{i \in \mathcal{I}} \mu_i(s,t).$$

Then equilibrium prices must satisfy:

$$(5) \qquad P(s,t,r) = \frac{1}{1+r} \left[\begin{array}{c} \mu_*(s,t)\left\{1 + P(s+1,t+1,r)\right\} \\ +\left(1-\mu_*(s,t)\right) P(s,t+1,r) \end{array} \right].$$

This condition states that the price of the asset after history (s,t) is equal to the highest expected discounted return (among all traders) of holding it to the next period. If the price of the risky asset was strictly higher than any trader's expected return from holding it to the next period, then no one will hold the asset and prices cannot be equilibrium prices. On the other hand, if the price was strictly lower than the highest expected return, then the trader with that highest expected return would want to hold infinite quantities, so markets would not clear.

It will useful to normalize prices by the interest rate: let $p(s,t,r) \equiv rP(s,t,r)$. Since the current dollar price of the riskless asset is $\frac{1}{r}$, $p(s,t,r)$ is the price of the risky asset in terms of the riskless asset. Substituting in equation (5) gives:

$$(6) \quad \left(\frac{1}{r}\right) p(s,t,r) = \frac{1}{1+r} \left[\begin{array}{c} \mu_*(s,t)\left\{1 + \left(\frac{1}{r}\right) p(s+1,t+1,r)\right\} \\ +\left(1-\mu_*(s,t)\right)\left(\frac{1}{r}\right) p(s,t+1,r) \end{array} \right],$$

or

$$(7) \qquad p(s,t,r) = \frac{1}{1+r} \left[\begin{array}{c} \mu_*(s,t)\left\{r + p(s+1,t+1,r)\right\} \\ +\left(1-\mu_*(s,t)\right) p(s,t+1,r) \end{array} \right].$$

Following Harrison and Kreps [1978], a price scheme satisfying equation (7) can be explicitly calculated as follows. Set $p^0(s,t,r) = 0$ for all $s \leq t, r \in \Re_{++}$; define $p^n(s,t,r)$ recursively by

$$(8) \qquad p^{n+1}(s,t,r) = \frac{1}{1+r}\left[\begin{array}{c} \mu_*(s,t)\{r + p^n(s+1,t+1,r)\} \\ \\ +(1-\mu_*(s,t))p^n(s,t+1,r) \end{array}\right].$$

An inductive argument shows that $p^n(s,t,r)$ is bounded above by 1 for all n, s, t, r: it is true for $n = 0$, by definition; if it is true for n, then $p^{n+1}(s,t,r) \leq (1/(r+1))(r+1) = 1$. Since $p^n(s,t,r)$ is non-decreasing in n, we also have that $p^n(s,t,r)$ converges to some limit for all s,t,r. Let $p^*(s,t,r) \equiv \lim_{n\to\infty} p^n(s,t,r)$. Since p^* is a fixed point of equation (8), it certainly satisfies equation (7).

Harrison and Kreps showed the existence of such a "minimal pricing scheme" in a more general setting and showed that the price is no less than any trader's valuation, i.e., in this model, $p^*(s,t,r) \geq \mu_i(s,t)$ for all histories (s,t), all traders $i \in \mathcal{I}$ and all interest rates r. I can use the extra structure of the learning model to prove some stronger results.

Theorem 2 *(Speculative Premiums). (i) If trader k is an optimist, then* $p^*(s,t,r) = \mu_k(s,t)$ *for all histories* (s,t) *and interest rates r; (ii) if there is no optimist, then* $p^*(s,t,r) > \mu_i(s,t)$ *for all histories* (s,t)*, interest rates r and traders* $i \in \mathcal{I}$*; (iii) as* $t \to \infty$, $p^*(\theta_0 t,t,r) \to \mu_i(\theta_0 t,t) \to \theta_0$, *for all* $\theta_0 \in [0,1]$*, interest rates r and traders* $i \in \mathcal{I}$*; (iv) as* $r \to \infty$, $p^*(s,t,r) \to \mu_*(s,t)$ *for all histories* (s,t).

The proof follows from the construction of p^* and elementary properties of learning, and is omitted. The idea of the proof is as follows (a full proof is given in Morris 1995a, section 8.2).

(i) If there is an optimist, he will always end up holding the asset after every history. Therefore there is no possibility of re-selling the asset and the asset price will always reflect his fundamental valuation of the asset. Note that in this case the price of the asset is independent of the interest rate by normalization.

(ii) If there is no optimist, then there is eternal switching. Thus at every date, every holder of the asset attaches positive probability to being able to re-sell the asset at a price higher than his own valuation in some future contingency. Thus the price is always strictly greater than *any* trader's fundamental valuation.

(iii) As $t \to \infty$, *all* traders' posteriors converge to s/t; thus in particular, the differences in fundamental valuations converge to 0. So the option value of being allowed to re-sell the asset goes to zero, and the price converges to $\mu_*(s,t)$, which converges to s/t. In particular, $p^* \to \theta_0$ with probability one if the true value of θ is θ_0.

(iv) As the interest rate increases, the speculative premium goes to zero as the discounted value of the expected dividend tomorrow swamps the discounted value of any option to re-sell.

This result obviously depends on some extreme assumptions. I will note how the result would be altered if we weakened them.

The risk neutrality assumption ensures that (with heterogeneous beliefs) exactly one trader will be holding the asset. With risk averse traders and binding short sales constraints, the exact number of traders willing to hold the asset (i.e., not short sale constrained) is part of the description of the equilibrium and the simple characterization above no longer holds [Morris 1992]. But notice that, for sufficiently heterogeneous prior beliefs and a sufficiently small supply of the risky asset, short sales constraints would have to bind in early periods. On the other hand, for any given supply of the risky asset, no short sales constraints would bind once traders' posteriors had converged close enough. Thus initially the price would exceed any trader's marginal valuation of holding the asset forever, but after some finite period, traders' marginal valuations would have merged and be equal to the price.

The short sales constraint for the risky asset is central to the theorem (and, in section V, I discuss short sales restrictions in U.S. stock markets). The particular form of the constraint is not important: for example, I could allow traders to short by some finite amount and the analysis would be unchanged. It is crucial, however, that traders cannot short the risky asset but can short the riskless asset. An alternative justification for this assumption is that there is only a small stock of the risky asset being traded but traders have an infinite endowment of the safe asset [Harrison and Kreps 1978]: in this case, no trader would ever want to short the riskless asset. In any case, the assumption is extreme. If traders were sometimes short of cash to buy the risky asset, the premium could go negative. However, the qualitative conclusion that more heterogeneity of beliefs leads to greater deviations of prices from traders' fundamental valuations would be robust.[1]

The pricing scheme p^* characterized in theorem 2 is not the only one satisfying equation 7: there exists in addition a continuum of pricing schemes with a "Ponzi scheme" added on top of the "minimal pricing scheme" [Harrison and Kreps 1978]. Such Ponzi price schemes are not robust; for example, the price scheme p^* is the limit as $n \to \infty$ of the unique equilibrium price scheme of an n period truncation of the economy [Morris 1995a, section 8.3].

When there is no optimist, I do not have an analytic solution for p^*. However, it is possible to numerically calculate p^* using equation (8): this is done in the next section.

IV Natural Priors Lead to Significant Speculative Premiums

Say that the *speculative premium* at history (s,t) is $p^*(s,t,r) - \mu_*(s,t)$. This is non-negative by theorem 2. If there is a strictly positive speculative premium, then $p^*(s,t,r) > \mu_*(s,t)$ and thus $p^*(s,t,r) > \mu_i(s,t)$ for all $i \in \mathcal{I}$. In this case, the price exceeds *every* trader's fundamental valuation of the asset. Theorem 2 establishes that - in the absence of an optimist - a speculative premium exists after *every* history. On the other hand, theorem 2 also establishes that at $t \to \infty$, that speculative premium tends to zero. The purpose of this section is to establish numerically that apparently innocuous differences in prior beliefs lead to significant speculative premiums. In particular, I return to the ignorance prior case (example 1) which was intended to capture reasonable possible priors in the face of ignorance.

Recall the two reasonable priors implied $\mu_1(s,t) = (s+1)/(t+2)$, $\mu_2(s,t) = (s+1/2)/(t+1)$ and thus $\mu_*(s,t) = \max\left\{\frac{s+1}{t+2}, \frac{s+1/2}{t+1}\right\}$. However we noted in section II that neither trader is an optimist, so by theorem 2, we must have $p^*(s,t,r) > \max\left\{\frac{s+1}{t+2}, \frac{s+1/2}{t+1}\right\}$ for all histories (s,t) and interest rates r. In particular, we must have $p^*(0,0,r) > \mu_*(0,0) = 1/2$ for all interest rates r. On the other hand, theorem 2 also shows that as $r \to \infty$, the speculative premium becomes insignificant in pricing the asset: thus $p^* (0,0,r) \to 1/2$ as $r \to \infty$. Figure I plots $p(0,0,r)$ as r varies from 0.05 to 40.

Note that, because of the numerical procedure, it is not possible to calculate $p^* (0,0,r)$ accurately for arbitrarily small r. However, $p^* (0,0,0.05) = 0.54$, so that at an interest rate of five percent, a speculative premium of eight percent is generated.

Another prediction of theorem 2 is that $p^* (\theta_0 t, t, r) \to \theta_0$ as $t \to \infty$. Figure II plots $p^* \left(\frac{t}{2}, t, 0.05\right)$ for t in the interval $[0, 50]$.

V Initial Public Offerings

In the model of section III, the price of an asset is determined by the unique marginal trader who is just willing to hold the asset. The marginal trader is atypical in that his valuation is the highest of any trader. An old "marginal opinion theory" strand of the finance literature advocated the view that in a world where short sales constraints bind, the marginal opinion which determines an asset's price will be above average [Williams 1938, chapter 3; Miller 1977]. This literature was written before economists started making a careful distinction between differences of opinion generated by heterogeneous prior beliefs and differences of opinion generated by asymmetric information: "marginal opinion theory" breaks down when differences of opinion reflect different information [Diamond and Verrecchia 1987]. But genuine heterogeneity of prior beliefs and binding short sales constraints should push asset prices above the average investor's fundamental valuation.

Although short sales are not banned in U.S. stock markets, they are extremely rare: the outstanding value of short sale obligations of S&P 500 firms averaged around one fifth of one percent of the total value of shares over the period 1974-1983. This is a consequence of the exceptional transaction costs related to shorting stock, including the loss of interest on proceeds held in escrow accounts; high collateral requirements; and the ban on short selling on a downtick (i.e. when the price is declining) [Figlewski and Webb 1993, footnote 5].[2]

Given that short sales constraints appear to effectively bind, the prediction of the learning model presented in this paper is that prices should initially exceed traders' valuations, but as learning occurs, the premium should disappear. To test this prediction, it is necessary to track the price of an asset back to a time when traders have had little opportunity to learn about the behavior of its returns. For publicly traded shares, we would like to track prices back to their initial public offering, when presumably minimal learning could have occurred. The learning model would predict that the prices of initial public offerings would exceed the fundamental valuations when first traded, but would converge to fundamental valuations over time. If we believe that actual returns will be closer to average opinion than to the marginal investor's opinion, then we would expect that initial public offerings would on average underperform the market in similar but older shares in their first few years. This is consistent with the empirical literature on initial public offerings: Ritter [1991] finds that initial public offerings [IPOs] underperform comparable stocks by 17% in their first three years.[3]

This explanation was presented some years ago by Miller [1977, page 1156]:

> The prices of new issues, as of all securities, are set not by the appraisal of the typical investor, but by the small number who think highly enough of the investment merits of the new issue to include it in their portfolio. The divergence of opinion about a new issue are greatest when the stock is issued. Frequently the company has not started operations, or there is uncertainty about the success of new products or the profitability of a major business expansion. Over time, this uncertainty is reduced.... With the passage of time, and the reduction of uncertainty, the appraisal of the top x percent of the investors is likely to decline even if the average assessment is not changed. This would explain the poor performance of a group of new issues when compared to a group of stocks about which the uncertainty does not decrease over time.[4]

The model of section III provides a formal version of this argument. In addition, the ignorance prior example makes clear that this explanation does not require even that traders have different prior valuations of the asset. It is enough that their priors over the unknown parameters have a different shape.

It is useful to compare this *learning* explanation of IPO underperformance with an alternative *selection* explanation. Suppose that there is a systematic component in traders' prior beliefs about initial public offerings, so that in one period average opinion about all initial public offerings is high relative to the true value, while in other periods, it is low. A contrived explanation for IPO overpricing would be that traders are more often optimists than pessimists with respect to IPOs. But Lee, Shleifer and Thaler [1991] argue that even if traders are as likely to be optimistic as pessimistic over time, issuers of initial public offerings have an incentive to issue when traders are optimistic.

There is evidence for both views in the cross-sectional and time series data of Ritter [1991, section III]. There is strong support for selection in the timing of initial public offerings. Partitioning initial public offerings in his 1974-1983 sample by date of issue, Ritter shows that overpricing occurs in exactly half those years; there is average overpricing in the data precisely because the volume of issues was higher during periods of overpricing. This evidence strongly supports the selection hypothesis that issuers are more likely to launch an IPO when there is a premium. It does not discriminate between alternative explanations of why there is a premium in the first place: a temporary market sentiment favoring IPOs or greater heterogeneity of beliefs in the presence of short sales constraints.

Another piece of evidence gives direct support to the learning explanation. Controlling for industry, the initial overpricing of initial public offerings decreases monotonically from thirty four percent for firms that are less than one year old to four percent for firms that are more than twenty years old. It is more plausible that traders have different beliefs on the basis of the same public information when the firm going public also has a shorter record under private ownership: there has been even less opportunity for learning. Since this effect is independent of the year of issue, and age as a private company is (presumably) not a decision variable for issuers, this phenomenon cannot be explained by a selection mechanism. The only viable alternative to the learning explanation is that traders are systematically more optimistic about newer firms.

VI The Common Prior Assumption in Economic Theory

Economists now make a clear distinction between differences in posterior beliefs that are explained by differential information and those that are unexplained by private information and thus represent a violation of the common prior assumption. The differences in beliefs in sections II through IV cannot be interpreted as differences in information. If they were, then a "no trade" theorem along the lines of Milgrom and Stokey [1982] would guarantee no trade and no speculative premiums.

I will argue that the model of this paper illustrates how economists might - selectively - allow differences in prior beliefs to be used to understand economic phenomena.[5] I will discuss some of the main arguments made in support of the common prior assumption and argue why they are not compelling, at least in this and certain other contexts.

There is a widespread intuition that differences in beliefs between rational people must be a consequence of private information. This intuition conflicts with the usual economists' notion of rationality as consistency [Savage 1954], and other attempts to formalize the intuition mathematically or philosophically have met with little success [Morris 1995b, section 3]. On a more practical level, I presented in section II the thought experiment of imagining traders forming priors about the dividends of an as yet unobserved asset. It was extremely hard to conceive of any criteria - rational or otherwise - that might require traders to have the same prior. A number of different priors seem entirely reasonable.

An alternative way of presenting the "rationality implies common priors" argument can be told in the context of initial public offerings. The explanation of the over-pricing anomaly has something of a "winner's curse" flavor. It is tempting to argue that any trader holding the asset should want to revise downward his valuation of the asset in the light of others' willingness to sell to him at that price. It is tempting, in other words, to interpret the different priors as different information. But while many apparent differences in prior beliefs may be explainable by different information at some level, we must surely allow for the possibility that the performance of initial public offerings depends at least in part on unique factors about which reasonable people could form different beliefs on the basis of the same information.

Another argument in favor of the common prior assumption appeals to rational learning. We are justified in assuming common priors, the argument goes, because past experience will have removed differences in beliefs unexplained by differences of information. The common prior assumption is then justified when learning has finished, so that everyone has learned the true underlying data-generating process. But presumably we live in a world where rational learning is still taking place. One reason why this is true is that there are new types of events whose distribution cannot be predicted from past experience. In the economy, there may be some data-generating processes that have been learned. Initial public offerings are presumably a situation where learning has not been completed.

Indeed, the argument that learning justifies the common prior assumption can be turned around. Suppose we want to test the idea that learning has (typically) led to a world in which all differences in posteriors are explained by information. Then consider those (rare) situations where there has not been a chance for complete learning to occur. Presumably we should expect to find distinctive behavior in those situations reflecting the heterogeneity of priors. In that sense, the over-pricing of initial public offerings is consistent with the learning rationale for the common prior assumption. But since it may take some time for full learning to occur, there remains a role for investigating what happens before learning is complete.

Perhaps the most popular argument in support of the common prior assumption is the odd claim that "anything can happen" when the common prior assumption is dropped. It is true that differences in prior beliefs - like differences in utility functions and information - introduce extra degrees of freedom into modelling. But - when combined with auxiliary hypotheses - it is possible to make explicit and interesting predictions using heterogeneous priors. Theorem 2 and section V illustrate this claim.

VII Conclusion

In the simple environment of this paper, traders' fundamental valuations of an asset can be identified with their point estimates of the single parameter of the dividend process. I showed that even a small amount of heterogeneity in traders' prior beliefs implied "eternal switching," so that every trader after every history attaches positive probability to someone else valuing the asset *strictly* more after some continuation history. This in turn implies that even if traders' posterior beliefs are converging to the true value, the speculative premium never disappears. I also showed that reasonable differences in prior beliefs lead to significant speculative premiums. In evaluating the existence and size of speculative premiums, what matters is not just who initially has the highest expected value of the dividend in the next period. The speculative premium depends on differences in beliefs at all possible future contingencies. This explains why intuitively small differences in prior beliefs matter a lot.

Using unexplained differences in prior beliefs in economics has been out of fashion for some time. Initial public offerings represent the canonical situation where past experience will not have removed differences in beliefs. They thus represent an ideal test of whether it is possible to make interesting predictions from economic models with unexplained - but reasonable - differences in prior beliefs.

APPENDIX (proof of theorem 1).

To prove theorem 1, I must show the equivalence of the following: (1) trader k is rate dominant; (2) trader k is a global optimist; and (3) trader k is a local optimist. By the definitions, (2) implies (3). So it suffices to show that (3) implies (1) and (1) implies (2).

(3) implies (1): Write $\nu_i(s,t)$ for the approximation of $\mu_i(s,t)$ obtained by the second order Taylor series expansion of $\pi_i(\theta)$, i.e. setting $\pi_i(\theta) = \pi_i(\theta_0) + (\theta-\theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)$. Thus:

$$\nu_i(s,t) = \frac{\int\limits_{\theta=0}^{1} \binom{t}{s}\theta^{s+1}(1-\theta)^{t-s}\left(\pi_i(\theta_0) + (\theta-\theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)\right)d\theta}{\int\limits_{\theta=0}^{1} \binom{t}{s}\theta^{s}(1-\theta)^{t-s}\left(\pi_i(\theta_0) + (\theta-\theta_0)\pi_i'(\theta_0) + \frac{1}{2}(\theta-\theta_0)^2\pi_i''(\theta_0)\right)d\theta}.$$

Writing $I(a,b) = \int\limits_{\theta=0}^{1} \theta^a(1-\theta)^b\,d\theta$, this equals:

$$\nu_i(s,t) = \frac{\left\{\begin{array}{c} \pi_i(\theta_0)I(s+1,t-s) + \pi_i'(\theta_0)I(s+2,t-s) - \theta_0\pi_i'(\theta_0)I(s+1,t-s) \\ +\frac{1}{2}\pi_i''(\theta_0)I(s+3,t-s) - \pi_i''(\theta_0)\theta_0 I(s+2,t-s) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 I(s+1,t-s) \end{array}\right\}}{\left\{\begin{array}{c} \pi_i(\theta_0)I(s,t-s) + \pi_i'(\theta_0)I(s+1,t-s) - \theta_0\pi_i'(\theta_0)I(s,t-s) \\ +\frac{1}{2}\pi_i''(\theta_0)I(s+2,t-s) - \pi_i''(\theta_0)\theta_0 I(s+1,t-s) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 I(s,t-s) \end{array}\right\}}.$$

Now substituting:

$$I(a,b) = \frac{a!\ b!}{(a+b+1)!},$$

we get

$$\nu_i(s,t) = \frac{\left\{\begin{array}{c} \pi_i(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} + \pi_i'(\theta_0)\frac{(s+2)!(t-s)!}{(t+3)!} - \theta_0\pi_i'(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+3)!(t-s)!}{(t+4)!} - \pi_i''(\theta_0)\theta_0\frac{(s+2)!(t-s)!}{(t+3)!} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{(s+1)!(t-s)!}{(t+2)!} \end{array}\right\}}{\left\{\begin{array}{c} \pi_i(\theta_0)\frac{s!(t-s)!}{(t+1)!} + \pi_i'(\theta_0)\frac{(s+1)!(t-s)!}{(t+2)!} - \theta_0\pi_i'(\theta_0)\frac{s!(t-s)!}{(t+1)!} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+2)!(t-s)!}{(t+3)!} - \pi_i''(\theta_0)\theta_0\frac{(s+1)!(t-s)!}{(t+2)!} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{s!(t-s)!}{(t+1)!} \end{array}\right\}}.$$

Cancelling out $\frac{s!(t-s)!}{(t+1)!}$ gives:

$$\nu_i(s,t) = \frac{\left\{\begin{array}{c} \pi_i(\theta_0)\frac{s+1}{t+2} + \pi_i'(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \theta_0\pi_i'(\theta_0)\frac{s+1}{t+2} \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)(s+3)}{(t+2)(t+3)(t+4)} - \pi_i''(\theta_0)\theta_0\frac{(s+1)(s+2)}{(t+2)(t+3)} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{s+1}{t+2} \end{array}\right\}}{\left\{\begin{array}{c} \pi_i(\theta_0) + \pi_i'(\theta_0)\frac{s+1}{t+2} - \theta_0\pi_i'(\theta_0) \\ +\frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 \end{array}\right\}}.$$

Thus $\nu_i(s,t) - \frac{s+1}{t+2} =$

$$\frac{\pi_i'(\theta_0)\frac{s+1}{t+2}\left(\frac{s+2}{t+3} - \frac{s+1}{t+2}\right) + \frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)}\left(\frac{s+3}{t+4} - \frac{s+1}{t+2}\right) - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2}\left(\frac{s+2}{t+3} - \frac{s+1}{t+2}\right)}{\pi_i(\theta_0) + \pi_i'(\theta_0)\frac{s+1}{t+2} - \theta_0\pi_i'(\theta_0) + \frac{1}{2}\pi_i''(\theta_0)\frac{(s+1)(s+2)}{(t+2)(t+3)} - \pi_i''(\theta_0)\theta_0\frac{s+1}{t+2} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2}.$$

Observe that as $t \to \infty$,

- $\frac{\theta_0 t+1}{t+2} \to \theta_0$; $\frac{\theta_0 t+2}{t+3} \to \theta_0$; $\frac{\theta_0 t+2}{t+3} - \frac{\theta_0 t+1}{t+2} = \frac{(1-\theta_0)t+1}{(t+2)(t+3)} \to \frac{1-\theta_0}{t}$.
- $\left(\frac{\theta_0 t+3}{t+4} - \frac{\theta_0 t+1}{t+2}\right) = \frac{\theta_0 t^2+3t+2\theta_0 t+6-\theta_0 t^2-t-4\theta_0 t-4}{(t+2)(t+4)} = \frac{2(1-\theta_0)t-2}{(t+2)(t+4)} \to \frac{2(1-\theta_0)}{t}$.
- $\frac{\theta_0 t+2}{t+3} - \frac{\theta_0 t+1}{t+2} = \frac{\theta_0 t^2+2t+2\theta_0 t+4-\theta_0 t^2-t-3\theta_0 t-3}{(t+2)(t+3)} = \frac{(1-\theta_0)t-1}{(t+2)(t+3)} \to \frac{1-\theta_0}{t}$.

Thus as $t \to \infty$,

$$\begin{aligned} \nu_i(\theta_0 t,t) &\to \theta_0 + \frac{\pi_i'(\theta_0)\theta_0\frac{1-\theta_0}{t} + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2\frac{2(1-\theta_0)}{t} - \pi_i''(\theta_0)\theta_0^2\frac{1-\theta_0}{t}}{\pi_i(\theta_0) + \pi_i'(\theta_0)\theta_0 - \theta_0\pi_i'(\theta_0) + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2 - \pi_i''(\theta_0)\theta_0^2 + \frac{1}{2}\pi_i''(\theta_0)\theta_0^2} \\ &= \theta_0 + \frac{\theta_0(1-\theta_0)}{t}\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)}. \end{aligned}$$

Now as $t \to \infty$, trader i's posterior $\xi_i(\theta|\theta_0 t,t)$ concentrates mass around θ_0 (recall that each π_i was uniformly bounded below), so

$$\mu_i(\theta_0 t,t) \to \nu_i(\theta_0 t,t) \to \theta_0 + \frac{\theta_0(1-\theta_0)}{t}\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)}. \qquad \text{(A1)}$$

Suppose π_k is *not* rate dominant. Then (by continuity) there exists $i \in \mathcal{I}$ and a rational number $\theta_0 \in (0,1)$ such that $\frac{\pi_i'(\theta_0)}{\pi_i(\theta_0)} > \frac{\pi_k'(\theta_0)}{\pi_k(\theta_0)}$. Let

$\theta_0 = \frac{a}{b}$, for two strictly positive integers a, b. But now for any (s,t), there exists T sufficiently large such that history (aT, bT) follows (s,t) and (by A1) $\mu_k(aT, bT) < \mu_i(aT, bT)$. Thus k is not a local optimist at (s,t).

(1) implies (2): Suppose trader k is rate dominant. Let

$$\beta_i(\theta) = \frac{\pi_i(\theta)}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}.$$

By trader k rate dominant, we have (for any $i \in \mathcal{I}$) $\frac{\pi_k(\theta)}{\pi_i(\theta)}$ non-decreasing in θ, so that

$$\frac{\pi_k(\theta')}{\pi_k(\theta)} \geq \frac{\pi_i(\theta')}{\pi_i(\theta)}, \text{ for all } \theta' \geq \theta.$$

Thus

$$\frac{\int_{\theta'=\theta}^{1} \pi_k(\theta')\, d\theta'}{\pi_k(\theta)} \geq \frac{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}{\pi_i(\theta)}, \text{ for all } \theta \in [0,1],$$

so

$$\beta_k(\theta) = \frac{\pi_k(\theta)}{\int_{\theta'=\theta}^{1} \pi_k(\theta')\, d\theta'} \leq \frac{\pi_i(\theta)}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'} = \beta_i(\theta), \text{ for all } \theta \in [0,1]. \tag{A2}$$

Let

$$\alpha_i(\theta) = \frac{\int_{\theta'=\theta}^{1} \theta' \pi_i(\theta')\, d\theta'}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'}.$$

Thus

$$\frac{d\alpha_i}{d\theta} = -\frac{\theta \pi_i(\theta)}{\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'} + \frac{\pi_i(\theta) \int_{\theta'=\theta}^{1} \theta' \pi_i(\theta')\, d\theta'}{\left(\int_{\theta'=\theta}^{1} \pi_i(\theta')\, d\theta'\right)^2} = \beta_i(\theta)\left(\alpha_i(\theta) - \theta\right). \tag{A3}$$

Now (A2) and (A3) imply that if $\alpha_k(\theta) = \alpha_i(\theta)$, then $\frac{d\alpha_k}{d\theta} \leq \frac{d\alpha_i}{d\theta}$. Thus since $\alpha_j(1) = 1$ for all $j \in \mathcal{I}$, α_k can never fall below α_i as θ goes from 1 to 0. Thus $\mu_k(0,0) = \alpha_k(0) \geq \alpha_i(0) = \mu_i(0,0)$. Thus a rate dominant trader is the most optimistic after the null history $(0,0)$. But traders' posteriors after history (s,t) are

$$\xi_i(\theta|s,t) = \frac{\theta^s(1-\theta)^{t-s}\pi_i(\theta)}{\int_{\zeta=0}^{1} \zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta},$$

so that

$$\xi_i'(\theta|s,t) = \frac{d\xi_i(\theta|s,t)}{d\theta} = \frac{s\theta^{s-1}(1-\theta)^{t-s}\pi_i(\theta) - (t-s)\,\theta^s(1-\theta)^{t-s-1}\pi_i(\theta) + \theta^s(1-\theta)^{t-s}\pi_i'(\theta)}{\int_{\zeta=0}^{1} \zeta^s(1-\zeta)^{t-s}\pi_i(\zeta)d\zeta}$$

Thus

$$\frac{\xi_i'(\theta|s,t)}{\xi_i(\theta|s,t)} = \frac{s}{\theta} - \frac{t-s}{1-\theta} + \frac{\pi_i'(\theta)}{\pi_i(\theta)},$$

so

$$\frac{\xi_k'(\theta|s,t)}{\xi_k(\theta|s,t)} - \frac{\xi_i'(\theta|s,t)}{\xi_i(\theta|s,t)} = \frac{\pi_k'(\theta)}{\pi_k(\theta)} - \frac{\pi_i'(\theta)}{\pi_i(\theta)}.$$

Thus ξ_k inherits the rate dominance property. So, by the above argument, $\mu_k(s,t) \geq \mu_i(s,t)$, for every history (s,t), so trader k is a global optimist.

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NOTES

1. Jarrow [1980] examines short sales constraints in the capital asset pricing model and shows that prices of some assets may be lower than in an unconstrained economy because of correlation between asset returns.

2. Figlewski [1981] uses the volume of short sale positions in a stock as a proxy for the demand for short sales and shows that binding short sales constraints are correlated with measures of over-pricing, giving one indirect confirmation of the "marginal investor theory."

3. It is important to distinguish this "hot issue anomaly" - that opening market prices tend be high relative to long-run prices - from the much studied but distinct "under pricing anomaly:" offer prices tend to be significantly lower than opening market prices.

4. Miller also suggests a partial but non-strategic explanation for the under-pricing anomaly, based on the marginal investor viewpoint:

> Incidently, if underwriters ignore the above arguments and price new issues on the basis of their own best estimates of the prices of comparable seasoned securities, they will typically underprice new issues. The mean of their appraisals will resemble the mean appraisal of the typical investor, and this will be below the appraisals of the most optimistic investors who actually constitute the market for the security. This may be a partial explanation for the underpricing of new issues by underwriters.

5. Following Lintner [1969], there have been a number of attempts to allow for heterogeneous prior beliefs in finance: see Biais and Boessarts [1993] and Harris and Raviv [1993] for recent contributions. Morris [1995b] contains a more detailed discussion of the role of the common prior assumption in economic theory.