Estimating Macroeconomic Models of Financial Crises: An Endogenous Regime Switching Approach

Gianluca Benigno
London School of Economics

Andrew Foerester
Federal Reserve Bank of Kansas City

Christopher Otrok
University of Missouri
Federal Reserve Bank of St. Louis

Alessandro Rebucci
Johns Hopkins University

Eric R. Young
University of Virginia

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Abstract This paper develops an endogenous regime switching approach to modeling financial crises. In the model there are two regimes, one a crisis regime, the second a regime for normal economic times. The switch between regimes is based on a probability determined by economic variables in the economy. Agents in the economy know how economic fundamentals affect the probability of moving in or out of the crisis state. That is, it is a rational expectations solution of the model. The solution then ensures that decisions made in the normal state fully incorporate how those decision affect the probability of moving into the crisis state as well as how the economy will operate in a crisis. The model developed captures all of the salient features one would want in an empirical model of financial crises. First, it captures the non-linear nature of a crisis. Second, the regime switching model is solved using perturbation methods and a second order solution. This allows the solution to capture the impact of risk on decision rules due both in an out of the crisis. Third, since the solution method is perturbation based it can handle a number of state variables and many shocks. That is, we are less constrained than current non-linear methods in terms of the size of the model. Fourth, the speed of the solution method means that non-linear filters (e.g. particle filter) can be used to calculate the likelihood function of the model for a full Bayesian estimation of the relevant shocks and frictions that are fundamental to models of financial crises. Fifth, the fully rational expectations nature of the solution allows one to ask key counterfactual policy questions. We adopt this approach to study sudden stop episodes in Mexico. Our model is an adaption of Mendoza (2010). In particular, we rewrite his occasionally binding collateral contraint model as a multi regime model and take it to the data using our new estimation procedures.

1 Introduction

This paper develops a new approach to estimating structural macroeconomic models of financial crises. Financial crises are rare but large events, implying that any reasonable model for analyzing crises must be non-linear. This non-linearity of course poses problems computationally, particularly when one wants to take these models to the data formally using likelihood based empirical methods. In this paper we combine two key types of non-linearity. The first writes models of financial crises as a two (or more) regime model. By viewing a financial crisis as a discrete regime we can capture the large and significant change in an economy in crisis. The second is to solve our model to a second order solution, allowing us to capture the impact of risk on decision making both in and out of the the crisis.

The core of the new methodology is an endogenous regime switching approach to modeling financial crises. In the model there are two regimes, one a crisis regime, the second a regime for normal economic times. In our model a crisis regime is a regime where an occasionally binding borrowing constraint binds, (Mendoza 2010). The switch between regimes is based on a probability determined by economic variables in the economy. Likewise, the switch back to normal times is based on economic fundamentals. In our model the probability of moving to the crisis regime where the borrowing constraint binds is a logistic function of the debt to output ratio. Agents in the economy know of this probability and how debt, output and other choices map into the probability of moving in or out of the crisis state. That is, it is a rational expectations solution of the model. Our solution then ensures that decisions made in the normal state fully incorporate how those decision affect the probability of moving into the crisis state as well how the economy will operate in a crisis (i.e the decision rules in this crisis).

The approach we develop allows us to capture all of the salient features one would want in an empirical model of financial crises. First, it captures the non-linear nature of a crisis: the crisis state can have very different properties/parameters from the normal state. Second, we solve the regime switching model using perturbation methods and a second order solution. This means that we can capture the change in decision rules as risk changes in a crisis. Third, since our solution method is perturbation based we can handle a number of state variables and many shocks. That is, we are less constrained than current non-linear methods in terms of the size of the model. Fourth, the speed of the solution method means that we can use non-linear filters to calculate the likelihood function of the model for a full Bayesian estimation of the relevant shocks and frictions that are fundamental to models of financial crises. Fifth, the fully rational expectations nature of the solution allows us to ask counterfactual policy questions.

The literature on Markov-switching linear rational expectations (MSLRE) is now well established (e.g Leeper and Zha (2003), Davig and Leeper (2007), and Farmer, Waggoner, and Zha (2009)). The MSLRE approach introduces an important nonlinearity into the standard linear rational expectations models. That nonlinearity is a discrete change in the parameters across regimes. The MSLRE approach has been widely used to model shifts in monetary and fiscal policy (e.g Bianchi 2014 or Davig, Leeper, and Walker 2010). Markov-switching models in general are useful because they provide a tractable way to model how agents form expectations over discrete changes in policies. Foerster et al (2014) argue that the MSLRE solution techniques, which impose Markov Switching after linearizing the model, may differ from the solution to a model where switching is present in the original model before linearization. In addition, the MSLRE model of course cannot be solved to higher orders, which may matter in economies where risk is important. To address these issues Foerster et al (2014) develop a perturbation methodology for constructing first or second order solutions of markov-switching DSGE (MSDSGE) models. A key innovation of their approach is to work with the original MSDSGE model directly, rather than starting with a system of linear rational expectations equations. Our approach builds on their approach, but solves a class of model when the probability of regime switch is an endogenous function of the economy. An exogenous switching model would not be interesting for financial crisis models as choices (for debt, production, policy) would all be unrelated to whether or not a crisis occurs

The application of the methodology is most closely related to the literature that has built on the seminal work of Mendoza (2010). This literature has studied the normative properties of model economies with endogenous financial crises (also labelled sudden stops or credit crunches). Some examples include Bianchi (2011) who uses an endowment version of such an economy and finds that the competitive equilibrium always entails more borrowing relative to the constrained social planner allocation, and that a prudential tax on debt (i.e., a prudential capital control) can replicate the social planner allocation. Benigno et al (2012) show that in a production economy agents can actually borrow too little relative to what is socially optimal. Benigno et al (2014) compare alternative tax instruments chosen by a Ramsey planner in the same economy analyzed by Bianchi (2011) and find that taxes on consumption (i.e., real exchange rate interventions) dominate capital controls as a policy tool because they can achieve the unconstrained allocation while capital controls can achieve only the constrained efficient one. Cespedes, Chang and Velasco (2012) compare the transmission mechanism of alternative policy interventions in a similar model. Jeanne and Korinek (2011) and Bianchi and Mendoza (2010) analyze models in which the price externality arises because agents fail to internalize the effect of their decisions on an asset price. Korinek and Mendoza (2013) provide a thorough review of the models, questions and results from this large literature. They conclude by stating that an important future step for this literature is the "development of numerical methods that combine the strenghts of global solution methods in describing non-linear dynamics with the power of perturbation methods in dealing with a large number of variables so as to analyze sudden stops in even richer macroeconomic models". This is exactly what the methodology developed in this paper delivers.

The economic innovation of the paper is then to take the lessons from this normative literature to the data and develop ways to implement these policies. Since the model will be estimated on the data we can test, in a counterfactual sense, how various policies would have worked in different historical episodes. This is an important innovation to the literature, as the current papers (including my own) largely discuss policies effectiviness while abstracting away from the historical sequence of shocks that occured in any given crisis period. The paper will also contribute by documenting the sources of financial crises by identifying the key shocks that drive us into a crisis.

There are many possible applications of our approach to other classes of models. For example, Bocola (2015) builds a model of sovereign default. His estimation procedure is to first estimate the model outside of the crisis period, using a solution technique that assumes a crisis will not occur. Conditional on those parameter estimates a crisis probability that is exogenous is appended to the model. Our approach allows one to estimate model parameters fully incorporating the possibility of a crisis, and allowing for that crisis to be a function of the economy.

Other methods have been developed to deal with occasionally binding constraints. Most recently Guerrieri and Iacoviello (2015) developed a set of procedures called OccBin. OccBin is a certainty equivalent solution method which requires agents to know precisely how long a regime (the one you are not currently in) will apply if there are no shocks, making it functionally quite similar to the perfect foresight methods used in the ZLB literature (Eggertsson and Woodford; Christiano, Eichenbaum, and Rebelo). These methods rule out precautionary effects, which we find to be important for the model we are looking at (we can't of course claim that these affects are important for all occasionally binding constraint models).

2 The Model and the Competitive Equilibrium

In this section, we describe our model set-up. The model is largely from Mendoza (2010). An exception is that we use a interest elastic debt function to pin down debt instead of an endogenous rate of time preference.

2.1 Representative Household-Firm

As in Mendoza (2010) there is a representative household that also makes decisions about production and capital accumulation. This structure ensures that the working capital constraint affects the borrowing constraint, a key feature of Mendoza's paper The households maximizes the utility function

$$U \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t \frac{1}{1-\rho} \left(C_t - \frac{H_t^{\omega}}{\omega} \right)^{1-\rho} \right\}, \tag{1}$$

with C_t denoting the individual consumption and H_t the individual supply of labor. The elasticity of labor supply is ω, while ρ is the coefficient of relative risk aversion.

Households maximize utility subject to their budget constraint. The constraint each household faces is:

$$C_t + I_t = A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{1-\alpha-\eta} - P_t V_t - \phi r_t \left(W_t H_t + P_t V_t \right) - \frac{1}{(1+r_t)} B_t + B_{t-1} \tag{2}$$

V_t are imported intermediate goods. P_t is the price of these imports, which will be a stochastic process specified below. This shock is interpreted as a terms of trade shock. B_t are one period international bonds with price net interest rate r_t. The interest rate is exogenous and equal to a stochastic process specified below. The ϕr_t term reflects a working capital constraint, both wages and intermediate goods must be paid for with borrowed funds. The price of labor and capital are given by w_t and q_t, both of which are endogenous variables, but taken as given by the household. Investment incurs an adjustment cost in terms of net investment:

$$I_t = \delta K_{t-1} + (K_t - K_{t-1})\left(1 + \frac{\iota}{2}\left(\frac{K_t - K_{t-1}}{K_{t-1}}\right)\right) \tag{3}$$

I_t is gross investment.

The agents faces a collateral constraint:

$$\frac{1}{(1+r_t)} B_t - \phi\,(1+r_t)\,(W_t H_t + P_t V_t) \geq -\kappa q_t K_t \tag{4}$$

Households maximize (1) subject to (2) and (4) by choosing C_t, B_t, K_t, V_t and H_t. The first-order conditions of this problem are the following:

$$C_t : \left(C_t - \frac{H_t^{\omega}}{\omega}\right)^{-\rho} - \mu_t = 0 \tag{5}$$

$$V_t : (1-\alpha-\eta)\, A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{-\alpha-\eta} = P_t\left(1 + \phi r_t + \frac{\lambda_t}{\mu_t}\phi\,(1+r_t)\right) \tag{6}$$

$$H_t : \alpha A_t K_{t-1}^{\eta} H_t^{\alpha-1} V_t^{1-\alpha-\eta} = \phi W_t\left(r_t + \frac{\lambda_t}{\mu_t}(1+r_t)\right) + H_t^{\omega-1} \tag{7}$$

$$B_t : \mu_t = \lambda_t + \beta\,(1+r_t)\,\mathbb{E}_t\mu_{t+1} \tag{8}$$

$$K_t : \mathbb{E}_t\mu_{t+1}\beta\left(\begin{array}{c} 1-\delta+\left(\frac{\iota}{2}\left(\frac{K_{t+1}}{K_t}\right)^2 - \frac{\iota}{2}\right) \\ +\eta A_{t+1}K_t^{\eta-1}H_{t+1}^{\alpha}V_{t+1}^{1-\eta-\alpha} \end{array}\right) = \mu_t\left(1-\iota+\iota\left(\frac{K_t}{K_{t-1}}\right)\right) - \lambda_t\kappa q_t \tag{9}$$

Market optimal prices for capital and labor are

$$q_t = 1 + \iota\left(\frac{K_t - K_{t-1}}{K_{t-1}}\right) \tag{10}$$

$$W_t = H_t^{\omega-1} \tag{11}$$

The last two conditions are the budget and complementary slackness conditions

$$C_t + I_t = A_t K_{t-1}^{\eta} H_t^{\alpha} V_t^{1-\alpha-\eta} - P_t V_t - \phi r_t\,(W_t H_t + P_t V_t) - \frac{1}{(1+r_t)} B_t + B_{t-1} \tag{12}$$

$$\left(\frac{1}{(1+r_t)} B_t - \phi\,(1+r_t)\,(w_t H_t + P_t V_t) + \kappa q_t K_t\right)\lambda_t = 0 \tag{13}$$

where λ_t is the multiplier on the international borrowing constraint.

We have three exogenous processes and 4 shocks. The interest rate has a debt elastic component as well as a stochastic shock to the elasticity (country specific risk premium) as well as to world interest rates.

$$r_t = r^* + (\psi_r + \sigma_r\varepsilon_{r,t})\left(e^{\overline{B}-B_t} - 1\right) + \sigma_w\varepsilon_{w,t}$$

laws of motion for the exogenous variables

$$\log A_t = a(s_t) + \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t} \tag{14}$$

$$\log P_t = p(s_t) + \rho_P \log P_{t-1} + \sigma_P \varepsilon_{P,t} \tag{15}$$

The borrowing cushion is given by the amount of borrowing over the debt limit

$$B_t^* = \frac{1}{(1+r_t)} B_t - \phi (1+r_t)(W_t H_t + P_t V_t) + \kappa q_t K_t \tag{16}$$

where $B_t^* \geq 0$.

2.2 Regime Switching

2.2.1 Re-writing the Slackness Condition

The complementary slackness condition (13) is summarized as

$$B_t^* \lambda_t = 0, \text{ with } B_t^*, \lambda_t \geq 0.$$

To re-interpret this condition, note there are two regimes: one in which the constraint binds ($B_t^* = 0$, $\lambda_t \geq 0$), and one in which it does not ($B_t^* \geq 0$, $\lambda_t = 0$). The variable $s_t \in \{0, 1\}$ denotes the regime, and whether the constraint binds ($s_t = 0$) or does not bind ($s_t = 1$). The regime switching variables $\varphi(s_t) = \gamma(s_t) = s_t$ turn "on" or "off" the relevant portions of the slackness condition. Having two parameters allows flexibility so that regime switching affects to a maximal extent both the level of the economy as well as how it responds to other state variables–in effect the slope and intercept of the decision rules.

The exact functional form is somewhat flexible, but one option that works well is

$$\varphi(s_t) B_{ss}^* + \gamma(s_t)(B_t^* - B_{ss}^*) = (1 - \varphi(s_t)) \lambda_{ss} + (1 - \gamma(s_t))(\lambda_t - \lambda_{ss}). \tag{17}$$

This functional form works well for several reasons. First, note that when $s_t = 0$, then $\varphi(0) = \gamma(0) = 0$ and so the equation simplifies to

$$\lambda_t = 0 \tag{18}$$

and when $s_t = 1$ then $\varphi(1) = \gamma(1) = 1$ and so the equation simplifies to

$$B_t^* = 0. \tag{19}$$

Further, the equation pins down the steady state values appropriately. As will be explained in more detail later, only the switching variable $\varphi(s_t)$ is perturbed, so the steady state satisfies

$$\bar{\varphi} B_{ss}^* = (1 - \bar{\varphi}) \lambda_{ss} \tag{20}$$

where $\bar{\varphi}$ denotes the ergodic mean of $\varphi(s_t)$. Then if only the non-binding regime occurs, then $\bar{\varphi} = 0$ and

$$\lambda_{ss} = 0 \tag{21}$$

whereas if only the binding regime occurs then $\bar{\varphi} = 1$ and

$$B_{ss}^{*} = 0. \tag{22}$$

Intermediate values of $\bar{\varphi}$ scale between these to cases according to

$$\bar{\varphi} B_{ss}^{*} = (1 - \bar{\varphi}) \lambda_{ss}. \tag{23}$$

Finally, due to the elasticity-type parameter $\gamma(s_t)$, regime switching will affect the first-order decision rules. To see this, note that the total derivative of the above and evaluated at steady state is

$$\gamma(s_t) \, dB_t^{*} = (1 - \gamma(s_t)) \, d\lambda_t.$$

Again, when $s_t = 0$ then

$$d\lambda_t = 0$$

which implies a constant λ_t, and when $s_t = 1$ then

$$dB_t^{*} = 0$$

which implies a constant B_t^{*}.

2.2.2 Endogenous Probabilities

In order to ensure the regime switching setup maps as closely to the original framework as possible, the transition probabilities are logistic functions. When the constraint is not binding, the probability that it binds the next period depends on the value of its debt relative to the credit limit (**??**):

$$\Pr(s_{t+1} = 1 | s_t = 0) = \frac{\exp(-\gamma_{0,1} B_t^{*})}{1 + \exp(-\gamma_{0,1} B_t^{*})} \tag{24}$$

where $\gamma_{0,0} = \gamma_{0,1} B_{ss}^{*}$.

When the constraint is binding, the probability that it does not bind the next period depends on the slackness multiplier

$$\Pr(s_{t+1} = 0 | s_t = 1) = \frac{\exp(-\gamma_{1,1} \lambda_t)}{1 + \exp(-\gamma_{1,1} \lambda_t)} \tag{25}$$

The transition matrix is then

$$P_t = \begin{bmatrix} p_{00,t} & p_{01,t} \\ p_{10,t} & p_{11,t} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\exp(-\gamma_{0,1} B_t^{*})}{1+\exp(-\gamma_{0,1} B_t^{*})} & \frac{\exp(-\gamma_{0,1} B_t^{*})}{1+\exp(-\gamma_{0,1} B_t^{*})} \\ \frac{\exp(-\gamma_{1,1} \lambda_t)}{1+\exp(-\gamma_{1,1} \lambda_t)} & 1 - \frac{\exp(-\gamma_{1,1} \lambda_t)}{1+\exp(-\gamma_{1,1} \lambda_t)} \end{bmatrix}.$$

Note that as $\gamma_{0,1}, \gamma_{1,1} \to \infty$ the probabilities achieve the threshold behavior of the original constraint.

In addition to the borrowing constraint, several other parameters change at the same time as the regime, which allows for capturing important aspects of crises. In particular, the laws of motion switch intercepts, so

$$\log A_t = a(s_t) + \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t} \tag{26}$$

$$\log P_t = p(s_t) + \rho_P \log P_{t-1} + \sigma_P \varepsilon_{P,t} \tag{27}$$

3 Competitive Equilibrium

The competitive equilibrium is given by first-order conditions for the representative household-firm (5 equations)

$$\left(C_t - \frac{H_t^\omega}{\omega}\right)^{-\rho} = \mu_t$$

$$(1-\alpha-\eta) A_t K_{t-1}^\eta H_t^\alpha V_t^{-\alpha-\eta} = P_t \left(1 + \phi r_t + \frac{\lambda_t}{\mu_t}\phi (1+r_t)\right)$$

$$\alpha A_t K_{t-1}^\eta H_t^{\alpha-1} V_t^{1-\alpha-\eta} = \phi W_t \left(r_t + \frac{\lambda_t}{\mu_t}(1+r_t)\right) + H_t^{\omega-1}$$

$$\mu_t = \lambda_t + \beta(s_t)(1+r_t)\mathbb{E}_t \mu_{t+1}$$

$$\mathbb{E}_t \mu_{t+1} \beta(s_t) \left(\begin{array}{c} 1-\delta + \left(\frac{\iota}{2}\left(\frac{k_{t+1}}{K_t}\right)^2 - \frac{\iota}{2}\right) \\ + \eta A_{t+1} K_t^{\eta-1} H_{t+1}^\alpha V_{t+1}^{1-\eta-\alpha} \end{array}\right) = \mu_t \left(1 - \iota + \iota\left(\frac{K_t}{K_{t-1}}\right)\right) - \lambda_t \kappa q_t$$

market price equations (2 equations)

$$q_t = 1 + \iota\left(\frac{K_t - K_{t-1}}{K_{t-1}}\right)$$

$$W_t = H_t^{\omega-1}$$

budget constraints (2 equations)

$$C_t + I_t = A_t K_{t-1}^\eta H_t^\alpha V_t^{1-\alpha-\eta} - P_t V_t - \phi r_t (W_t H_t + P_t V_t) - \frac{1}{(1+r_t)} B_t + B_{t-1}$$

$$I_t = \delta K_{t-1} + (K_t - K_{t-1})\left(1 + \frac{\iota}{2}\left(\frac{K_t - K_{t-1}}{K_{t-1}}\right)\right)$$

the debt cushion and borrowing limit constraints (2 equations)

$$B_t^* = \frac{1}{(1+r_t)} B_t - \phi(1+r_t)(W_t H_t + P_t V_t) + \kappa q_t K_t$$

$$\varphi(s_t) B_{ss}^* \left(\frac{B_t^*}{B_{ss}^*}\right)^{\gamma(s_t)} = (1-\varphi(s_t)) \lambda_{ss} \left(\frac{\lambda_t}{\lambda_{ss}}\right)^{1-\gamma(s_t)}$$

and laws of motion (3 equations)

$$r_t = r^* + (\psi_r + \sigma_r \varepsilon_{r,t})\left(e^{\overline{B} - B_t} - 1\right) + \sigma_w \varepsilon_{w,t}$$

$$\log A_t = a(s_t) + \rho_A \log A_{t-1} + \sigma_A \varepsilon_{A,t}$$

$$\log P_t = p(s_t) + \rho_P \log P_{t-1} + \sigma_P \varepsilon_{P,t}$$

and an auxiliary equation connecting capital (1 equation)

$$k_t = K_t$$

3.1 Regime Switching Equilibrium

The 15 equilibrium conditions are written as

$$\mathbb{E}_t f\left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \theta_{t+1}, \theta_t\right) = 0$$

where the variables are separated into the predetermined variables $\mathbf{x}_{t-1}$ and the non-predetermined variables $\mathbf{y}_t$. The variables are 4 predetermined variables

$$\mathbf{x}_{t-1} = [K_{t-1}, B_{t-1}, A_{t-1}, P_{t-1}]$$

and 11 non-predetermined variables

$$\mathbf{y}_t = [C_t, H_t, V_t, I_t, k_t, r_t, q_t, W_t, \mu_t, \lambda_t, B_t^*]$$

with 2 shocks

$$\varepsilon_t = [\varepsilon_{r,t}, \varepsilon_{w,t}, \varepsilon_{A,t}, \varepsilon_{P,t}]$$

and 4 switching variables

$$\theta_t = \left[\varphi\left(s_t\right), a\left(s_t\right), p\left(s_t\right), \gamma\left(s_t\right)\right].$$

These variables are partitioned into some that affect the steady state, $\theta_{1,t}$, and some that do not, $\theta_{2,t}$. The partition in this case is

$$\begin{aligned}
\theta_{1,t} &= \left[\varphi\left(s_t\right), a\left(s_t\right), p\left(s_t\right)\right] \\
\theta_{2,t} &= \left[\gamma\left(s_t\right)\right]
\end{aligned}$$

For solving the model, the functional forms are

$$\begin{aligned}
\theta_{1,t+1} &= \bar{\theta}_1 + \chi\hat{\theta}_1\left(s_{t+1}\right) \\
\theta_{1,t} &= \bar{\theta}_1 + \chi\hat{\theta}_1\left(s_t\right)
\end{aligned}$$

$$\begin{aligned}
\theta_{2,t+1} &= \theta_2\left(s_{t+1}\right) \\
\theta_{2,t} &= \theta_2\left(s_t\right)
\end{aligned}$$

$$\mathbf{x}_t = h_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)$$

$$\begin{aligned}
\mathbf{y}_t &= g_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right) \\
\mathbf{y}_{t+1} &= g_{s_{t+1}}\left(\mathbf{x}_t, \chi\varepsilon_{t+1}, \chi\right)
\end{aligned}$$

$$p_{s_t,s_{t+1},t} = \pi_{s_t,s_{t+1}}\left(\mathbf{y}_t\right)$$

Using these in the equilibrium conditions and being more explicit about the expectation operator, given $(\mathbf{x}_{t-1}, \varepsilon_t, \chi)$ and s_t, the

$$\begin{aligned}
F_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right) &= \\
\int \sum_{s'=0}^{1} \pi_{s_t,s'}\left(g_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)\right) f\begin{pmatrix} g_{s_{t+1}}\left(h_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right), \chi\varepsilon', \chi\right), \\ g_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right), \\ h_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right), \\ \mathbf{x}_{t-1}, \chi\varepsilon', \varepsilon_t, \\ \bar{\theta} + \chi\hat{\theta}\left(s'\right), \bar{\theta} + \chi\hat{\theta}\left(s_t\right) \end{pmatrix} d\mu\varepsilon' &= 0
\end{aligned}$$

Stacking these conditions for each regime produces

$$\mathbb{F}\left(\mathbf{x}_{t-1},\varepsilon_t,\chi\right)=\left[\begin{array}{c} F_{s_t=1}\left(\mathbf{x}_{t-1},\varepsilon_t,\chi\right) \\ F_{s_t=2}\left(\mathbf{x}_{t-1},\varepsilon_t,\chi\right)\end{array}\right]$$

3.2 Steady State

The model has switching in parameters that would affect the steady state of the economy in a fixed parameter case. In other words, the switching parameters $\varphi\left(s_t\right)$, $\beta\left(s_t\right)$, $a\left(s_t\right)$, and $p\left(s_t\right)$ all affect the level of the economy directly, and will thus matter for steady state calculations. In steady state, the transition matrix satisfies

$$P_{ss}=\left[\begin{array}{cc} p_{00,ss} & p_{01,ss} \\ p_{10,ss} & p_{11,ss}\end{array}\right]=\left[\begin{array}{cc} 1-\frac{\exp(-\gamma_{0,1}B^*_{ss})}{1+\exp(-\gamma_{0,1}B^*_{ss})} & \frac{\exp(-\gamma_{0,1}B^*_{ss})}{1+\exp(-\gamma_{0,1}B^*_{ss})} \\ \frac{\exp(-\gamma_{1,1}\lambda_{ss})}{1+\exp(-\gamma_{1,1}\lambda_{ss})} & 1-\frac{\exp(-\gamma_{1,1}\lambda_{ss})}{1+\exp(-\gamma_{1,1}\lambda_{ss})}\end{array}\right].$$

Let $\xi=[\xi_0,\xi_1]$ denote the ergodic vector of P_{ss}. Then define the ergodic means of the switching parameters as

$$\bar{\varphi}=\xi_0\varphi\left(0\right)+\xi_1\varphi\left(1\right)$$
$$\bar{\beta}=\xi_0\beta\left(0\right)+\xi_1\beta\left(1\right)$$
$$\bar{a}=\xi_0 a\left(0\right)+\xi_1 a\left(1\right)$$
$$\bar{p}=\xi_0 p\left(0\right)+\xi_1 p\left(1\right)$$

The steady state of the economy depends on these ergodic means, and satisfies the following equations

$$\left(C_{ss}-\frac{H_{ss}^{\omega}}{\omega}\right)^{-\rho}=\mu_{ss}$$

$$\left(1-\alpha-\eta\right)A_{ss}K_{ss}^{\eta}H_{ss}^{\alpha}V_{ss}^{-\alpha-\eta}=P_{ss}\left(1+\phi r_{ss}+\frac{\lambda_{ss}}{\mu_{ss}}\phi\left(1+r_{ss}\right)\right)$$

$$\alpha A_{ss}K_{ss}^{\eta}H_{ss}^{\alpha-1}V_{ss}^{1-\alpha-\eta}=\phi W_{ss}\left(r_{ss}+\frac{\lambda_{ss}}{\mu_{ss}}\left(1+r_{ss}\right)\right)+H_{ss}^{\omega-1}$$

$$\mu_{ss}=\lambda_{ss}+\bar{\beta}\left(1+r_{ss}\right)\mu_{ss}$$

$$\mu_{ss}\bar{\beta}\left(\begin{array}{c} 1-\delta+\left(\frac{\iota}{2}\left(\frac{k_{ss}}{K_{ss}}\right)^2-\frac{\iota}{2}\right) \\ +\eta A_{ss}K_{ss}^{\eta-1}H_{ss}^{\alpha}V_{ss}^{1-\eta-\alpha}\end{array}\right)=\mu_{ss}\left(1-\iota+\iota\left(\frac{K_{ss}}{K_{ss}}\right)\right)-\lambda_{ss}\kappa q_{ss}$$

$$q_{ss}=1+\iota\left(\frac{K_{ss}-K_{ss}}{K_{ss}}\right)$$

$$W_{ss}=H_{ss}^{\omega-1}$$

$$C_{ss}+I_{ss}=A_{ss}K_{ss}^{\eta}H_{ss}^{\alpha}V_{ss}^{1-\alpha-\eta}-P_{ss}V_{ss}-\phi r_{ss}\left(W_{ss}H_{ss}+P_{ss}V_{ss}\right)-\frac{1}{\left(1+r_{ss}\right)}B_{ss}+B_{ss}$$

$$I_{ss}=\delta K_{ss}+\left(K_{ss}-K_{ss}\right)\left(1+\frac{\iota}{2}\left(\frac{K_{ss}-K_{ss}}{K_{ss}}\right)\right)$$

$$B^*_{ss} = \frac{1}{(1+r_{ss})} B_{ss} - \phi\left(1+r_{ss}\right)\left(W_{ss}H_{ss} + P_{ss}V_{ss}\right) + \kappa q_{ss}K_{ss}$$

$$\bar{\varphi} B^*_{ss} = \left(1-\bar{\varphi}\right)\lambda_{ss}$$

$$r_{ss} = r^* + \psi_r\left(e^{\overline{B}-B_{ss}} - 1\right)$$

$$\log A_{ss} = \bar{a} + \rho_A \log A_{ss}$$

$$\log P_{ss} = \bar{p} + \rho_P \log P_{ss}$$

$$k_{ss} = K_{ss}$$

We can partially solve some of these directly

$$A_{ss} = \exp\frac{\bar{a}}{1-\rho_A}$$

$$P_{ss} = \exp\frac{\bar{p}}{1-\rho_P}$$

$$q_{ss} = 1$$

Suppose know r_{ss}

$$\Omega_v = \frac{P_{ss}\left(1+\phi r_{ss} + \left(1-\bar{\beta}\left(1+r_{ss}\right)\right)\phi\left(1+r_{ss}\right)\right)}{(1-\alpha-\eta)}$$

$$\Omega_w = \left(\frac{\phi\left(r_{ss} + \left(1-\bar{\beta}\left(1+r_{ss}\right)\right)\left(1+r_{ss}\right)\right)+1}{\alpha}\right)$$

$$\Omega_k = \frac{\frac{1-\left(1-\bar{\beta}(1+r_{ss})\right)\kappa}{\bar{\beta}} - 1 + \delta}{\eta}$$

$$H_{ss} = \left(\frac{A_{ss}}{\Omega_k^{\eta}\Omega_w^{\alpha}\Omega_v^{1-\alpha-\eta}}\right)^{\frac{1}{\alpha(\omega-1)}}$$

$$V_{ss} = \frac{\Omega_w}{\Omega_v}H_{ss}^{\omega}$$

$$Y_{ss} = \Omega_w H_{ss}^{\omega}$$

$$K_{ss} = \frac{\Omega_w}{\Omega_k}H_{ss}^{\omega}$$

$$W_{ss} = H_{ss}^{\omega-1}$$

$$I_{ss} = \delta K_{ss}$$

$$k_{ss} = K_{ss}$$

$$B_{ss} = \bar{B} - \log\left(1 + \frac{r_{ss}-r^*}{\psi_r}\right)$$

$$C_{ss} = Y_{ss} - (1+\phi r_{ss}) P_{ss} V_{ss} - \phi r_{ss} W_{ss} H_{ss} - \delta K_{ss} + \left(\frac{r_{ss}}{1+r_{ss}}\right) B_{ss}$$

$$\mu_{ss} = \left(C_{ss} - \frac{H_{ss}^{\omega}}{\omega}\right)^{-\rho}$$

$$\lambda_{ss} = \left(1 - \bar{\beta}(1+r_{ss})\right)\mu_{ss}$$

$$B_{ss}^{*} = \frac{1}{(1+r_{ss})} B_{ss} - \phi(1+r_{ss})(W_{ss}H_{ss} + P_{ss}V_{ss}) + \kappa K_{ss}$$

And then r_{ss} solves

$$\bar{\varphi} B_{ss}^{*} - (1-\bar{\varphi})\lambda_{ss} = 0$$

4 Calibration

We start the calibration by taking these parameters from Mendoza (2010).

Parameter	Value
Risk Aversion	$\rho = 2$
Labor Share	$\alpha = 0.592$
Capital Share	$\eta = 0.306$
Wage Elasticity of Labor Supply	$\omega = 1.846$
Capital Depreciation (8.8% Annually)	$\delta = 0.022766$
Capital Adjustment Cost	$\iota = 2.75$
Collateral Constraint Parameter (Weakest Case)	$\kappa = 0.15$
Working Capital Parameter	$\phi = 0.2579$
Persistence of Processes	$\rho_A = \rho_P = 0.9$
Shock Standard Deviations	$\sigma_r = \sigma_w = \sigma_a = \sigma_p = 0.01$

First consider the steady state of the model in the case when only the non-binding regime occurs. We normalize $a(0) = p(0) = 1$. Mendoza targets an annualized real rate of 8.57%. In the regime where the constraint does not bind, the steady state interest rate in this case is $r_{ss} = r^{*} = \frac{1}{\beta} - 1$, and the debt level is $B_{ss} = \bar{B}$. Setting $\beta = 0.97959$ yields $r^{*} = 0.0208352$, which matches the target annualized rate. Mendoza also targets a debt-to-output ratio of -0.86, which requires $\bar{B} = -1.7517$.

$$\bar{B} = \left(\frac{B}{Y}\right)_{ss} \Omega_w^{\frac{1}{1-\omega}} \left(\Omega_k^{\eta}\Omega_v^{1-\alpha-\eta}\right)^{\frac{\omega}{\alpha(1-\omega)}}$$

Now consider the steady state of the model in the case when only the binding regime occurs. In line with Mendoza's estimates on the Mexican sudden stop, we set $a(1) = -0.005$ and $p(1) = 0.005$, which, combined with ρ_a and ρ_p, lead to a roughly 5% decrease in TFP and a 5% increase in import prices. We set the interest rate elasticity $\psi_r = 0.05$, which implies the real rate is increasing in debt.

The following summarizes the rest of the parameterization

Parameter	Non-Binding	Binding
TFP	$a\left(0\right)=0$	$a\left(1\right)=-0.005$
Import Prices	$p\left(0\right)=0$	$p\left(1\right)=0.005$
Discount Factor	$\beta=0.97959$	
Interest Rate Intercept	$r^{*}=0.0208352$	
Interest Rate Elasticity	$\psi_{r}=0.05$	
Neutral Debt Level	$\bar{B}=-1.7517$	

Finally, we compare four variants of the model that differ in the calibration of the probabilities. In order to avoid circularity of finding the steady state, which in turn might depend on the steady state of the transition probabilities, we calibrate the steady state probabilities and back out the associated parameters of the probability function. That is

$$\begin{aligned}\gamma_{0,0} &= \log\left(\frac{1}{p_{00,ss}}-1\right)+\gamma_{0,1}B_{ss}^{*} \\ \gamma_{1,0} &= \log\left(\frac{1}{p_{11,ss}}-1\right)-\gamma_{1,1}\lambda_{ss}\end{aligned}$$

and then we can directly calibrate $p_{00,ss}$ and $p_{11,ss}$.

Model	$p_{00,ss}$	$\gamma_{0,1}$	$p_{11,ss}$	$\gamma_{1,1}$
Endogenous	0.98	1000	0.98	1000
Exogenous	0.98	0	0.98	0
Nonbinding Only	1	0	0	0
Binding Only	0	0	1	0

The following table shows the steady state values for the variables in steady state. Note that these are the deterministic steady states associated with each model.

Variable	Endogenous	Exogenous	Nonbinding Only	Binding Only
K	2.6163	2.6163	2.6599	2.5843
B (level)	-1.7320	-1.7320	-1.7517	-1.6787
P	0.0125	0.0125	0.0000	0.0250
A	-0.0125	-0.0125	0.0000	-0.0250
C	0.3322	0.3322	0.3794	0.2929
H	0.0730	0.0730	0.0985	0.0509
V	-1.6362	-1.6362	-1.5767	-1.6895
I	-1.1662	-1.1662	-1.1226	-1.1982
k	2.6163	2.6163	2.6599	2.5843
r (pp)	0.0199	0.0199	0.0208	0.0173
Q	0.0000	0.0000	0.0000	0.0000
W	0.0618	0.0618	0.0833	0.0431
μ	0.5119	0.5119	0.4171	0.5882
λ (level)	0.0016	0.0016	-0.0000	0.0062
B* (level)	0.0016	0.0016	0.0581	0.0000
Y	0.6644	0.6644	0.7114	0.6236
B/Y (level)	-0.8912	-0.8912	-0.8600	-0.8998

5 Perturbation

For perturbation, we take the stacked equilibrium conditions $\mathbb{F}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)$, and differentiate with respect to $\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)$. In general regime-switching models, the first-order derivative with respect to $\mathbf{x}_{t-1}$ produces a complicated polynomial system denoted

$$\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0.$$

Often this system needs to be solved via Gröbner bases, which finds all possible solutions in order to check them for stability. In our case, all the regime switching parameters show up in the steady state, and we write $\theta_t = \bar{\theta} + \chi \hat{\theta}\left(s_t\right)$ so the steady state can be solved. This is the *Partition Principle* of Foerster, Rubio-Ramirez, Waggoner, and Zha (2015). Given these parameters, the regime switching in $\mathbb{F}_{\mathbf{x}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right)$ disappears and simplifies to the standard no-switching case that can be solved via a generalized eigenvalue procedure.

After solving the eigenvalue problem, the other systems to solve are

$$\begin{aligned} \mathbb{F}_{\varepsilon}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) &= 0 \\ \mathbb{F}_{\chi}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) &= 0 \end{aligned}$$

and second order systems of the form (can apply equality of cross-partials)

$$\mathbb{F}_{\mathbf{i},\mathbf{j}}\left(\mathbf{x}_{ss}, \mathbf{0}, 0\right) = 0,\ \mathbf{i}, \mathbf{j} \in \{\mathbf{x}, \varepsilon, \chi\}.$$

Recall the decision rules have the form

$$\mathbf{x}_t = h_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)$$

$$\mathbf{y}_t = g_{s_t}\left(\mathbf{x}_{t-1}, \varepsilon_t, \chi\right)$$

and so the second-order approximation takes the form

$$\mathbf{x}_t \approx \mathbf{x}_t + H_{s_t}^{(1)} S_t + \frac{1}{2} H_{s_t}^{(2)} \left(S_t \otimes S_t\right)$$

$$\mathbf{y}_t \approx \mathbf{y}_t + G_{s_t}^{(1)} S_t + \frac{1}{2} G_{s_t}^{(2)} \left(S_t \otimes S_t\right)$$

where $S_t = \left[\begin{array}{ccc} (\mathbf{x}_{t-1} - \mathbf{x}_{ss})' & \varepsilon_t' & 1 \end{array}\right]'$.

6 Results

6.1 Stochastic Steady States

This table shows the stochastic steady state conditional on the nonbinding regime.

Variable	Endogenous	Exogenous	Nonbinding Only
K	1.6757	2.6830	2.6605
B (level)	-0.8653	-1.7540	-1.7510
P	0.0000	0.0000	0.0000
A	0.0000	0.0000	0.0000
C	0.0322	0.3893	0.3798
H	-0.1736	0.1051	0.0987
V	-2.0789	-1.5646	-1.5764
I	-2.1068	-1.0995	-1.1220
k	1.6757	2.6830	2.6605
r (pp)	-0.0226	0.0210	0.0208
Q	0.0000	-0.0000	-0.0000
W	-0.1468	0.0889	0.0835
μ	0.8635	0.4011	0.4165
λ (level)	-0.0000	-0.0000	-0.0000
B* (level)	-0.9710	0.1013	0.0600
Y	0.1980	0.7236	0.7117
B/Y (level)	-0.8618	-0.8507	-0.8594

This table shows the steady state conditional on the binding regime.

Variable	Endogenous	Exogenous	Binding Only
K	2.7412	2.5703	2.5845
B (level)	-1.9987	-1.6517	-1.6793
P	0.0250	0.0250	0.0250
A	-0.0250	-0.0250	-0.0250
C	0.3354	0.2888	0.2930
H	0.0973	0.0468	0.0510
V	-1.6039	-1.6972	-1.6894
I	-1.0413	-1.2122	-1.1979
k	2.7412	2.5703	2.5845
r (pp)	0.0331	0.0161	0.0173
Q	-0.0000	0.0000	0.0000
W	0.0823	0.0396	0.0432
μ	0.5721	0.5907	0.5882
λ (level)	-0.0285	0.0095	0.0061
B* (level)	0.0000	0.0000	0.0000
Y	0.7078	0.6161	0.6237
B/Y (level)	-0.9888	-0.8920	-0.9000

7 Estimation of the Endogenous Regime-Switching Model

The advantage of writing the occasionally binding constraint model as an endogenous regime switching model is that we can take the model to the data using likelihood based estimation methods. Our estimation procedure is a Bayesian approach, based on a second-order solution of the model.

7.1 Priors for Financial Crises

A problem that we face with financial crisis models, such as the Mendoza (2010) sudden stop model, is that for any given country the number of periods spent in the constrained state is relatively small, and the number of crisis events is small as well. While this fact is good for the countries in question, it is a problem for the econometrician trying to estimate a model of these events. This contrasts with the existing regime switching literature where you typically switch between long lived regimes. For example, both regimes in hawk/dove monetary policy models typically last a decade or more. In the case of financial crises, with few observations the likelihood function may not be very informative for all of the parameters in the constrained state. To address this problem we build a prior that incorporates information from a panel of countries that have experienced sudden stops. This dramatically increases the information we have on sudden stop episodes. We then combine this prior with the likelihood function from the endogenous regime switching model to form the posterior for a single country.

We build our prior with a preliminary estimation of the model based on a GMM limited-information approach. Andreasen, Fernandez-Villaverde and Rubio-Ramirez (2014) provide analytical solutions for the moments of a model that has been computed to the second (or third) order. Given that computing the solution to the model is rapid with our perturbation approach, the availability of analytic moments means that is is computational feasible to quickly build a prior for the model. We choose moments to exploit the information from a large panel of countries to form our prior for the estimation of our target country. These moments include, means, variances, autocorrelations, unconditional probabilities of a crisis, and potentially higher order moments. Importantly, we calculate these moments by state, so we can develop prior information for parameters specific to each state.

The prior is formed by minimizing the distance between the model-implied moments and the empirical counterparts from the data. To build the prior we start by denoting by $\hat{\Psi}$ a vector of empirical moments from the data. Likewise, denote by $\Psi(\zeta)$ the same vector of moments implied by the MSDSGE model, where ζ contains all the structural parameters to be estimated, including state specific parameters. Our estimate of ζ will be the parameters that minimizes the distance between $\hat{\Psi} - \Psi(\zeta)$, weighted with the usual GMM weighting matrix. The properties and specification of this estimator is by now well known. There is also a literature that concerns itself with the finite sample properties of this type of estimate. Here we are unconcerned if the asymptotics hold (though they should since these are simple moments we calculate.) We are primarily interest in bringing in information outside of the likelihood function. We can increase or decrease the variance of this prior, depending on how influential we want these moments to be. For example, we expect to want to use a stronger prior for the crisis period since there are few observations time series observations for the likelihood function for a single country. While the procedure here is to develop our prior, the results themselves may be interesting, as they represent an estimate of the model that can be thought of as a 'typical' emerging market economy.

For the non-binding regime the regime duration is quite long lived (in Mexico there are 3 short crisis periods). The moments we match then are growth rates in this regime. Since the regime is long lived it is safe to use the moments implied by the model solution for that regime. These are ergodic moments that essentially imply that you always stay in that regime. Construction of these moments are described below. For the crisis regime the economy spends little time in the crisis regime. This means that using unconditional moments from the decision rules in that regime makes little sense. Instead we use conditional moments for the crisis regime. In particular, we look at the implied declined in output, consumption etc. in the first period of the crisis. Construction of these moments for the model are described below.

7.1.1 Variance-Covariance for Growth Rates

Note that via pruning, we can focus on the first-order terms for computing second moments. In this case, assume that $s_t = s$ for all t, and then supress the regime dependency of the coefficient matrices for notational simplicity. Basic analysis of the shocks produces the following first and second moments

$$\mu_\varepsilon = \mathbb{E}\left[\varepsilon_t\right] = 0$$

$$\Sigma_\varepsilon = Var\left(\varepsilon_t\right) = I_{n_\varepsilon}$$

$$\mu_{\Delta\varepsilon} = \mathbb{E}\left[\Delta\varepsilon_t\right] = 0$$

$$\begin{aligned} \Sigma_{\Delta\varepsilon} &= Var\left(\Delta\varepsilon_t\right) \\ &= Var\left(\varepsilon_t\right) + Var\left(\varepsilon_{t-1}\right) - 2Cov\left(\varepsilon_t, \varepsilon_{t-1}\right) \\ &= 2I_{n_e} \end{aligned}$$

Now for the predetermined variables x_t, the

$$\begin{aligned} \mu_x &= \mathbb{E}\left[\hat{x}_t\right] = h_x\left(\mathbb{E}\left[\hat{x}_{t-1}\right]\right) + h_\chi \\ &= \left(I - h_x\right)^{-1} h_\chi \end{aligned}$$

Deriving the variance-covariance matrix of x

$$\begin{aligned} \Sigma_x &= Var\left(x_t\right) \\ &= \mathbb{E}\left[\hat{x}_t\hat{x}_t'\right] - \mu_x\mu_x' \\ &= h_x\mathbb{E}\left[\hat{x}_{t-1}\hat{x}_{t-1}'\right] h_x' - h_x\mu_x\mu_x'h_x' + h_\varepsilon h_\varepsilon' \\ &= h_x\Sigma_x h_x' + h_\varepsilon h_\varepsilon' \end{aligned}$$

The properties of the vec operator include $vec\left(ABC\right) = \left(C' \otimes A\right) vec\left(B\right)$ which imply

$$vec\left(\Sigma_x\right) = \left(h_x \otimes h_x\right) vec\left(\Sigma_x\right) + vec\left(h_\varepsilon h_\varepsilon'\right)$$

so the variance-covariance matrix of x_t is given by

$$vec\left(\Sigma_x\right) = \left[I_{n_x^2} - \left(h_x \otimes h_x\right)\right]^{-1} vec\left(h_\varepsilon h_\varepsilon'\right)$$

Now to derive the autocovariance matrix $\Sigma_{x_t,x_{t-1}}$

$$\begin{aligned} \Sigma_{x_t,x_{t-1}} &= Cov\left(x_t, x_{t-1}\right) \\ &= \mathbb{E}\left[x_t x_{t-1}'\right] - \mu_x\mu_x' \\ &= \mathbb{E}\left[h_x\hat{x}_{t-1}x_{t-1}' + h_\chi x_{t-1}'\right] - \mathbb{E}\left[h_x\hat{x}_{t-1} + h_\chi\right]\mathbb{E}\left[x_{t-1}'\right] \\ &= h_x\mathbb{E}\left[\hat{x}_{t-1}x_{t-1}'\right] - h_x\mu_x\mu_x' \\ &= h_x\Sigma_x \end{aligned}$$

and the autocovariance matrix $\Sigma_{x_t,x_{t-2}}$

$$\begin{aligned} \Sigma_{x_t,x_{t-2}} &= Cov\left(x_t, x_{t-2}\right) \\ &= \mathbb{E}\left[x_t x_{t-2}'\right] - \mu_x\mu_x' \\ &= \mathbb{E}\left[h_x\hat{x}_{t-1}x_{t-2}' + h_\chi x_{t-2}'\right] - \mathbb{E}\left[h_x\hat{x}_{t-1} + h_\chi\right]\mathbb{E}\left[x_{t-2}'\right] \\ &= h_x\mathbb{E}\left[\hat{x}_{t-1}x_{t-2}'\right] + h_\chi\mathbb{E}\left[x_{t-2}'\right] - \left(h_x\mathbb{E}\left[\hat{x}_{t-1}\right]\mathbb{E}\left[x_{t-2}'\right] + h_\chi\mathbb{E}\left[x_{t-2}'\right]\right) \\ &= h_x\mathbb{E}\left[\hat{x}_{t-1}x_{t-2}'\right] - h_x\mu_x\mu_x' \\ &= h_x\Sigma_{x_t,x_{t-1}} \\ &= h_x h_x\Sigma_x \end{aligned}$$

Now the growth of x_t has the moments

$$\mu_{\Delta x} = \mathbb{E}\left[\Delta x_t\right] = 0$$

$$\begin{aligned}
\Sigma_{\Delta x} &= Var\left(\Delta x_t\right) \\
&= Var\left(\Delta x_t\right) + Var\left(\Delta x_{t-1}\right) - 2Cov\left(x_t, x_{t-1}\right) \\
&= 2\Sigma_x - 2\Sigma_{x_t,x_{t-1}} \\
&= 2\Sigma_x - 2h_x\Sigma_x
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\Delta x_t,\Delta x_{t-1}} &= Cov\left(\Delta x_t, \Delta x_{t-1}\right) \\
&= Cov\left(x_t, x_{t-1}\right) - Cov\left(x_t, x_{t-2}\right) - Cov\left(x_{t-1}, x_{t-1}\right) + Cov\left(x_{t-1}, x_{t-2}\right) \\
&= \Sigma_{x_t,x_{t-1}} - \Sigma_{x_t,x_{t-2}} - \Sigma_x + \Sigma_{x_{t-1},x_{t-2}} \\
&= 2h_x\Sigma_x - h_xh_x\Sigma_x - \Sigma_x
\end{aligned}$$

Now turning to y_t, the mean is

$$\begin{aligned}
\mu_y &= \mathbb{E}\left[\hat{y}_t\right] = g_x\mathbb{E}\left[\hat{x}_{t-1}\right] + g_\chi \\
&= g_x\mu_x + g_\chi
\end{aligned}$$

the variance-covariance is

$$\begin{aligned}
\Sigma_y &= Var\left(y_t\right) \\
&= \mathbb{E}\left[\hat{y}_t\hat{y}_t'\right] - \mu_y\mu_y' \\
&= g_x\mathbb{E}\left[\hat{x}_{t-1}\hat{x}_{t-1}'\right]g_x' - g_x\mu_x\mu_x'g_x' + g_\varepsilon g_\varepsilon' \\
&= g_x\nprec_x g_x' + g_\varepsilon g_\varepsilon'
\end{aligned}$$

and the autocovariance between y_t and y_{t-1} is

$$\begin{aligned}
\Sigma_{y,y_{t-1}} &= Cov\left(y_t, y_{t-1}\right) \\
&= \mathbb{E}\left[y_ty_{t-1}'\right] - \mu_y\mu_y' \\
&= g_x\mathbb{E}\left[\hat{x}_{t-1}\hat{x}_{t-2}'\right]g_x' - g_x\mu_x\mu_x'g_x' + g_\varepsilon g_\varepsilon' \\
&= g_x\left(\Sigma_{x_t,x_{t-1}}\right)g_x' + g_\varepsilon g_\varepsilon'
\end{aligned}$$

and between y_t and y_{t-2} is

$$\begin{aligned}
\Sigma_{y,y_{t-2}} &= Cov\left(y_t, y_{t-2}\right) \\
&= \mathbb{E}\left[y_ty_{t-2}'\right] - \mu_y\mu_y' \\
&= g_x\mathbb{E}\left[x_{t-1}x_{t-3}'\right]g_x' - g_x\mu_x\mu_x'g_x' \\
&= g_x\Sigma_{x_t,x_{t-2}}g_x'
\end{aligned}$$

The growth of y_t has variance-covariance matrix

$$\begin{aligned}
\Sigma_{\Delta y} &= Var\left(\Delta y_t\right) \\
&= Var\left(y_t\right) + Var\left(y_{t-1}\right) - 2Cov\left(y_t, y_{t-1}\right) \\
&= 2\left(g_x\nprec_x g_x' + g_\varepsilon g_\varepsilon'\right) - 2\left(g_x\left(\Sigma_{x_t,x_{t-1}}\right)g_x' + g_\varepsilon g_\varepsilon'\right) \\
&= g_x\left(2\nprec_x - 2\Sigma_{x_t,x_{t-1}}\right)g_x' \\
&= g_x\left(\Sigma_{\Delta x}\right)g_x'
\end{aligned}$$

and auto-covariance matrix

$$
\begin{aligned}
\Sigma_{\Delta y_t,\Delta y_{t-1}} &= Cov\left(\Delta y_t, \Delta y_{t-1}\right) \\
&= Cov\left(y_t, y_{t-1}\right) - Cov\left(y_t, y_{t-2}\right) - Cov\left(y_{t-1}, y_{t-1}\right) + Cov\left(y_{t-1}, x_{t-2}\right) \\
&= \Sigma_{y_t,y_{t-1}} - \Sigma_{y_t,y_{t-2}} - \Sigma_y + \Sigma_{y_{t-1},y_{t-2}} \\
&= 2g_x\left(\Sigma_{x_t,x_{t-1}}\right)g_x' - g_x\Sigma_{x_t,x_{t-2}}g_x' - g_x\not\prec_x g_x' + g_\varepsilon g_\varepsilon' \\
&= g_x\left(2h_x\Sigma_x - h_xh_x\Sigma_x - \not\prec_x\right)g_x' + g_\varepsilon g_\varepsilon'
\end{aligned}
$$

7.1.2 Means Conditional on First Period of Crisis

First, note that pruning implies a second order is needed for means. The first- and second-order contributions are

$$\hat{x}_t^f = H_{s_t}^{(1)}\left[\begin{array}{ccc} \hat{x}_{t-1}^{f\prime} & 0 & 1 \end{array}\right]'$$

$$\hat{x}_t^s = H_{s_t}^{(1)}\left[\begin{array}{ccc} \hat{x}_{t-1}^{s\prime} & 0 & 0 \end{array}\right]' + H_{s_t}^{(2)}\left(\left[\begin{array}{ccc} \hat{x}_{t-1}^{f\prime} & 0 & 1 \end{array}\right]' \otimes \left[\begin{array}{ccc} \hat{x}_{t-1}^{f\prime} & 0 & 1 \end{array}\right]'\right)$$

which imply a total effect of

$$x_t = x_{ss} + \hat{x}_t^f + \hat{x}_t^s.$$

So the stochastic steady state associated with s_t is given by

$$\bar{x}_{s_t}^f = \left(I_{n_x} - h_x\left(s_t\right)\right)^{-1} h_\chi\left(s_t\right)$$

$$\bar{x}_{s_t}^s = \left(I_{n_x} - h_x\left(s_t\right)\right)^{-1} H_{s_t}^{(2)}\left(\left[\begin{array}{ccc} \bar{x}_{t-1}^{f\prime} & 0 & 1 \end{array}\right]' \otimes \left[\begin{array}{ccc} \bar{x}_{t-1}^{f\prime} & 0 & 1 \end{array}\right]'\right)$$

$$\bar{x}_{s_t} = x_{ss} + \bar{x}_{s_t}^f + \bar{x}_{s_t}^s$$

Starting from the stochastic steady state of s_{t-1}, if the realization of the regime is s_t, then

$$\hat{x}_t^f = H_{s_t}^{(1)}\left[\begin{array}{ccc} \bar{x}_{s_{t-1}}^f & 0 & 1 \end{array}\right]'$$

$$\hat{x}_t^s = H_{s_t}^{(1)}\left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{s\prime} & 0 & 0 \end{array}\right]' + H_{s_t}^{(2)}\left(\left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{f\prime} & 0 & 1 \end{array}\right]' \otimes \left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{f\prime} & 0 & 1 \end{array}\right]'\right)$$

$$x_t = x_{ss} + \hat{x}_t^f + \hat{x}_t^s.$$

Similarly, for the non-predetermined variables

$$\hat{y}_t^f = G_{s_t}^{(1)}\left[\begin{array}{ccc} \bar{x}_{s_{t-1}}^f & 0 & 1 \end{array}\right]'$$

$$\hat{y}_t^s = G_{s_t}^{(1)}\left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{s\prime} & 0 & 0 \end{array}\right]' + G_{s_t}^{(2)}\left(\left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{f\prime} & 0 & 1 \end{array}\right]' \otimes \left[\begin{array}{ccc} \hat{x}_{s_{t-1}}^{f\prime} & 0 & 1 \end{array}\right]'\right)$$

$$y_t = y_{ss} + \hat{y}_t^f + \hat{y}_t^s.$$

7.2 State Space Representation

For likelihood estimation, the state space representation is

$$\mathcal{X}_t = \mathcal{H}_{s_t}(\mathcal{X}_{t-1}, \epsilon_t)$$

$$\mathcal{Y}_t = \mathcal{G}_{s_t}(\mathcal{X}_t, \mathcal{U}_t)$$

The observables are

$$\mathcal{Y}_t = \begin{bmatrix} \Delta y_t & \Delta c_t & \Delta i_t & r_t \end{bmatrix}'$$

Given s_t and ε_t, We can construct a first order approximation to $\Delta \mathbf{y}_t$ by

$$\begin{aligned} \Delta \mathbf{y}_t &= \mathbf{y}_t - \mathbf{y}_{t-1} \\ &= G^{(1)}_{s_t} \begin{bmatrix} \hat{\mathbf{x}}'_{t-1} & \varepsilon_t & 1 \end{bmatrix}' - \mathbf{y}_{t-1} \end{aligned}$$

and the first order approximation to $\mathbf{x}_t$ is

$$\mathbf{x}_t = \mathbf{x}_{ss} + H^{(1)}_{s_t} \begin{bmatrix} \hat{\mathbf{x}}'_{t-1} & \varepsilon_t & 1 \end{bmatrix}'$$

Therefore, the state equation is

$$\mathcal{X}_t = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{ss} + H^{(1)}_{s_t} \begin{bmatrix} \hat{\mathbf{x}}'_{t-1} & \varepsilon_t & 1 \end{bmatrix}' \\ \mathbf{y}_{ss} + G^{(1)}_{s_t} \begin{bmatrix} \hat{\mathbf{x}}'_{t-1} & \varepsilon_t & 1 \end{bmatrix}' \\ G^{(1)}_{s_t} \begin{bmatrix} \hat{\mathbf{x}}'_{t-1} & \varepsilon_t & 1 \end{bmatrix}' - \mathbf{y}_{t-1} \end{bmatrix}$$

and the observation equation is

$$\mathcal{Y}_t = \begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{bmatrix} = D \begin{bmatrix} \hat{\mathbf{x}}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{bmatrix} + \mathcal{U}_t$$

where D denotes a selection matrix.

In matrix form, the above are

$$\begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \\ \Delta \mathbf{y}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{ss} + H^{(1)}_{\chi,s_t} \\ \mathbf{y}_{ss} + G^{(1)}_{\chi,s_t} \\ G^{(1)}_{\chi,s_t} \end{bmatrix} + \begin{bmatrix} H^{(1)}_{x,s_t} & 0 & 0 \\ G^{(1)}_{x,s_t} & 0 & 0 \\ G^{(1)}_{x,s_t} & -I & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}_{t-1} \\ \mathbf{y}_{t-1} \\ \Delta \mathbf{y}_{t-1} \end{bmatrix} + \begin{bmatrix} H^{(1)}_{\varepsilon,s_t} \\ G^{(1)}_{\varepsilon,s_t} \\ G^{(1)}_{\varepsilon,s_t} \end{bmatrix} \varepsilon_t$$

and

$$\begin{bmatrix} \Delta y_t \\ \Delta c_t \\ \Delta i_t \\ r_t \end{bmatrix} = S \Delta \mathbf{y}_t + \mathcal{U}_t$$

which can be denoted as

$$\mathcal{X}_t = A_{s_t} + B_{s_t} \mathcal{X}_{t-1} + C_{s_t} \varepsilon_t$$

$$\mathcal{Y}_t = D \mathcal{X}_t + E \mathcal{U}_t$$