Bitcoin's Fatal Flaw: The Limited Adoption Problem[*]

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Abstract

Bitcoin remains sparsely adopted even a decade after its birth. We demonstrate theoretically that this limited adoption arises as an inescapable equilibrium outcome rather than as a transient feature. We establish such a result for a wide class of blockchains that employ Proof-of-Work. Our results arise due to three features: (1) an artificial supply constraint, (2) free entry to the validator network, and (3) a need for consensus. Network delay precludes relaxing the supply constraint as a solution. Nonetheless, we demonstrate that permissioned blockchains may obtain widespread adoption, thereby highlighting the need for research on alternatives to Bitcoin.

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1 Introduction

A question remains whether Bitcoin's limited usage arises due to its infancy or because of its underlying economic structure. This paper answers that question by demonstrating that limited adoption constitutes an endogenous characteristic of not only Bitcoin but also Proof-of-Work (PoW) payments blockchains more generally. We demonstrate that the economics of PoW payments blockchains make limited adoption an inescapable equilibrium outcome. Our critique does not apply to other blockchains such as smart contract platforms and permissioned platforms. In fact, our analysis explicitly highlights that permissioned blockchains may overcome limited adoption. Recently, that insight has become particularly salient with Facebook's announcement of the Libra blockchain, a permissioned platform with the explicit goal of widespread adoption. Nonetheless, our analysis does not explicitly endorse any particular project; rather, our work highlights the need for research on alternatives to Bitcoin in the nascent field of blockchain economics.

PoW dates back to Dwork and Naor (1992) and later gained mainstream attention when Nakamoto (2008) popularized the concept by employing it to allegedly induce good validator behavior within a permissionless blockchain setting.[1,2] Nakamoto (2008) envisioned a decentralized network that admits free entry and perfect competition among validators. To achieve that vision while creating appropriate validator incentives, Nakamoto (2008) specified that agents must solve a verifiable puzzle to update the blockchain.[3] Nakamoto (2008) specified the puzzle difficulty as a parameter so that the block arrival rate (i.e., rate of blockchain updating) may be targeted. The motivation for this targeting feature arises from the premise that blockchain updates occur-

[1] Validators on a Proof-of-Work blockchain are called miners.

[2] A permissionless blockchain constitutes a blockchain that admits free entry with respect to the validator network.

[3] The interested reader may consult Biais, Bisière, Bouvard, and Casamatta (2019) for further reference.

ring faster than the network delay undermines validators agreeing on ledger contents.[4] Narayanan, Bonneau, Felten, Miller, and Goldfeder (2016) argue that the block rate "should be [targeted as] a fixed amount" because "blocks [coming] very close together [induces] a lot of inefficiency." The block arrival rate targeting, however, artificially constrains ledger space. We demonstrate that this artificial supply constraint interacts with network delay and PoW's permissionless nature to make limited adoption endemic to PoW payments blockchains.

Due to PoW's supply constraint, an increase in transaction demand endogenously generates an increase in fees. That fee increase in turn induces validators to enter the PoW network. The PoW network expansion then exacerbates network delay and protracts the validator agreement process. For users, this delay amounts to increased payment confirmation times which drives users away from the blockchain platform towards traditional payment systems. In equilibrium, the blockchain maintains only users relatively insensitive to payment confirmation delays. Thus, our analysis demonstrates that PoW payments blockchains cannot simultaneously sustain large volumes and a non-negligible payments market share - we term this problem the *limited adoption problem*.

To overcome the limited adoption problem, we consider dynamic adjustment of PoW's block rate. That putative solution corresponds economically to expanding supply. However, it falls short as a remedy due to the need for validators to obtain consensus. If the block rate fails to keep pace with transaction demand, then demand outpaces supply and prohibitive wait times drive users from the blockchain. Alternatively, if the block rate keeps pace with transaction demand, then supply meets demand but the rapid block rate leaves validators insufficient time to communicate across the network thereby protracting the validator agreement process. The protracted validator agreement process elongates payment confirmation times and drives users away from the blockchain.

[4] Network delay references the time required for information to travel across the network. We provide further detail within Section 2.

Increasing block sizes yields similar results as increasing block-rates because network delay increases approximately linearly in block sizes for non-trivial block sizes (see Decker and Wattenhofer (2013)). Thus, dynamic supply fails to overcome the limited adoption problem. This reasoning breaks down only if the PoW blockchain features a single validator. A single validator network allows simultaneously for arbitrarily fast block rates and an expedient validator agreement process.

The necessity of centralization to break PoW's limited adoption problem motivates us to consider permissioned blockchains. A permissioned blockchain offers a semi-centralized setting with neither an artificial supply constraint nor free entry among validators. We demonstrate that a permissioned blockchain induces lower payment confirmation times than a PoW blockchain and overcomes the limited adoption problem. Nonetheless, we acknowledge that a permissioned blockchain may not dominate a PoW blockchain because malicious validator behavior may arise in equilibrium for a permissioned blockchain. We, therefore, turn to examining validator incentives for this class of blockchains.

We begin by analyzing a standard majority rule consensus protocol. Such a protocol creates a coordination game with multiple equilibria. All validators behave honestly in one equilibrium and maliciously in another equilibrium. These results arise because a validator gains from successfully attacking the blockchain but faces a reputation cost from an unsuccessful attack. The majority-rule consensus protocol thus raises security concerns for a permissioned blockchain.

To resolve the aforementioned concerns, we propose an alternative consensus protocol. That protocol weights votes by each validators' stake in the cryptocurrency native to the blockchain. Such a protocol aligns validator incentives in a way that precludes malicious validator behavior. Validators internalize that prices negatively reflect the probability that the blockchain incurs a successful attack. An attack equilibrium cannot exist because validators respond optimally to a potential attack by acquiring a stake in the cryptocurrency sufficiently large to become marginal and thwart the attack.

A permissioned blockchain with a stake-based consensus protocol escapes the limited adoption problem and induces honest validator behavior. This has important implications for the introduction of blockchain as a payment system. While PoW may not be viable due to the limited adoption problem, a well-designed permissioned alternative may be suitable for widespread adoption. Notably, Facebook recently announced plans for a permissioned blockchain with the explicit goal of widespread adoption. While we demonstrate that permissioned blockchains may overcome limited adoption, our results do not demonstrate that arbitrary implementations of permissioned blockchains necessarily obtain widespread adoption.

This paper relates to a large literature that studies PoW economics and cryptoassets. Eyal and Sirer (2014), Nayak, Kumar, Miller, and Shi (2015), Carlsten, Kalodner, Weinberg, and Narayanan (2016), Cong, He, and Li (2018), Alsabah and Capponi (2019) and Biais et al. (2019) analyze PoW mining strategies. Huberman, Leshno, and Moallemi (2019) and Easley, O'Hara, and Basu (2019) analyze transaction fees and wait times for users under a PoW protocol. Foley, Karlsen, and Putnins (2019) examine the extent to which cryptocurrencies facilitate illegal activities. Raskin, Saleh, and Yermack (2019) analyze the relationship between private digital currencies and government policy. Kroeger and Sarkar (2017), Biais, Bisière, Bouvard, Casamatta, and Menkveld (2018), Liu and Tsyvinski (2018), Makarov and Schoar (2019), Pagnotta and Buraschi (2018), Li, Shin, and Wang (2019b) and Shams (2019) study the determinants of cryptoasset prices. Other notable works include Gandal and Halaburda (2016), Harvey (2016), Chiu and Koeppl (2017), Abadi and Brunnermeier (2018), Griffin and Shams (2018), Jermann (2018) and Chiu and Koeppl (2019) and Fernández-Villaverde and Sanches (2019).

This paper highlights an important shortcoming of PoW payments blockhains. In doing so, our work adds to the literature that highlights PoW's economic limitations. Budish (2018) argues that the possibility of an attack limits Bitcoin's economic size.

Yermack (2015) documents exorbitant bitcoin price volatility. Pagnotta (2018) and Saleh (2019b) theoretically demonstrate that PoW contributes to that price volatility; Saleh (2019b) also demonstrates that PoW induces welfare losses.

This paper also contributes to a growing literature that considers alternatives to PoW payments blockchains. We provide one of the first analyses of permissioned blockchains and show that a properly designed consensus protocol yields desirable validator behavior. Cao, Cong, and Yang (2018) and Chod, Trichakis, Tsoukalas, Aspegren, and Weber (2018) predate our work and also study permissioned blockchains but for auditing and supply chain purposes respectively. Cong, Li, and Wang (2019b), Sockin and Xiong (2018), Tinn (2018), Cong and He (2019) and Cong, Li, and Wang (2019a) depart from the Bitcoin paradigm by examining a blockchain platform that possesses functionality beyond payment processing. Falk and Tsoukalas (2018) provide a theoretical analysis of blockchain-based token weighted voting platforms. Chod and Lyandres (2018), Lee, Li, and Shin (2018), Li and Mann (2018), Malinova and Park (2018), Howell, Niessner, and Yermack (2018), Catalini and Gans (2019) and Davydiuk, Gupta, and Rosen (2019) study initial coin offerings. Basu, Easley, O'Hara, and Sirer (2019) propose an alternative fee setting mechanism to that employed by Bitcoin. Saleh (2019a) formally analyzes Proof-of-Stake (PoS) and establishes that such a protocol induces consensus under certain conditions. Fanti, Kogan, and Viswanath (2019) provide a valuation framework for PoS payments systems. Rosu and Saleh (2019) study the evolution of shares in a PoS cryptocurrency. Liu, Tsyvinski, and Wu (2019) study cryptoasset risk factors in general and find novel empirical evidence highlighting that more cost-efficient cryptoassets possess better return characteristics than PoW cryptoassets.

Also notable, there exists a large literature within computer science that studies security of various blockchain protocols. Prominent papers within that literature include Miller and LaViola (2014), Chen and Micali (2016), Kiayias, Russell, David, and Oliynykov (2017) and Daian, Pass, and Shi (2019). Our paper differs from those works in that we do not establish security of any protocol. Rather, we assume security of PoW and establish limited adoption despite this generous assumption. Our paper also analyzes security of a permissioned blockchain protocol. However, our notion of security equates with incentive compatibility of validators, whereas the computer science security notion equates to robustness in the presence of an exogenously motivated attacker.

This paper proceeds as follows. Section 2 discusses relevant institutional details. Section 3 presents the PoW model, defines a PoW Equilibrium and establishes both existence and uniqueness of such an equilibrium. Section 4 analyzes payment confirmation times and formalizes the limited adoption problem. Section 5 discusses permissioned blockchains and offers a stake-based consensus protocol as an alternative to PoW. Section 6 concludes. All proofs appear in Appendix B.

2 Institutional Background

For a block to enter a PoW blockchain, that block must solve a puzzle. Hereafter, we refer to that puzzle as the PoW puzzle and any block that solves the PoW puzzle as a valid block. Being valid constitutes a necessary, but not a sufficient condition, for a block to enter the blockchain. Block validity is not a sufficient condition due to PoW's permissionless nature which requires that any validator may propose a block. If multiple validators propose valid blocks at the same height, then only one such block may enter the blockchain, thereby precluding block validity as a sufficient condition for a block to enter the blockchain.

Validators may propose valid blocks at the same height for various reasons. Biais et al. (2019) consider such events arising from validator incentives. We abstract from validator incentives and assume each validator follows the longest-chain rule described by Nakamoto (2008). A key ingredient of our model is that, even with such a generous security assumption, multiple blocks may be proposed at the same height due to network delay. We show that network delay has grave economic implications that prevent PoW payments blockchains, such as Bitcoin, from becoming widely adopted.

Network delay refers to the time required for information to travel across the network. The presence of network delay implies that validators may perceive different longest chains at a given point in time. If Validator A proposes a valid block at a given height, other validators may nonetheless continue searching for a valid block at that same height, because news of Validator A's valid block has not propagated through the entire network. With a positive probability, some other validator, Validator B, may find a valid block before receiving news regarding Validator A's valid block. Then, Validators A and B perceive different blockchains which we refer to as a fork.

The propensity of such forks arising thus depends on the extent of network delay which in turn is a function of the structure of the validator network. Since PoW blockchains are permissionless, such blockchains generally adopt a random network topology in which case network delay is approximately a logarithmic function of the number of nodes (see Chung and Lu (2002) and Riordan and Wormald (2010)). In our analysis, we specify network delay in more general terms so that a logarithmic function constitutes a special case. In practice, forks generated by network delay constitute the majority of forks arising on the Bitcoin blockchain (see Decker and Wattenhofer (2013)), yet the economics literature has largely ignored such forks. Although these forks arise for non-economic reasons, our work highlights that they possess significant economic implications in that they generate the limited adoption problem.

3 PoW Model

We model an infinite horizon economy that evolves in continuous time. Our model consists of a validator network that stores the blockchain and a finite number of potential blockchain users.

3.1 Users

Our model involves finitely many users, $i \in \{1, ..., N\}$. Each user possesses only one transaction. We model user preferences akin to Easley et al. (2019) and Huberman et al. (2019). At $t = 0$, User i learns her type, $c_i \sim U[0, 1]$. c_i denotes the delay cost for User i, which remains unknown to others.[5] After learning her type, User i selects a fee level, f_i, that solves the problem in (1) below.

$$\max_{f \geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i}) \mid c_i] - f \tag{1}$$

$W(f, f_{-i})$ represents the wait time for User i's transaction to earn confirmation when User i pays f as a fee, while the other users pay fee f_{-i}. R represents the utility of User i having her transaction processed. If $\max_{f \geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i}) \mid c_i] - f < 0$ then User i opts to transact via traditional payment systems rather than on the blockchain.

3.2 Validators

Because PoW blockchains admit free entry among validators, we determine the number of validators, V, endogenously. Each potential validator must pay some cost $\beta > 0$ to acquire validation technology and join the network. Each validating node represents a single processor, and we assume that each processor possesses identical hashing power so that each validator expects to earn an equal share of fees. We assume validators possess risk-neutral preferences. Then, free entry yields Equation (2) with V being the equilibrium number of validators.

$$V = \frac{\mathbb{E}[\sum_i f_i]}{\beta} \tag{2}$$

For exposition, we assume that each block contains only one transaction.[6] We fur-

[5] We model c_i as independent of all else.

[6] Decker and Wattenhofer (2013) establishes that network delay increases linearly in block size for

ther assume that no coinbase transactions exist so that validators receive compensation exclusively through fees. Validators optimally service transactions in descending order of fees.

3.3 Blockchain

Blocks arrive according to a compound Poisson process with rate $\Lambda > 0$. We assume that each arrival occurs at a new block height, but we allow that network delay may yield multiple blocks at the same height. Multiple blocks at the same height constitute a fork and correspond to disagreement regarding the blockchain's content. A fork arises if different validators solve the same PoW puzzle before communicating with each other. Given an arrival at time t, a Poisson process with rate Λ produces at least one more arrival within the next Δ time units with probability $1 - e^{-\Lambda\Delta}$. Accordingly, we assume that an arrival corresponds to multiple blocks at a given height with probability $1 - e^{-\Lambda\Delta(V)}$. $\Delta(V)$ denotes the delay for a network of size V. We impose $\Delta(1) = 0$, $\lim_{V\to\infty} \Delta(V) = \infty$, and $\Delta'(V) > 0$ for $V > 1$.[7,8]

We assume that payments cannot be confirmed during a fork because, in such a case, validators disagree regarding the ledger's contents. Once a fork arises, we require a "k-blocks" rule to resolve the fork. Specifically, we require k consecutive arrivals without multiple blocks at the same height to return the blockchain to consensus.

non-trivial block sizes so that increasing block-rates and increasing block-sizes produce similar results. We allow arbitrary block-rates, so our results hold approximately for arbitrary block sizes.

[7] We model network delay in such generality to capture various potential validator network structures. Co-ordination may reduce network delay's sensitivity to network size, but our results nonetheless hold due to our general specification of $\Delta(V)$.

[8] $\Delta(V)$ lacks real-world meaning if $V \in [0, 1)$. Nonetheless, we specify $\forall V \in [0, 1) : \Delta(V) = 0$ for technical reasons. Our results do not depend upon this assumption.

3.4 Equilibrium

Definition 3.1. PoW Equilibrium

A PoW Equilibrium is an entrant cut-off, $c^* \in [0,1]$, a fee function, $\phi : [0,1] \mapsto \mathbb{R}_+$, a set of fee choices, $\{f_i\}_{i=1}^N$, and a validator network size, $V \geqslant 0$, given a number of users, $N \geqslant 2$, a blockchain utility, $R > 0$, and a block arrival rate, $\Lambda > 0$, such that:

(i) $\forall i : \phi(c_i)$ solves the problem in (1) if $c_i \leqslant c^*$ and $\phi(c_i) = 0$ otherwise

(ii) $\forall i : c_i \leqslant c^* \Leftrightarrow \max\limits_{f \geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i}) \mid c_i] - f \geqslant 0$

(iii) $\forall i : f_i = \phi(c_i)$

(iv) $W(f, f_{-i}) = \sum\limits_{j:f<f_j} H_j + H_i + Z_i, H_j \sim \exp(\Lambda), \mathbb{E}[Z_i] = \tau(\Lambda, V)$.

(v) $\beta V = \mathbb{E}[\sum\limits_i f_i]$.

Definition 3.1 characterizes the equilibrium. Without further reference, we assume that the blockchain's stationary distribution characterizes its initial state. The interested reader may consult Appendix A for the explicit stationary distribution and associated technical details. Condition 3.1 (i) asserts that users select an optimal fee schedule. Condition 3.1 (ii) states that a user transacts on the blockchain if and only if she derives weakly higher utility from transacting on the blockchain over the traditional payment systems. Condition 3.1 (iii) states that a user pays a fee only if she transacts on the blockchain. Condition 3.1 (iv) characterizes wait times as decomposed into three components; the wait for higher priority transactions, $\sum\limits_{j:f<f_j} H_j$, for personal service, H_i, and for fork resolution, Z_i. Due to block arrival according to a compound Poisson process, wait times for individual blocks are independently and identically distributed following an exponential distribution with rate Λ. We let $\tau(\Lambda, V)$ denote the expected fork-resolution time and characterize this function explicitly in Appendix A. Condition

3.1 (v) imposes no profits for validators in equilibrium because free entry characterizes the validator network.

Proposition 3.1. *Existence and Uniqueness of a PoW Equilibrium*
There exists a PoW Equilibrium. There exists no other equilibrium for which ϕ *constitutes a strictly increasing and differentiable function on the interval* $(0, c^*)$*. The following conditions characterize the equilibrium:*

(A) $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$ *if* $c_i \leqslant c^*$ *and* $\phi(c_i) = 0$ *otherwise*

(B) $R < \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} \implies R = c^*\Psi(\Lambda, V) + \frac{(c^*)^2(N-1)}{2\Lambda}$

(C) $R \geqslant \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} \implies c^* = 1$

(D) $\beta V = (N-1)N\frac{(c^*)^3}{6\Lambda}$.

Proposition 3.1 establishes existence and uniqueness of a PoW Equilibrium with $\Psi(\Lambda, V) \equiv \frac{1}{\Lambda} + \tau(\Lambda, V)$ denoting the expected wait time of the highest priority user. Proposition 3.1 (A) characterizes the equilibrium fee function. Proposition 3.1 (B) characterizes the entrant cut-off in the case that there exists a user indifferent between using the blockchain and a traditional alternative. Proposition 3.1 (C) characterizes the entrant cut-off in the case that all users weakly prefer transacting via the blockchain. Proposition 3.1 (D) characterizes the equilibrium number of validators.

4 PoW Results

Having established existence and uniqueness of a PoW Equilibrium, we turn to analyzing the properties of that equilibrium. Section 4.1 analyzes payment confirmation times. Section 4.2 establishes the limited adoption problem.

4.1 Payment Confirmation Times

We define $W_i \equiv \mathbb{E}[W(f_i, f_{-i}) \mid c_i]$ as the expected confirmation time for User i if she uses the blockchain. Equation (3) decomposes payment confirmation times into three parts.[9] $(N-1)\frac{(c^*-c_i)}{\Lambda}$ equals the expected service time for higher priority users. $\frac{1}{\Lambda}$ equals the expected service time for User i. $\tau(\Lambda, V)$ references the expected fork resolution time.

$$W_i = (N-1)\frac{(c^* - c_i)}{\Lambda} + \frac{1}{\Lambda} + \tau(\Lambda, V) \tag{3}$$

Fork resolution time constitutes a feature distinct from a traditional setting. This feature arises because blockchain payment confirmation requires agreement by all validators within the network. That agreement becomes harder to achieve when blocks arrive quickly relative to the time needed for a given validator to communicate her ledger to the network. Accordingly, disagreement arises more frequently as the network grows or as the block rate rises so that increasing the block rate need not expedite conformation times. In the absence of forks, confirmation times decrease as the block rate rises. Nonetheless, in the presence of forks, as the block rate rises so too does the fork frequency which counteracts the aforementioned effect.

Proposition 4.1. *Payment Confirmation Lower Bound*
Network delay bounds below all user payment confirmation times (*i.e.*, $\forall i : W_i \geqslant \tau(\Lambda, V) \geqslant \Delta(V)$).

Proposition 4.1 establishes that PoW induces network delay as a lower bound for confirmation times. Intuitively, a slow block rate yields a low fork frequency whereas a fast block rate yields a high fork frequency. Since forks delay validator agreement, arbitrarily fast payment confirmation cannot obtain for a decentralized PoW blockchain.

[9] Equation (3) follows from Definition 3.1 (iv) and Proposition 3.1 (A)

Proposition 4.2. *Arbitrarily Large Payment Confirmation Time*
All user payment confirmation times diverge as demand diverges, (i.e., $\forall i : \lim_{N\to\infty} W_i = \infty$*). This result holds in particular for the marginal user (i.e., i such that $c_i = c^*$), who is serviced with highest priority (i.e., $\forall j : f_i \geqslant f_j$).*

Next, we turn our attention to how payment confirmation times vary with increases in transaction demand. Proposition 4.2 establishes that payment confirmation times diverge for all users, including the highest priority user, as transaction demand grows.[10]

A PoW blockchain imposes an artificial supply constraint via a fixed block rate. As transaction demand rises, the artifical supply constraint induces higher fees which in turn causes more validators to enter the network. The larger validator network increases network delay which in turn increases fork frequency and yields arbitrarily large payment confirmation times even for the highest priority user. Although the highest priority user receives service first (with probability one), her expected confirmation time diverges because expected fork resolution time diverges.

4.2 Limited Adoption Problem

The aforementioned elongated payment confirmation times have important implications for the viability of a PoW payments blockchain. Specifically, a PoW payments blockchain cannot simultaneously sustain a large volume and a non-negligible market share. Proposition 4.3 formalizes that result.

Proposition 4.3. *An Adoption Problem*
Adoption decreases as demand rises (i.e., c^* decreases in N). Moreover, the blockchain faces limited adoption (i.e., $\lim_{N\to\infty} c^* = 0$*).*

Section 4.1 demonstrates that increases in transaction demand eventually yield increases in expected confirmation times for all blockchain users. These increased payment

[10] We refer to User i such that $c_i = c^*$ as the highest priority user. Any such user receives service first with probability one.

confirmation times drive users from the blockchain to traditional payment systems. If the blockchain sustains a large volume, then congestion induces fees which leads to validator entry. That validator entry prolongs payment confirmation times and thereby drives away all but the most dogmatic blockchain fanatics (i.e., Users i such that $c_i \leqslant c^*$). Therefore, PoW payments blockchains such as Bitcoin cannot obtain widespread adoption; rather, limited adoption constitutes an intrinsic and endogenous characteristic of such blockchains.

To highlight the role of endogenous network delay, we compare adoption associated with a variable network delay function to that associated with a constant network delay function via Proposition 4.4. Proposition 4.4 establishes that adoption for a network with constant delay eventually dominates that for a network with variable delay. The constant network delay may initially exceed the variable network delay, but network size diverges with transaction demand so that the variable nature of network delay in practice (see Chung and Lu (2002) and Riordan and Wormald (2010)) exacerbates the limited adoption problem.

Proposition 4.4. *Endogenous Network Delay*
Let c_v^* denote the adoption rate of a network with variable network delay that satisfies the regularity discussed within Section 3. Let c_c^* denote the adoption rate of a network with constant network delay. Then, $c_v^* < c_c^*$ for large transaction demands (i.e., $\exists \underline{N} : \forall N > \underline{N} : c_v^* < c_c^*$).

One may conjecture that a relaxation of PoW's artificial supply constraint (i.e., increasing Λ) may provide a solution to the limited adoption problem. Proposition 4.5, however, demonstrates that such an approach succeeds only in so far as it induces centralization. This result arises because relaxing PoW's artificial supply constraint implies a faster block rate which in turn increases disagreement among validators because blocks arrive too rapidly relative to network delay. A faster block rate paradoxically eventually increases wait times by prolonging the validator agreement process. This difficulty may be overcome only if the network possesses one validator which eliminates the need for communication among validators. Thus, even allowing dynamic supply achieves widespread adoption only at the expense of decentralization. The notion of sacrificing decentralization to obtain widespread adoption motivates one alternative solution: a semi-centralized permissioned blockchain. We analyze that setting in Section 5.

Proposition 4.5. *Decentralization implies Limited Adoption*
For exposition, we assume that $\lim_{N\to\infty} c^*$ *exist. The blockchain necessarily faces either centralization* $\left(i.e., \limsup_{N\to\infty} V \leqslant 1\right)$ *or limited adoption* $\left(i.e., \lim_{N\to\infty} c^* = 0\right)$.

The supply constraint can also be relaxed by increasing the number of transactions that can be recorded on any single block. Our model can be generalized to capture an alternative increase through larger block size. As noted by Decker and Wattenhofer (2013), the network delay increases linearly in the block size. Thus, a larger block size increases the fork propensity due to higher network delay and thereby also fails to remedy the limited adoption problem.

Our results may be interpreted as an economic parallel of Vitalik Buterin's Blockchain Trilemma.[11] Buterin's Trilemma pits decentralization, scalability and security against one another. Our analysis assumes security and demonstrates that a secure PoW payments blockchain cannot simultaneously achieve both scalability and decentralization. Proposition 4.3 demonstrates that a secure PoW payments blockchain cannot scale in the sense that such a blockchain cannot realize high transaction volumes and a non-negligible payments market share. Proposition 4.5 then highlights that increasing the blockchain's throughput resolves the scalability issue only if that increased throughput induces centralization. Hence, a PoW payments blockchain cannot simultaneously achieve decentralization, scalability and security as Buterin suggested.

[11] The interested reader may consult https://github.com/ethereum/wiki/wiki/Sharding-FAQs for further details.

Proposition 4.6. *No Adoption Problem Without Network Delay*
Both widespread adoption (i.e., $\lim_{N\to\infty} c^* > 0$*) and decentralization (i.e.,* $\lim_{N\to\infty} V = \infty$*) can be obtained simultaneously under the counterfactual assumption of no network delay (i.e.,* $\Delta(V) = 0$*).*

Before transitioning to a discussion surrounding permissioned blockchains, we offer a final PoW result to demonstrate the importance of network delay in generating our results. Proposition 4.6 assumes, counterfactually, that network delay does not exist (i.e., $\Delta(V) = 0$). Under this assumption a PoW payments blockchain can overcome the limited adoption problem. Widespread adoption becomes possible for a decentralized PoW system in the absence of network delay which establishes that network delay constitutes a critical factor for our results.

Our results highlight that limited adoption constitutes an endogenous and endemic characteristic of PoW payments blockchains. PoW combines an artificial supply constraint, free entry among validators and network delay that collectively make the system intrinsically impractical for widespread adoption. Our results do not argue against the potential for blockchain more broadly. In fact, we subsequently offer an alternative blockchain solution that overcomes the limited adoption problem.

5 A Permissioned Alternative

Proposition 4.5 highlights that a PoW payments blockchain must centralize to overcome the limited adoption problem. In this section, we consider a semi-centralized alternative: a permissioned blockchain.[12] Section 5.1 formally puts forth the permissioned blockchain model. Section 5.2 establishes that permissioned blockchains can obtain low

[12] Our focus upon permissioned blockchains does not imply that a permissionless setting cannot overcome the limited adoption problem. Some promising permissionless protocols include Byzantine Consensus PoS (e.g., Chen and Micali (2016)), delegated PoS (e.g., Kiayias et al. (2017)) and off-chain solutions (e.g., Poon and Dryja (2016) and Li, Wang, Xiei, and Zou (2019a)).

confirmation times and widespread adoption.

Nonetheless, those benefits are insufficient for a blockchain to be viable. Establishing blockchain security constitutes a necessary condition for blockchain viability. We consider that topic for a permissioned blockchain in Sections 5.3 and 5.4. Section 5.3 introduces a standard consensus protocol and demonstrates that this protocol may incur successful attacks in equilibrium. Section 5.4 introduces an alternative protocol that overcomes both the limited adoption problem and blockchain attacks.

5.1 Permissioned Blockchain Model

We model users as in Section 3 since the blockchain itself does not affect transaction demand. Unlike Section 3, we exogenously specify a set of validators, $V_P \in \mathbb{N}$.[13] All transactions enter at $t = 0$ at a single node so that all validators observe the full set of transactions by $t = \Delta(V_P)$. As with a PoW setting, validators instantly validate transactions. However, unlike a PoW setting, they need not solve any puzzle to partake in the consensus process so that no artificial supply constraint exists.

PoW attempts to create incentives for validators to not maliciously attack the blockchain. Thus, in offering an alternative, we focus on not only user adoption but also validator incentives. Validator i selects $a_i \in \{0, 1\}$ with $a_i = 0$ corresponding to malicious behavior and $a_i = 1$ corresponding to honest behavior. Malicious behavior yields some profit, $\Pi > 0$, if the attack succeeds. In contrast, a failed attack imposes a cost, $\kappa > 0$, on a malicious validator. For simplicity, we assume that an honest validator earns neither a profit nor a loss. The success of an attack depends upon the blockchain's consensus protocol which we discuss later in this section.

A permissioned blockchain may possess a cryptocurrency which enables a blockchain designer to shape validator incentives. We invoke a cryptocurrency when designing our own consensus protocol and denote Validator i's holding of that cryptocurrency by

[13] For exposition, we impose $V_P \geqslant 3$ in the equilibrium analysis.

$\alpha_i \in \mathbb{R}$.

We define a consensus protocol as a function $\omega : \{0,1\}^{V_P} \times \mathbb{R}^{V_P} \mapsto \{p \in [0,1]^{V_P} : \sum_{i=1}^{V_P} p_i = 1\}$ with ω_i corresponding to the probability that Validator i's ledger becomes the consensus ledger.[14] We further define $\Gamma(a_1, ..., a_{V_P}, \alpha_1, ..., \alpha_{V_P}) \equiv \sum_{i=1}^{V_P} \omega_i(a_1, ..., a_{V_P}, \alpha_1, ..., \alpha_{V_P})\, a_i$ so that Γ gives the probability that the blockchain does not suffer a successful attack.

Saleh (2019a) demonstrates that a cryptocurrency's price depends upon validator behavior on the associated blockchain. Taking such a premise as given, we assume that $P_{\Delta(V_P)} = P_H$ if the blockchain does not suffer a successful attack and $P_{\Delta(V_P)} = P_L$ otherwise with $P_t, t \in \{0, \Delta(V_P)\}$, denoting the time-$t$ cryptocurrency price and $P_H > P_L > 0$.

Definition 5.1. Permissioned Equilibrium

A Permissioned Equilibrium is an entrant cut-off, $c_P^* \in [0,1]$, a cryptocurrency price, P_0, a set of validator decisions, $\{a_i\}_{i=1}^{V_P} \in \{0,1\}^{V_P}$ and a set of validator cryptocurrency holdings, $\{\alpha_i\}_{i=1}^{V_P} \in \mathbb{R}^{V_P}$, given a validator network size, $V_P \geqslant 3$, a number of users, $N \geqslant 2$, a blockchain utility, $R_P > 0$, and a consensus protocol, ω, such that:

(i) $\forall i : c_i \leqslant c_P^* \Leftrightarrow R_P - c_i \Delta(V_P) \geqslant 0$

(ii) $(a_i, \alpha_i) \in \underset{(a,\alpha)}{\arg\sup}\ \Phi(a, \alpha; a_{-i}, \alpha_{-i})$
with $\Phi(a, \alpha; a_{-i}, \alpha_{-i}) \equiv (\Pi - (\Pi + \kappa)\mathbb{E}[\Gamma(a, a_{-i}, \alpha, \alpha_{-i})])\mathcal{I}_{a=0} + \alpha(\mathbb{E}[P_{\Delta(V_P)}] - P_0)$

(iii) $P_0 = \Gamma P_H + (1 - \Gamma)P_L$.

Definition 5.1 defines a Permissioned Equilibrium.[15] Definition 5.1 (i) asserts that a user employs the blockchain if and only if she (weakly) gains from employing the blockchain instead of a traditional payment system. Definition 5.1 (ii) requires that validators act optimally. We assume that all validators possess risk neutral preferences

[14] Our consensus protocol specification arises as a simplification of the more general construct.
[15] For exposition, we restrict our attention to pure strategies.

with perfect patience so that Definition 5.1 (iii) constitutes a necessary condition for equilibrium.

5.2 Permissioned Blockchain Benefits

Proposition 5.1. *Lower Payment Confirmation Times*

For any PoW protocol, there exists a permissioned blockchain which induces (weakly) lower payment confirmation times.

Section 4 demonstrates that PoW suffers from large payment confirmation times. This issue arises due to an artificial supply constraint and network delay which can be exacerbated by the permissionless nature of a PoW blockchain. A permissioned blockchain that omits PoW's artificial supply constraint enables lower payment confirmation times. Proposition 5.1 formalizes that assertion.

Proposition 5.2. *No Limited Adoption Problem*

In any Permissioned Equilibrium, widespread adoption (i.e., $\lim_{N \to \infty} c_P^* = \min\{\frac{R_P}{\Delta(V_P)}, 1\} > 0$*) obtains.*

Section 4 establishes that PoW faces the limited adoption problem. Proposition 5.2 highlights that a permissioned blockchain does not face that problem. This result arises because the lack of an artificial supply constraint facilitates timely service even for high transaction volumes. Thus, as Proposition 5.2 posits, a permissioned blockchain may obtain widespread adoption.

5.3 Majority Rule Consensus

Definition 5.2. Majority Rule Permissioned Equilibrium (MRPE)

A Majority Rule Permissioned Equilibrium (MRPE) is a Permissioned Equilibrium such that voting power is equally distributed among the majority.[16] More formally, $\omega_i \equiv \mathcal{I}\{|S_{a_i}| > |S_{1-a_i}| \vee |S_{a_i}| = |S_{1-a_i}| \wedge a_i = 0\} \times \frac{1}{|S_{a_i}|}$. Moreover, $S_a \equiv \{i : a_i = a\}$.

Lemma 5.3. *Majority Rule Permissioned Equilibrium (MRPE)*
For a Majority Rule Permissioned Equilibrium (MRPE), the blockchain does not suffer a successful attack if and only if honest validators strictly outnumber malicious validators (i.e., $\Gamma = \mathcal{I}\{|S_1| > |S_0|\}$).

Definition 5.2 specializes Definition 5.1 to a standard permissioned blockchain protocol. This standard permissioned blockchain protocol determines blockchain updates by a simple majority rule. Lemma 5.3 formalizes that assertion.

As established by Proposition 5.2, a majority rule permissioned blockchain overcomes the limited adoption problem. Nonetheless, the viability of a blockchain requires also that it overcomes attacks. We discuss this issue subsequently.

Proposition 5.4. *Honest MRPE*
There exists an MRPE in which all validators behave honestly and the blockchain does not suffer a successful attack (i.e., $\exists$MRPE s.t. $\forall i : a_i = 1, \Gamma = 1$).

Proposition 5.4 establishes the existence of an equilibrium in which all validators behave honestly. This result arises because a single validator cannot successfully attack the blockchain by behaving maliciously if all other validators behave honestly. Malicious behavior yields a cost to reputation with no off-setting gain so that honest behavior constitutes the unique best response to all other validators behaving honestly.

Proposition 5.5. *Malicious MRPE*
There exists an MRPE in which all validators behave maliciously and the blockchain suffers a successful attack (i.e, $\exists$MRPE s.t. $\forall i : a_i = 0, \Gamma = 0$).

[16] In case of a tie, we treat the malicious validators as the majority.

Proposition 5.5 establishes the existence of a second equilibrium in which all validators behave maliciously. This result arises because a single validator cannot unilaterally thwart a blockchain attack by behaving honestly. Honest behavior forgoes a reward from colluding to attack the blockchain when all other validators behave maliciously. Consequently, malicious behavior constitutes the unique best response to all other validators behaving maliciously.

Proposition 5.5 raises concern about employing a permissioned blockchain with a majority rule consensus protocol. Ideally, we wish a blockchain to both overcome the limited adoption problem and possess no equilibria in which a blockchain attack succeeds. Section 5.4 offers an alternative protocol that achieves both the desired goals.

5.4 Stake-Based Consensus

Definition 5.3. Stake-Based Permissioned Equilibrium (SBPE)

A Stake-Based Permissioned Equilibrium (SBPE) is a Permissioned Equilibrium such that voting power is equally distributed among the validators with majority stake.[17] More formally, $\omega_i \equiv \mathcal{I}\{T_{a_i} > T_{1-a_i} \vee T_{a_i} = T_{1-a_i} \wedge a_i = 0\} \times \frac{1}{|S_{a_i}|}$ with $T_a \equiv \sum\limits_{i \in S_a} \alpha_i^+$.

Lemma 5.6. *Stake-Based Permissioned Equilibrium (SBPE)*

For a Stake-Based Permissioned Equilibrium (SBPE), the blockchain does not suffer a successful attack if and only if the cumulative stake of honest validators strictly outweighs that of malicious validators (i.e., $\Gamma = \mathcal{I}\{T_1 > T_0\}$*).*

Definition 5.3 specializes Definition 5.1 to a permissioned blockchain protocol that we refer to as a stake-based protocol. This protocol determines blockchain updates by majority stake (with zero weights given to short-sale positions) rather than majority rule. By majority stake we refer to a protocol under which a validators' vote is weighted by her holding in the native cryptocurrency. Lemma 5.6 formalizes that result.

[17] In case of a tie, we treat the malicious validators as having the larger stake.

Proposition 5.7. *Honest SBPE*

There exists an SBPE in which all validators behave honestly and the blockchain does not suffer a successful attack (*i.e.,* $\exists$*SBPE s.t.* $\forall i : a_i = 1, \Gamma = 1$).

Proposition 5.7 establishes the existence of an equilibrium in which all validators behave honestly. This equilibrium arises for similar reasons as that described within Proposition 5.4, so we omit further discussion.

Proposition 5.8. *No Malicious SBPE*

There exists no SBPE in which an attack succeeds with strictly positive probability (*i.e.,* $\Gamma = 1$ *for all equilibria*).

Proposition 5.8 highlights the non-existence of an equilibrium in which a blockchain attack succeeds. This result arises because a single validator may become marginal by acquiring a sufficiently large stake. Since a validator's profit varies with her cryptocurrency position, she opts to become marginal and prevent a blockchain attack if she believes that an attack succeeds otherwise. Thus, a blockchain attack cannot succeed in equilibrium. A stake-based permissioned blockchain overcomes both blockchain attacks and the limited adoption problem.

6 Conclusion

Bitcoin has been envisioned as an alternative to traditional payment systems. While individual vendors have adopted Bitcoin and other PoW payment platforms, no such platform has obtained widespread adoption. We demonstrate that this lack of widespread adoption constitutes an intrinsic property of PoW payments blockchains. PoW imposes an artificial supply constraint on transactions. As transaction demand grows, fees increase endogenously. Due to the permissionless nature of PoW blockchains, more validators engage in the validation process. That entry expands the network size thereby protracting the consensus process and generating increased payment confirmation times. Thus, only users extremely insensitive to wait times transact via the blockchain in equilibrium and limited adoption arises. We demonstrate that this limited adoption cannot be overcome by relaxing the artificial supply constraint. Rather, network delay ensures limited adoption for decentralized PoW payment blockchains.

We consider permissioned blockchains as an alternative to PoW blockchains. We demonstrate that permissioned blockchain may overcome the limited adoption problem. Permissioned blockchains, however, may generate malicious validator behavior. In fact, under a simple permissioned consensus protocol, an equilibrium with malicious validator behavior and a successful blockchain attack exists. We propose an alternative protocol that overcomes this undesirable feature. This protocol employs a cryptocurrency native to the blockchain to align validator incentives such that a blockchain attack cannot succeed in equilibrium.

This paper has important policy implications. It directly concerns adoption of blockchain as a payment system. The limited adoption problem makes PoW blockchains impractical for widespread adoption as a payment system. Our work highlights the need for research examining alternative protocols.

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Appendices

A CTMC Blockchain Model

We model the blockchain as a Continuous Time Markov Chain (CTMC), $\{X_t\}_{t \geqslant 0}$, with states $x \in X \equiv \{0, 1, ..., k\}$ with $x < k$ denoting that the blockchains last x heights contain single block and $x = k$ denoting the complement. Given the discussion in Section 3, $x < k$ corresponds to the blockchain being in the midst of a fork and $x = k$ corresponds to the complement. This section offers background results including the stationary distribution and sojourn times.

Formally, the CTMC rate matrix, $Q \in \mathbb{R}^{X \times X}$, characterizes our model. For exposition, we define $p(x, y) = 1 - e^{-xy}$ and abuse notation by setting $p \equiv p(\Lambda, \Delta(V)) = 1 - e^{-\Lambda \Delta(V)} \in (0, 1)$. Then, $\forall x \in X/\{0, k\} : Q_{x,x} = -\Lambda$, $\forall x \in X/\{0\} : Q_{x,0} = \Lambda p$, $\forall x \in X/\{k\} : Q_{x,x+1} = \Lambda(1 - p)$, $Q_{K,K} = -\Lambda p$, $Q_{0,0} = -\Lambda(1 - p)$ and all other entries equal 0.

Lemma A.1. *Stationary Distribution*

$\{\pi_x\}_{x \in X}$ corresponds to the unique stationary distribution with $\forall x < k : \pi_x = p(1 - p)^x$ and $\pi_k = (1 - p)^k$

Proof.

Any stationary distribution, $\tilde{\pi} \in \mathbb{R}^X$, must satisfy $\tilde{\pi} Q = 0$. The result follows from algebra. □

For exposition, we uniformize our CTMC. We let $\{Y_t\}_{t \in \mathbb{N}}$ denote the associated Discrete Time Markov Chain (DTMC) and $P \in \mathbb{R}^{X \times X}$ denote the associated transition matrix. Then, $X_t = Y_{N(t)}$ with $\{N(t)\}_{t \geqslant 0}$ being a Poisson Process with rate λV.

Lemma A.2. *Fork Resolution Times*

We define $T_k \equiv \inf\{t \in \mathbb{N} : Y_t = k\}$. Then, The expected block heights until fork *resolution,* $s_x = \mathbb{E}[T_k|Y_0 = x]$*, conditional upon initial state,* $x \in X$*, satisfies* $\forall x \in X$: $s_x = (1 + s_0 p)\,\frac{1-(1-p)^{k-x}}{p} \quad \forall x \in X$ *so that* $s_0 = \frac{1-(1-p)^k}{p(1-p)^k}$.

Proof.

We prove the result by induction. $s_{k-j} = (1 + s_0 p)\sum_{i=0}^{j-1}(1-p)^i$ holds for $j = 1$ by definition. Then, $s_{k-(j+1)} = 1 + (1-p)s_{k-j} + ps_0 = (1 + s_0 p)\sum_{i=0}^{(j+1)-1}(1-p)^i$ with the last equality following from the inductive hypothesis. The conclusion then follows from algebra. □

Subsequently, we provide results useful for establishing existence of a PoW equilibria.

Lemma A.3. *Monotone Fork Resolution Times*

$\forall x \in X/\{k\} : s_x > s_{x+1} \geqslant 0$

Proof.

We prove the result by induction. By definition, $\forall x \in X/\{k\} : s_x = 1 + (1-p)s_{x+1} + ps_0$ so that $s_0 > s_1$ follows by taking $x = 0$. Then, by induction, $s_x = 1 + (1-p)s_{x+1} + ps_0 > 1 + (1-p)s_{x+1} + ps_x$ which implies $s_x > s_{x+1}$ as desired. $\forall x \in X/\{k\} : s_{x+1} \geqslant 0$ follows from $s_K = 0$. □

Hereafter, we define $\forall x \in X : s_x(\Lambda, \Delta(V)) \equiv s_x(p) \equiv s_x(p(\Lambda, \Delta(V)))$ and abuse notation by using s_x to mean the multivariate function. Similarly, we define $\forall x \in X$: $\pi_x(\Lambda, \Delta(V)) \equiv \pi_x(p) \equiv \pi_x(p(\Lambda, \Delta(V)))$ and abuse notation by using π_x to mean the multivariate function.

Lemma A.4. *Monotone Fork Resolution Derivatives*

$\forall x \in X/\{k\} : \frac{\partial s_x}{\partial \Lambda} > \frac{\partial s_{x+1}}{\partial \Lambda} \geqslant 0, \frac{\partial s_x}{\partial \Delta(V)} > \frac{\partial s_{x+1}}{\partial \Delta(V)} \geqslant 0$

Proof.

We prove the result by induction. By definition, $\forall x \in X/\{k\} : s_x = 1 + (1-p)s_{x+1} + ps_0$ so that $s_0 = e^{\Lambda\Delta(V)} + s_1$ so that $\frac{\partial s_0}{\partial \Lambda} > \frac{\partial s_1}{\partial \Lambda}$ follows immediately. Then, $s_x = 1 + e^{-\Lambda\Delta(V)}s_{x+1} +$

$(1-e^{-\Lambda\Delta(V)})s_0$ so that $\frac{\partial s_x}{\partial \Lambda} = e^{-\Lambda\Delta(V)}\frac{\partial s_{x+1}}{\partial \Lambda} + \Delta(V)e^{-\Lambda\Delta(V)}(s_0 - s_{x+1}) + (1-e^{-\Lambda\Delta(V)})\frac{\partial s_0}{\partial \Lambda} > \frac{\partial s_{x+1}}{\partial \Lambda}$ with the last inequality following by induction and Lemma A.3 which implies $\frac{\partial s_x}{\partial \Lambda} > \frac{\partial s_{x+1}}{\partial \Lambda}$ as desired. $\forall x \in X/\{k\} : \frac{\partial s_{x+1}}{\partial \Lambda} \geqslant 0$ follows from $\frac{\partial s_K}{\partial \Lambda} = 0$. Symmetry of the functions, $\{s_X\}_{x \in X}$, implies $\forall x \in X/\{k\} : \frac{\partial s_x}{\partial \Delta(V)} > \frac{\partial s_{x+1}}{\partial \Delta(V)} \geqslant 0$ which completes the proof. □

We define $\tau \equiv \mathbb{E}[\sum_{t=1}^{T_k} A_t]$ as the expected fork resolution time under the stationary distribution with $\{A_t\}_{t=1}^{\infty}$ independent and exponentially distributed with parameter Λ and initial distribution $\{\pi_x\}_{x \in X}$. Then, by definition, $\tau = \tau(\Lambda, \Delta(V)) = \sum_{x \in X} \frac{s_x(\Lambda,\Delta(V))}{\Lambda}\pi_x(\Lambda, \Delta(V))$.

Lemma A.5. *Lower Bound for* τ

$\tau(\Lambda, \Delta(V)) \geqslant \Delta(V)\frac{e^{\Lambda\Delta(V)k}-1}{\Lambda\Delta(V)}$

Proof.

$\tau(\Lambda, \Delta(V)) \geqslant \Delta(V)\frac{s_0(\Lambda,\Delta(V))}{\Lambda\Delta(V)}\pi_0(\Lambda, \Delta(V)) = \Delta(V)\frac{e^{\Lambda\Delta(V)k}-1}{\Lambda\Delta(V)}$ as desired.

□

We define $\Psi(\Lambda, V) \equiv \tau(\Lambda, \Delta(V)) + \frac{1}{\Lambda}$ which equates with the expected wait time for the marginal user (i.e., Type $c_i = c^*$). Then, trivially, $\frac{\partial \Psi}{\partial V} = \frac{\partial \Psi}{\partial V}$.

Lemma A.6. *Increasing Wait Time in* V

$\forall V' > V \geqslant 0 : \Psi(\Lambda, V') - \Psi(\Lambda, V) = \tau(\Lambda, \Delta(V')) - \tau(\Lambda, \Delta(V)) > 0$

Proof.

$$
\begin{aligned}
&\Psi(\Lambda, V') - \Psi(\Lambda, V)\\
&= \tau(\Lambda, \Delta(V')) - \tau(\Lambda, \Delta(V))\\
&= \sum_{x \in X}\{\tfrac{s_x(\Lambda,\Delta(V'))}{\Lambda}\pi_x(\Lambda, \Delta(V')) - \tfrac{s_x(\Lambda,\Delta(V))}{\Lambda}\pi_x(\Lambda, \Delta(V))\}\\
&\geqslant \sum_{x \in X} \tfrac{s_x(\Lambda,\Delta(V'))-s_x(\Lambda,\Delta(V))}{\Lambda}\pi_x(\Lambda, \Delta(V))\\
&= \sum_{x \in X} \tfrac{1}{\Lambda}\int_V^{V'} \tfrac{\partial s_x}{\partial \Delta(V)}\Delta'(v)dv\ \pi_x(\Lambda, \Delta(V))\\
&> 0
\end{aligned}
$$

□

Lemma A.7. *Zero Wait*

$\tau(\Lambda, 0) = 0$

Proof.

$\tau(\Lambda, 0) = s_k(\Lambda, 0) = 0$ □

B Proofs

Proposition 3.1 *Existence and Uniqueness of a PoW Equilibrium*

There exists a PoW Equilibrium. There exists no other equilibrium for which ϕ constitutes a strictly increasing and differentiable function on the interval $(0, c^*)$. The following conditions characterize the equilibrium:

(A) $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$ *if* $c_i \leqslant c^*$ *and* $\phi(c_i) = 0$ *otherwise*

(B) $R < \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} \implies R = c^*\Psi(\Lambda, V) + \frac{(c^*)^2(N-1)}{2\Lambda}$

(C) $R \geqslant \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} \implies c^* = 1$

(D) $\beta V = (N-1)N\frac{(c^*)^3}{6\Lambda}$.

Proof.

For coherence of our discussion, we must specify an initial distribution for our Blockchain CTMC model. We specify that distribution as the stationary distribution. The interested reader may consult Appendix A for details. For exposition, we define $V^*(N, c^*, \Lambda, \beta) \equiv \frac{(N-1)N(c^*)^3}{6\beta\Lambda}$.

As a preliminary step, we rule out the existence of any equilibrium such that $c^* = 0$. By contradiction, we suppose there exists an equilibrium such that $c^* = 0$. Definition 3.1 (iv) implies $\max_{f\geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i}) \mid c_i] - f \geqslant R - c_i \cdot \mathbb{E}[W(0, f_{-i}) \mid c_i] \geqslant R - c_i(\frac{N}{\Lambda} + \tau(\Lambda, V))$. Then, Definition 3.1 (ii) yields $\forall c_i > 0 : R - c_i(\frac{N}{\Lambda} + \tau(\Lambda, V)) \leqslant 0$ which in turn implies $R \leqslant 0$. $R \leqslant 0$ contradicts our assumption $R > 0$ and thereby eliminates the possibility of an equilibrium such that $c^* = 0$.

Problem 1 and Definitions 3.1 (iii) and (iv) yield $\max_{f \geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i})|c_i] - f = \max_{f \geqslant 0} R - c_i \frac{(N-1)}{\Lambda} \mathbb{P}(\phi(c_j) \geqslant f \wedge c^* \geqslant c_j) - c_i \Psi(\Lambda, V) - f$. $\phi(c_i)$ being a strictly increasing function enables us to rewrite the latter problem as $\max_{f \geqslant 0} R - c_i \frac{(N-1)}{\Lambda} \max\{c^* - \phi^{-1}(f), 0\} - c_i \Psi(\Lambda, V) - f$. Differentiability of ϕ then yields $\frac{c_i(N-1)}{\Lambda} \frac{1}{\phi'(\phi^{-1}(f))} = 1$ as a first-order condition for $c_i \in (0, c^*)$. In equilibrium, $f_i = \phi(c_i)$ for $c_i \in [0, c^*]$ so that the latter condition simplifies to $\frac{c_i(N-1)}{\Lambda} = \phi'(c_i)$ for $c_i \in [0, c^*]$. In turn, that result implies $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$ over $c \in [0, c^*]$. This result demonstrates that Proposition 3.1 (A) is necessary for the class of equilibria considered. Sufficiency for satisfying Definition 3.1 (i) follows from negativity of the objective's second derivative for Problem 1.

To establish existence and uniqueness of an equilibrium, we must establish the existence of some $V > 0$ and $c^* \in [0, 1]$ such that Definitions 3.1 (ii) and (v) hold with $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$ given and Definitions 3.1 (iii) and (iv) taken as definitions for f_i and $W(f, f_i)$ respectively.

For $c^* \in (0, 1)$, Definition 3.1 (iii) and $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$ imply Definition 3.1 (v) equates with $V^*(N, c^*, \Lambda, \beta) = V$. Moreover, 3.1 (ii) implies $\forall c_i > c^* : 0 > \max_{f \geqslant 0} R - c_i \cdot \mathbb{E}[W(f, f_{-i}) \mid c_i] - f \geqslant R - c_i \Psi(\Lambda, V) - \frac{(c^*)^2(N-1)}{2\Lambda}$ so that another application of 3.1 (ii) implies $R = c^* \Psi(\Lambda, V) + \frac{(c^*)^2(N-1)}{2\Lambda}$. Thus, existence and uniqueness equates with finding a unique solution, $c^* \in (0, 1)$, to $R = c^* \Psi(\Lambda, V^*(N, c^*, \Lambda, \beta)) + \frac{(c^*)^2(N-1)}{2\Lambda} \equiv G(c^*; N, \Lambda, \beta)$. Lemma A.7 yields $G(0; N, \Lambda, \beta) = 0 < R$ so that if $G(1; N, \Lambda, \beta) = \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} > R$ then continuity and strict monotonicity of G in c^* imply existence and uniqueness of an equilibrium with $c^* \in (0, 1)$ and $V = V^*(N, c^*, \Lambda, \beta)$.

To conclude, we need demonstrate only non-existence of an equilibrium with $c^* = 1$ if $\Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda} > R$ and existence of a unique equilibrium with $c^* = 1$ otherwise. If $c^* = 1$ then $V = \frac{(N-1)N}{6\beta\Lambda}$ uniquely satisfies Definition 3.1 (v) so that $R \geqslant \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda}$ by Definitions 3.1 (ii) - (iv) and $\phi(c_i) = (N-1)\frac{c_i^2}{2\Lambda}$. Thus, no equilibrium with

$c^* = 1$ exists if $R < \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda}$. Existence of a unique equilibrium with $c^* = 1$ follows because $c^* = 1$ and $V = \frac{(N-1)N}{6\beta\Lambda}$ satisfy all conditions for Definition 3.1 and all other choices for V violate Definition 3.1 (v).

□

Proposition 4.1 *Payment Confirmation Lower Bound*

Network delay bounds below all user payment confirmation times (i.e., $\forall i : W_i \geqslant \tau(\Lambda, V) \geqslant \Delta(V)$*).*

Proof.

Follows immediately from Lemma A.5 □

Lemma B.1. *Increasing V*

V increases in N and $\lim_{N\to\infty} V(N) = \infty$

Proof.

If $R \geqslant \Psi(\Lambda, \frac{(N-1)N}{6\beta\Lambda}) + \frac{N-1}{2\Lambda}$ then $\frac{dV}{dN} > 0$ follows from Proposition 3.1 (D). Otherwise, Proposition 3.1 (B) and (D) imply $R = \sqrt[3]{\frac{6\beta\Lambda V}{N(N-1)}}\Psi(\Lambda, V) + \sqrt[3]{\frac{9\beta^2 V^2(N-1)}{2\Lambda N^2}} \equiv H(V, N; \beta, \Lambda) \equiv H(V, N)$. Proposition 3.1 implies the existence of a non-negative function $V(N)$ that uniquely satisfies $R = H(V(N), N)$. By the implicit function theorem, $\frac{dV}{dN} = -\frac{\frac{\partial H}{\partial N}}{\frac{\partial H}{\partial V}} > 0$ which in turn implies the existence of $\lim_{N\to\infty} V(N)$. $0 \leqslant \lim_{N\to\infty} V(N) < \infty$ implies $\lim_{N\to\infty} H(V, N) = 0$ so that $\lim_{N\to\infty} H(V, N) = R > 0$ yields the desired conclusion. □

Proposition 4.2 *Arbitrarily Large Payment Confirmation Time*

All user payment confirmation times diverge as demand diverges, (i.e., $\forall i : \lim_{N\to\infty} W_i = \infty$*). This result holds in particular for the marginal user (i.e., i such that $c_i = c^*$), who is serviced with highest priority (i.e.,* $\forall j : f_i \geqslant f_j$*).*

Proof.

Proposition 4.1 yields $W_i \geqslant \Psi(\Lambda, V) \geqslant \tau(\Lambda, V) \geqslant \Delta(V)$ so that Lemma B.1 and

$\lim_{V\to\infty} \Delta(V) = \infty$ delivers the result. □

Proposition 4.3 *An Adoption Problem*

Adoption decreases as demand rises (i.e., c^* decreases in N). Moreover, the blockchain faces limited adoption (i.e., $\lim_{N\to\infty} c^* = 0$).

Proof.

Proposition 3.1 and Lemma B.1 imply that c^* decreases in N so that $\lim_{N\to\infty} c^* \in [0,1]$ exists. $\lim_{N\to\infty} c^* \in (0,1]$ implies $\lim_{N\to\infty}\{c^*\Psi(\Lambda,V)+\frac{(c^*)^2(N-1)}{2\Lambda}\} = \infty$ so that $\lim_{N\to\infty}\{c^*\Psi(\Lambda,V)+\frac{(c^*)^2(N-1)}{2\Lambda}\} = R < \infty$ via Proposition 3.1 (B) yields $\lim_{N\to\infty} c^* = 0$ as desired. □

Proposition 4.4 *Endogenous Network Delay*

Let c_v^* denote the adoption rate of a network with variable network delay that satisfies the regularity discussed within Section 3. Let c_c^* denote the adoption rate of a network with constant network delay. Then, $c_v^* < c_c^*$ for large transaction demands (i.e., $\exists \underline{N} : \forall N > \underline{N} : c_v^* < c_c^*$).

Proof.

From Appendix A, recall that $\Psi(\Lambda,V) \equiv \tau(\Lambda,\Delta(V)) + \frac{1}{\Lambda}$. Let Δ_c denote the constant network delay (associated with c_c^*) and $\Delta_v(V)$ denote the variable network delay (associated with c_v^*). Then, Proposition 3.1 implies that $\forall N > 2R\Lambda{+}1 : c_v^*(\tau(\Lambda,\Delta_v(V)) + \frac{1}{\Lambda}) + \frac{(c_v^*)^2(N-1)}{2\Lambda} = c_c^*(\tau(\Lambda,\Delta_c) + \frac{1}{\Lambda}) + \frac{(c_c^*)^2(N-1)}{2\Lambda}$. Lemmas A.5 and B.1 imply $\exists N_1 > 0 : \forall N \geqslant N_1 : \tau(\Lambda,\Delta_v(V)) \geqslant \tau(\Lambda,\Delta_c)$ so that $\forall N > \max\{N_1, 2R\Lambda{+}1\} \equiv \underline{N} : c_v^*(\tau(\Lambda,\Delta_c) + \frac{1}{\Lambda}) + \frac{(c_v^*)^2(N-1)}{2\Lambda} \leqslant c_c^*(\tau(\Lambda,\Delta_c) + \frac{1}{\Lambda}) + \frac{(c_c^*)^2(N-1)}{2\Lambda}$ which implies $\forall N > \underline{N} : c_v^* < c_c^*$ as desired. □

Proposition 4.5 *Decentralization implies Limited Adoption*

For exposition, we assume that $\lim_{N\to\infty} c^*$ exist. The blockchain necessarily faces either centralization (i.e., $\limsup_{N\to\infty} V \leqslant 1$) or limited adoption (i.e., $\lim_{N\to\infty} c^* = 0$).

Proof.

Formally, we consider a sequence of parameters $\{(N_n, \Lambda_n, R, \beta)\}_{n\in\mathbb{N}}$ with $R, \beta > 0$, $2 \leqslant N_n \nearrow \infty$. Then, following Proposition 3.1, there exists a sequence $\{(c^*_n, V_n)\}_{n\in\mathbb{N}}$ such that (c^*_n, V_n) corresponds to the equilibrium solution for a model with parameters $(N_n, \Lambda_n, R, \beta)$.

We proceed by contradiction. We assume that $L \equiv \limsup_{n\to\infty} V_n > 1$ and $M \equiv \lim_{n\to\infty} c^*_n > 0$. We take a subsequence, $\{(N_{n_j}, \Lambda_{n_j}, c^*_{n_j}, V_{n_j})\}_{j\in\mathbb{N}}$, such that $\forall j : V_{n_j} \geqslant \frac{1+L}{2}$. Then, Proposition 3.1 (B) and (C) yield $\Lambda_{n_j} \geqslant \frac{(c^*_{n_j})^2(N_{n_j}-1)}{2R}$ so that $\lim_{j\to\infty} \Lambda_{n_j} = \infty$. Lemma A.5 and Proposition 3.1 (B) - (C) then give $R \geqslant c^*_{n_j}\Delta(V_{n_j})\frac{e^{\Lambda_{n_j}\Delta(V_{n_j})}-1}{\Lambda_{n_j}\Delta(V_{n_j})}$ so that monotonicity of Δ coupled with $\forall j : V_{n_j} \geqslant \frac{1+L}{2}$ yields $R \geqslant c^*_{n_j}\Delta(\frac{1+L}{2})\frac{e^{\Lambda_{n_j}\Delta(\frac{1+L}{2})}-1}{\Lambda_{n_j}\Delta(\frac{1+L}{2})}$. Finally, invoking $\lim_{j\to\infty} \Lambda_{n_j} = \infty$ gives $R \geqslant \lim_{j\to\infty} c^*_{n_j}\Delta(\frac{1+L}{2})\frac{e^{\Lambda_{n_j}\Delta(\frac{1+L}{2})}-1}{\Lambda_{n_j}\Delta(\frac{1+L}{2})} = \infty$ delivering the desired contradiction and thereby completing the proof. □

Proposition 4.6 *No Adoption Problem Without Network Delay*

Both widespread adoption (i.e., $\lim_{N\to\infty} c^* > 0$*) and decentralization (i.e.,* $\lim_{N\to\infty} V = \infty$*) can be obtained simultaneously under the counterfactual assumption of no network delay (i.e.,* $\Delta(V) = 0$*).*

Proof.

Formally, we take a sequence of parameters $\{(N_n, R, \beta)\}_{n\in\mathbb{N}}$ such that $R, \beta > 0$, $2 \leqslant N_n \nearrow \infty$ and construct a sequence $\{\Lambda_n\}_{n=1}^{\infty}$. Then, we provide a sequence $\{(c^*_n, V_n)\}_{n\in\mathbb{N}}$ such that (c^*_n, V_n) corresponds to equilibrium solutions for a model with parameters $(N_n, \Lambda_n, R, \beta)$. We demonstrate that, given our choice, $\{\Lambda_n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} c^*_n > 0$ and $\lim_{n\to\infty} V_n = \infty$ if $\Delta(V) = 0$ (i.e., no network delay). Note that this result does not contradict Proposition 4.5 as all parts of the paper (except this proposition) preclude $\Delta(V) = 0$ (i.e., we assume existence of network delay outside of this proposition).

Let $\Lambda_n \equiv \frac{N_n-1}{2}$. Let $c^*_n \equiv \min\{c_n, 1\}$ with c_n being the unique positive solution for $R = \frac{c_n}{\Lambda_n} + c_n^2$ and let $V_n = \frac{N_n(c^*)^3}{3}$. Then, $\{(c^*_n, V_n)\}_{n\in\mathbb{N}}$ satisfies all conditions from Definition 3.1 thereby constituting an equilibrium for $\{(N_n, R, \beta)\}_{n\in\mathbb{N}}$. Moreover, $\lim\limits_{n\to\infty} c_n^* = c^* = \min\{\sqrt{R}, 1\} > 0$ and $\lim\limits_{n\to\infty} V_n = \infty$ as desired. □

Proposition 5.1 *Lower Payment Confirmation Times*

For any PoW protocol, there exists a permissioned blockchain which induces (weakly) lower payment confirmation time.

Proof.

Let $V_P = V$. Then, the result follows from Proposition 4.1. □

Proposition 5.2 *No Limited Adoption Problem*

In any Permissioned Equilibrium, widespread adoption (i.e., $\lim\limits_{N\to\infty} c_P^* = \min\{\frac{R_P}{\Delta(V_P)}, 1\} > 0$*) obtains.*

Proof.

$R_P - c_i\Delta(V_P)$ decreases in c_i so that Definition 5.1 (i) implies $c_P^* = \min\{\frac{R_P}{\Delta(V_P)}, 1\}$ so that $\lim\limits_{N\to\infty} c_P^* = \min\{\frac{R_P}{\Delta(V_P)}, 1\}$ follows trivially. □

Lemma 5.3 *Majority Rule Permissioned Blockchain Equilibrium (MRPBE)*

For a Majority Rule Permissioned Equilibrium (MRPE), the blockchain does not suffer a successful attack if and only if honest validators strictly outnumber malicious validators (i.e., $\Gamma = \mathcal{I}\{|S_1| > |S_0|\}$*).*

Proof.

$\Gamma(x) = \sum\limits_{i=1}^{V_P} \omega_i(x)a_i = \sum\limits_{i\in S_1} \omega_i(x) = \mathcal{I}(|S(1)| > |S(0)|)$ □

Proposition 5.4 *Honest MRPBE*

There exists an MRPE in which all validators behave honestly and the blockchain does not suffer a successful attack (i.e., $\exists$*MRPE s.t.* $\forall i : a_i = 1, \Gamma = 1$*).*

Proof.

We demonstrate the existence of a symmetric equilibrium in which $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$,

$P_0 = P_H$ and $\forall i : (a_i, \alpha_i) = (1, 0)$. In such an equilibrium, all validators behave honestly since $\forall i : a_i = 1$ and $\Gamma = 1$ so that the blockchain does not sustain a successful attack.

Direct verification shows that $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$ satisfies Definition 5.1 (i) and $P_0 = P_H$ satisfies Definition 5.1 (iii). As such, to prove the result, we need only demonstrate that $\forall a \in \{0,1\}, \alpha \in \mathbb{R} : \Phi(1, 0; a_{-i}, \alpha_{-i}) \geqslant \Phi(a, \alpha, a_{-i}, \alpha_{-i})$ with $\forall j \neq i : (a_j, \alpha_j) = (1, 0)$. $V_P \geqslant 3$ implies $\Gamma = 1$ so that $a \in \{0,1\}, \alpha \in \mathbb{R} : \Phi(a, \alpha; a_{-i}, \alpha_{-i}) = -\kappa \mathcal{I}_{a=0} \leqslant 0 = \Phi(1, 0; a_{-i}, \alpha_{-i})$ as desired. □

Proposition 5.5 *Malicious MRPBE*

There exists an MRPE in which all validators behave maliciously and the blockchain suffers a successful attack (*i.e,* $\exists$*MRPE s.t.* $\forall i : a_i = 0, \Gamma = 0$).

Proof.

We demonstrate the existence of a symmetric equilibrium in which $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$, $P_0 = P_L$ and $\forall i : (a_i, \alpha_i) = (0, 0)$. In such an equilibrium, all validators behave maliciously since $\forall i : a_i = 0$ and $\Gamma = 0$ so that the blockchain sustains a successful attack with probability 1.

Direct verification shows that $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$ satisfies Definition 5.1 (i) and $P_0 = P_L$ satisfies Definition 5.1 (iii). As such, to prove the result, we need only demonstrate that $\forall a \in \{0,1\}, \alpha \in \mathbb{R} : \Phi(0, 0; a_{-i}, \alpha_{-i}) \geqslant \Phi(a, \alpha, a_{-i}, \alpha_{-i})$ with $\forall j \neq i : (a_j, \alpha_j) = (0, 0)$. $V_P \geqslant 3$ implies $\Gamma = 0$ so that $a \in \{0,1\}, \alpha \in \mathbb{R} : \Phi(a, \alpha; a_{-i}, \alpha_{-i}) = \Pi \mathcal{I}_{a=0} \leqslant \Pi = \Phi(0, 0; a_{-i}, \alpha_{-i})$ as desired. □

Lemma 5.6 *Stake-Based Permissioned Equilibrium (SBPE)*

For a Stake-Based Permissioned Equilibrium (*SBPE*)*, the blockchain does not suffer a successful attack if and only if the cumulative stake of honest validators strictly outweighs that of malicious validators* (*i.e.,* $\Gamma = \mathcal{I}\{T_1 > T_0\}$).

Proof.

$$\Gamma(x) = \sum_{i=1}^{V_P} \omega_i(x) a_i = \sum_{i \in S_1} \omega_i(x) = \mathcal{I}(\sum_{i \in S_1} \alpha_i^+ > \sum_{i \in S_0} \alpha_i^+) \qquad \square$$

Proposition 5.7 *Honest SBPE*

There exists an SBPE in which all validators behave honestly and the blockchain does not suffer a successful attack (*i.e.*, $\exists$*SBPE s.t.* $\forall i : a_i = 1, \Gamma = 1$).

Proof.

We demonstrate the existence of a symmetric equilibrium in which $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$, $P_0 = P_H$ and $\forall i : (a_i, \alpha_i) = (1, \frac{\Pi}{P_H - P_L})$. In such an equilibrium, all validators behave honestly since $\forall i : a_i = 1$ and $\Gamma = 1$ so that the blockchain does not sustain a successful attack.

Direct verification shows that $c_P^* = \min\{1, \frac{R_P}{\Delta(V_P)}\}$ satisfies Definition 5.1 (i), and $P_0 = P_H$ satisfies Definition 5.1 (iii). As such, to prove the result, we need only demonstrate that $\forall a \in \{0,1\}, \alpha \in \mathbb{R} : \Phi(1, \frac{\Pi}{P_H - P_L}; a_{-i}, \alpha_{-i}) \geqslant \Phi(a, \alpha, a_{-i}, \alpha_{-i})$ with $\forall j \neq i : (a_j, \alpha_j) = (1, \frac{\Pi}{P_H - P_L})$. We define $\underline{\alpha} \equiv \frac{\Pi(V_P - 1)}{P_H - P_L} \geqslant \frac{2\Pi}{P_H - P_L} > 0$.

Then, $\forall a \in \{0,1\}, \alpha \in \mathbb{R}$:

$$\begin{aligned}
&\Phi(a, \alpha; a_{-i}, \alpha_{-i}) \\
&\leqslant \max\{ \sup_{\alpha^* < \underline{\alpha}} \Phi(a, \alpha^*; a_{-i}, \alpha_{-i}), \sup_{\alpha^* \geqslant \underline{\alpha}} \Phi(a, \alpha^*; a_{-i}, \alpha_{-i}) \ \} \\
&\leqslant \max\{ -\kappa \mathcal{I}_{a=0}, \max\{0, \Pi + (P_L - P_H)\underline{\alpha} \ \} \ \} \\
&\leqslant 0
\end{aligned}$$

$\Phi(1, \frac{\Pi}{P_H - P_L}; a_{-i}, \alpha_{-i}) = 0$ completes the proof. □

Proposition 5.8 *No Malicious SBPE*

There exists no SBPE in which an attack succeeds with strictly positive probability (*i.e.*, $\Gamma = 1$ *for all equilibria*).

Proof.

We proceed by contradiction. We assume that there exists an equilibrium in which an attack succeeds with strictly positive probability (i.e., $\Gamma < 1$). Via Lemma 5.6, $\Gamma < 1 \implies \Gamma = 0$ which in turn implies $P_0 = P_L$ via Definition 5.1 (iii). Then, defining $\alpha_* \equiv \sum_{j \in S_0, j \neq 1} \alpha_j - \sum_{j \in S_1, j \neq 1} \alpha_j + 1$ implies $\sup_{(a,\alpha)} \Phi(a, \alpha; a_{-1}, \alpha_{-1}) \geqslant \sup_{\alpha \geqslant \alpha_*} \Phi(1, \alpha; a_{-1}, \alpha_{-1}) = \sup_{\alpha \geqslant \alpha_*} \alpha(P_H - P_0)$ so that $P_0 \geqslant P_H$ constitutes a necessary condition for equilibrium. $P_H > P_L = P_0 \geqslant P_H$ gives the desired contradiction thereby completing the proof. $\square$