A Tractable Circular City Model with Endogenous Internal Structure[1]

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Abstract

A production-externality-based circular city model in which both firms and workers choose location as well as intensity of land use is presented. Depending on parameter values, the equilibrium structure of the city is either decentralized (with employment spread out over the city) or monocentric (with all employment concentrated within a central business district of positive radius). Regardless of which form prevails, the intra-city variation in all endogenous variables are negative exponentials. This feature implies a unique spatial equilibrium given city size. The tractability of the model leads to easily interpretable conditions under which it displays an inverted-U relationship between welfare and city size.

Keywords: Production externalities, land use, density gradients, commuting costs
JEL Codes: R30, R52

1 Introduction

We present a model of a production-externality-based circular city in which both firms and workers choose location as well as intensity of land use. In our model, the internal structure of the city is endogenous and can take exactly one of two forms. Either the city is decentralized with workers residing at the same location as the firm they work for or it is monocentric with a circular central business district (CBD) and a surrounding residential ring. Regardless of which form prevails, the intra-city variation in all endogenous variables – residential and commercial rents, employment and residential densities, and wages – display (over their relevant domains) the negative exponential form: $x(r) = x(0)e^{-\phi_x r}$, where r is distance from the city center (which is indexed by 0) and ϕ_x depends only on preference and technology parameters.

Our model is related to but distinct from the linear city model of Fujita and Ogawa (1982) and the circular city model of Lucas and Rossi-Hansberg (2002). In these earlier studies, proximity is taken to mean Euclidean distance between two points (in one or two dimensions).[1] In our model, proximity between two points is defined in relation to the city center and is measured as the sum of the lengths of the rays connecting the two points to the city center.[2] Coupled with standard assumptions on preferences and technology, this choice ensures a unique spatial equilibrium for a city of any given size, which facilitates comparative static analysis. In contrast, uniqueness of within-city spatial equilibrium is not assured in these earlier studies.

We also show that our distance concept is the only distance concept that implies exponentially declining density and rent gradients for our setting. In addition to making our model tractable, this feature makes our model empirically useful. Beginning with Clark (1951) and Mills (1969), many researchers have used the negative exponential specification in em-

[1]While any production-externality based city model must take a stand on how proximity to other firms enhances firm productivity, there is no consensus on the measure of proximity that is relevant for agglomeration economies.

[2]Thus our model takes as given that there is one city center which makes it different from Fujita and Ogawa (1982) who are mainly interested in understanding the emergence of multiple city centers.

pirical studies.[3] By providing a structural interpretation to the coefficient on distance (in the log-linear form), our model sheds light on the determinants of population, employment and land price variation within a city and why these patterns change over time. Along this dimension, our model generalizes Anas, Arnott, and Small (2000) notion of a *panexponential* city model, which is defined to be a city in which rent and density gradients are negative exponentials. While Anas, Arnott and Small offered their definition in the context of the classic monocentric city model (where all employment, by assumption, occurs at the city center), panexponentiality arises in our model for the more general setting where firms and workers compete for land at each point in the city.

Finally, the paper sharpens our understanding of the equilibrium relationship between the utility deliverable by a city and its population. In the standard monocentric city model, higher population leads to lower utility (see for instance, Brueckner (1987)) because there is no explicit benefit for firms to locate together at the center of the city. In contrast, studies that focus on the creation of cities imply that the relationship between utility and population is an inverted-U (Henderson (1974)). These models posit some benefit from agglomeration but assume that firms must locate at the city center and, typically, also assume that intensity of land use by households is exogenously given (see, for instance, the survey by Abdel-Rahman and Anas (2004)). Thus an important question left open in the existing literature is the conditions under which the inverted-U result is maintained when intra-city land use is analyzed in more detail.[4] We will show that for the inverted-U to emerge, the agglomeration parameter γ must be bounded above by a quantity that depends on parameters that govern the intensity of land use by businesses *and* households.

The paper is organized as follows. Section 2 describes the environment. Sections 3 shows that the internal structure of the city can take one of two forms. Section 4 develops the

[3]The first of these pioneering studies established that the negative exponential form gave a remarkably good description of population density in urban areas and the second showed that the same was true for land values in Chicago during most of the 19th century. Eden and Sclar (1975) and Atack and Margo (1998) establish similar patterns in historical land values for Boston and New York City, respectively.

[4]This question was not addressed in Lucas and Rossi-Hansberg (2002), although the structure of their model is certainly suited to answering it.

equilibrium implications for either type of city and gives comparative statics results that reveal how the model works. Section 5 gives conditions under which the utility delivered by the city is an inverted U.

2 Environment

Space is modeled as a flat plain extending infinitely in all directions, with a point marked off as the city center. Given that each point other than the center is physically indistinguishable from any other, we focus on allocations that are symmetric relative to the center. A location is then described fully by its distance r from the center.

Utility function of a worker depends on the consumption of the single *numeraire* good avaliable in this economy and on the service flow from land. A worker who resides in location r has utility

$$U = c^{\beta}(r)l(r)^{1-\beta}, \;\; \beta \in (0,1), \tag{1}$$

where $l(r)$ is the consumption of land in location r and $c(r)$ is consumption at location r.

A firm has a technology to produce the single consumption good. The production function of a firm that uses one unit of land at location s is

$$Y(s) = Az(s)^{\gamma}n^{\alpha}(s), \;\; \alpha \in (0,1), \gamma > 0, \tag{2}$$

where $n(s)$ is the number of workers per unit of land at location s, A is a TFP term that is common to all firms in the city, and $z(s)$ is a variable—defined more precisely below—that captures the how many other workers are in close proximity to the firm.

A key assumption is that the proximity between any two firms is measured by the sum of the distance of the two firms from the city center. In other words, if one firm is located on a circle of radius r and the other firm is located on a circle of radius s, the distance of the firms to each other is simply $(r+s)$. The assumption that distance between two firms is measured by the sum of the lengths to the city center is reasonable if communication between workers in different firms requires travel to a central meeting place and the road system is radial. A second justification of this assumption is given below.

If we let $N(s)$ denote the number of workers employed by a firm at location s, the level of the production externality enjoyed by a firm at location r is

$$z(r) = \int_0^\infty 2\pi s \exp\left(-\delta\left(r+s\right)\right) N(s) ds.$$

Since $z(0) = \int_0^\infty 2\pi s \exp\left(-\delta s\right) N(s) ds$, the above definition implies

$$z(r) = z(0) \exp\left(-\delta r\right). \tag{3}$$

Thus, irrespective of the distribution of employment across the city, the level of the production externality decays at the rate δ with distance from the city center. The spatial distribution of employment affects the level of the production externality at any location only through the $z(0)$ term. As will become evident, (3) is the reason why our model predicts that all density and price gradients follow exponential functions (and it is also the reason why our model is tractable).

Given the importance of (3), we might ask, what other distance measures generate (3)? If we denote the general distance function as $\nu(r,s)$ and require that $\nu(r,s) = \nu(s,r)$ (symmetry), then it is straightforward to show that any symmetric distance function that generates (3) must be a linear transform of $r+s$. To see this, observe that for the general distance function $z(r) = \int_0^S \exp(-\nu(r,s))N(s)ds$. We require that $z(r) = z(0)\exp(-\delta r)$, where δ is some positive constant. Then

$$z(0) = \int_0^S \exp(-\nu(r,s) + \delta r) N(s) ds. \tag{4}$$

Since this relationship must hold for any r, it follows that $\nu(r,s)$ must be of the form

$a + \delta r + f(s)$. From symmetry $a + \delta r + f(s) = a + f(r) + \delta s$, which in turn implies $f(s) - f(r) = \delta \cdot (s - r)$. Hence $\nu(r, s) = A + \delta \cdot (r + s)$. Thus a second justification for our distance measure is that it is the only (symmetric) measure that is consistent with (3) and, therefore, with exponential density and price gradients.

There is a technology for commuting. This technology allows workers to commute to any firm that is located on the straight line that connects the worker's residential location to the city center. We follow Anas, Arnott, and Small (2000) and Lucas and Rossi-Hansberg (2002) and assume that a worker who resides in location s and commutes to a firm at location r has $\exp(-\kappa|s - r|)$ unit of time to devote to production, where $\kappa > 0$.[5]

There is also a technology for converting land from its natural state into land that can be used by workers and firms. The cost of converting a unit of natural land into developed land is d units of the consumption good.

Finally, following convention, it is assumed that all land in the economy is owned by entities outside of the model. These entities decide whether to convert any given unit of natural land into developed land and then rent the developed land to workers and firms.

3 On the Internal Structure of the City

In this section, we show that the spatial organization of the city is consistent with only one of two forms, depending on technology and preference parameters. Either the city is *decentralized* with workers choosing to live next to their firms, or it is *monocentric* with a CBD of positive radius and a surrounding residential ring. This result can be rigorously established by applying the method described in Lucas and Rossi-Hansberg (2002) (section

[5]As noted in Anas, Arnott, and Small (2000), this assumption is key to obtaining an exponentially declining land rent and population density function without making counterfactual assumptions on the structure of preferences for land. Coupled with our assumption regarding how proximity between firms is calculated, we can extend the negative exponential form to commercial rents as well as employment density. Note also that, to a first-order approximation, the (net) income of a commuter is $w(r)[1 - \kappa|s - r|]$, which corresponds to the common assumption that the commuting cost is proportional to the hourly wage and linear in the distance traveled.

3, pp. 1453-1462) to the case where $z(r)$ is of the form in (3), namely, $z(0)\exp(-\delta r)$. In the interest of brevity, what we do here is simply derive conditions on parameter values under which the two forms can prevail and show that these conditions are mutually exclusive and exhaust the parameter space.

We will assume that the set of developed locations comprises all points on and inside of a circle of radius S (this circle defines the city boundary). The question we want to answer is, how is this developed land allocated between commercial and/or residential use? It is customary in urban economic theory to approach land use in terms of bid rent functions (Alonso (1964) and Fujita (1989)). Let $w(r)$ be the market wage at location r. Turning first to firms, we let $q_F(r)$ be the maximum rent a firm would be willing to pay for a unit of land at location r. This quantity is simply $Az(r)^\gamma n^*(r)^\alpha - w(r)n^*(r)$, where $n^*(r)$ is the optimal choice of n conditional on locating at r, and is given by

$$n^*(r) = \left[A\alpha z(r)^\gamma / w(r)\right]^{1/(1-\alpha)}. \tag{5}$$

Then,

$$q_F(r) = [(1-\alpha)/\alpha]\left[\alpha A z(r)^\gamma w(r)^{-\alpha}\right]^{1/(1-\alpha)}. \tag{6}$$

As is intuitive, the maximum rent a firm is willing to pay depends positively on the location's productivity and negatively on the location's wage.

Turning to households, we let $q_H(r,s)$ be the maximum rent a worker would be willing to pay for a unit of land at location r, given that he will work at location s. Conditional on paying $q_H(r,s)$ in rent, a worker's optimal choices of c and l at location r are

$$c^*(r,s) = \beta w(s)\exp\left(\kappa|s-r|\right) \text{ and } l^*(r,s) = (1-\beta)w(s)\exp\left(\kappa|s-r|\right)/q_H(r,s), \tag{7}$$

and his optimal utility is $\beta^\beta(1-\beta)^{1-\beta}w(s)\exp(\kappa|s-r|)q_H(r,s)^{-(1-\beta)}$. If U is the maximum utility a worker can obtain from locating somewhere else, then

$$q_H(r,s) = (1-\beta)\,\beta^{\beta/(1-\beta)}(w(s)\exp(\kappa|s-r|)/U)^{1/(1-\beta)}. \tag{8}$$

As is intuitive, the maximum rent a worker is willing to pay for land at r depends positively on the wage he earns and negatively on the utility he can get elsewhere.

Consider first the decentralized city where firms and their workers co-locate. In this case, the bid rent functions $q_F(r)$ and $q_H(r,r)$ must coincide for all $r \in [0,S]$. Setting s equal to r in the bid rent function for households, setting the resulting bid rent function equal to the bid rent function for firms, and using the expression in (3) for $z(r)$ implies

$$w(r) = w(0)\exp\left(-\frac{\delta\gamma(1-\beta)}{1-\alpha\beta}r\right) \text{ for } r \in [0,S]. \tag{9}$$

Thus wages decline exponentially from the city center, reflecting the fact that the production externality is felt most strongly at the center. However, for this wage profile to be an equilibrium, it must be the case that workers do not have an incentive to commute to a job closer to the city center to take advantage of higher wages. This requires that the rise in wages as a worker commutes toward the center not exceed the loss in working time due to commuting, namely,

$$\frac{\delta\gamma(1-\beta)}{1-\alpha\beta} \leq \kappa. \tag{10}$$

If commuting costs are high (κ is large), if the production externality is weak (γ is small), and if communication between workers in different locations is not too difficult (δ is low), then decentralized urban form can be sustained in equilibrium. When (9) holds, (8) implies that

$$q(r) = q(0)\exp\left(-\frac{\delta\gamma}{1-\alpha\beta}r\right) \text{ for } r \in [0,S]. \tag{11}$$

We now turn to the case in which the city has a monocentric structure. In this case, there is an endogenously determined boundary $S_F < S$ such that all $r \in [0, S_F)$ are devoted to production and all $s \in (S_F, S]$ are devoted to residential use. The boundary S_F can be devoted to either use. If there is a CBD, workers must be indifferent between working at different locations within this district. This implies that in the business district the wages must satisfy the condition

$$w(r) = w(0) \exp(-\kappa r) \text{ for } r \in [0, S_F]. \tag{12}$$

Substituting this into the expression for $n^*(r)$ and using the expression for $z(r)$ in (3) yields

$$q_F(r) = q_F(0) \exp\left(-\frac{\delta\gamma - \kappa\alpha}{1-\alpha} r\right) \text{ for } r \in [0, S_F), \tag{13}$$

which is declining in r provided $\delta\gamma - \kappa\alpha > 0$.

Given that workers earn the same regardless of where they work, we do not need to know their place of work in order to determine their bid rent for a particular location in the city. It is convenient, however, to imagine that the place of work is in the city center. Then, the maximum rent a worker is willing to pay for land at location $r \in [0, S]$ and still get a utility of U is

$$q_H(r) = (1-\beta)\, \beta^{\frac{\beta}{1-\beta}} \left(\frac{w(0)\exp(-\kappa r)}{U}\right)^{\frac{1}{1-\beta}} = q_H(0) \exp\left(-\frac{\kappa}{1-\beta}\right) \text{ for } r \in [0, S]. \tag{14}$$

For the monocentric structure to be an equilibrium outcome, the two bid rents must be the same at the boundary of the CBD, and the slope of the firm's bid rent function must be steeper than the slope of the worker's bid rent function. These requirements impose a constraint on the admissible value of κ. Observe that the slope of the worker's bid rent function at S_F is $[-\kappa/(1-\beta)]q_H(S_F)$ and the slope of the firm's bid rent function at S_F is $[(\kappa\alpha - \delta\gamma)/(1-\alpha)]q_F(S_F)$. Since at the boundary of the CBD $q_H(S_F) = q_F(S_F)$, the necessary slope condition boils down to $-(\kappa\alpha - \delta\gamma)/(1-\alpha) > \kappa/(1-\beta)$. This implies that

$$\kappa < \frac{(1-\beta)\gamma\delta}{(1-\beta\alpha)}, \tag{15}$$

which is the exact complement of the condition (10). Since both α and β are less than unity, (15) implies that $\gamma\delta > \alpha\kappa$. Therefore, when (15) holds, the firm's bid rent function is downward sloping, as assumed. This shows that the internal structure of the city can be only one of these two types. For completeness, we note the CBD analog of equation (11):

$$q(r) = \begin{cases} q_F(0)\exp\left(-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}r\right) & \text{for } r \in [0, S_F) \\ q_H(0)\exp\left(-\frac{\kappa}{1-\beta}r\right) & \text{for } r \in [S_F, S]. \end{cases} \tag{16}$$

4 Equilibrium

The goal of this section is to show how the equilibrium of the model is determined. We will approach this discussion in terms of a *closed city*, wherein the city's population, P, is taken as given and the equilibrium determines the city's geographic size, S, and the utility it can deliver to its residents, U. We will develop the equilibrium conditions for the decentralized and the monocentric cities in parallel since the arguments are (for the most part) very similar.

The determination of equilibrium can be broken down into two parts. In the first part, P and S are taken as given and the equilibrium employment and residential density functions along with the equilibrium wage and rent functions are uniquely determined as functions of P and S. In the second part, S as well as U is uniquely determined as functions of P.

The task of determining the various equilibrium functions is made very simple by the fact that all these functions are negative exponentials, where the only unknowns are the values of these functions at $r = 0$ (the city center). Furthermore, these unknown values are all determined once $n(0)$ and $z(0)$ are determined. To see this, note that, in either type of city,

$w(0)$ is simply the marginal product of labor at the city center. Therefore

$$w(0) = \alpha A z(0)^{\gamma} n(0)^{\alpha-1}. \tag{17}$$

And, in any type of city, there are businesses operating at the city center and therefore $q(0)$ must be output at 0 minus the wage bill at 0 (since all "surplus" must go to the owners of land). Therefore $q(0) = Az(0)^{\gamma} n(0)^{\alpha} - w(0)n(0)$. This implies

$$q(0) = [(1-\alpha)/\alpha] w(0)n(0) = (1-\alpha) A z(0)^{\gamma} n(0)^{\alpha}. \tag{18}$$

For the decentralized city, $q(0)$ is the only unknown for the rent function since the bid rent functions for businesses and workers coincide. For the monocentric city, (18) determines $q_F(0)$. To determine $q_H(0)$, we use the fact that the bid rents for businesses and workers are the same at S_F, which implies $q_F(0)\exp(-[\delta\gamma - \kappa\alpha]/[1-\alpha]S_F) = q_H(0)\exp(-[\kappa/(1-\beta)]S_F)$. Therefore

$$q_H(0) = q_F(0) \exp\left(\frac{\kappa - \delta\gamma + \beta\delta\gamma - \beta\kappa\gamma}{(1-\alpha)(1-\beta)} S_F\right). \tag{19}$$

While this equation depends on S_F, we will show below that S_F is, in fact, pinned down by S alone (recall that we are taking both P and S as parametrically given in this part). Therefore, the first part of the equilibrium problem boils down to simply determining $n(0)$ and $z(0)$.

To proceed, we observe that the expression for $n^*(r)$, along with the expressions for $w(r)$ in (9) and $z(r)$ in (3), gives the following employment density equation for the decentralized city:

$$n(r) = n(0) \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) \quad \text{for } r \in [0, S] \tag{20}$$

and, using (12), gives the following employment density equation for the monocentric city:

$$n(r) = n(0)\exp\left(-\frac{\delta\gamma-\kappa}{1-\alpha}r\right) \text{ for } r \in [0, S_F]. \tag{21}$$

For either type of city, the values of $n(0)$ and $z(0)$ are determined by invoking two market-clearing conditions. First, there is the labor-market-clearing condition. For the decentralized city, since a firm and its workers co-locate, each location is a labor market in which demand and supply for labor have to match. Letting $\theta(r)$ denote the fraction of land devoted to production in location r, we can express this location-by-location labor market balance requirement as

$$n^*(r)\theta(r) = [1-\theta(r)]/l^*(r). \tag{22}$$

Since $n^*(r) = [\alpha q(r)]/[(1-\alpha)w(r)]$ and $l^*(r) = (1-\beta)w(r)/q(r)$, we find that $\theta(r) = [1-\beta]/[1-\alpha\beta] = \theta$.[6] Thus, the proportion of land devoted to production is constant across all locations in the city, and therefore the *level* of employment in location r, $N(r)$, is simply $\theta n(r)$.

For the monocentric city, we already know that $\theta(s) = 1$ for $s \in [0, S_F]$ and $\theta(s) = 0$ for $s \in (S_F, S]$. What the labor-market-clearing condition determines for this case is the location of the commercial district boundary, namely, S_F. To develop this condition, we note that the total supply of labor time available at the border of the CBD, taking into account the time lost in commuting, is $\int_{S_F}^{S}[2\pi r/l(r)]e^{-\kappa(r-S_F)}dr$. If the employment density at a CBD location r is $n(r)$, the labor time needed at the border of the commercial district to fulfill this demand is $e^{\kappa(S_F-r)}n(r)$. Therefore, the total time needed at the border of the CBD to satisfy total labor demand inside the commercial district is $\int_0^{S_F} 2\pi r n(r)e^{\kappa(S_F-r)}dr$. Equality

[6]The decentralized city case has also been analyzed in Wheaton (2004) for an exogenously given productivity gradient and exogenously given land use intensities for firms and workers. Wheaton does not impose the local labor-market-clearing condition (22). Instead, the fraction of land in use by firms (or workers) at any location is determined by the relative magnitude of the rent levels for each use.

of labor demand and supply then requires

$$\int_0^{S_F} 2\pi r n(r) \exp\left(\kappa(S_F - r)\right) dr = \int_{S_F}^{S} \frac{2\pi r}{l(r)} \exp\left(-\kappa(r - S_F)\right) dr,$$

which, using the fact that $l(r) = (1-\beta)w(0)e^{-\kappa r}/q_H(r)$ and the expressions for $n(r)$ and $q_H(r)$ derived earlier, simplifies to

$$n(0)w(0)(1-\beta)\int_0^{S_F} r\exp\left(-\frac{\delta\gamma - \kappa\alpha}{1-\alpha}r\right) dr = q_H(0)\int_{S_F}^{S} r\exp\left(-\frac{\kappa}{(1-\beta)}r\right) dr.$$

Using (18) and (19) we can further simplify this equation to

$$\left[\int_{S_F}^{S} r\exp\left(-\frac{\kappa}{1-\beta}r\right) dr\right] =$$
$$\frac{(1-\beta)}{(1-\alpha)}\alpha\left[\int_0^{S_F} r\exp\left(-\frac{\gamma\delta - \alpha\kappa}{1-\alpha}r\right) dr\right]\exp\left(\frac{-\kappa + \delta\gamma + \beta\kappa\alpha - \beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F\right). \quad (23)$$

Observe that this is an equation that implicitly defines S_F as a function of S. The following Lemma establishes that there is a unique S_F corresponding to each S that is strictly increasing in S and converging to a finite limit as S increases unboundedly.

Lemma 1 *For each $S > 0$, (23) uniquely determines $S_F(S) \in (0, S)$. Furthermore, $S_F(S)$ is strictly increasing in S and $\lim_{S\to\infty} S_F(S) = \bar{S}_F > 0$.*

Proof. See Appendix.

The second market-clearing condition requires that the total number of residents in the city must equal the total population of the city, P. For the decentralized city, this requires

$$P = \int_0^S 2\pi r[1-\theta]/l^*(r)dr.$$

Using (20) and (22), we see that the above implies

$$n(0) = \frac{P}{2\pi\theta \int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{24}$$

Knowing $n(0)$ and the fact that $\theta(s) = \theta$ also allows us to pin down $z(0)$:

$$z(0) = 2\pi \int_0^S r \exp(-\delta r) N(r) ds = 2\pi\theta n(0) \int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr, \tag{25}$$

or

$$z(0) = P \frac{\int_0^S r \exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta} + \delta\right] r\right) dr}{\int_0^S r \exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta} r\right) dr}. \tag{26}$$

For the monocentric city, the analogous requirement is $P = \int_{S_F}^S [2\pi r / l(r)] dr$. Since $l(r) = (1-\beta) w(0) \exp(-\kappa r) / q_H(0) \exp\left(-\frac{\kappa}{1-\beta} r\right)$, this implies

$$P = \frac{q_H(0)}{(1-\beta) w(0)} \int_{S_F}^S 2\pi r \exp\left(-\frac{\beta\kappa}{(1-\beta)} r\right) dr.$$

Using (18), (19), and (23), we obtain

$$n(0) = \frac{P}{2\pi} \frac{1}{\left[\int_0^{S_F(S)} r \exp\left(-\frac{\gamma\delta - \alpha\kappa}{1-\alpha} r\right) dr\right]} \frac{\left[\int_{S_F(S)}^S r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int_{S_F(S)}^S r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{27}$$

Again, knowing $n(0)$ allows us to pin down the level of the external effect at the city center.

Since $z(0) = n(0) \int_0^{S_F} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right]\right) rdr$, we have

$$z(0) = \frac{P}{2\pi} \frac{\left[\int\limits_0^{S_F} 2\pi r \exp\left(-\left[\frac{\delta\gamma-\kappa}{1-\alpha}+\delta\right] r\right) dr\right]}{\left[\int\limits_0^{S_F} r \exp\left(-\frac{\gamma\delta-\alpha\kappa}{1-\alpha} r\right) dr\right]} \frac{\left[\int\limits_{S_F}^{S} r \exp\left(-\frac{\kappa}{1-\beta} r\right) dr\right]}{\left[\int\limits_{S_F}^{S} r \exp\left(-\frac{\kappa\beta}{1-\beta} r\right) dr\right]}. \tag{28}$$

This completes the first part of the equilibrium determination problem.

Before proceeding to the second part, it is useful to report how the equilibrium is affected by changes in the demand and supply of urban land, separately considered.[7] Consider, first, the effects of a change in the general demand for urban land. In the model, this could come about through a change in A, which changes the productivity of firms located in the city, or a change in P, which changes the numbers of city residents. If P and S are held constant, a change in A will leave both $n(0)$ and $z(0)$ unchanged, since A does not appear in (24)-(28). Given this, it follows that a change in A will simply shift the wage and bid rent functions proportionally, and there will be no change in any relative price or in the intensity of land use in any location. If A and S are held constant, a change in P will change $n(0)$ and $z(0)$ proportionally since both quantities depend proportionally on P for both types of cities. From this fact, we can infer, using (17), (18) and the fact that $U = \beta^\beta(1-\beta)^{1-\beta}w(r)q(r)^{-(1-\beta)}$, the following:

Proposition 1 *(The Effects of a Change in Population): If A and S are held constant, (i) employment density and the level of the production externality change proportionately with P, (ii) the elasticity of rents in any location with respect to P is $\alpha+\gamma$, (iii) the elasticity of wage in any location with respect to P is $\alpha+\gamma-1$, and (iv) elasticity of U with respect to P is $\beta(\alpha+\gamma)-1$.*

We turn now to the effects of change in the supply of urban land. Consider, first, a change in S for the decentralized city, holding A and P constant. From (24) we see immediately that

[7] Of course, in full equilibrium, changes in the demand for urban land will induce changes in its supply. This interaction is the focus of the next section of this paper.

$n(0)$ is decreasing in S: Employment density at the city center is lower in a more spread-out city. The effect on $z(0)$ is not so clear because an increase in S increases the geographic reach of the externality—the numerator term in (26). Notice, however, that both integrals calculate a "mean distance" with weights that decline exponentially with distance and the weights decline *faster* for the numerator term (since $\delta > 0$). Intuitively, we would expect an increase in S to increase the numerator proportionately less than the denominator, and that, indeed, is true. Since this sort of logic will be used repeatedly to sign expressions, we state it as a Lemma here:

Lemma 2 *Let* $0 \leq s_L < s_U$. *Let* $\Lambda(s_L, s_U) = [\int_{s_L}^{s_U} se^{k_2 s} ds]/[\int_{s_L}^{s_U} se^{k_1 s} ds]$. *Then,* $\Lambda(s_L, s_U)$ *is increasing (decreasing) in both* S_U *and* S_L *if* $k_1 < (>) k_2$.

Proof. See Appendix.

Therefore, by virtue of Lemma 2, it is the case that $z(0)$ is declining in S as well.

Turning to the monocentric city, recall that $S_F(S)$ is increasing in S (Lemma 1). Therefore, by Lemma 2 again, the ratio of the integrals in the expression for $n(0)$ in (27) is decreasing in S. Since the remaining fractional term is clearly decreasing in S, employment density at the city center is decreasing in S for the monocentric city as well. The effect on $z(0)$ is potentially ambiguous for the same reason as in the decentralized city: While employment density is decreasing in S, the geographic reach of the external effect is increasing in S_F and therefore in S. However, if $\delta > \kappa$ (communication is harder than commuting) then, by Lemma 2 again, the first of the two ratios of integrals in (28) is decreasing in S. And, since $\beta < 1$, the second ratio of integrals is also decreasing in S. Henceforth, we will always operate under the assumption that $\delta > \kappa$. Then, it is easy to verify that the following is true:

Proposition 2 *(The Effect of Change in City Size* S*): If* A *and* P *are held constant, (i) employment density (and employment), the level of the production externality, and rents at the city center are decreasing in* S*, (ii) if* $\alpha+\gamma \leq 1$*, wages at the city center are increasing in*

S, otherwise the effect is ambiguous and (iii) if $\beta(\alpha+\gamma) \leq 1$, U is increasing in S, otherwise the effect is ambiguous.

We now turn to the second part of equilibrium determination, namely, the determination of S and U, given A and P. Since it costs d units of the consumption good to convert one unit of undeveloped land into urban land, developers (the entities that own all land in this economy) will continue to develop urban land until the rent at the city boundary S is equal to the cost of development. Therefore, S is determined by

$$q(S; A, P) = d, \tag{29}$$

where $q(S; A, P)$ is the rent at the city boundary when TFP is A and population is P. The following Lemma establishes how $q(S; A, P)$ varies with S.

Lemma 3 *$q(S; A, P)$ is strictly decreasing in S and strictly increasing in A and P. Furthermore, $\lim_{S \to 0} q(S; A, P) = \infty$ and $\lim_{S \to \infty} q(S; A, P) = 0$.*

Proof. See Appendix.

All else the same, rents fall with S because workers who live at the boundary earn the least. Complementing this effect is the fact that, recorded in Proposition 2, rents at the city *center* are also declining with S. The latter effect pushes down rents in all locations in the city, including the boundary. The "Inada-type" conditions of $q(S; A, P)$ are also intuitive: Rents in locations very far from the city center must be very low to compensate for very low wages in those locations (for the decentralized city) or for the very large amount of time lost in commuting to a job (for the monocentric city). If the boundary is very close to the city center, employment density at the center must be very high, which would require very high rents there and, by extension, at the city boundary. Given Lemma 3, it follows that, for any A, P, and d, there is a unique S, denoted $S_d(A, P)$ that solves (29).

The following proposition describes how S is affected by changes in TFP, population, and costs of development. These properties follow directly from Lemma 3.

Proposition 3 *$S_d(A, P)$ is strictly increasing in A and P and strictly decreasing in d. Furthermore,* $\lim_{P\to 0} S_d(A, P) = 0$ *and* $\lim_{P\to\infty} S_d(A, P) = \infty$.

5 Welfare and City Population

Finally, we come to the relationship between U, the utility deliverable by a city, and A and P when the city boundary adjusts so that rent at the boundary is d. We will denote this relationship by the function $U_d(A, P)$. We are primarily interested in understanding how this function behaves with respect to variations in P, since migration in or out of the city is the key adjustment mechanism for cities.

It is a convenient feature of the model that this function can be expressed as a composition of two functions: An "outer" function, denoted $V_d(A, S)$, which gives the utility deliverable by a city given A and S and rent at the boundary of d, and an "inner" function, which is just $S_d(A, P)$. Thus, $U_d(A, P) = V_d(A, S_d(A, P))$. The benefit of this decomposition is that the $V_d(A, S)$ function has a closed-form expression that allows easy assessment of its shape with respect to variations in S. And, since $S_d(A, P)$ is strictly increasing in P (Proposition 3), the shape of $U_d(A, P)$ with respect to P is simply a shape-preserving rescaling of $V_d(A, S)$.

To develop the $V_d(A, S)$ function, we use two conditions. The first condition is that rent at the city boundary must be d. For the decentralized city, this condition implies $d = q(0)\exp(-\delta\gamma/(1-\alpha\beta)S)$, and for the monocentric city it implies $d = q_H(0)\exp(-\kappa/(1-\beta))S$. This condition implies that S and d pin down rents in the city center. We have already seen, however, that rents at the city center are determined by A, $n(0)$, and $z(0)$. Since $z(0)$ is itself pinned down by $n(0)$, it follows that the first condition fully determines $n(0)$ as a function of A, S, and d.

The second condition equates the utility obtained by a worker who resides at the city boundary when the city size is S and rent at the boundary is d to the utility delivered by the city to any worker, which is U. For the decentralized city this condition is $U = \beta^\beta(1 -$

$\beta)^{1-\beta}d^{-(1-\beta)}w(0)\exp(-[\delta\gamma(1-\beta]/[1-\alpha\beta]S)$, and for the monocentric city the condition is $U = \beta^{\beta}(1-\beta)^{1-\beta}d^{-(1-\beta)}w(0)\exp(-\kappa/(1-\beta)S)$. These conditions imply that U, S, and d completely determine wages at the city center. Since wages at the city center are also fully determined by A, $n(0)$, $z(0)$, the second condition fully determines $n(0)$ as a function of A, S, d, and U.

Equating the two expressions for $n(0)$ and rearranging terms yields $V_d(A,S)$. To determine the shape of this function with respect to S, it is convenient to examine $\ln(V_d(A,S))$. Collecting terms that do not depend on S into a "constant" D, for the decentralized city we have

$$\ln(V_d(A,S)) = \\ D + \frac{\gamma}{\gamma+\alpha}\left\{\ln\left[\int_0^S r\exp\left(-\frac{\delta\left(1-\beta\alpha+\gamma\beta\right)}{1-\beta\alpha}r\right)dr\right] - \frac{\delta\left(1-\beta(\alpha+\gamma)\right)}{(1-\beta\alpha)}S\right\}. \quad (30)$$

Thus, on the logarithmic scale, $V_d(A,S)$ has a component that starts at 0 and declines linearly with S provided $1-\beta(\alpha+\gamma)>0$, and a component that starts at $-\infty$ and rises at most logarithmically with S. Since the rate of change of $\ln(x)$ is infinite at $x=0$, $\ln(V_d(A,S))$ must be increasing at $S=0$. Furthermore, since the derivative of the ln term declines monotonically to 0 with S, there is some $\hat{S}>0$ at which $\ln(V_d(A,S))$ peaks and then declines monotonically, asymptoting to $-\infty$. It follows that $V_d(A,S)$ is hump-shaped, with $\lim_{S\to 0} V_d(A,S) = \lim_{S\to\infty} V_d(A,S) = 0$.

For the monocentric city, the corresponding expression is

$$\ln(V_d(A,S)) = \quad (31) \\ D + \frac{\gamma}{\alpha+\gamma}\ln\left[\int_0^{S_F(S)} r\exp\left(-\frac{\delta\left(\gamma+1-\alpha\right)-\kappa}{1-\alpha}r\right)dr\right] + \\ -\kappa\frac{1-\beta\left(\alpha+\gamma\right)}{\left(1-\beta\right)\left(\gamma+\alpha\right)}S + \left(\frac{\gamma+\alpha-1}{\gamma+\alpha}\right)\frac{-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma}{\left(1-\alpha\right)\left(1-\beta\right)}S_F(S).$$

The presence of $S_F(S)$ introduces a new element that is not present in the decentralized city. From Lemma 1, however, we know that $\lim_{S\to 0} S_F(S) = 0$ and $\lim_{S\to\infty} S_F(S) = \bar{S}_F$. Therefore, it is still true that $\lim_{S\to 0} \ln(V_d(A,S)) = \lim_{S\to\infty} \ln(V_d(A,S)) = -\infty$. Whether the function generally has a single peak is not easy to establish, but, for the region of the parameter space that matters empirically, it is likely to be monotonically declining beyond some value of S. Empirically, $\alpha+\gamma \approx 1$ and $\delta >> \kappa$. Assume for the moment that $\alpha+\gamma = 1$. Then the r.h.s. of (31) simplifies to $D + \ln[\int_0^{S(F)} r \exp[\kappa/(1-\alpha)r]dr - [\kappa(1-\beta(\alpha+\gamma))/(1-\beta)(1-\alpha)]S$. If $\delta >> \kappa$, then (23) implies that S_F changes very little in response to any change in S. In this case, the behavior of the r.h.s. of (31) is effectively dominated by the term involving S. Therefore, beyond some initial (potentially non-monotone) segment, the function will decline with S. To summarize:

Lemma 4 *Assume* $1 - \beta(\alpha+\gamma) > 0$. *Then,* $\lim_{S\to 0} V_d(A,S) = \lim_{S\to\infty} V_d(A,S) = 0$. *In addition, for the decentralized city,* $V_d(A,S)$ *is single-peaked. For the monocentric city,* $V_d(A,S)$ *is eventually monotonically declining in* S*, provided* $\alpha+\gamma \approx 1$ *and* $\delta >> \kappa$.

As mentioned earlier, because $S_d(A,P)$ is strictly increasing in P, $U_d(A,P)$ inherits all the properties of $V_d(A,S)$. Therefore, we have the following proposition:

Proposition 4 *Assume* $1 - \beta(\alpha+\gamma) > 0$. *Then,* $\lim_{P\to 0} U_d(A,P) = \lim_{P\to\infty} U_d(A,P) = 0$. *In addition, for the decentralized city,* $U_d(A,P)$ *is single-peaked. For the monocentric city,* $U_d(A,P)$ *is eventually monotonically declining in* P*, provided* $\alpha+\gamma \approx 1$ *and* $\delta >> \kappa$.

It is worth noting that Proposition 4 is consistent with the results in Propositions 1 and 2. Proposition 1 states that if $1-\beta(\alpha+\gamma) > 0$, utility in the city is decreasing in P (when S is fixed), while Proposition 2 states that, under the same condition, utility is increasing in S (when P is fixed). Thus, as S increases and the city fills up with people so that the rent at the boundary is d, there are two offsetting forces working on utility obtained by residents of the city. When the city is physically small, the utility-enhancing effect of S is stronger than the utility-decreasing effect of higher population. Eventually, though, the utility-depressing effect of higher population dominates and utility declines with P.

To understand why welfare is increasing when P is low but declining when P is high (the inverted U shape), it is helpful to think of the case in which the city cannot expand at all. In this case, utility deliverable by the city declines as population increases. With population growth, even if the wages increase (which happens when $\alpha + \gamma > 1$), utility declines because the increase in wages, and the implied increase in c, is not large enough to compensate for the lower consumption of residential space. This comes from the condition $\beta(\alpha + \gamma) < 1$. In the case in which the city can expand at the cost of d, the city does expand with higher P and allows workers to increase their consumption of land, but only when the city is small. As we see from equation (24), employment density at the center (which is inversely proportional to the consumption of land per worker at the center) becomes increasingly insensitive to an increase in S as S gets higher. In the limit, equilibrium allocations at the center become similar to the case in which S does not change. Although some people move to the outskirts when S goes up, they form an increasingly small portion of the general population, so this reshuffling has little effect on employment and residential densities in the center.

The condition $1 - \beta(\alpha + \gamma) > 0$ is our analog of what Fujita, Krugman, and Venables (1999) call the "no-black-hole condition." If this condition is violated, then, as is evident from the expression of $\ln(V_d(A, S))$, utility deliverable by the city would be increasing in S. Since $S_d(A, P)$ is strictly increasing in P, utility deliverable by the city would be strictly increasing in P. The model would then imply that the entire population of an economy would tend to gravitate to one giant city—the "black hole," so to speak. To rule this out, the strength of increasing returns must be bounded above.[8]

[8] Lucas and Rossi-Hansberg (2002) (and also Lucas (2001)) assume a condition that is stronger, namely, $\alpha + \gamma < 1$. Although this condition is also labeled a "no-black-hole condition," it is needed to rule out a different kind of black hole, one in which all firms pile up at 0 (the city center) with each firm using a vanishingly small amount of land but enjoying unboundedly high external effect, i.e., it is needed to rule out the case where $z(0)$ diverges to ∞. However, this case is not a concern for us because $z(r)$ is known to have the negative exponential form and, hence, productivity at the city center is naturally bounded above by city size and total population, as seen in (26).

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APPENDIX

Proof of Lemma 1

Given any $S > 0$, (15) (the upper bound on κ) implies that the r.h.s. of (23) is increasing in S_F. The l.h.s. of (23) is clearly decreasing in S_F. Furthermore, the r.h.s. is 0 for $S_F = 0$ while the l.h.s. is strictly positive, and the r.h.s. is strictly positive for $S_F = S$ while the l.h.s. is 0. Therefore, for each $S > 0$ there is a unique $S_F \in (0, S)$ that ensures (23) is satisfied. Observe also that as S goes up and S_F does not change, the integral on the l.h.s. goes up. Since the r.h.s. is increasing in S_F, the equilibrium S_F must be strictly higher. Thus $S_F(S)$ is strictly increasing in S.

To prove the second part, we observe that since $S_F(S) < S$ for all S, it must be the case that $\lim_{S\to 0} S_F(S) = 0$. To prove the other limiting result, we will first establish that $\lim_{S\to\infty} S_F(S)$ is bounded above. Let S_n be an increasing sequence diverging to ∞. Let $S_F(S_n)$ be a corresponding sequence of S_F that satisfies (23). Then $S_F(S_n)$ is also a strictly increasing sequence. Next, observe that

$$\int\limits_{S_F(S_n)}^{S_n} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds = -\left[\frac{(1-\beta)}{\kappa}\right]^2 \left[e^{-\frac{\kappa}{(1-\beta}s}(ks+1)\right]_{S_F(S_n)}^{S_n}.$$

If $S_F(S_n)$ diverges to infinity along with S_n, the above integral will converge to 0. This will imply that the l.h.s. of (23) will converge to 0 while the r.h.s. will diverge to ∞, which is impossible. Hence, $S_F(S_n)$ must be bounded above. Since $S_F(S)$ is strictly increasing, it follows that $\lim S_F(S)$ must converge to some number $\bar{S}_F > 0$. ■

Proof of Lemma 2

We will first establish the following two sets of inequalities. If $k_1 < k_2$, then

$$e^{(k_2-k_1)s_L} < \frac{\int\limits_{s_L}^{s_U} se^{k_2 s}ds}{\int\limits_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_U}, \tag{32}$$

and if $k_2 < k_1$, then

$$e^{(k_2-k_1)s_U} < \frac{\int\limits_{s_L}^{s_U} se^{k_2 s}ds}{\int\limits_{s_L}^{s_U} se^{k_1 s}ds} < e^{(k_2-k_1)s_L}. \tag{33}$$

Turning first to the l.h.s. inequality in 32, we observe that $se^{k_2 s} = se^{s_L k_2 + (s-s_L)k_2}$and $se^{k_1 s} = se^{s_L k_1 + (s-s_L)k_1}$. Multiplying both sides of the latter equation by $e^{(k_2-k_1)s_L}$ yields $e^{(k_2-k_1)s_L}se^{k_1 s} = se^{s_L k_2 + (s-s_L)k_1} \leq se^{s_L k_2 + (s-s_L)k_2} = se^{k_2 s}$, where the inequality follows because $k_2 > k_1$ and $s - s_L \geq 0$. Furthermore, the inequality is strict for all $s \in (s_L, s_U]$. Therefore, integrating the first and last expressions in the chain with respect to s, we have

$$e^{(k_2-k_1)s_L} \int\limits_{s_L}^{s_U} se^{k_1 s}ds < \int\limits_{s_L}^{s_U} se^{k_2 s}ds.$$

Turning to the r.h.s. of the inequality, we observe that $se^{k_2 s} = se^{s_U k_2 + (s-s_U)k_2}$and $se^{k_1 s} = se^{s_U k_1 + (s-s_U)k_1}$. Multiplying both sides of the latter equation by $e^{(k_2-k_1)s_U}$ yields

$$e^{(k_2-k_1)s_U}se^{k_1 s} = se^{k_2 s_U + (s-s_U)k_1} \geq se^{s_U k_2 + (s-s_U)k_2} = se^{k_2 s},$$

where the inequality follows since $k_2 > k_1$ and $s - s_U \leq 0$. Furthermore, the inequality is strict for all $s \in [s_L, s_U)$. Therefore, integrating the first and last terms in the chain with respect to s, we have

$$e^{(k_2-k_1)s_U} \int\limits_{s_L}^{s_U} se^{k_1 s}ds > \int\limits_{s_L}^{s_U} se^{k_2 s}ds. \quad \blacksquare$$

The proof of 33 is entirely analogous.

We now turn to the proof of the Lemma. We begin with the case in which $k_1 < k_2$.

Observe that

$$\frac{\partial \ln(\Lambda(s_L, s_U))}{\partial s_U} = \frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} s e^{k_2 s} ds} - \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} s e^{k_1 s} ds}.$$

Suppose, to get a contradiction, that $\partial \Lambda(s_L, s_U)/\partial s_U \leq 0$. Then, we must have

$$\frac{s_U \exp(k_2 s_U)}{\int_{s_L}^{s_U} s e^{k_2 s} ds} \leq \frac{s_U \exp(k_1 s_U)}{\int_{s_L}^{s_U} s e^{k_1 s} ds}.$$

Or, given that all elements are positive, we have

$$\exp([k_2 - k_1] s_U) = \frac{s_U \exp(k_2 s_U)}{s_U \exp(k_1 s_U)} \leq \frac{\int_{s_L}^{s_U} s e^{k_2 s} ds}{\int_{s_L}^{s_U} s e^{k_1 s} ds}.$$

But this contradicts the r.h.s. inequality in Lemma 1. Therefore, $\partial \Lambda(s_L, s_U)/\partial s_U > 0$. Analogous proof can be given for the case in which $k_2 < k_1$. ■

Remark: Let $I(s_U, s_L, k) = \int_{s_L}^{s_U} s \exp(-ks)\, ds$. Then (i) $\lim_{s_U, s_L \to \infty} I(s_U, s_L, k) = 0$ and (ii) $\lim_{s_U \to \infty, s_L \to \underline{s}} I(s_U, s_L, k) = \bar{I} > 0$.

Observe that

$$\int_{s_L}^{s_U} s e^{-ks} ds = \frac{s_U e^{-k s_U} - s_L e^{-k s_L}}{-k} - \frac{e^{-k s_U} - e^{-k s_L}}{k^2}.$$

To prove (i), we notice that, as s_U and s_L go to infinity, the second term goes to 0, and the first term (on an application of L'Hospital's Rule to s/e^{ks}) also goes to 0. To prove (ii), we observe that if s_U goes to infinity and s_L converges to $\underline{s}$, then $I(s_U, s_L, k)$ converges to

$$\frac{-\underline{s} e^{-k\underline{s}}}{-k} + \frac{e^{-k\underline{s}}}{k^2} > 0. \quad ■$$

Proof of Lemma 3

Decentralized City: From (11) we have that

$$q(S;A,P) = q(0)\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right).$$

Holding fixed A and P, we see that $q(0)$ is decreasing in S by Proposition 2. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right)$ is strictly decreasing in S, it follows that $q(S;A,P)$ is strictly decreasing in S. And, holding fixed S and P, we see that $q(0)$ is proportional to A and therefore $q(S;A,P)$ is increasing in A. And, if S and A are held fixed, $q(0)$ is increasing in P by Proposition 1. Therefore $q(S;A,P)$ is increasing in P.

To establish the limit properties, we use (18), (24), and (26) to express $q(0)$ in terms of P and S:

$$q(0) = \\ (1-\alpha)AP^{(\alpha+\gamma)}\left[\frac{\int_0^S r\exp\left(-\left[\frac{\delta\gamma\beta}{1-\alpha\beta}+\delta\right]r\right)dr}{\int_0^S r\exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right)dr}\right]^{\gamma} \times \left[2\pi\theta\int_0^S r\exp\left(-\frac{\delta\gamma\beta}{1-\alpha\beta}r\right)dr\right]^{-\alpha}.$$

As S approaches 0, the term involving the ratio of integrals approaches 1 (this follows from an application of L'Hospital's Rule) and the remaining integral term approaches infinity. Since $\exp\left(-\frac{\gamma\delta}{1-\beta\alpha}S\right)$ approaches 1, it follows that $\lim_{S\to 0} q(S;A,P) = \infty$. Going the other way, as S approaches ∞, by Lemma 2 the ratio of integrals term approaches 0 and the integral term approaches a positive constant. Hence, $\lim_{S\to\infty} q(S;A,P) = 0$.

Monocentric City: To prove the first part, we note that $q_H(S;A,P) = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S}$. Since $e^{-\frac{\kappa}{(1-\beta)}S}$ is decreasing in S, it is sufficient to show that, if we hold A and P constant, $q_H(0)$ is decreasing in S. To begin, note that $q_F(0)e^{-\frac{\delta\gamma-\kappa\alpha}{1-\alpha}S_F} = q_H(0)e^{-\frac{\kappa}{(1-\beta)}S_F}$, which implies that $q_F(0)/q_H(0) = e^{-\frac{-\kappa+\delta\gamma+\alpha\beta\kappa-\beta\delta\gamma}{(1-\alpha)(1-\beta)}S_F}$. By (15), the r.h.s. of the latter equation is increasing in S_F. Since $S_F(S)$ is increasing in S, it follows that $q_F(0)/q_H(0)$ is increasing in S. From Proposition 2 we know, holding A and P constant, that $q_F(0)$ is decreasing in S. Therefore

$q_H(0)$ must be decreasing in S. And, if we hold fixed S and P, $q_H(0)$ is proportional to A and therefore $q(S;A,P)$ is increasing in A. And, holding fixed S and A, we see that $q_H(0)$ is increasing in P by Proposition 1. Therefore $q(S;A,P)$ is increasing in P.

We now turn to limiting behavior of $q_H(S;A,P)$.

Part (i): $\lim_{S\to\infty} q_H(S;A,P) = 0$. Consider

$$q_H(S;A,P) = (1-\beta)\,\beta^{\frac{\beta}{1-\beta}} \left(\frac{w(0)\exp(-\kappa S)}{U}\right)^{\frac{1}{1-\beta}}.$$

Using (17), (19), (27), and (28), we can express the ratio of $w(0)$ to U as

$$\frac{w(0)}{U} = KP^{(1-\beta)(\gamma+\alpha)}A^{-1}\left(\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds\right)^{-(1-\beta)(\gamma+\alpha)} \times$$
$$\left(\int_0^{S_F} s\exp\left(\frac{\kappa-\delta(\gamma+1-\alpha)}{1-\alpha}s\right)ds\right)^{\gamma(1-\beta)} \times$$
$$\exp\left(\frac{(-\kappa+\delta\gamma+\beta\kappa\alpha-\beta\delta\gamma)(\gamma+\alpha-1)}{(1-\alpha)}S_F\right),$$

where K is a positive constant. Given that $\lim_{S\to\infty} S_F(S) = \bar{S}_F$, the last two terms approach finite numbers. And, by Lemma 2, $\int_{S_F}^{S} s\left(\exp\frac{-\kappa\beta}{1-\beta}s\right)ds$ appoaches a strictly positive finite number. Thus, we can conclude that, as $S\to\infty$, the ratio $w(0)/U$ approaches a finite number as well. Therefore, the limiting behavior of $q_H(S;A,P)$ is governed by the limiting behavior of $\exp(-\kappa S)$. Hence, $\lim_{S\to\infty} q_H(S;A,P) = 0$.

Part (ii): $\lim_{S\to 0} q_H(S;A,P) = \infty$

Since $S > S_F(S), S\to 0$ implies $S_F(S)\to 0$. Then, it is easiest to show that $q_F(0) = (1-\alpha)\,z(0)^{\gamma}n(0)^{\alpha}$ goes to infinity, which would imply that $q_H(S;A,P)$ goes to infinity also.

Turning first to $n(0)$, we observe that

$$n(0) = \left[\frac{\int\limits_{S_F}^{S} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds}{\int\limits_{S_F}^{S} s \left(\exp \frac{-\kappa\beta}{1-\beta}s\right) ds} \right] \frac{P}{2\pi \left[\int\limits_{0}^{S_F} s \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right) ds \right]}.$$

We know from Lemma 2 that

$$\exp\left(\kappa S_F\right) < \frac{\left[\int\limits_{S_F}^{S} s \left(\exp \frac{-\kappa\beta}{1-\beta}s\right) ds \right]}{\left[\int\limits_{S_F}^{S} s \exp\left(-\frac{\kappa}{1-\beta}s\right) ds \right]} < \exp\left(\kappa S\right).$$

This implies that as S and S_F converge to 0 (and so both $\exp\left(\kappa S_F\right)$ and $\exp\left(\kappa S\right)$ converge to 1) the term in square brackets converges to 1. We also know that $\left[\int\limits_{0}^{S_F} s \exp\left(\frac{\alpha\kappa-\gamma\delta}{1-\alpha}s\right) ds \right]$ goes to zero as S_F goes to zero, so $n(0)$ goes to infinity as S goes to zero.