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Public and Private Money: Inflation Dynamics and Optimal Policy*

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Abstract

The total money supply of a modern economy is a combination of publicly and privately issued monies. This paper develops a theory of inflation dynamics and optimal policy for a system in which public fiat money and collateral-backed private liabilities jointly circulate. Private money finances investment projects, and its supply depends on the issuer's net worth and collateral constraints. The resulting feedback mechanism can generate persistent oscillations and stable banking cycles without aggregate shocks. For some public money growth rates, a dual-money steady state coexists with a boundary regime in which private money displaces fiat money. Policy affects liquidity and investment. When the collateral constraint is slack, the Friedman rule attains the first best; when it binds, a public money growth rate above the Friedman rule can be optimal. The Ramsey problem considers the transitional dynamics in the net worth of banks.

Keywords: Private money; public money; inflation dynamics; collateral constraints; banking cycles; Friedman rule; optimal monetary policy.

JEL Classification: E42, E51, E52, E31, C62.

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1 Introduction

In the account of U.S. monetary history given by Friedman and Schwartz (1963), the stock of money that mattered for prices and economic activity was, over most of the period studied, predominantly created by private institutions. Government-issued currency circulated alongside the liabilities of commercial banks, and the sharpest movements in the money stock originated from the interaction between its public and private components. The lesson of the *Monetary History* is that the money supply of a modern economy is the joint product of a public issuer and a private, profit-maximizing issuing sector, and that the dynamics of this mix govern inflation and monetary stability. Friedman (1960) draws a sharp policy conclusion from this reading of history: private money creation is inherently unstable, so the government should closely control the stock of money.

These questions have returned with force. Privately issued digital currencies (e.g., stablecoins backed by portfolios of assets, tokenized bank deposits, and other related instruments) now circulate alongside public money and traditional collateral-backed private monies (e.g., bank deposits) at scale, and legislatures have recently written regulatory frameworks for their issuers. At the same time, central banks debate whether public money should occupy a larger share of the payment system in the form of a central bank digital currency. Under the new technology lie old questions. When collateral-backed private liabilities circulate as media of exchange, what determines the aggregate quantity of money, its composition, and the price level? Does the growth rate of public money still anchor long-run inflation? What is the optimal rate of public money creation when part of the money supply responds elastically to market incentives?

This paper develops a tractable general-equilibrium framework to answer these questions. The economy has the trading structure of Lagos and Wright (2005): in each period, anonymous buyers and sellers first trade in decentralized bilateral meetings, where the absence of commitment and record-keeping makes a medium of exchange essential, and then rebalance their portfolios in a frictionless centralized market. In this environment, we introduce overlapping generations of entrepreneurs who are heterogeneous in the quality of an indivisible investment project. Entrepreneurs cannot fully fund their projects internally, so those with sufficiently productive projects borrow by issuing one-period debt claims backed by a pledgeable fraction of the project's return. Because these claims are perfectly recognizable and freely transferable in bilateral meetings, they circulate as media of exchange: we refer to issuing entrepreneurs as banks and their circulating liabilities as private money. The government issues fiat money that grows at a constant rate, and both assets can serve as media of exchange.

The equilibrium of the model reduces to a planar dynamical system in two aggregate variables: the mid-life wealth of the entrepreneurs, which determines the net worth of the banking sector, and aggregate real balances. This reduction delivers a transparent link between banking and inflation: in any equilibrium in which both monies circulate, the gross inflation rate equals the price of private money claims and is inversely proportional to the net worth of banks. The dynamics of inflation and the dynamics of private money creation are, therefore, a single object of analysis. The key feedback mechanism operates through the market price of private money. Predetermined entrepreneurial wealth determines both the external finance required by each active bank and the set of projects that can be funded. Aggregate issuance then determines the price and return on private claims. That return, in turn, determines the wealth with which the next generation of banks enters the credit market in the following period.

The first set of results concerns endogenous dynamics. In the private-money-only economy, the wealth law of motion implies the following dynamics: low current net worth raises the return on private claims and thereby increases next-period net worth, while high current net worth has the opposite effect. When this correction is stable, the economy converges through damped oscillations. For other parameter values, the same mechanism generates a stable period-two banking cycle. These fluctuations are deterministic and arise with constant fundamentals. Introducing fiat money does not necessarily eliminate them. Depending on policy and on which financing constraint binds, convergence to the dual-money steady state can be monotonic or oscillatory and highly persistent; the dual-money steady state can also lose local saddle-path stability. A change in the public money growth rate can thus alter the qualitative form of the transition, not only its steady-state value.

The model also yields a conditional limit to the public nominal anchor. For a range of money-growth rates, a dual-money steady state coexists with a boundary steady state at which only private money circulates as a medium of exchange. In many cases, the two steady states are close, even at low levels of the public money growth rate. Near the point where the two steady states meet, the boundary steady state is a sink, and the dual-money steady state is a saddle; its stable manifold locally separates trajectories converging to the two regimes. Along paths that converge to the boundary steady state, private money displaces public money, and inflation converges to a limiting finite value that is unrelated to the public money growth rate. Finally, we show that a stable period-two banking cycle organizes the dynamics of the two-money economy in the same way that the boundary steady state does. Fiat money can also remain valued in the long run in an economy that cycles.

The second set of results concerns policy. By comparing dual-money steady states indexed by the constant money-growth factor, we find that the Friedman rule attains the

unconstrained first best and is globally optimal when collateral constraints are slack. When the collateral constraints bind, however, the Friedman rule entails under-investment. Raising the money growth rate above the Friedman rule reduces decentralized-market output but can improve project selection. Under sufficient conditions, this tradeoff produces an interior optimum strictly above the Friedman rule. The Ramsey problem considers the transition in the net worth of banks. Under sufficient conditions, it can be shown that the optimal public money growth rate is set above its long-run Ramsey value during the transitional dynamics when banks begin under-capitalized and below it when banks begin over-capitalized. The Ramsey planner manages aggregate liquidity by setting the money growth rate in such a way that the capitalization of the banking system approaches its long-run optimum value, either from above or below, depending on the initial banking capital.

The remainder of the paper proceeds as follows. After a discussion of the related literature, Section 2 describes the environment, and Section 3 develops the monetary economy and defines equilibrium. Section 4 characterizes stationary and non-stationary equilibrium trajectories. Section 5 studies optimal policy, first in terms of stationary allocations and then accounting for transitional dynamics. Section 6 concludes. All proofs that are not directly derived below can be found in Appendix A.

Related Literature

This paper builds on the search-theoretic approach to monetary economics initiated by Kiyotaki and Wright (1989, 1993) and developed by Trejos and Wright (1995), Shi (1995), and Shi (1997). It adopts the alternating decentralized-centralized market structure of Lagos and Wright (2003) and Lagos and Wright (2005), which keeps the distribution of asset holdings degenerate and delivers tractable dynamics. Shi (2006), Williamson and Wright (2010), and Lagos et al. (2017) survey this research program. Relative to the standard framework, the present model identifies an essential component of the money supply as an equilibrium object: liquid assets are created by a sector of collateral-constrained issuers whose net worth evolves endogenously and induces aggregate fluctuations.

A branch of the New Monetarist literature studies the creation and circulation of inside money. Cavalcanti and Wallace (1999a,b) show that note issuance by a subset of monitored agents expands the set of implementable allocations relative to outside money alone, while Williamson (1999) examines the efficiency of private money issuance and Berentsen et al. (2007) integrate banks that take deposits and make loans into the Lagos-Wright framework. Gu et al. (2013a) provide a mechanism-design foundation for banking as liquidity transformation. Williamson (2012) analyzes monetary policy when intermediated assets serve as

media of exchange, and Venkateswaran and Wright (2014) study the macroeconomic effects of limited pledgeability on liquidity provision. More recently, Choi and Rocheteau (2023) develop a model with fiat money and bank deposits to study the effects of monetary policy. This paper studies the joint dynamics of the composition of the money supply, the behavior of inflation during the transition, and the optimal rate of public money creation in an environment in which the net worth of banks is an endogenous state variable.

The oscillatory equilibria and period-two cycles relate the paper to the literature on endogenous credit cycles, including Gu et al. (2013b), where cycles arise from limited commitment, as well as Azariadis and Smith (1998) and Matsuyama (2013). Boldrin and Woodford (1990) survey equilibrium models with endogenous fluctuations, and Grandmont (1985) is the classic treatment of dynamics in monetary overlapping-generations economies. Ferraris and Watanabe (2011) study endogenous fluctuations in an economy with monetary and real assets. Rocheteau and Wright (2013) generate equilibrium cycles from liquidity considerations in asset markets. The distinctive feature of the present model is that the constrained sector issues the economy's medium of exchange, so fluctuations in the issuer's net worth are, at the same time, fluctuations in the money supply and the inflation rate.

The coexistence of government and private circulating liabilities is a classical theme in monetary theory, much of which developed in the overlapping-generations tradition of Samuelson (1958) and Wallace (1980). Sargent and Wallace (1982) contrast the real-bills and quantity-theory regimes, and Smith (1988) reexamines the associated policy debates; Azariadis et al. (2001) and Bullard and Smith (2003) study economies in which public and private liabilities circulate together, Cavalcanti et al. (1999) analyze reserve management by private note issuers, and Champ et al. (1996) connect the elasticity of private note issuance to banking panics. Holmström and Tirole (1998) and Farhi and Tirole (2012) study the public and private supply of liquidity under limited pledgeability, although liquid assets in their framework do not serve as media of exchange. The contribution here is a dynamic characterization of a two-asset monetary system with collateral-constrained private issuers, together with a normative analysis of the money growth rate that accounts for transitional dynamics.

A modern literature investigates the coexistence of private and public money in digital form. Fernández-Villaverde and Sanches (2019) show that competition among profit-maximizing issuers of intrinsically worthless currencies generally fails to deliver price stability and efficiency, and Schilling and Uhlig (2019) and Benigno et al. (2022) study the interaction between cryptocurrencies and monetary policy. Brunnermeier and Niepelt (2019) establish conditions under which public and private money are equivalent; those conditions fail here because private issuance is tied to investment through the collateral constraint, so the com-

position of the money supply matters for real allocations. The paper also relates to work on central bank digital currency and its effects on private intermediation, investment, consumption, and welfare, including Fernández-Villaverde et al. (2021), Andolfatto (2021), Keister and Sanches (2023), and Chiu et al. (2023), and to the discussion of stablecoin regulation by Gorton and Zhang (2023), who draw a parallel between stablecoins and free-banking-era notes. Viewed through that discussion, the model provides a laboratory in which collateral-backed private money coexists with public money, and the pledgeability parameter in the model maps naturally onto the collateral and reserve requirements imposed on private issuers.

On the normative side, the paper contributes to the literature on the optimum quantity of money descending from Friedman (1969). Departures from the Friedman rule can arise when money growth interacts with investment or credit margins. Altermatt and Wipf (2024) show that the Friedman rule can be suboptimal when higher inflation produces a first-order welfare gain through greater capital investment, while the associated loss from reduced liquidity-intensive trade is only second order. Their mechanism is a Mundell–Tobin portfolio shift between fiat money and capital, whereas ours operates through the endogenous interaction of public and private money, with monetary policy affecting the net worth of banks, collateral capacity, and project selection.

2 Model

Environment. Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. The planning horizon is infinite. Each period is divided into two sub-periods in which economic activity differs. A perishable commodity is produced and consumed within each sub-period. The economy is populated by three types of agents: buyers, sellers, and entrepreneurs. Buyers and sellers are infinitely-lived agents who discount future periods at the discount factor $\beta \in (0, 1)$. There is a unit mass of each type of infinitely-lived agent. Entrepreneurs live for three periods and consume only in old age. Each generation contains a mass $\eta > 0$ of entrepreneurs.

Markets. In the first sub-period, a buyer is randomly matched with a seller in a decentralized market. Let $\alpha \in (0, 1)$ denote the probability of finding a trading partner. The buyer wants to consume but cannot produce, while the seller can produce but does not want to consume. In particular, the seller can produce a perishable commodity, termed the decentralized market (DM) good, using a linear technology that requires labor as input. Entrepreneurs do not participate in the DM. In the second sub-period, all agents trade in a centralized market. Buyers and sellers can produce a perishable commodity, called the centralized market (CM) good, using a linear technology that requires labor as input. The

entrepreneurial production technology is described below. All agents derive utility from the consumption of the CM good.

Preferences. The buyer has preferences represented by the utility function

$$\mathcal{U}^b(q,x) = u(q) + x,$$

where $q \in \mathbb{R}_+$ denotes the consumption of the DM good and $x \in \mathbb{R}$ denotes the net consumption of the CM good. The DM utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^2$ on $(0,\infty)$, strictly increasing, and concave, with $u(0) = 0$ and $u'(0) = \infty$. The seller has preferences represented by the utility function

$$\mathcal{U}^s(q,x) = -w(q) + x,$$

where $q \in \mathbb{R}_+$ denotes the production of the DM good and $x \in \mathbb{R}$ denotes the net consumption of the CM good. The DM cost function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^2$, strictly increasing, and convex, with $w(0) = 0$. Entrepreneurs derive linear utility from the consumption of the CM good in old age. Let $q^* \in \mathbb{R}_+$ be defined by $u'(q^*) = w'(q^*)$.

Investment. A new generation of entrepreneurs is born each period. An entrepreneur is endowed at birth with $A \in (0,\beta)$ units of the CM good. A middle-aged entrepreneur j is endowed with an indivisible and non-tradable project that requires exactly one unit of the CM good as input and pays off $\gamma_j \in \mathbb{R}_+$ units of the CM good in the following period. Project returns are publicly observable. The distribution of returns is uniform across the support $[0,\bar{\gamma}]$. The upper bound $\bar{\gamma} \in \mathbb{R}_+$ satisfies $\bar{\gamma} > \beta^{-1}$, so some projects are socially efficient to operate, while others are not.

Collateral. An entrepreneur has an opportunity to invest in a project in middle age. If an entrepreneur does not have enough funds to meet the required investment amount, they can borrow from external creditors by issuing a one-period debt claim. However, the entrepreneur's ability to borrow is limited by a collateral constraint. Only a fraction $\theta \in (0,1)$ of the project's return can be pledged to external debt holders.

Assets and exchange. Buyers and sellers are anonymous and cannot commit to their future actions, which precludes credit in the DM. Because there is no scope for trading future promises in this market, a medium of exchange is essential to achieve allocations that could not be accomplished without it. In contrast, entrepreneurs can have their trading history publicly recorded. If an entrepreneur satisfies a collateral constraint, they can borrow in the credit market. Privately issued debt claims can be perfectly recognized and transferred in a bilateral match. Therefore, they can be used as a medium of exchange. We state that an entrepreneur who borrows from external creditors by issuing a debt claim is operating a *bank*, since the entrepreneur's liabilities are backed by the project's return and constitute a

liquid asset for their holders. We refer to the debt claims issued by banks as *private money*. Government-issued fiat money can also be used as a medium of exchange.

3 Monetary Economy

3.1 Government

The government issues fiat money and injects new money via lump-sum transfers in the CM. Let $M_t \in \mathbb{R}_+$ denote the period-t money supply. Let $\phi_t \in \mathbb{R}_+$ denote the period-t value of money in terms of the CM good. Suppose that fiat money is positively valued at all dates. The government's budget constraint (in real terms) is given by

$$m_t = \frac{\phi_t}{\phi_{t-1}} m_{t-1} + \tau_t,$$

where $m_t \equiv \phi_t M_t$ denotes real balances, and $\tau_t \in \mathbb{R}$ is the real value of transfers to private agents. The government expands the money supply at a constant rate: $M_t = \mu M_{t-1}$ for all $t \in \{0, 1, 2, ...\}$. Assume that $\mu \geqslant \beta$.

3.2 Asset Returns

It is helpful to define the real rate of return on fiat money in terms of aggregate real balances. If fiat money is positively valued at each date, the money-growth rule allows us to write the real return on fiat money as

$$\frac{\phi_t}{\phi_{t-1}} = \frac{1}{\mu} \frac{m_t}{m_{t-1}}.$$

Let $f_t \in \mathbb{R}_+$ be the real price of a privately issued debt claim in the period-t CM. This price is determined by the market-clearing condition in the period-t private money market. A debt claim is a promise to pay one unit of the CM good in the following period. If private money is positively valued at date t, its real return is f_t^{-1}.

Suppose that both assets are positively valued in equilibrium. In this case, it will be shown that asset returns satisfy the following boundary conditions:

$$0 < \frac{1}{\mu} \frac{m_{t+1}}{m_t} \leqslant \frac{1}{\beta} \tag{1}$$

and

$$0 < \frac{1}{f_t} \leqslant \frac{1}{\beta}. \tag{2}$$

In other words, the expected real return on each asset is bounded above by the subjective interest rate.

3.3 Entrepreneurs

Let $n_t^1 \geqslant 0$ denote the young entrepreneur's holdings of private claims, and let $m_t^1 \geqslant 0$ denote their real fiat money balances. Their budget constraint is

$$f_t n_t^1 + m_t^1 \leqslant A. \tag{3}$$

The young entrepreneur invests all their wealth in external assets. If both assets are positively valued in equilibrium, their optimal portfolio satisfies (3) with equality and

$$n_t^1 \begin{cases} = 0 \text{ if } \frac{1}{f_t} < \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ \in [0, \frac{A}{f_t}] \text{ if } \frac{1}{f_t} = \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ = \frac{A}{f_t} \text{ if } \frac{1}{f_t} > \frac{1}{\mu}\frac{m_{t+1}}{m_t} \end{cases} \qquad m_t^1 \begin{cases} = 0 \text{ if } \frac{1}{f_t} > \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ \in [0, A] \text{ if } \frac{1}{f_t} = \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ = A \text{ if } \frac{1}{f_t} < \frac{1}{\mu}\frac{m_{t+1}}{m_t}. \end{cases}$$

If only private money is positively valued, $n_t^1 = \frac{A}{f_t}$ and $m_t^1 = 0$. If only fiat money is valued, $n_t^1 = 0$ and $m_t^1 = A$. When fiat money and private money yield the same expected real return, agents are indifferent between those assets, so the composition of their portfolio is indeterminate. Otherwise, a corner solution is obtained.

Next, we consider the decision problem of a middle-aged entrepreneur. Let $\omega_t \geqslant 0$ denote the wealth of a middle-aged entrepreneur. When both assets are positively valued, we have

$$\omega_t = \begin{cases} \frac{A}{f_{t-1}} \text{ if } \frac{1}{f_{t-1}} \geqslant \frac{1}{\mu}\frac{m_t}{m_{t-1}} \\ \frac{A}{\mu}\frac{m_t}{m_{t-1}} \text{ if } \frac{1}{f_{t-1}} \leqslant \frac{1}{\mu}\frac{m_t}{m_{t-1}}. \end{cases} \tag{4}$$

If only private money is positively valued, $\omega_t = \frac{A}{f_{t-1}}$. If only fiat money is valued, $\omega_t = \frac{A}{\mu}\frac{m_t}{m_{t-1}}$. Since $A < \beta$ and given (1)-(2), we have $\omega_t < 1$, which implies that an entrepreneur does not have enough wealth to fund their project. Therefore, an entrepreneur who wants to invest in their project must raise external funds by issuing private money to the public.

The decision to invest in a project implies two ex-post types of middle-aged entrepreneurs: those who issue private money to fund their projects and those who do not invest. The latter will reinvest their wealth in external assets to save for old age. An entrepreneur finds

it optimal to invest in their internal project, provided that the *return condition* holds:

$$\gamma_j - \frac{1}{f_t}(1-\omega_t) \geqslant (1+\rho_t)\omega_t,$$

where $1+\rho_t$ denotes the maximum expected real return available from t to $t+1$. An entrepreneur who operates a bank must be better off investing in their project than reinvesting their wealth in external assets.

To operate a bank, the entrepreneur must also satisfy the *collateral condition*:

$$\theta\gamma_j - \frac{1}{f_t}(1-\omega_t) \geqslant 0.$$

Otherwise, the entrepreneur is credit-constrained. If an entrepreneur is credit-constrained, their only option is to reinvest their wealth in external assets. Note that an entrepreneur who satisfies the collateral constraint can choose to reinvest their wealth in external assets if the market return is higher than the entrepreneur's internal return.

Let $\gamma_t^m \geqslant 0$ denote the marginal type above which investment is undertaken and below which it is not. The expression for the marginal type is given by

$$\gamma_t^m = max\left\{(1+\rho_t)\omega_t + \frac{1-\omega_t}{f_t}, \frac{1-\omega_t}{\theta f_t}\right\}. \tag{5}$$

We say that the economy is in the *return regime* when the marginal type is determined by the first term on the right-hand side of (5), and that the economy is in the *collateral regime* when the marginal type is determined by the second term on the right-hand side of (5).

The threshold (5) for the marginal type determines the issuance of private money and the size of the banking system in equilibrium. In period t, the measure of banks operated by entrepreneurs is given by

$$\eta\left[1-\frac{\gamma_t^m}{\bar{\gamma}}\right]_+.$$

Each bank has a net worth equal to ω_t, so it needs to issue $\frac{1-\omega_t}{f_t}$ to fund its single project.

Let $n_t^2 \geqslant 0$ denote the private money holdings of a non-bank entrepreneur, and let $m_t^2 \geqslant 0$ denote their real balances. Their optimal portfolio satisfies

$$f_t n_t^2 + m_t^2 = \omega_t. \tag{6}$$

When both assets are positively valued, we have

$$n_t^2 \begin{cases} = 0 \text{ if } \frac{1}{f_t} < \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ \in [0, \frac{\omega_t}{f_t}] \text{ if } \frac{1}{f_t} = \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ = \frac{\omega_t}{f_t} \text{ if } \frac{1}{f_t} > \frac{1}{\mu}\frac{m_{t+1}}{m_t} \end{cases} \qquad m_t^2 \begin{cases} = 0 \text{ if } \frac{1}{f_t} > \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ \in [0, \omega_t] \text{ if } \frac{1}{f_t} = \frac{1}{\mu}\frac{m_{t+1}}{m_t} \\ = \omega_t \text{ if } \frac{1}{f_t} < \frac{1}{\mu}\frac{m_{t+1}}{m_t}. \end{cases}$$

If only private money is positively valued, $n_t^2 = \frac{\omega_t}{f_t}$ and $m_t^2 = 0$. If only fiat money is positively valued, $n_t^2 = 0$ and $m_t^2 = \omega_t$. Non-bank entrepreneurs exhaust their budget constraint (6) and choose a portfolio that maximizes the expected real return on their wealth. Entrepreneurs who operate a bank do not hold external assets over time.

If both assets coexist in equilibrium, which is referred to as a dual-money equilibrium, it must be the case that

$$\frac{1}{f_t} = \frac{1}{\mu}\frac{m_{t+1}}{m_t}.$$

Fiat money and private money yield the same expected real return, and entrepreneurs are indifferent between these assets in a dual-money equilibrium.

3.4 Buyers

Let $\mathbf{a}_b = (m_b, n_b)$ denote a portfolio consisting of $m_b \geqslant 0$ real fiat money balances and $n_b \geqslant 0$ units of private money. Let $W_t^b(\mathbf{a}_b)$ denote the CM value function of a buyer with portfolio $\mathbf{a}_b \in \mathbb{R}_+^2$, and let $V_t^b(\mathbf{a}_b)$ denote the DM value function. The value functions satisfy the Bellman equation:

$$W_t^b(\mathbf{a}_b) = \max_{(x,\hat{\mathbf{a}}_b)\in\mathbb{R}\times\mathbb{R}_+^2} \left\{x + \beta V_{t+1}^b(\hat{\mathbf{a}}_b)\right\} \; s.t. \; x + \mathbf{p}_t \cdot \hat{\mathbf{a}}_b = \mathbf{R}_{t-1} \cdot \mathbf{a}_b + \tau_t^b.$$

The price vector $\mathbf{p}_t = (1, f_t)$ measures the cost of acquiring real balances and private money in terms of the CM good, the vector

$$\mathbf{R}_{t-1} = \left(\frac{1}{\mu}\frac{m_t}{m_{t-1}}, 1\right)$$

measures the gross real returns on assets carried over from the previous period, and $\tau_t^b \in \mathbb{R}$ is the real value of transfers from the government. Recall that $m_t \geqslant 0$ denotes aggregate real balances, and we continue to assume that fiat money is positively valued at each date.

The buyer's DM value function satisfies:

$$V_t^b(\mathbf{a}_b) = \alpha\left[u(q_t(\mathbf{a}_b)) + W_t^b(\mathbf{a}_b - \mathbf{h}_t(\mathbf{a}_b))\right] + (1-\alpha)W_t^b(\mathbf{a}_b),$$

where $q_t(\mathbf{a}_b) \in \mathbb{R}_+$ denotes the quantity of DM goods the seller produces for the buyer in a match, and $\mathbf{h}_t(\mathbf{a}_b) \in \mathbb{R}_+^2$ denotes the vector of asset transfers from the buyer to the seller. Because of quasi-linear preferences, it is a property of the model that the terms of trade in the DM depend only on the buyer's asset holdings. The fact that the CM value function is affine allows us to write the DM value function as

$$V_t^b(\mathbf{a}_b) = \alpha\left[u(q_t(\mathbf{a}_b)) - \mathbf{R}_{t-1}\cdot\mathbf{h}_t(\mathbf{a}_b)\right] + \mathbf{R}_{t-1}\cdot\mathbf{a}_b + \tau_t^b + \max_{\hat{\mathbf{a}}_b\in\mathbb{R}_+^2}\left\{-\mathbf{p}_t\cdot\hat{\mathbf{a}}_b + \beta V_{t+1}^b(\hat{\mathbf{a}}_b)\right\}.$$

This equation makes explicit the link between the DM value function in the current period and the DM value function in the following period.

3.5 Sellers

Let $W_t^s(\mathbf{a}_s)$ denote the CM value function of a seller with portfolio $\mathbf{a}_s \in \mathbb{R}_+^2$, and let $V_t^s(\mathbf{a}_s)$ denote the DM value function. The value functions satisfy the Bellman equation:

$$W_t^s(\mathbf{a}_s) = \max_{(x,\hat{\mathbf{a}}_s)\in\mathbb{R}\times\mathbb{R}_+^2}\left\{x + \beta V_{t+1}^s(\hat{\mathbf{a}}_s)\right\}\ s.t.\ x + \mathbf{p}_t\cdot\hat{\mathbf{a}}_s = \mathbf{R}_{t-1}\cdot\mathbf{a}_s + \tau_t^s.$$

The seller's DM value function satisfies

$$V_t^s(\mathbf{a}_s) = \alpha\left[-w(q_t(\mathbf{a}_b)) + W_t^s(\mathbf{a}_s + \mathbf{h}_t(\mathbf{a}_b))\right] + (1-\alpha)W_t^s\left(\mathbf{a}_s\right),$$

where $\mathbf{a}_b \in \mathbb{R}_+^2$ denotes the asset holdings of the buyer with whom the seller is paired in the DM. Following the same steps as before, we can show that the CM value function $W_t^s(\mathbf{a}_s)$ is affine, so the DM value function can be written as

$$V_t^s(\mathbf{a}_s) = \alpha\left[-w(q_t(\mathbf{a}_b)) + \mathbf{R}_{t-1}\cdot\mathbf{h}_t(\mathbf{a}_b)\right] + \mathbf{R}_{t-1}\cdot\mathbf{a}_s + \tau_t^s + \max_{\hat{\mathbf{a}}_s\in\mathbb{R}_+^2}\left\{-\mathbf{p}_t\cdot\hat{\mathbf{a}}_s + \beta V_{t+1}^s(\hat{\mathbf{a}}_s)\right\}.$$

It is clear from this expression that the seller's DM value function is also affine, given that the seller's asset holdings do not affect the terms of trade in a match.

3.6 Bargaining

The terms of trade in a bilateral match are determined by the generalized Nash bargaining solution. Let $\nu \in (0,1]$ denote the buyer's bargaining power. Define the following functions:

$$g_\nu(q) \equiv \frac{\nu u'(q)w(q) + (1-\nu)w'(q)u(q)}{\nu u'(q) + (1-\nu)w'(q)}$$

and

$$\rho_\nu(q) \equiv \frac{u'(q)}{g'_\nu(q)}.$$

Let $\hat{q}_\nu \in \mathbb{R}_+$ be the unique solution of $\rho_\nu(\hat{q}_\nu) = 1$. The solution to the bargaining problem implies the output schedule:

$$q_t(\mathbf{a}_b) = \begin{cases} g_\nu^{-1}(\mathbf{R}_{t-1} \cdot \mathbf{a}_b) \text{ if } \mathbf{R}_{t-1} \cdot \mathbf{a}_b < g_\nu(q^*) \\ q^* \text{ otherwise} \end{cases}$$

and the payment schedule:

$$\mathbf{R}_{t-1} \cdot \mathbf{h}_t(\mathbf{a}_b) = \begin{cases} \mathbf{R}_{t-1} \cdot \mathbf{a}_b \text{ if } \mathbf{R}_{t-1} \cdot \mathbf{a}_b < g_\nu(q^*) \\ g_\nu(q^*) \text{ otherwise.} \end{cases}$$

Note that $\hat{q}_\nu < q^*$ when $0 < \nu < 1$, whereas $\hat{q}_1 = q^*$.

3.7 Optimal Portfolio

Let $\mathbf{a}_t^b = (m_t^b, n_t^b) \in \mathbb{R}_+^2$ denote the optimal portfolio of the buyer in the CM. Given the properties of the value functions and the solution to the bargaining problem, we can write the buyer's optimization problem in the CM as

$$\max_{\mathbf{a}_t^b \in \mathbb{R}_+^2} \left\{ -\mathbf{p}_t \cdot \mathbf{a}_t^b + \beta\{\alpha\left[u(q_{t+1}(\mathbf{a}_t^b)) - \mathbf{R}_t \cdot \mathbf{h}_{t+1}(\mathbf{a}_t^b)\right] + \mathbf{R}_t \cdot \mathbf{a}_t^b\} \right\}.$$

Suppose that the upper bounds in (1) and (2) are strict inequalities. In that case, the necessary and sufficient first-order conditions are

$$-1 + \frac{\beta}{\mu}\frac{m_{t+1}}{m_t}\left[\alpha\rho_\nu(g_\nu^{-1}(\mathbf{R}_t \cdot \mathbf{a}_t^b)) + 1 - \alpha\right] \leqslant 0, \text{with} = \text{if } m_t^b > 0, \tag{7}$$

$$-f_t + \beta\left[\alpha\rho_\nu(g_\nu^{-1}(\mathbf{R}_t \cdot \mathbf{a}_t^b)) + 1 - \alpha\right] \leqslant 0, \text{with} = \text{if } n_t^b > 0. \tag{8}$$

If only private money is positively valued, (8) holds with equality. If only fiat money is valued, (7) holds with equality. Consider an interior solution in which the agent holds both assets. Although the real value of the asset portfolio is determined by (7) and (8), the composition of the portfolio is indeterminate because an interior solution requires that both assets yield the same expected real return.

The equilibrium analysis below is derived under CRRA utility and the power cost function: $u(q) = \frac{\zeta}{\sigma} q^{\sigma}$ and $w(q) = \frac{1}{\epsilon} q^{\epsilon}$, where $\zeta > 0$, $0 < \sigma < 1$, and $\epsilon \geqslant 1$. Define the function $\mathcal{L}_{\nu} : (0, \hat{q}_{\nu}] \to [1, \infty)$ by

$$\mathcal{L}_{\nu}(q) = \alpha \rho_{\nu}(q) + 1 - \alpha.$$

The function $\mathcal{L}_{\nu}$ maps the real value of the buyer's asset portfolio to a liquidity premium term. Note that $\mathcal{L}_{\nu}(q) > 1$ for any $0 < q < \hat{q}_{\nu}$. When the real value of the buyer's portfolio is sufficiently large to induce the seller to produce $\hat{q}_{\nu}$, the liquidity premium disappears (that is, the value of $\mathcal{L}_{\nu}$ equals one). Next, define the quantity $q_{\nu}(z)$ implicitly by

$$\mathcal{L}_{\nu}(q_{\nu}(z)) = \frac{1}{\beta z}$$

and the demand function $\mathcal{D}_{\nu}(z)$ by

$$\mathcal{D}_{\nu}(z) \equiv \frac{g_{\nu}(q_{\nu}(z))}{z}.$$

Its endpoints are $\mathcal{D}_{\nu}(0) = 0$ and $\mathcal{D}_{\nu}(\beta^{-1}) = \beta g_{\nu}(\hat{q}_{\nu})$. Given (7)–(8), the buyer's total expenditure on liquid assets satisfies

$$m_t^b + f_t n_t^b = \mathcal{D}_{\nu}\left(1 + \rho_t\right), \tag{9}$$

where $1 + \rho_t = \frac{1}{f_t}$ or $1 + \rho_t = \frac{1}{\mu} \frac{m_{t+1}}{m_t}$.

Let $\mathbf{a}_t^s = (m_t^s, n_t^s) \in \mathbb{R}_+^2$ denote the optimal portfolio of the seller in the CM. The seller's optimization problem in the CM is

$$\max_{\mathbf{a}_t^s \in \mathbb{R}_+^2} \left\{ -\mathbf{p}_t \cdot \mathbf{a}_t^s + \beta \left\{ \alpha \left[-w(q_{t+1}(\mathbf{a}_t^b)) + \mathbf{R}_t \cdot \mathbf{h}_{t+1}(\mathbf{a}_t^b) \right] + \mathbf{R}_t \cdot \mathbf{a}_t^s \right\} \right\},$$

where $\mathbf{a}_t^b \in \mathbb{R}_+^2$ denotes the portfolio of the buyer with whom the seller is paired in the following period. Suppose that the upper bounds of (1) and (2) are strict inequalities. In that case, it is clear that

$$\mathbf{a}_t^s = (0, 0). \tag{10}$$

The seller does not acquire assets in the CM when the upper bounds of (1) and (2) are strict inequalities, regardless of their initial wealth in the CM. Liquid assets are costly to hold over time and do not bring any additional benefit to the seller in the DM.

3.8 Market Clearing

Suppose that $\gamma_t^m \leqslant \bar{\gamma}$. The clearing condition in the private money market is

$$\eta\left(1-\frac{\gamma_t^m}{\bar{\gamma}}\right)(1-\omega_t)=\eta\left(f_t n_t^1+\frac{\gamma_t^m}{\bar{\gamma}}f_t n_t^2\right)+f_t n_t^b+f_t n_t^s.$$

The left-hand side gives the aggregate supply of private money by the banking system, and the right-hand side represents the aggregate demand for private money. The clearing condition in the fiat money market is

$$m_t=\eta\left(m_t^1+\frac{\gamma_t^m}{\bar{\gamma}}m_t^2\right)+m_t^b+m_t^s.$$

The left-hand side gives the real balances supplied by the government, and the right-hand side represents the aggregate demand for real balances. Using (3), (6), (9), and (10), the previous market-clearing conditions can be combined to yield a single condition:

$$\eta\left(1-\frac{\gamma_t^m}{\bar{\gamma}}\right)+m_t=\eta\left(A+\omega_t\right)+\mathcal{D}_\nu\left(1+\rho_t\right).$$

3.9 Equilibrium

The wealth equation (4) implies $\omega_{t+1} = (1+\rho_t)A$. The fraction of middle-aged entrepreneurs who do not operate banks can be written as

$$\min\left\{\frac{\omega_{t+1}K(\omega_t)}{\Gamma},1\right\},$$

where $\Gamma \equiv A\bar{\gamma}$ and

$$K(\omega)\equiv\max\left\{1,\frac{1-\omega}{\theta}\right\}.$$

We restrict attention to monetary equilibria with active private-money issuance. Accordingly, every equilibrium trajectory satisfies

$$\omega_{t+1}K(\omega_t)<\Gamma.$$

Under this restriction, a strictly positive mass of entrepreneurs operates banks, and the cutoff lies in the interior of the productivity distribution. Fiat-money-only equilibria, which may arise when the cutoff reaches or exceeds $\bar{\gamma}$, are outside the scope of the analysis.

Let $\bar{\omega} \equiv \frac{A}{\beta}$. An equilibrium is a nonnegative sequence $\{\omega_t, m_t\}_{t=0}^{\infty}$ satisfying $\omega_t \leqslant \bar{\omega}$,

$$\frac{\omega_{t+1} K(\omega_t)}{\Gamma} + \frac{\mathcal{D}_\nu\left(\frac{\omega_{t+1}}{A}\right)}{\eta} = 1 - A - \omega_t + \frac{m_t}{\eta} \tag{11}$$

and

$$m_{t+1} = \frac{\mu}{A} \omega_{t+1} m_t, \tag{12}$$

given the initial value $\omega_0 \in (0, \bar{\omega})$. The equilibrium can be described as a planar system containing one predetermined variable, the mid-life wealth of entrepreneurs, and one jump variable, the real balances. If fiat money is positively valued at all dates, the gross inflation rate is given by

$$\Pi_{t+1} = \frac{A}{\omega_{t+1}}. \tag{13}$$

In what follows, we characterize the possible solutions to the planar system in (ω_t, m_t) and use the previous equation to compute the associated inflation trajectory.

4 Equilibrium Analysis

In this section, we characterize stationary and non-stationary trajectories for the single-money economy and the dual-money economy. In the single-money economy, only private money is positively valued. In the dual-money economy, both fiat and private money are positively valued. Equation (11) can be rearranged as

$$m_t = \Psi(\omega_{t+1}, \omega_t), \tag{14}$$

where

$$\Psi(\omega', \omega) \equiv \mathcal{D}_\nu\left(\frac{\omega'}{A}\right) + \eta\left[A + \omega - 1 + \frac{\omega' K(\omega)}{\Gamma}\right].$$

It can be shown that $\Psi_{\omega'}(\omega', \omega) > 0$ for any feasible values (ω', ω). Therefore, for any feasible pair (ω_t, m_t), equation (14) implicitly defines the map $\omega_{t+1} = F(\omega_t, m_t)$. The implicit function theorem gives $F_m(\omega, m) > 0$.

For a given current wealth ω, the next-period wealth ω' must satisfy $0 \leqslant \omega' \leqslant \hat{\omega}(\omega)$, where the state-dependent upper bound for the next-period wealth is

$$\hat{\omega}(\omega) \equiv min\left\{\bar{\omega}, \frac{\Gamma}{K(\omega)}\right\}.$$

Since $\Psi(\omega', \omega)$ is strictly increasing in ω', the real balances satisfy $\Psi(0, \omega) \leqslant m \leqslant \Psi(\hat{\omega}(\omega), \omega)$. Define the lower bound on real balances as $\underline{m}(\omega) \equiv \max\{0, \eta(A + \omega - 1)\}$, and define the upper bound as $\overline{m}(\omega) \equiv \Psi(\hat{\omega}(\omega), \omega)$.

The one-period-ahead feasible region is

$$\mathcal{F} = \{(\omega, m) : 0 \leqslant \omega \leqslant \bar{\omega},\ \underline{m}(\omega) \leqslant m \leqslant \overline{m}(\omega)\}.$$

The equilibrium map is defined by

$$T_\mu(\omega, m) = \left(F(\omega, m), \frac{\mu}{A} F(\omega, m) m\right).$$

The viability kernel that ensures forward invariance is given by

$$\mathcal{Q}_\mu = \{(\omega, m) \in \mathcal{F} : T_\mu^t(\omega, m) \in \mathcal{F} \text{ for every } t \geqslant 0\}.$$

4.1 Steady states

Define $H(\omega) \equiv \Psi(\omega, \omega)$. Then, the mapping $m = H(\omega)$ defines the locus on which $\omega_{t+1} = \omega_t$. The properties of $\mathcal{D}(z)$ and the non-negativity of $K(\omega)$ imply that

$$m_t > H(\omega_t) \Leftrightarrow \omega_{t+1} > \omega_t$$
$$m_t < H(\omega_t) \Leftrightarrow \omega_{t+1} < \omega_t.$$

This means that the mid-life wealth of entrepreneurs increases above the locus $\omega_{t+1} = \omega_t$ and decreases below it.

4.1.1 Single-money steady state

Let (ω_N, m_N) denote a steady state of the single-money economy. We have $m_N = 0$, and the value ω_N is defined by $H(\omega_N) = 0$. If $\omega_N < 1 - \theta$, then $K(\omega_N) > 1$. In that case, the collateral constraint is binding at $(\omega_N, 0)$, so it determines the size of the banking system, the aggregate issuance of private money, and aggregate investment. If $\omega_N > 1 - \theta$, then $K(\omega_N) = 1$. In that case, the return constraint is binding at $(\omega_N, 0)$.

Because $H(0) = \eta(A-1) < 0$, a sufficient condition for the existence of a single-money steady state is

$$H(\bar{\omega}) > 0. \tag{15}$$

Note that

$$H'(\omega) = \frac{1}{A}\mathcal{D}'_\nu\left(\frac{\omega}{A}\right) + \eta\left[1 + \frac{K(\omega) + \omega K'(\omega)}{\Gamma}\right]$$

away from the kink. On the collateral branch (when $\omega < 1-\theta$), we have $K(\omega) + \omega K'(\omega) = \theta^{-1}(1-2\omega)$. On the return branch (when $\omega > 1-\theta$), we have $K(\omega) + \omega K'(\omega) = 1$, which means that $H'(\omega) > 0$ always holds on the return branch.

Only the collateral branch can potentially be non-monotone. A sufficient condition for monotonicity on the collateral branch is

$$\frac{A}{\beta} \leqslant \frac{1+\theta\Gamma}{2}. \tag{16}$$

These observations allow us to prove the following proposition.

Proposition 1 *If conditions* (15) *and* (16) *hold, there is exactly one single-money steady state:* $(\omega_N, 0)$ *defined by* $H(\omega_N) = 0$.

4.1.2 Dual-money steady state

Let (ω_M, m_M) denote a dual-money steady state of the dual-money economy. In this equilibrium, we have $m_M > 0$; thus, equation (12) requires

$$\omega_M = \frac{A}{\mu} \equiv \omega(\mu). \tag{17}$$

The corresponding steady-state real balances are

$$m_M = \mathcal{D}_\nu\left(\frac{1}{\mu}\right) + \eta\left[A + \frac{A}{\mu} - 1 + \frac{K\left(\frac{A}{\mu}\right)}{\mu\bar{\gamma}}\right] \equiv m\left(\mu\right). \tag{18}$$

A dual-money steady state exists when $H\left(\omega(\mu)\right) > 0$. It must also satisfy

$$\frac{K\left(\frac{A}{\mu}\right)}{\mu} < \bar{\gamma}, \tag{19}$$

so the marginal project type is strictly below the upper bound on project return $\bar{\gamma}$.

If $H(\omega)$ satisfies conditions (15) and (16), then $H(\omega(\mu)) > 0 \Leftrightarrow \omega(\mu) > \omega_N$. Therefore, the money growth rate must satisfy

$$\beta < \mu < \frac{A}{\omega_N} \equiv \mu_N. \tag{20}$$

We also consider the allocation in the limiting case $\mu \downarrow \beta$, which is associated with the Friedman rule. These observations allow us to prove the following proposition.

Proposition 2 *Suppose that conditions (15) and (16) hold. There exists a unique dual-money steady state $(\omega_M, m_M) = (\omega(\mu), m(\mu))$ if and only if the money growth rate satisfies (19) and (20). The steady-state values are given by (17) and (18). As $\mu \uparrow \mu_N$, we have $(\omega_M, m_M) \to (\omega_N, 0)$. As $\mu \downarrow \beta$, we have $(\omega_M, m_M) \to (\bar{\omega}, m(\beta))$.*

Define the threshold value $\mu_R \equiv A(1-\theta)^{-1}$. The economy is in the return regime if $\mu < \mu_R$ and in the collateral regime if $\mu > \mu_R$. The steady-state wealth and the steady-state real balances decrease as the money growth rate rises. The previous result also shows that the dual-money steady state and the single-money steady state collide in the limit $\mu \uparrow \mu_N$.

4.2 Dynamics

Consider now the isocline $m_{t+1} = m_t$. For $m_t > 0$, we have $m_{t+1} > m_t \Leftrightarrow \omega_{t+1} > \omega(\mu)$. Since $F(\omega, m)$ is strictly increasing in m, define $\mathcal{P}_\mu(\omega) \equiv \Psi(\omega(\mu), \omega)$. Then, it can be shown that

$$m_t > \mathcal{P}_\mu(\omega_t) \Leftrightarrow m_{t+1} > m_t$$
$$m_t < \mathcal{P}_\mu(\omega_t) \Leftrightarrow m_{t+1} < m_t.$$

It is also straightforward to show that the wealth isocline and the money isocline have a unique intersection. A global trajectory for the wealth of entrepreneurs can be represented by the following second-order difference equation:

$$\Psi(\omega_{t+2}, \omega_{t+1}) = \frac{\mu}{A}\omega_{t+1}\Psi(\omega_{t+1}, \omega_t).$$

Given ω_0 and $\omega_1 = F(\omega_0, m_0)$, we can use the previous equation to generate global trajectories. Let $m = \mathcal{S}_\mu(\omega)$ denote the stable manifold in the feasible set. If $\mathcal{S}_\mu(\omega)$ is a graph, then

it satisfies

$$\mathcal{S}_\mu(F(\omega, \mathcal{S}_\mu(\omega))) = \frac{\mu}{A} F(\omega, \mathcal{S}_\mu(\omega)) \mathcal{S}_\mu(\omega).$$

The boundary condition is $\mathcal{S}_\mu(\omega(\mu)) = m(\mu)$. Figures 1 and 2 illustrate the phase diagram for different parametric values. They also include the linearized saddle path around the dual-money steady state, which is derived below.

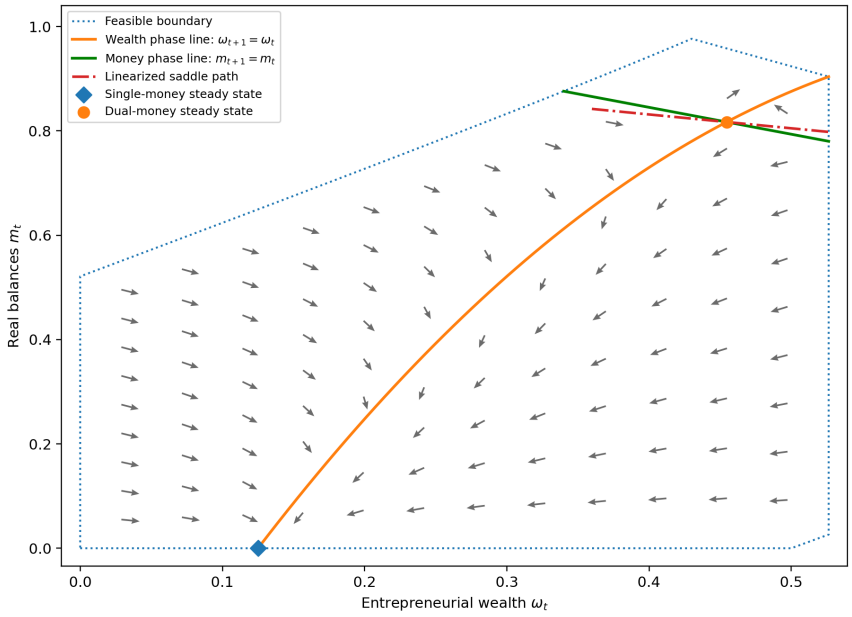


Figure 1: An example of the phase diagram for the economy. The locus $m_{t+1} = m_t$ is negatively sloped. The parametric values are: $A = 0.50$; $\beta = 0.95$; $\eta = 1.00$; $\theta = 0.30$; $\bar{\gamma} = 2.00$; $\alpha = 0.50$; $\zeta = 0.15$; $\sigma = 0.40$; $\nu = 1$; $\epsilon = 1.25$; and $\mu = 1.10$.

Next, we can establish that a feasible, bounded trajectory with monotone ω_t converges to either the single-money steady state or the dual-money steady state.

Proposition 3 *Assume that $H(A\mu^{-1}) > 0$ and $\mu^{-1}K(A\mu^{-1}) < \bar{\gamma}$. Suppose that $H(\omega) = 0$ has a unique root ω_N. Every feasible, bounded trajectory with monotone ω_t must converge to one of the two steady states.*

Non-monotonic trajectories can also be an equilibrium outcome. In fact, we later show the existence of equilibrium trajectories that imply persistent aggregate fluctuations. Next, we establish that the single-money steady state is a sink for an open region around it. Later in the analysis, we establish the mode of convergence to this steady state by studying the local behavior of trajectories.

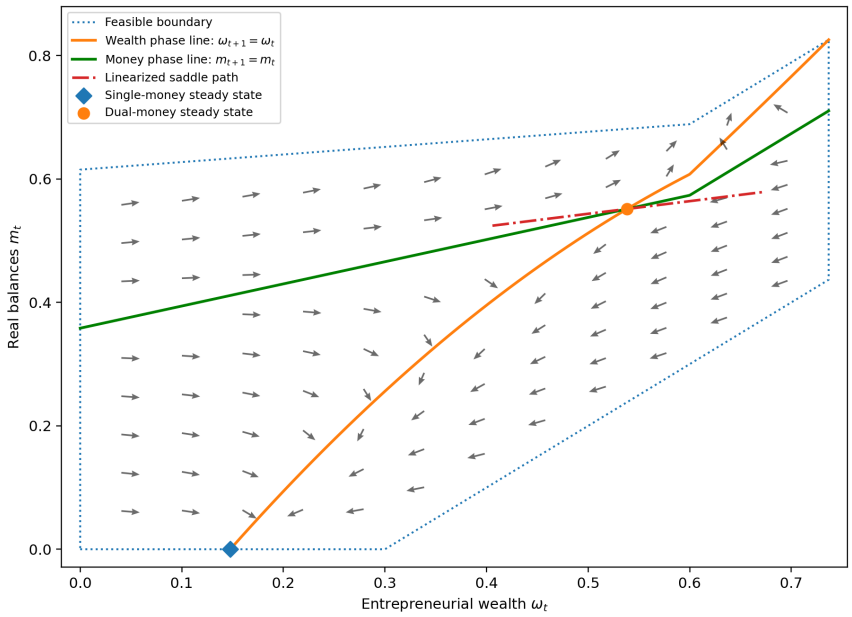


Figure 2: An example of the phase diagram for the economy. The locus $m_{t+1} = m_t$ is positively sloped. The parametric values are: $A = 0.70$; $\beta = 0.95$; $\eta = 1.00$; $\theta = 0.40$; $\bar{\gamma} = 3.00$; $\alpha = 0.50$; $\zeta = 0.20$; $\sigma = 0.50$; $\nu = 1$; $\epsilon = 1.00$; and $\mu = 1.30$.

Proposition 4 *Suppose that $\omega_N \neq 1 - \theta$, $-1 < \varphi'(\omega_N) < 0$, and $\beta \leqslant \mu < \mu_N$, where $\varphi(\omega) \equiv F(\omega, 0)$. Then there exists $\delta, \bar{m} > 0$ such that $[\omega_N - \delta, \omega_N + \delta] \times [0, \bar{m}]$ is a forward-invariant trapping rectangle, and every trajectory beginning in it converges to $(\omega_N, 0)$.*

Any trajectory in a neighborhood of $(\omega_N, 0)$ converges to that steady state. The real balances converge to zero, and the gross inflation rate converges to a value higher than μ. Along a path converging to $(\omega_N, 0)$, we have $\Pi_{t+1} = \frac{A}{\omega_{t+1}} \to \mu_N > \mu$. Although m_t converges to zero, the ratio m_{t+1}/m_t converges to a finite value. The monetary authority expands the nominal money stock at a constant gross rate μ, but the long-run inflation rate is higher than μ and is independent of μ conditional on convergence to $(\omega_N, 0)$.

The following proposition establishes that the stable manifold is a graph (locally) and that the dual-money steady state is an interior local saddle. It also defines the attraction region of the single-money steady state.

Proposition 5 *Suppose that $\beta < \mu_N$, $\omega_N \neq 1 - \theta$, and $-1 < \varphi'(\omega_N) < 0$. Also, assume that the equilibrium map T_μ admits a $\mathcal{C}^3$ extension to a neighborhood of $(\omega_N, 0)$. There is $\varepsilon > 0$ such that, for any $\mu \in (\mu_N - \varepsilon, \mu_N)$,*

1. *$(\omega_N, 0)$ is a sink;*

2. *there is a nearby dual-money steady state $(\omega(\mu), m(\mu))$, and it is an interior saddle;*

3. *its local stable manifold is a graph* $m = \mathcal{S}_\mu(\omega)$*;*

4. *in a sufficiently small neighborhood* U_μ *containing both steady states, feasible points between* $m = 0$ *and this graph converge to* $(\omega_N, 0)$*, points on the graph converge to* $(\omega(\mu), m(\mu))$*, and feasible points immediately above it leave* U_μ*.*

Moreover, there is an interval J_μ *of predetermined initial states* ω_0 *on which* $\mathcal{S}_\mu > 0$*. For every* $\omega_0 \in J_\mu$ *, the initial condition* $m_0 = \mathcal{S}_\mu(\omega_0)$ *converges to* $(\omega(\mu), m(\mu))$*, whereas every feasible initial condition in* U_μ *satisfying* $0 < m_0 < \mathcal{S}_\mu(\omega_0)$ *converges to* $(\omega_N, 0)$*.*

In the following, we characterize the mode of convergence to $(\omega(\mu), m(\mu))$ by using local methods. But before we proceed to the study of the local dynamics of the dual-money economy, it is useful to understand the mechanics of the single-money economy.

4.3 Single-money dynamics

The dynamics of the single-money economy are helpful in understanding the mechanism of private money creation that will drive the dynamics of the dual-money equilibrium. In the single-money economy, we have $m_t = 0$ for all $t \geqslant 0$. In this equilibrium, the wealth sequence satisfies $\omega_{t+1} = \varphi(\omega_t)$, where $\varphi(\omega)$ is implicitly defined by $\Psi(\varphi(\omega), \omega) = 0$. Feasibility requires $\omega \leqslant 1 - A$. The feasible set is defined by $\mathcal{I} = [0, \Omega]$, where $\Omega \equiv min\{1 - A, \bar{\omega}\}$. It follows that $\omega \in \mathcal{I} \Rightarrow \varphi(\omega) \in \mathcal{I}$. One way to verify invariance is to show that, for every $\omega \in \mathcal{I}$, we have $\Psi(0, \omega) \leqslant 0$ and $\Psi(\Omega, \omega) \geqslant 0$. The first inequality follows from the feasibility requirement: $\omega \leqslant 1 - A$. In what follows, we assume that the second inequality holds for any $\omega \in \mathcal{I}$.

It can be easily shown that, away from the kink,

$$\varphi'(\omega) = -\frac{\eta\left[1 + \frac{\varphi(\omega)K'(\omega)}{\Gamma}\right]}{\frac{1}{A}\mathcal{D}'_\nu\left(\frac{\varphi(\omega)}{A}\right) + \frac{\eta K(\omega)}{\Gamma}} < 0,$$

given that $\varphi(\omega) < \theta\Gamma$ holds on the collateral branch. If conditions (15) and (16) hold, the single-money steady state is unique, and

$$\omega_t > \omega_N \Leftrightarrow \omega_{t+1} < \omega_N$$
$$\omega_t < \omega_N \Leftrightarrow \omega_{t+1} > \omega_N.$$

The equilibrium trajectory alternates around ω_N. Why does this happen? Suppose that the current state is $\omega_t < \omega_N$. The mid-life wealth of entrepreneurs is below its steady-state level.

Lower entrepreneurial wealth increases the required external finance per active bank and changes the project cutoff and the number of active banks. Under the model's monotonicity conditions, the net market-clearing effect is a lower price and a higher return on private money, which raises next-period entrepreneurial wealth. This behavior can be self-correcting, leading to diminishing deviations from the steady state and, therefore, convergence.

The previously described interplay between the mid-life wealth of entrepreneurs (which translates into the net worth of the banking system) and the market-clearing price of private money determines the process of money creation by private banks, leading to fluctuations in the quantity of private money and other aggregate variables. Next, we show that these oscillations can be damped or can form perpetual banking cycles.

4.3.1 Local stability

The single-money steady state is locally stable if and only if $-1 < \varphi'(\omega_N) < 0$. If the steady state occurs on the collateral branch (when $\omega_N < 1 - \theta$), the condition for local stability is

$$\frac{1}{A}\mathcal{D}'_\nu\left(\frac{\omega_N}{A}\right) + \eta\left(\frac{1}{\theta\Gamma} - 1\right) > 0.$$

If the steady state occurs on the return branch (when $\omega_N > 1 - \theta$), the condition for local stability is

$$\frac{1}{A}\mathcal{D}'_\nu\left(\frac{\omega_N}{A}\right) + \eta\left(\frac{1}{\Gamma} - 1\right) > 0.$$

These conditions imply that any trajectory starting near the steady state converges to the single-money steady state through damped oscillations. The fluctuations in the mid-life wealth of entrepreneurs can be persistent, depending on the parametric configuration.

4.3.2 Global stability and period-two cycles

A continuous decreasing map of a compact interval has the property that its even and odd subsequences converge, possibly to different limits. Those limits either coincide at the unique fixed point or form a period-two orbit. Therefore, the uniqueness of the fixed point alone does not ensure global convergence. If it is possible to find a pair (ω_L, ω_H) in the feasible set such that $\omega_L \neq \omega_H$, $\omega_H = \varphi(\omega_L)$, and $\omega_L = \varphi(\omega_H)$, then (ω_L, ω_H) forms a two-cycle, so the economy is in a perpetual banking cycle. The fluctuations inherent in the mechanism of private money creation are not self-correcting in a two-cycle.

Proposition 6 *Suppose that* (15) *and* (16) *hold and that* $\varphi : \mathcal{I} \to \mathcal{I}$, *where* $\mathcal{I} = [0, \Omega]$. *If* $\varphi'(\omega_N) < -1$, *there is a nontrivial two-period cycle for the single-money economy.*

4.4 Local dynamics around the dual-money steady state

Now we consider the dynamics of the dual-money equilibrium in a neighborhood of $M_\mu \equiv (\omega(\mu), m(\mu))$. The behavior of the economy in this region is the focus of analysis in the optimal policy problem. Assume that the money growth rate satisfies $\mu > \beta$, $\mu \neq \mu_R$, and $m(\mu) > 0$. Define the following coefficients:

$$\delta(\mu) \equiv \mathcal{D}'_\nu\left(\frac{1}{\mu}\right) > 0, \qquad \Delta(\mu) \equiv \frac{\delta(\mu)}{A} + \frac{\eta K\left(\frac{A}{\mu}\right)}{\Gamma} > 0, \qquad N(\mu) \equiv \eta\left[1 + \frac{A}{\mu}\frac{K'\left(\frac{A}{\mu}\right)}{\Gamma}\right].$$

Also, define $\gamma(\mu) \equiv \mu^{-1} K(\omega(\mu))$. The set of feasible values for the money growth rate is given by

$$\mathcal{A} = \{\mu > 0 : m(\mu) > 0,\, \gamma(\mu) < \bar{\gamma}\}.$$

Linearizing equation (11) around the dual-money steady state yields

$$d\omega_{t+1} = a(\mu) d\omega_t + b(\mu) dm_t,$$

where $a(\mu) \equiv -\frac{N(\mu)}{\Delta(\mu)}$ and $b(\mu) \equiv \frac{1}{\Delta(\mu)}$. Linearizing equation (12) around the dual-money steady state yields

$$dm_{t+1} = \frac{\mu m(\mu) a(\mu)}{A} d\omega_t + \left[1 + \frac{\mu m(\mu) b(\mu)}{A}\right] dm_t.$$

The Jacobian matrix of the linearized system is given by

$$DT_\mu(\omega_M, m_M) = \begin{pmatrix} a(\mu) & b(\mu) \\ \frac{\kappa(\mu) a(\mu)}{b(\mu)} & 1 + \kappa(\mu) \end{pmatrix},$$

where $\kappa(\mu) \equiv \mu m(\mu) b(\mu) A^{-1} > 0$.

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - [1 + a(\mu) + \kappa(\mu)]\lambda + a(\mu).$$

It is straightforward to show that the roots of the characteristic polynomial are real. Since

$P(1) = -\kappa(\mu) < 0$, one of the roots is always greater than one. The candidate stable eigenvalue is

$$\lambda(\mu) = \frac{1 + a(\mu) + \kappa(\mu) - \sqrt{[1 + a(\mu) + \kappa(\mu)]^2 - 4a(\mu)}}{2}.$$

If $a(\mu) \geqslant 0$, then $0 \leqslant \lambda(\mu) < 1$; therefore, the dual-money steady state is a local saddle. If $a(\mu) < 0$, then $\lambda(\mu) < 0$. It is stable in this case if and only if it can be shown that $\lambda(\mu) > -1$.

The exact local saddle condition is $P(-1) = 2[1 + a(\mu)] + \kappa(\mu) > 0$. This condition can be written as

$$2[\Delta(\mu) - N(\mu)] + \frac{m(\mu)}{\omega(\mu)} > 0. \tag{21}$$

If equality holds in (21), then $\lambda(\mu) = -1$, which is the local loss of stability condition and can indicate a flip bifurcation. On the other hand, if the left-hand side of (21) is strictly negative, then $\lambda(\mu) < -1$, and both eigenvalues are outside the unit circle. In that case, the dual-money steady state is a local source rather than a saddle. Thus, the money growth rate determines the local saddle-path stability of the dual-money equilibrium.

4.4.1 Monotone versus damped saddle-path convergence

Define the threshold $\mu_P \equiv \frac{1}{\theta\bar{\gamma}}$. If the dual-money steady state occurs on the collateral branch, then we have

$$a(\mu) = \frac{\eta\left(\frac{1}{\theta\bar{\gamma}\mu} - 1\right)}{\Delta(\mu)} > 0 \Leftrightarrow \mu < \mu_P. \tag{22}$$

If (22) holds, it follows that $0 < \lambda(\mu) < 1$, and the convergence of the economy along the local saddle path is monotonic. If the stable eigenvalue is close to one, the convergence to the steady state occurs slowly; thus, deviations from the steady state can be persistent in that case.

It is possible that damped oscillations occur around M_μ. Note that

$$a(\mu) = \frac{\eta\left(\frac{1}{\theta\bar{\gamma}\mu} - 1\right)}{\Delta(\mu)} < 0 \Leftrightarrow \mu > \mu_P. \tag{23}$$

If (21) and (23) hold, then $-1 < \lambda(\mu) < 0$, and the convergence of the economy takes the form of damped oscillations. If the stable eigenvalue is close to -1, the fluctuations are more

persistent.

We can provide a sufficient condition for the existence of a monotone-convergence region. The dual-money steady state lies on the collateral branch only if $\mu > \mu_R$. Thus, a non-empty monotone-convergence region requires

$$max\{\beta, \mu_R\} < min\{\mu_P, \mu_N\}. \tag{24}$$

If the dual-money steady state occurs on the return branch, trajectories along the local saddle path can converge only through damped oscillations. To verify this claim, note that $K'(\omega) = 0$ for any ω on the return branch, so

$$a(\mu) = -\frac{\eta}{\Delta(\mu)} < 0.$$

Necessary and sufficient conditions for the existence of the return regime are

$$\beta < \mu < min\{\mu_R, \mu_N\}. \tag{25}$$

These findings can be summarized in the following proposition.

Proposition 7 *Assume that* (15) *and* (16) *hold.*

1. *Suppose M_μ occurs on the collateral branch. If the money growth rate satisfies* (22)*, the convergence of the dual-money economy along the saddle path is monotonic. If the money growth rate satisfies* (21) *and* (23)*, the convergence of the dual-money economy along the saddle path takes the form of damped oscillations. Condition* (24) *ensures a non-empty monotone-convergence region.*

2. *Suppose M_μ occurs on the return branch. If* (21) *holds, the convergence of the dual-money economy along the saddle path takes the form of damped oscillations. Condition* (25) *ensures the existence of the return regime.*

We have seen that $\mu = \mu_P$ is the threshold at which the stable root crosses zero and the stable map reverses orientation. The actual loss-of-stability occurs when $\lambda(\mu) = -1$, namely when the money growth rate is such that condition (21) holds with equality. If the left-hand side of (21) is negative, the dual-money steady state is unstable. The following proposition characterizes all possible binding regimes (collateral or return) and modes of convergence (monotone or oscillatory) for non-stationary trajectories in a neighborhood of M_μ.

Proposition 8 *Suppose that (15) and (16) hold. Any $\mu \in (\beta, min\{\mu_R, \mu_N\}) \cap \mathcal{A}$ implies the return regime and has a negative candidate stable root. Any $\mu \in (max\{\beta, \mu_R\}, min\{\mu_P, \mu_N\}) \cap \mathcal{A}$ implies the collateral regime with a positive stable root. Any $\mu \in (max\{\beta, \mu_R, \mu_P\}, \mu_N) \cap \mathcal{A}$ implies the collateral regime with a negative candidate stable root. Condition (21) ensures the stability of the negative root.*

The interaction between public and private money in a dual-money equilibrium can change the dynamics of private money creation previously described for the single-money non-stationary equilibrium. The pattern of convergence can be monotonic or oscillatory in the dual-money equilibrium as the composition of the total money supply changes over time. The adjustment of the net worth of banks along the transition path remains gradual, even though the mode of convergence can be different.

An interesting case is the following: $\beta < \mu_R < \mu_P < \mu_N$. If $\mu \in (\beta, \mu_R)$, the economy is in the return regime with oscillatory fluctuations around the steady state (assuming the proper stability condition holds). If $\mu \in (\mu_R, \mu_P)$, the economy switches to the collateral regime with monotonic convergence. If $\mu \in (\mu_P, \mu_N)$, the economy stays in the collateral regime, but the convergence to the steady state is now oscillatory, provided the stability condition holds. Therefore, a small change in policy can cause a drastic change in the dynamic properties of the dual-money economy.

4.4.2 Saddle-path slope

To characterize the inflation trajectories in a neighborhood of M_μ, it is necessary to further characterize the linearized system. The linearized saddle path is given by

$$m_t - m(\mu) = s(\mu)\left[\omega_t - \omega(\mu)\right] + \mathcal{O}(||\omega_t - \omega(\mu)||^2),$$

where $s(\mu) \equiv b(\mu)^{-1}\left[\lambda(\mu) - a(\mu)\right]$. If $a(\mu) > 0$, then $0 < \lambda(\mu) < 1$ and $s(\mu) < 0$. Thus, along the local saddle path, wealth and real balances move on opposite sides of their steady-state values. If $a(\mu) < 0$ and (21) hold, then $-1 < \lambda(\mu) < 0$ and $s(\mu) > 0$. Thus, wealth and real balances are initially on the same side of the steady state, but both alternate signs over time.

Locally, for any deviation on the local saddle path,

$$\begin{pmatrix} \omega_t - \omega(\mu) \\ m_t - m(\mu) \end{pmatrix} \approx c_0 \lambda(\mu)^t \begin{pmatrix} 1 \\ s(\mu) \end{pmatrix}.$$

This equation fully describes the local behavior of the planar system around M_μ.

4.4.3 Inflation dynamics

Given (13), it follows that

$$\Pi_{t+1} \approx \frac{A}{\omega(\mu) + c_0 \lambda(\mu)^{t+1}}.$$

Figure 3 depicts the case of monotonic convergence: The inflation rate converges from above if $\omega_0 < \omega(\mu)$ and from below if $\omega_0 > \omega(\mu)$. In both cases, the economy approaches the dual-money steady state gradually. Figure 4 shows the case of oscillatory convergence. These examples show two economies with the same policy but different dynamics. The economies differ in the fundamentals of the private money issuing sector.

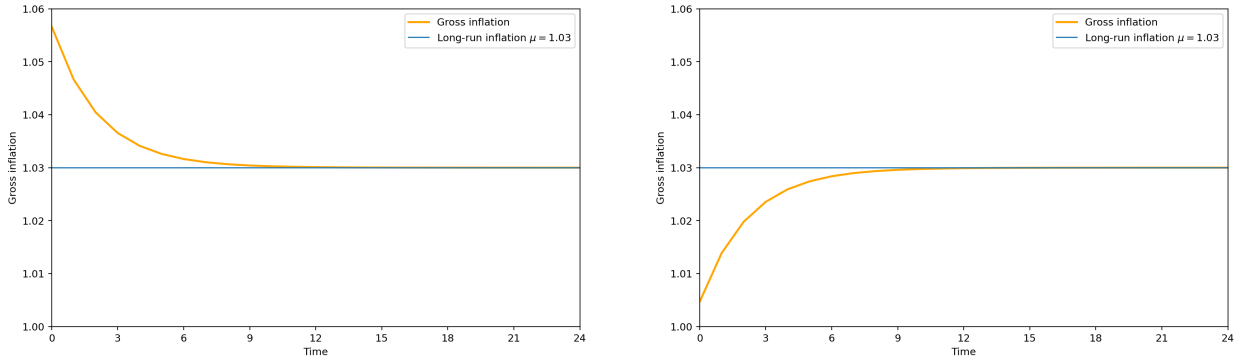


Figure 3: An example of monotonic transitional dynamics. The parametric values are: $A = 0.85$; $\beta = 0.95$; $\eta = 0.80$; $\theta = 0.10$; $\bar{\gamma} = 2.00$; $\alpha = 0.40$; $\zeta = 0.15$; $\sigma = 0.60$; $\nu = 1$; $\epsilon = 1.10$; and $\mu = 1.03$.

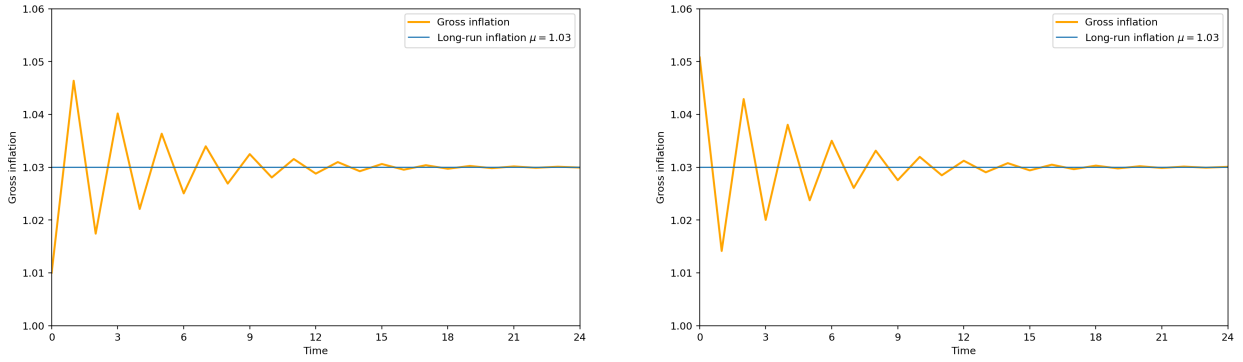


Figure 4: An example of oscillatory transitional dynamics. The parametric values are: $A = 0.41$; $\beta = 0.925$; $\eta = 0.82$; $\theta = 0.57$; $\bar{\gamma} = 4.28$; $\alpha = 0.70$; $\zeta = 0.11$; $\sigma = 0.53$; $\nu = 1$; $\epsilon = 1.83$; and $\mu = 1.03$.

4.5 Local dynamics around the single-money steady state

In this section, we characterize the behavior of the planar system in the neighborhood of the single-money steady state. The local dynamic analysis presented in this section allows us to derive sharp results for the binding regime and the modes of convergence around $(\omega_N, 0)$. Define $a_N \equiv \varphi'(\omega_N)$ and $b_N \equiv F_m(\omega_N, 0) > 0$. The Jacobian matrix at the single-money steady state is

$$DT_\mu(\omega_N, 0) = \begin{pmatrix} a_N & b_N \\ 0 & \frac{\mu}{A}\omega_N \end{pmatrix}.$$

The two eigenvalues are $\lambda_N^\omega = a_N$ and $\lambda_N^m = \mu/\mu_N$. The following proposition provides the conditions for the local stability of the single-money steady state.

Proposition 9 *Suppose that* (15) *and* (16) *hold. Assume that* $\omega_N \neq 1-\theta$. *If* $-1 < \varphi'(\omega_N) < 0$ *and* $\mu < \mu_N$, *then both eigenvalues lie inside the unit circle. The single-money steady state is a two-dimensional local sink.*

Note that the inflation rate rises monotonically towards μ_N only if the mid-life wealth of entrepreneurs eventually declines monotonically towards ω_N. This is not automatic because the wealth eigenvalue a_N is typically negative. It is possible that the inflation trajectory is oscillatory around $(\omega_N, 0)$. The local system is given by

$$\omega_t - \omega_N \approx C_a a_N^t + \frac{b_N C_m}{\lambda_N^m - a_N}(\lambda_N^m)^t,$$

$$m_t \approx C_m(\lambda_N^m)^t.$$

Since $a_N < 0$ and $\lambda_N^m > 0$, there are two generic cases and one knife-edge case.

4.5.1 Case 1: $\lambda_N^m > |a_N|$

For any $C_m > 0$,

$$\omega_t - \omega_N \sim \frac{b_N C_m}{\lambda_N^m - a_N}(\lambda_N^m)^t.$$

Since $b_N > 0$, $C_m > 0$, and $\lambda_N^m - a_N > 0$, the coefficient on the right-hand side is positive. Hence, $\omega_t > \omega_N$ and

$$\frac{\omega_{t+1} - \omega_N}{\omega_t - \omega_N} \to \lambda_N^m \in (0, 1).$$

Therefore, we have $\omega_t \downarrow \omega_N$. In other words, the mid-life wealth of entrepreneurs eventually approaches its steady-state value monotonically from above.

4.5.2 Case 2: $\lambda_N^m < |a_N|$

In this case, the negative eigenvalue dominates if $C_a \neq 0$:

$$\omega_t - \omega_N \sim C_a a_N^t.$$

Because $a_N < 0$, the gap $\omega_t - \omega_N$ oscillates around zero. The exception is the special case: $C_a = 0$. In the linearized system, this means

$$m_0 = \frac{(\lambda_N^m - a_N)}{b_N}(\omega_0 - \omega_N).$$

This is precisely the λ_N^m-eigenvector direction. Along it, we have

$$\omega_t - \omega_N = (\omega_0 - \omega_N)(\lambda_N^m)^t,$$

so the mid-life wealth of entrepreneurs approaches the steady state monotonically from above, even if $\lambda_N^m < |a_N|$.

4.5.3 Case 3: $\lambda_N^m = |a_N|$

In this knife-edge case, neither mode dominates. Note that $a_N = -\lambda_N^m$. The two modes have equal absolute decay rates. The linearized equation is

$$\omega_t - \omega_N = (\lambda_N^m)^t \left[B + C(-1)^t\right].$$

Whether wealth is eventually monotonic or alternating depends on the coefficients B and C.

4.6 Banking cycles in the dual-money economy

Proposition 6 establishes perpetual banking cycles in the single-money economy. We now show that these cycles also organize the dynamics of the dual-money economy. Two

questions arise. First, does the private-money cycle attract equilibrium trajectories along which fiat money is valued, in the same way that the single-money steady state does in Proposition 4? Second, can fiat money remain valued in the long run in an economy that cycles?

Note that the set $\{(\omega, m) : m = 0\}$ is invariant under T_μ. Hence, a two-cycle (ω_L, ω_H) of the single-money economy is also a two-cycle $\{(\omega_L, 0), (\omega_H, 0)\}$ of the dual-money economy. Let $\Phi \equiv \varphi'(\omega_L)\varphi'(\omega_H)$ denote the multiplier of the cycle in the single-money economy and define the threshold

$$\mu_C \equiv \frac{A}{\sqrt{\omega_L \omega_H}}. \tag{26}$$

By (13), the limiting inflation rate along the cycle alternates between A/ω_L and A/ω_H, so μ_C is the geometric mean of the limiting inflation rate over one cycle. It plays the role that μ_N plays for the single-money steady state. Indeed, (12) implies

$$\frac{m_{t+2}}{m_t} = \left(\frac{\mu}{A}\right)^2 \omega_{t+2}\omega_{t+1}.$$

So, along the cycle, fiat money loses value faster than the government creates it if and only if $\mu < \mu_C$.

Proposition 10 *Let (ω_L, ω_H) be a two-cycle of the single-money economy whose points lie away from the kink and satisfy $0 < \omega_L < \omega_H < \bar{\omega}$ and $\omega_H K(\omega_L) < \Gamma$. Then $\{(\omega_L, 0), (\omega_H, 0)\}$ is a two-cycle of the dual-money economy, and the eigenvalues of $DT^2_\mu(\omega_L, 0)$ are Φ and $\Lambda_m(\mu) \equiv \mu^2\mu_C^{-2}$. Suppose that $|\Phi| < 1$. If $\mu < \mu_C$, the cycle is a local sink of the dual-money economy. There exists a neighborhood U of the cycle such that every initial condition $(\omega_0, m_0) \in U$ with $m_0 > 0$ generates a dual-money equilibrium along which m_t converges to zero and (ω_t, ω_{t+1}) converges to (ω_L, ω_H) or to (ω_H, ω_L).*

Proposition 10 is the counterpart of Proposition 4 when $\varphi'(\omega_N) < -1$. In that case, the single-money steady state is a saddle of the dual-money system for $\mu < \mu_N$. Proposition 10 implies that the fluctuations in the mid-life wealth of entrepreneurs, the quantity of private money, and the inflation rate are permanent, and fiat money is displaced asymptotically by private money, even though it is valued at every date. The threshold μ_C exceeds μ_N if and only if $\sqrt{\omega_L\omega_H} < \omega_N$.

We now turn to the second question. A two-cycle of the dual-money economy with valued fiat money must reproduce real balances every two periods, which, by (12), pins down the geometric mean of wealth along the cycle. A sequence $\{(\omega_a, m_a), (\omega_b, m_b), (\omega_a, m_a), \ldots\}$ with

$m_a > 0$ is a two-cycle of the dual-money economy if and only if

$$\omega_a \omega_b = [\omega(\mu)]^2, \tag{27}$$

$$A\,\Psi(\omega_b, \omega_a) = \mu\,\omega_a\,\Psi(\omega_a, \omega_b), \tag{28}$$

and $\Psi(\omega_b, \omega_a) > 0$. Condition (27) states that the geometric mean of wealth over the cycle equals the wealth at the dual-money steady state or, equivalently, by (13), that the geometric mean of the gross inflation rate over the cycle equals the money growth rate: $\Pi_{t+1}\Pi_{t+2} = \mu^2$. The cycle implies $\omega_a < \omega(\mu) < \omega_b$.

Define the function $Z : [\bar{\omega}^{-1}\omega(\mu)^2, \bar{\omega}] \times (\beta, \bar{\mu}) \to \mathbb{R}$ by

$$Z(\omega, \mu) \equiv A\,\Psi\left(\frac{\omega(\mu)^2}{\omega}, \omega\right) - \mu\,\omega\,\Psi\left(\omega, \frac{\omega(\mu)^2}{\omega}\right),$$

where $\bar{\mu} > \beta$ is sufficiently large. The steady state $\omega(\mu)$ is always a root of $Z(\omega, \mu)$. Dual-money two-cycles are the other roots. They come in pairs $\{\omega_a, \omega(\mu)^2/\omega_a\}$. The single-money cycle satisfies (28) for every μ, since $\Psi(\omega_H, \omega_L) = \Psi(\omega_L, \omega_H) = 0$, and it satisfies (27) exactly at $\mu = \mu_C$. The following proposition applies the implicit function theorem to show that a branch of dual-money two-cycles emerges from the single-money cycle at that point.

Proposition 11 *Let (ω_L, ω_H) satisfy the hypotheses of Proposition 10, and suppose that $Z_\omega(\omega_L, \mu_C) \neq 0$. Then there exist $\varepsilon > 0$ and $\mathcal{C}^1$ functions $\omega_a(\mu)$, $\omega_b(\mu) = \omega(\mu)^2/\omega_a(\mu)$, and $m_a(\mu) = \Psi(\omega_b(\mu), \omega_a(\mu))$ on $(\mu_C - \varepsilon, \mu_C + \varepsilon)$ that satisfy (27)–(28) and*

$$(\omega_a(\mu_C), \omega_b(\mu_C), m_a(\mu_C)) = (\omega_L, \omega_H, 0).$$

The dual-money two-cycle converges to the single-money two-cycle as $\mu \to \mu_C$. If $|\Phi| < 1$, the dual-money cycle is asymptotically stable when $Z_\omega(\omega_L, \mu_C) > 0$, and it has a saddle-path structure when $Z_\omega(\omega_L, \mu_C) < 0$.

Lagos and Wright (2003) have shown that in the model with fiat money only, monetary two-cycles can emerge only if the demand for liquidity is non-monotonic in inflation. In the model with private and fiat money developed here, monetary two-cycles emerge with a monotonic demand for liquidity. In the case of CRRA utility, the dual-money cycle can occur even if the coefficient of relative risk aversion is less than one.

Figure 5 provides an example of the different steady states and two-cycles of the model. The value μ_F is the threshold value for the money growth factor that implies the loss of stability in the eigenvalue of the stable root for the dual-money economy. The system

bifurcates, and the periodic points of the dual-money two-cycle enclose the dual-money steady state in the range $\mu_F < \mu < \mu_N$. In this range, the dual-money steady state is a source. For $\mu > \mu_C$, the single-money two-cycle has a saddle-path structure.

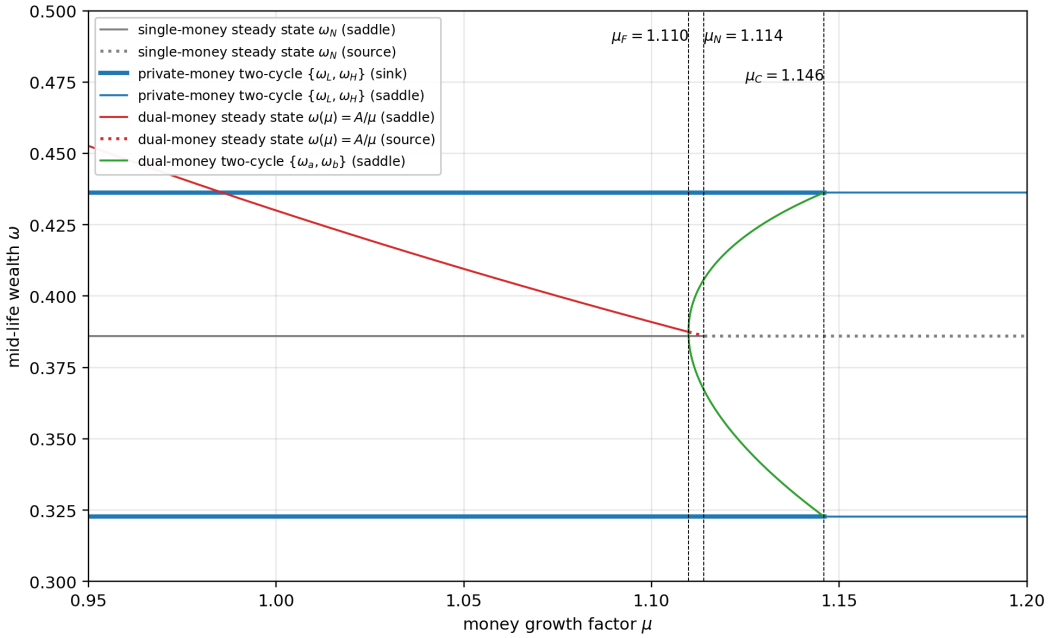


Figure 5: Steady states and two-cycles. The parametric values are: $A = 0.43$; $\beta = 0.95$; $\eta = 1.00$; $\theta = 0.53$; $\bar{\gamma} = 6.30$; $\alpha = 0.46$; $\zeta = 0.47$; $\sigma = 0.75$; $\nu = 1$; and $\epsilon = 1.02$.

4.7 Feedback policy rule

In this subsection, we allow the government to implement a feedback rule of the form: $\mu = \mu(\omega_t, m_t)$, where $\mu : \mathbb{R}^2_+ \to (0, \infty)$ is a smooth function. In that case, the dynamic mapping becomes:

$$\hat{T}(\omega, m) = \left(F(\omega, m), \frac{\mu(\omega, m)}{A} F(\omega, m) m \right).$$

Although the rule changes the viability kernel, it does not change the locus $\omega_{t+1} = \omega_t$. Appendix B provides the details about the dynamic system under a feedback rule. The key result is to show that the policy rule can impact the dynamic properties of the dual-money and single-money steady states.

Proposition 12 *Assume that $H(\omega)$ is strictly increasing with a unique root and that $|\varphi'(\omega_N)| < 1$. Assume also that $\omega_N \neq 1 - \theta$. Suppose that the feedback rule is locally affine: $\mu(\omega, m) =$*

$a\omega + bm + \bar{\mu}$ in a neighborhood of $(\omega_N, 0)$. Let

$$C \equiv \frac{a + bH'(\omega_N) + \frac{A}{\omega_N^2}}{\mu_N H'(\omega_N)} \qquad \text{and} \qquad \delta \equiv \frac{\bar{\mu} - \bar{\mu}_C}{\mu_N},$$

where $\bar{\mu}_C \equiv \mu_N - a\omega_N$. Hold both parameters a and b fixed.

1. *Suppose that $C > 0$. There is $\varepsilon > 0$ such that whenever $\bar{\mu} \in (\bar{\mu}_C - \varepsilon, \bar{\mu}_C)$, (ω_M, m_M) has a saddle-path structure, and $(\omega_N, 0)$ is a local sink.*

2. *Suppose that $C < 0$. There is $\varepsilon > 0$ such that whenever $\bar{\mu} \in (\bar{\mu}_C, \bar{\mu}_C + \varepsilon)$, (ω_M, m_M) is a local sink, and $(\omega_N, 0)$ has a saddle-path structure.*

Figure 6 shows an example of the local phase diagram under the two cases described in Proposition 12. A feedback rule can alter the dynamic properties of the two nearby steady states, depending on the parameters set by the monetary authority.

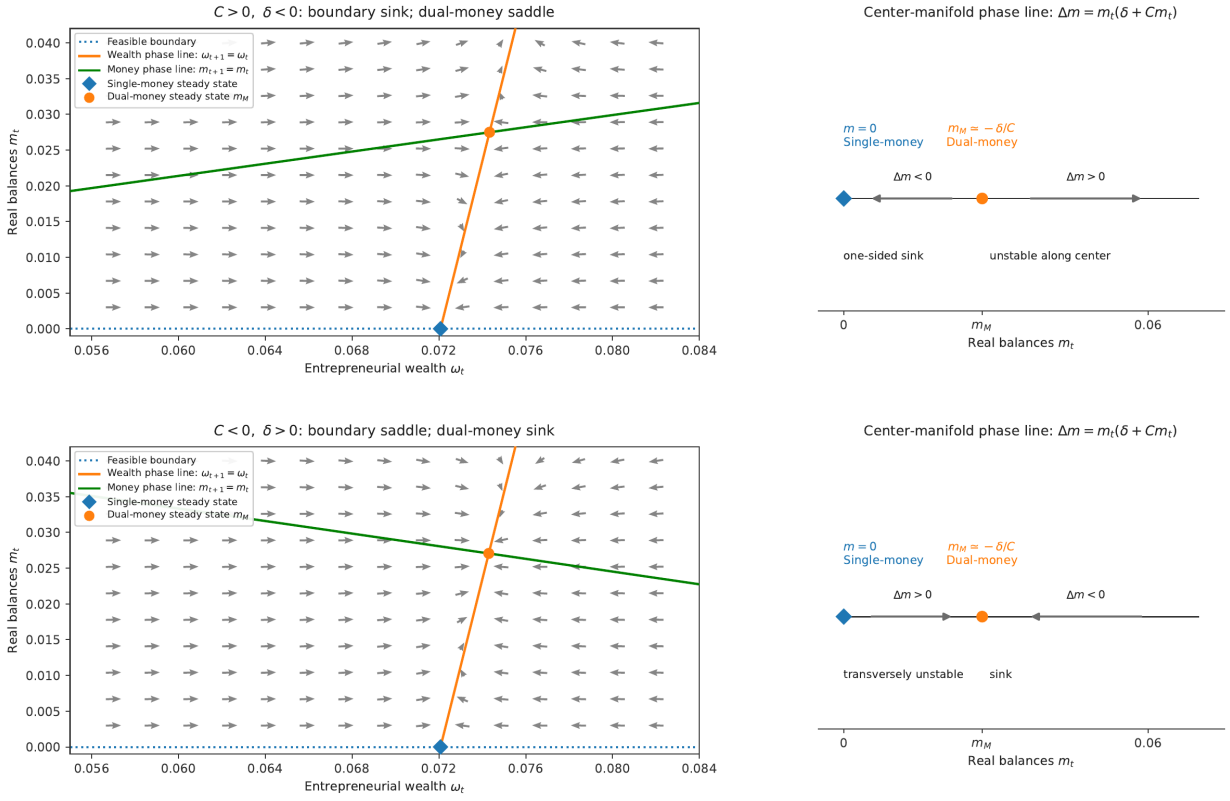


Figure 6: Dynamic properties under a feedback rule. The parametric values are: $A = 0.08$; $\beta = 0.95$; $\eta = 1.00$; $\theta = 0.30$; $\bar{\gamma} = 3.43$; $\alpha = 0.50$; $\zeta = 0.15$; $\sigma = 0.40$; $\nu = 1$; and $\epsilon = 1.25$. The upper panel has $a = -2.00$, $b = 3.7830$, and $\bar{\mu} = 1.1209$. The lower panel has $a = -2.00$, $b = -5.9850$, and $\bar{\mu} = 1.3873$.

5 Optimal Policy

An important question in monetary economics is the optimum quantity of money. We have developed a model with a dual monetary system in which both fiat money and private money coexist as media of exchange, so the quantity of money in this economy is a combination of government-issued money and privately created money. The optimum quantity of money in this hybrid public-private monetary system is presented next.

Let $X_t^b \in \mathbb{R}$, $X_t^s \in \mathbb{R}$, and $X_t \in \mathbb{R}_+$ denote the total period-t CM consumption of buyers, sellers, and entrepreneurs, respectively. Social welfare is given by

$$\sum_{t=0}^{\infty} \beta^t \{X_t^b + X_t^s + X_t + \alpha [u(q_t) - w(q_t)]\},$$

where $q_t \in \mathbb{R}_+$ denotes period-t DM output (symmetric across all matches). We focus on allocations characterized by a cutoff value $\hat{\gamma}_t$ above which the banker's project is operated and below which it is not. The feasibility constraint in period t is given by

$$X_t^b + X_t^s + X_t \leqslant \eta \left[\int_{\hat{\gamma}_{t-1}}^{\bar{\gamma}} \frac{\gamma}{\bar{\gamma}} d\gamma - \left(1 - \frac{\hat{\gamma}_t}{\bar{\gamma}}\right) + A \right].$$

The right-hand side of this resource constraint gives the aggregate output of CM goods, net of the investment cost.

Throughout this section, we set $\nu = 1$. This allows us to isolate the interaction between monetary policy and private-money creation without adding the separate generalized-Nash hold-up distortion. Section 5.1 characterizes the optimal stationary-allocation problem. Section 5.2 characterizes the Ramsey problem in which the planner chooses the money growth factor sequence to maximize welfare, subject to the initial state of the economy ω_0.

5.1 Steady-state comparison

Here, the planner uses the dual-money steady state mapping to choose the stationary allocation that maximizes long-run welfare. We consider a discounted stationary-policy comparison from a common inherited project distribution. Throughout this section, we assume that $H(\omega)$ is strictly increasing with a unique root, and that $\beta < \mu_N$. At the dual-money steady state, the marginal project type is given by

$$\gamma(\mu) = \frac{1}{\mu} max \left\{ 1, \frac{1}{\theta} \left(1 - \frac{A}{\mu}\right) \right\}.$$

From the equilibrium analysis, it follows that

$$\gamma(\mu) = \begin{cases} \frac{1}{\mu} \text{ if } \beta \leqslant \mu < \mu_R \\ \frac{1}{\theta\mu}\left(1 - \frac{A}{\mu}\right) \text{ if } \mu > \mu_R. \end{cases}$$

Given that the planner discounts future periods at the discount factor β, a project produces a net social present value equal to $\beta\gamma - 1$. The socially efficient project threshold is given by $\gamma^* = \frac{1}{\beta}$. Therefore, we find that $\gamma(\mu) > \frac{1}{\beta}$ means under-investment, whereas $\gamma(\mu) < \frac{1}{\beta}$ means that projects with a negative social surplus are being operated, leading to over-investment.

Recall that $m(\mu) > 0 \Leftrightarrow \mu < \mu_N$. We should also require $\gamma(\mu) < \bar{\gamma}$ for a positive measure of active banks. Therefore, the stationary policy set is given by

$$\mathcal{J} = \{\mu \geqslant \beta : m(\mu) > 0,\, \gamma(\mu) < \bar{\gamma}\}.$$

Define the stationary DM welfare as $\alpha\Sigma(q(\mu))$, where $\Sigma(q) \equiv u(q) - w(q)$ denotes the surplus in a match. The quantity $q(\mu)$ is defined by

$$q(\mu) = \psi^{-1}\left(1 + \frac{\mu - \beta}{\alpha\beta}\right),$$

where $\psi(q) \equiv \frac{u'(q)}{w'(q)}$ is strictly decreasing in q.

Therefore, up to constants independent of μ, stationary social welfare is given by

$$\mathcal{W}(\mu) = \alpha\Sigma(q(\mu)) + \frac{\eta}{\bar{\gamma}}\int_{\gamma(\mu)}^{\bar{\gamma}} (\beta\gamma - 1)d\gamma.$$

This allows us to write the optimal policy problem simply as: $\max_{\mu\in\mathcal{J}} \mathcal{W}(\mu)$. After dropping the constant terms, we can write the objective function as:

$$\tilde{\mathcal{W}}(\mu) \equiv \mathcal{W}_D(\mu) + \mathcal{W}_I(\mu),$$

where $\mathcal{W}_D(\mu) \equiv \alpha\Sigma(q(\mu))$ and $\mathcal{W}_I(\mu) \equiv \frac{\eta}{\bar{\gamma}}\left[\gamma(\mu) - \frac{\beta}{2}\gamma(\mu)^2\right]$ denote the two relevant components of the objective function.

Note that

$$q'(\mu) = \frac{1}{\alpha\beta\psi'(q(\mu))} < 0.$$

Therefore, we have

$$\mathcal{W}_D'(\mu) = \frac{u'(q(\mu)) - w'(q(\mu))}{\beta\psi'(q(\mu))}.$$

Define the Friedman rule allocation as that associated with the limit: $\mu \downarrow \beta$. At the Friedman rule, we have $u'(q^*) = w'(q^*)$, so $\mathcal{W}_D'(\beta) = 0$. For any $\mu > \beta$, we have $q(\mu) < q^*$ and $\psi' < 0$. Therefore, $\mathcal{W}_D'(\mu) < 0$ for $\mu > \beta$.

Note that

$$\mathcal{W}_I'(\mu) = \frac{\eta}{\bar{\gamma}}[1 - \beta\gamma(\mu)]\gamma'(\mu) > 0 \Leftrightarrow \left[\gamma(\mu) - \frac{1}{\beta}\right]\gamma'(\mu) < 0.$$

Thus, the welfare of CM investment rises when:

- there is under-investment and $\gamma(\mu)$ falls;
- there is over-investment and $\gamma(\mu)$ rises.

On the return branch, we have $\gamma'(\mu) = -\mu^{-2}$. Consequently, the slope of the welfare function for investment is

$$\mathcal{W}_I'(\mu) = -\frac{\eta(\mu - \beta)}{\bar{\gamma}\mu^3}.$$

For $\mu > \beta$, it follows that $\mathcal{W}_I'(\mu) < 0$. The slope of the total welfare function is

$$\mathcal{W}'(\mu) = \frac{u'(q(\mu)) - w'(q(\mu))}{\beta\psi'(q(\mu))} - \frac{\eta(\mu - \beta)}{\bar{\gamma}\mu^3}$$

for every $\mu > \beta$ in the return regime. Therefore, there is no interior welfare optimum in the return regime.

5.1.1 When the Friedman rule is globally optimal

The previous result allows us to characterize the case in which the Friedman rule ($\mu \downarrow \beta$) attains the first-best allocation. Suppose that

$$\beta \leqslant \mu_R. \tag{29}$$

When $\mu \downarrow \beta$, the dual-money steady state is in the return regime. This means that the collateral constraint is not binding at the Friedman rule. Therefore, we have $q(\beta) = q^*$ and $\gamma(\beta) = \frac{1}{\beta}$. Thus, the Friedman rule achieves the unconstrained first-best allocation. The

optimality of the Friedman rule is indeed global in this case because no other policy can improve either the DM allocation or the CM project-selection allocation.

Proposition 13 *If (29) holds, the Friedman rule is the optimal monetary policy. It attains the unconstrained first-best allocation.*

5.1.2 Collateral regime at the Friedman rule

Suppose that

$$\beta > \mu_R. \tag{30}$$

Then the collateral constraint binds at the Friedman rule. This means that some entrepreneurs who own socially productive projects are credit-constrained under the Friedman rule, which implies that the first-best allocation is no longer available.

The expression for the first derivative of $\gamma(\mu)$ implies that the investment threshold is increasing for $\mu < 2A$, maximized at $\mu = 2A$, and decreasing for $\mu > 2A$. An interior optimum μ^* satisfies

$$-\frac{u'(q(\mu^*)) - w'(q(\mu^*))}{\beta\psi'(q(\mu^*))} = \frac{\eta}{\bar{\gamma}}\left[1 - \frac{\beta(\mu^* - A)}{\theta(\mu^*)^2}\right]\frac{(2A - \mu^*)}{\theta(\mu^*)^3}. \tag{31}$$

The left-hand side of (31) is positive. An interior optimum is possible only where

$$\left[\gamma(\mu) - \frac{1}{\beta}\right]\gamma'(\mu) < 0.$$

Note that $\gamma(\beta) > \beta^{-1}$ since (30) holds. This means that there is under-investment at the Friedman rule.

Since the DM welfare derivative is zero at β, we have

$$\mathcal{W}'(\beta^+) = \frac{\eta}{\bar{\gamma}}\frac{[A - \beta(1-\theta)](2A - \beta)}{\theta^2\beta^4}.$$

Because $A - \beta(1-\theta) < 0$, we find that: $sgn[\mathcal{W}'(\beta^+)] = sgn[\beta - 2A]$. Therefore, the following must be true: $\beta > 2A \Rightarrow \mathcal{W}'(\beta^+) > 0$; $\beta = 2A \Rightarrow \mathcal{W}'(\beta^+) = 0$; $\beta < 2A \Rightarrow \mathcal{W}'(\beta^+) < 0$. These properties allow us to evaluate the implications of the Friedman rule under the collateral regime.

Proposition 14 *Assume that $\gamma(\beta^+) < \bar{\gamma}$. Suppose that (30) holds. If $\beta > 2A$, then the Friedman rule is not optimal. If $\beta < 2A$, then the Friedman rule is locally optimal, but it*

may not be globally optimal.

5.1.3 A sufficient condition for the existence of an optimum

It is helpful to characterize the efficient project selection, denoted by μ_E, that satisfies

$$\gamma(\mu_E) = \frac{1}{\beta} \Leftrightarrow \theta\mu_E^2 - \beta\mu_E + \beta A = 0.$$

When the collateral constraint binds at $\mu \downarrow \beta$, the relevant root is

$$\mu_E = \frac{\beta + \sqrt{\beta^2 - 4\theta\beta A}}{2\theta}.$$

The social surplus from project selection is maximized at $\mu = \mu_E$, where the marginal project has a zero social net present value. However, DM output is below q^*, so μ_E generally is not the overall welfare optimum.

Proposition 15 *Assume that $\bar{\gamma} > 1/(4\theta A)$. Suppose that (30) holds. If $\beta > 2A$ and $\mu_E < \mu_N$, then there exists an interior optimum $\mu^* \in (\beta, \mu_E)$ in $\mathcal{J}$.*

5.2 Ramsey problem

In this section, we solve the Ramsey problem of finding the money-growth factor sequence that implies maximum welfare, given the economy's initial state (ω_0, m_0). Then, ω_1 is determined by $m_0 = \Psi(\omega_1, \omega_0)$. Because the DM occurs before the CM in each period, q_0 is inherited through f_{-1} and ω_0. It is constant across candidate policies for a fixed ω_0.

Note that in period t, DM output is determined by the inherited entrepreneurial wealth. Since $f_{t-1} = \frac{A}{\omega_t}$, the buyer's first-order condition implies

$$\alpha\psi(q_t) + 1 - \alpha = \frac{A}{\beta\omega_t} \Leftrightarrow q_t = \psi^{-1}\left(\frac{\frac{A}{\beta\omega_t} - (1-\alpha)}{\alpha}\right) \equiv q(\omega_t). \tag{32}$$

The period-t marginal project is given by

$$\gamma_t^m = \frac{\omega_{t+1}K(\omega_t)}{A}. \tag{33}$$

The flow-welfare measure is therefore

$$\Lambda(\omega_t, \omega_{t+1}) \equiv \alpha\left[u(q_t) - w(q_t)\right] + \frac{\eta}{\bar{\gamma}}\int_{\gamma_t^m}^{\bar{\gamma}} (\beta\gamma - 1)d\gamma,$$

where q_t satisfies (32) and γ_t^m satisfies (33).

Define $\mu_t \equiv \frac{M_{t+1}}{M_t}$. Along any strictly positive feasible path, we have

$$\mu_t = \frac{A\Psi(\omega_{t+2}, \omega_{t+1})}{\omega_{t+1}\Psi(\omega_{t+1}, \omega_t)}. \tag{34}$$

Once the planner selects $\{\omega_t\}_{t=0}^{\infty}$, equation (34) tells us which time-varying money-growth factor sequence implements it.

The primal Ramsey problem is defined by

$$\max_{\{\omega_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \Lambda(\omega_t, \omega_{t+1}),$$

subject to $0 \leqslant \omega_{t+1} \leqslant \hat{\omega}(\omega_t)$, $\Psi(\omega_{t+1}, \omega_t) > 0$, and $A\Psi(\omega_{t+2}, \omega_{t+1}) \geqslant \beta\omega_{t+1}\Psi(\omega_{t+1}, \omega_t)$, given ω_0. The Ramsey problem has a recursive formulation. Let the state be represented by two consecutive wealth levels, $(\omega, \omega') = (\omega_t, \omega_{t+1})$. Then, we can write the Bellman equation:

$$\hat{J}(\omega, \omega') = \Lambda(\omega, \omega') + \beta \max_{\zeta} \hat{J}(\omega', \zeta),$$

subject to $0 \leqslant \zeta \leqslant \hat{\omega}(\omega')$, $\Psi(\zeta, \omega') > 0$, and $A\Psi(\zeta, \omega') \geqslant \beta\omega'\Psi(\omega', \omega)$.

Next, it is helpful to formulate a relaxed one-state problem as follows. First, solve the problem without the constraint $\mu_t \geqslant \beta$. If the recovered money-growth sequence satisfies $\mu_t \geqslant \beta$ at every date, the relaxed solution also solves the constrained Ramsey problem. When the recovered sequence $\{\mu_t\}_{t=0}^{\infty}$ from (34) is such that $\mu_t \geqslant \beta$ holds for all $t \geqslant 0$, the problem reduces to

$$J(\omega) = \max_{\omega' \in \Theta(\omega)} \{\Lambda(\omega, \omega') + \beta J(\omega')\},$$

where the feasible set is

$$\Theta(\omega) = \{\zeta : 0 \leqslant \zeta \leqslant \hat{\omega}(\omega),\ \Psi(\zeta, \omega) > 0\}.$$

The value, conditional on the inherited ω_0, is

$$V(\omega_0) = \max_{\omega_1 \in I_R(\omega_0)} \hat{J}(\omega_0, \omega_1),$$

where $I_R(\omega_0)$ contains initial choices that allow for an infinite feasible continuation.

To apply the contraction mapping theorem, it is helpful to take the following steps. Let

$I = [a, b] \subset (0, \bar{\omega}]$ be a set containing the initial state ω_0. Define

$$\Theta_I(\omega) = \{\zeta \in I : \zeta \leqslant \hat{\omega}(\omega),\ \Psi(\zeta, \omega) \geqslant 0\}.$$

Assume that the set $\Theta_I(\omega)$ is nonempty for every $\omega \in I$, that the correspondence Θ_I is compact-valued and continuous, and that Λ is continuous on the graph of Θ_I.

Under these conditions, the relaxed Bellman operator, defined on the space of continuous and bounded functions,

$$TJ(\omega) = \max_{z \in \Theta_I(\omega)} \{\Lambda(\omega, z) + \beta J(z)\},$$

is a contraction with modulus β. It therefore has a unique fixed point. Let $h(\omega)$ denote the relaxed Ramsey policy function: $\omega_{t+1} = h(\omega_t)$. If the maximizer is unique, $h(\omega)$ is continuous.

Define $r(\omega) \equiv \Psi(h(\omega), \omega)$. If $r(\omega)$ is continuous and positive everywhere, then it must be that

$$\min_{\omega \in I} r(\omega) > 0.$$

Therefore, the money-growth factor that implements this policy is defined by

$$k(\omega) \equiv \frac{Ar(h(\omega))}{h(\omega)r(\omega)}.$$

If $k(\omega) \geqslant \beta$ holds, the relaxed optimizer satisfies all the constraints of the original Ramsey problem. Because the original feasible set is contained in the relaxed one, a relaxed optimizer that is feasible for the original problem must also be optimal for the original problem.

The Euler equation for an interior optimum is

$$\Lambda_2(\omega_{t-1}, \omega_t) + \beta \Lambda_1(\omega_t, \omega_{t+1}) = 0,$$

where $\Lambda_1(\omega, \omega')$ is the partial derivative of $\Lambda(\omega, \omega')$ with respect to the first argument, and $\Lambda_2(\omega, \omega')$ is the partial derivative with respect to the second argument. An interior Ramsey steady state, denoted by ω^{**}, satisfies

$$\Lambda_2(\omega^{**}, \omega^{**}) + \beta \Lambda_1(\omega^{**}, \omega^{**}) = 0.$$

The long-run money growth factor is $\mu^{**} \equiv \frac{A}{\omega^{**}}$. Using the functional form of $\Lambda(\omega, \omega')$, an

interior Ramsey steady state can be written as

$$\beta\hat{S}'(\omega^{**}) + \frac{\eta}{\Gamma}(1-\beta\gamma^{**})[K(\omega^{**}) + \beta\omega^{**}K'(\omega^{**})] = 0,$$

where $\gamma^{**} \equiv \hat{\gamma}(\omega^{**}, \omega^{**})$, $\hat{S}(\omega) \equiv \alpha\Sigma(q(\omega))$, and $\hat{\gamma}(\omega, \zeta) \equiv \frac{\zeta K(\omega)}{A}$.

Note that the Ramsey steady state generically differs from the optimum steady state obtained by comparing the dual-money steady state indexed by μ. The reason is analogous to the modified golden rule: The Ramsey planner discounts the DM welfare gain that ω_{t+1} delivers next period and the date-$(t+1)$ cutoff effect through $K(\omega_{t+1})$. The next proposition derives some properties of the Ramsey solution. The results are, admittedly, conditional on the required smoothness of the value and policy functions.

Proposition 16 *Define $Q(\omega, z) \equiv \Lambda(\omega, z) + \beta J(z)$. Assume that $h(\omega) < \theta\Gamma$ holds on the collateral branch. Suppose that on a compact invariant interval $I = [a, b]$ the following holds: (i) the optimum is unique, interior, and remains within one smooth regime; (ii) $J, Q \in \mathcal{C}^2$; (iii) $Q_{zz} \leqslant -c < 0$ for some $c > 0$; (iv) $0 \leqslant \Lambda_{12} \leqslant d < c$; and (v) the feasibility constraints are slack. Then, $h : I \to I$ and $r : I \to (0, \infty)$ satisfy:*

1. *h is non-decreasing;*
2. *h has a unique fixed point ω^{**};*
3. *r is strictly increasing.*

Moreover, we have: (i) $\omega < \omega^{**} \Rightarrow k(\omega) > \mu^{**}$ and (ii) $\omega > \omega^{**} \Rightarrow k(\omega) < \mu^{**}$. If, in addition, $k(\omega) \geqslant \beta$ holds for every $\omega \in I$, the relaxed policy also solves the constrained Ramsey problem.

This proposition produces the following corollary:

$$\omega_t < \omega^{**} \Rightarrow \begin{cases} \mu_t > \mu^{**} \\ m_{t+1} > m_t \\ \omega_{t+1} > \omega_t \\ \Pi_{t+1} < \Pi_t \end{cases} \qquad \omega_t > \omega^{**} \Rightarrow \begin{cases} \mu_t < \mu^{**} \\ m_{t+1} < m_t \\ \omega_{t+1} < \omega_t \\ \Pi_{t+1} > \Pi_t. \end{cases}$$

When the net worth of banks is low ($\omega_t < \omega^{**}$), the Ramsey planner sets the money growth path above the steady state to relax external-financing constraints and accelerate the development of the banking sector. When the net worth of banks is high ($\omega_t > \omega^{**}$), the Ramsey planner chooses the money growth path below the long-run growth rate, allowing the net

worth of banks to decline towards their Ramsey steady-state values. Under sufficient conditions, we find that the optimal policy is initially expansionary (with money growth above its steady-state value) when issuers begin under-capitalized and initially contractionary (with money growth below its steady-state value) when issuers begin over-capitalized. Figure 7 provides an example.

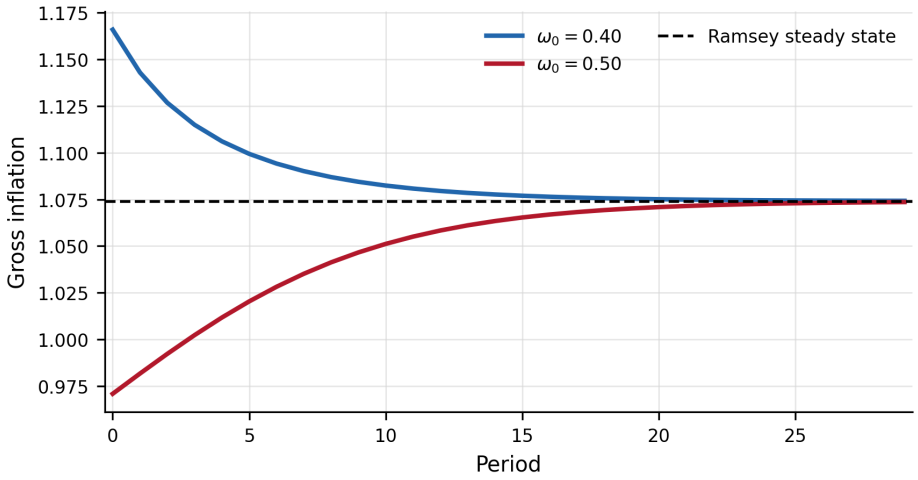


Figure 7: Ramsey inflation paths. The parametric values are: $A = 0.48$; $\beta = 0.95$; $\eta = 1.00$; $\theta = 0.485$; $\bar{\gamma} = 2.50$; $\alpha = 0.50$; $\zeta = 0.04$; $\sigma = 0.50$; $\nu = 1$; and $\epsilon = 1.20$.

6 Conclusion

This paper studies a monetary system in which public fiat money and collateral-backed private liabilities jointly provide liquidity services. The central mechanism is a feedback loop between bank capitalization and the market value of private money. The net worth of banks determines individual funding needs and the allocation of resources across productive projects. Private-money issuance determines its return, and that return influences the net worth of banks in the following period. This mechanism generates aggregate fluctuations without external shocks or sticky prices. The fundamentals of the private money issuing sector determine the type of dynamics in the economy. A private-money economy can converge through persistent oscillations or settle into a stable banking cycle. When public and private money coexist, the same policy can lead to fundamentally different dynamic behaviors if the primitives of the private money issuing sector differ across two economies. For the same primitives, we find that a small change in policy can result in a drastic change in the properties of the dynamic system. We have also shown that banking cycles are robust to the presence of public money.

This paper also offers valuable insights into the optimum quantity of money under a hybrid monetary system. The Friedman rule implements the first best when collateral constraints are slack. When they bind, policy trades off decentralized-market output against the allocation of resources across productive projects, and a money-growth factor above the Friedman rule can be optimal. Along the Ramsey transition path, the policymaker manages the capitalization of money issuers by choosing a non-stationary path for the money growth rate to maximize welfare, subject to the economy's initial state.

These results do not rely on defaults, runs, or informational opacity; private claims are recognizable and collateralized. The balance sheets of money issuers alone are sufficient to generate endogenous fluctuations and a state-dependent role for policy. The broader implication is that, in a hybrid monetary system, inflation is jointly determined by public issuance and the capitalization of money-issuing private institutions.

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A Proofs

Proof of Proposition 1. If condition (16) holds, then $H(\omega)$ is strictly increasing everywhere in its domain. Since $H(\bar{\omega}) > 0$ and $H(0) < 0$, there is a unique $\omega_N \in (0, \bar{\omega})$ that satisfies $H(\omega_N) = 0$. ■

Proof of Proposition 2. Conditions (15) and (16) imply that $H(\omega)$ is strictly increasing everywhere in its domain and that $H(\bar{\omega}) > 0$. We have shown that $H(\omega) \to 0$ as $\omega \downarrow \omega_N$. Also, note that $H(\omega) \to H(\bar{\omega})$ as $\omega \uparrow \bar{\omega}$. Since $\omega(\mu) = \frac{A}{\mu}$, a unique dual-money steady state exists if and only if $\beta < \mu < \mu_N$. The limit $\mu \downarrow \beta$ implies that

$$m(\mu) = H\left(\frac{A}{\mu}\right) \to H\left(\frac{A}{\beta}\right) = m(\beta) > 0.$$

The limit $\mu \uparrow \mu_N$ implies that

$$m(\mu) = H\left(\frac{A}{\mu}\right) \to H\left(\frac{A}{\mu_N}\right) = 0.$$

■

Proof of Proposition 3. We show the proof for the case in which ω_t is monotonically increasing. The case in which ω_t is monotonically decreasing is analogous. Suppose that $(\omega_0, m_0) \in \mathcal{F}$ and that the trajectory derived from it satisfies $\omega_{t+1} > \omega_t$ for all $t \geqslant 0$. Given that the wealth sequence is bounded, there exists a limit: $\omega_t \to \tilde{\omega}$. By the continuity of Ψ, we have $m_t \to H(\tilde{\omega})$. If we take the limits on both sides of (12), we get

$$H(\tilde{\omega}) = \frac{\mu}{A}\tilde{\omega}H(\tilde{\omega}).$$

Consequently, either $H(\tilde{\omega}) = 0$ or $\tilde{\omega} = \omega(\mu)$. There are, therefore, only two possible cases. First, suppose that $H(\tilde{\omega}) = 0$. This implies that $\tilde{\omega} = \omega_N$, so the limit is $(\omega_N, 0)$. Second, suppose that $H(\tilde{\omega}) > 0$. Then, we have $\tilde{\omega} = \omega(\mu)$. Therefore, it must be that $H(\omega(\mu)) = m(\mu)$, which means that the limit is $(\omega(\mu), m(\mu))$. ■

Proof of Proposition 4. Let $\varphi(\omega) \equiv F(\omega, 0)$. Suppose that $(\omega_N, 0)$ is feasible. By hypothesis, $|\varphi'(\omega_N)| < 1$. Choose $\delta > 0$ such that there is $0 < c < 1$ that satisfies $I_\delta = [\omega_N - \delta, \omega_N + \delta]$, $\omega_N + \delta < \omega(\mu)$, and $\sup_{\omega \in I_\delta} |\varphi'(\omega)| \leqslant c < 1$. Let B be an upper bound for $F_m(\omega, m)$ on a sufficiently small rectangle and choose $\bar{m} > 0$ so that $c\delta + B\bar{m} \leqslant \delta$ and $I_\delta \times [0, \bar{m}] \subset \mathcal{F}$. Then, the rectangle $\mathcal{R}_N = I_\delta \times [0, \bar{m}]$ is forward invariant.

Note that $|\omega_{t+1} - \omega_N| \leqslant c|\omega_t - \omega_N| + Bm_t$ and $m_{t+1} \leqslant \rho(\mu) m_t$, where $\rho(\mu) \equiv [\omega(\mu)]^{-1}(\omega_N + \delta)$. Consequently, $m_t \leqslant \rho(\mu)^t m_0 \to 0$ and $\omega_t \to \omega_N$. For every $m_0 > 0$ in this open rectangle, fiat money remains positively valued at every finite date, but $\Pi_{t+1} \to \mu_N > \mu$. ■

Proof of Proposition 5. Define $\upsilon \equiv \frac{\mu}{\mu_N} - 1$, $a_N \equiv \varphi'(\omega_N)$, and $b_N \equiv F_m(\omega_N, 0) > 0$. Let $\hat{\omega}_t \equiv \omega_t - \omega_N$. Note that $\omega(\mu) = \frac{\omega_N}{1+\upsilon}$. Define the function $\hat{F}(\hat{\omega}, m) \equiv F(\omega_N + \hat{\omega}, m) - \omega_N$. Since $F(\omega_N, 0) = \omega_N$, its first-order expansion is

$$\hat{\omega}_{t+1} = a_N \hat{\omega}_t + b_N m_t + \mathcal{O}(||(\hat{\omega}_t, m_t)||^2).$$

Substituting $\hat{\omega}_t$ in (12), we get the exact relation:

$$m_{t+1} = (1 + \upsilon) m_t + \frac{1 + \upsilon}{\omega_N} \hat{\omega}_{t+1} m_t. \tag{35}$$

At $\upsilon = 0$ the eigenvalues of the Jacobian matrix are a_N and 1. Under $|a_N| < 1$, $\hat{\omega}$ is the stable direction.

Represent the parameter-dependent center manifold by $\hat{\omega} = g(m, \upsilon)$. Because $(\omega_N, 0)$ is a fixed point for every μ, we have $g(0, \upsilon) = 0$. Its first-order expansion is $g(m, \upsilon) = cm + \mathcal{O}(m^2 + |\upsilon| m)$. To obtain the coefficient c, impose invariance at $\upsilon = 0$. Along the center manifold, $\hat{\omega}_t = cm_t$. Then,

$$\hat{\omega}_{t+1} = (a_N c + b_N) m_t + \mathcal{O}(m_t^2).$$

At the same time, we have $\hat{\omega}_{t+1} = \hat{\omega}_t + \mathcal{O}(m_t^2)$ when $\upsilon = 0$. Hence, it must be that $c = \frac{b_N}{1 - a_N}$. Thus, locally on the center manifold, we have

$$\hat{\omega}_{t+1} = \frac{b_N}{1 - a_N} m_t + \mathcal{O}(m_t^2 + |\upsilon| m_t).$$

Since $\Psi(F(\omega, m), \omega) = m$, it follows that $H'(\omega_N) = \frac{1-a_N}{b_N}$. Substituting the expression for $\hat{\omega}_{t+1}$ into (35), we get

$$m_{t+1} = (1+\upsilon)m_t + \frac{1}{\omega_N H'(\omega_N)} m_t^2 + \mathcal{O}(m_t^3 + |\upsilon| m_t^2).$$

The coefficient on the quadratic term is positive. The two nearby fixed points of this reduced equation satisfy

$$m\left[\upsilon + \frac{m}{\omega_N H'(\omega_N)} + ...\right] = 0.$$

Hence, they are $m_N = 0$ and $m_M = -\upsilon\omega_N H'(\omega_N) + \mathcal{O}(\upsilon^2)$. For $\upsilon < 0$, the second fixed point is positive. This is the dual-money steady state, while $m = 0$ is the single-money steady state. Their collision at $\upsilon = 0$ produces a boundary transcritical bifurcation. On the center manifold, the reduced map is $m \mapsto (1+\upsilon)m + \frac{m^2}{\omega_N H'(\omega_N)}$; for $\upsilon < 0$, its derivative is $1+\upsilon$ at 0 and $1 - \upsilon + \mathcal{O}(\upsilon^2) > 1$ at (ω_M, m_M), which with the attracting $\hat{\omega}$-direction gives the saddle and the separatrix. ■

Proof of Proposition 6. Define $G(\omega) \equiv \varphi^2(\omega) - \omega$. Because ω_N is a fixed point, $G(\omega_N) = 0$. Moreover, $G'(\omega_N) = [\varphi'(\omega_N)]^2 - 1 > 0$. There is $\varepsilon > 0$ such that $G(\omega) > 0$ for all $\omega \in (\omega_N, \omega_N + \varepsilon]$. Therefore, we have $\varphi^2(\omega) > \omega$ for any ω slightly above the steady state. Now consider the upper bound Ω. Since $\mathcal{I}$ is invariant, $\varphi^2(\Omega) \in \mathcal{I}$. Hence, $G(\Omega) \leqslant 0$.

Suppose that $G(\Omega) = 0$. Then, $\varphi^2(\Omega) = \Omega$. Since $\Omega > \omega_N$ and φ is strictly decreasing, $\varphi(\Omega) < \varphi(\omega_N) = \omega_N < \Omega$. Therefore, setting $\omega_L = \varphi(\Omega)$ and $\omega_H = \Omega$ gives a nontrivial two-period cycle. Suppose that $G(\Omega) < 0$. We can find $\omega_N < \hat{\omega} < \Omega$ sufficiently close to Ω such that $G(\hat{\omega}) < 0$. Hence, $\varphi^2(\hat{\omega}) < \hat{\omega}$. Since φ is strictly decreasing, we have $\hat{\omega} > \varphi(\hat{\omega})$. Then, we can apply Theorem 8.2 from Azariadis (1993) to show the existence of a two-period cycle. ■

Proof of Proposition 8. Let $M_\mu \equiv (\omega(\mu), m(\mu))$ denote the dual-money steady state. Suppose that $\bar{\omega} < 1 - \theta$. For any $\mu \in \mathcal{A}$ satisfying (20), we have $\omega(\mu) = \frac{A}{\mu} < 1 - \theta$. Thus, M_μ is in the collateral regime. Suppose that $\bar{\omega} > 1 - \theta$. For any $\mu \in \mathcal{A}$ satisfying (20) and $\frac{A}{\mu} < 1 - \theta$, M_μ is in the collateral regime. For any $\mu \in \mathcal{A}$ satisfying (20) and $\frac{A}{\mu} > 1 - \theta$, M_μ is in the return regime.

The threshold μ_P determines whether the coefficient $a(\mu)$ is positive or negative. If $\mu \in \mathcal{A}$ and $\mu < min\{\mu_P, \mu_N\}$, then $a(\mu) > 0$, which implies $\lambda(\mu) \in (0, 1)$, so convergence to M_μ is monotonic. If $\mu \in \mathcal{A}$ and $\mu > \mu_P$, then $a(\mu) < 0$. If (21) holds, then $-1 < \lambda(\mu) < 0$, so convergence to M_μ is oscillatory. ■

Proof of Proposition 10. Since $m_{t+1} = A^{-1}\mu\omega_{t+1}m_t$, the set $\{(\omega, m) : m = 0\}$ is invariant under T_μ, and $F(\omega, 0) = \varphi(\omega)$. Hence, $T_\mu(\omega_L, 0) = (\omega_H, 0)$ and $T_\mu(\omega_H, 0) = (\omega_L, 0)$. Away from the kink, we have

$$DT_\mu(\omega, 0) = \begin{pmatrix} \varphi'(\omega) & F_m(\omega, 0) \\ 0 & \frac{\mu}{A}\varphi(\omega) \end{pmatrix}.$$

By the chain rule, $DT_\mu^2(\omega_L, 0) = DT_\mu(\omega_H, 0)\, DT_\mu(\omega_L, 0)$. The product of two upper triangular matrices is upper triangular. Its diagonal entries are the products of the corresponding diagonal entries:

$$\Phi \equiv \varphi'(\omega_H)\varphi'(\omega_L) \qquad \text{and} \qquad \Lambda_m(\mu) \equiv \frac{\mu^2\omega_L\omega_H}{A^2}.$$

These are the eigenvalues of $DT_\mu^2(\omega_L, 0)$ and, by the same argument, of $DT_\mu^2(\omega_H, 0)$.

If $|\Phi| < 1$ and $\mu < \mu_C$, both eigenvalues lie inside the unit circle, so $(\omega_L, 0)$ is a sink of T_μ^2 and the orbit $\{(\omega_L, 0), (\omega_H, 0)\}$ is a local sink of T_μ. Choose a neighborhood U of the orbit that is forward invariant under T_μ^2, on which every point converges to the orbit, and on which the constraints $\omega \leqslant \bar{\omega}$ and $F(\omega, m)K(\omega) \leqslant \Gamma$ hold. Points of U with $m > 0$ are mapped into points with $m > 0$, since $F(\omega, m)$ is close to ω_L or ω_H and hence strictly positive. A trajectory starting in U with $m_0 > 0$ satisfies $m_t > 0$ for all $t \geqslant 0$, $m_t \to 0$, and $(\omega_t, \omega_{t+1}) \to (\omega_L, \omega_H)$ or (ω_H, ω_L). It satisfies (11)–(12) by construction and the feasibility constraints because it never leaves U. It is therefore a dual-money equilibrium. ■

Proof of Proposition 11. Consider the equation $Z(\omega, \mu) = 0$. Away from the kink, Z is continuously differentiable in (ω, μ). At $\mu = \mu_C$, we have

$$\frac{\omega(\mu_C)^2}{\omega_L} = \frac{A^2}{\mu_C^2\omega_L} = \omega_H$$

by (26), so $Z(\omega_L, \mu_C) = 0$. Since $Z_\omega(\omega_L, \mu_C) \neq 0$, the implicit function theorem yields $\varepsilon > 0$ and a $\mathcal{C}^1$ function $\omega_a(\mu)$ on $(\mu_C - \varepsilon, \mu_C + \varepsilon)$ with $\omega_a(\mu_C) = \omega_L$ and $Z(\omega_a(\mu), \mu) = 0$. Setting

$$\omega_b(\mu) = \frac{\omega(\mu)^2}{\omega_a(\mu)} \qquad \text{and} \qquad m_a(\mu) = \Psi(\omega_b(\mu), \omega_a(\mu))$$

gives $\mathcal{C}^1$ functions satisfying (27)–(28), with $\omega_b(\mu_C) = \omega_H$ and $m_a(\mu_C) = \Psi(\omega_H, \omega_L) = 0$. Under $|\Phi| < 1$, we have $m_a'(\mu_C) \neq 0$. Then, $m_a(\mu)$ has the sign of $m_a'(\mu_C)(\mu - \mu_C)$ for μ close to μ_C. Whenever $m_a(\mu) > 0$, it follows that $(\omega_a(\mu), m_a(\mu))$ and $(\omega_b(\mu), m_b(\mu))$, with $m_b = \frac{\mu}{A}\omega_b m_a > 0$, form a two-cycle of the dual-money economy. The feasibility constraints hold at both points for μ close to μ_C, because they hold with strict inequality at the private-money cycle. Continuity of the branch gives the convergence to (ω_L, ω_H) as $\mu \to \mu_C$.

Let $\mathbf{X}_t = (\omega_t, m_t)$ denote the state at date t. Define $\mathbf{X}_a = (\omega_a, m_a)$ and $\mathbf{X}_b = (\omega_b.m_b)$. The cycle conditions imply $F(\mathbf{X}_a) = \omega_b$ and $F(\mathbf{X}_b) = \omega_a$. To study local stability, consider deviations from the two-cycle every two periods:

$$\hat{\mathbf{X}}_{t+2} = M_a \hat{\mathbf{X}}_t + \mathcal{O}(||\hat{\mathbf{X}}_t||^2),$$

where $M_a = DT_\mu^2(\mathbf{X}_a) = DT_\mu(\mathbf{X}_b)DT_\mu(\mathbf{X}_a)$. The corresponding matrix at $\mathbf{X}_b$ is $M_b = DT_\mu^2(\mathbf{X}_b) = DT_\mu(\mathbf{X}_a)DT_\mu(\mathbf{X}_b)$.

It can be shown that $\det M_a = a_a a_b$, where $a_a \equiv -\frac{\Psi_m(\omega_b,\omega_a)}{\Psi_\omega(\omega_b,\omega_a)}$ and $a_b \equiv -\frac{\Psi_m(\omega_a,\omega_b)}{\Psi_\omega(\omega_a,\omega_b)}$. The trace of M_a is

$$trM_a = a_a a_b + \frac{\mu}{A} m_a a_a b_b + \frac{\mu}{A} m_b a_b b_a + \left(\frac{\mu}{A}\right)^2 (\omega_a + m_b b_b)(\omega_b + m_a b_a).$$

The characteristic polynomial is $P(\lambda) = \lambda^2 - trM_a\lambda + \det M_a$. The dual-money two-cycle is a local sink if and only if $P(1) > 0$, $P(-1) > 0$, and $\det M_a < 1$. If $P(1)P(-1) < 0$, it has a saddle-path structure. Finally, note that

$$P(1) = \frac{\mu m_a b_a b_b}{A^2} Z_\omega(\omega_a, \mu).$$

Since $\mu > 0$, $m_a > 0$, $b_a > 0$, and $b_b > 0$, we have $sgn[P(1)] = sgn[Z_\omega(\omega_a, \mu)]$. Given these properties, the analysis near μ_C delivers the desired local stability result. ■

Proof of Proposition 12. Holding a, b fixed, define $\bar{\mu}_C \equiv \mu_N - a\omega_N$ and $\delta \equiv \frac{\bar{\mu}-\bar{\mu}_C}{\mu_N}$. At $(\omega_N, 0) = \mu_N(1+\delta)$. For small positive m, we have $\frac{m_{t+1}}{m_t} \to 1 + \delta$. The steady state $(\omega_N, 0)$ exists for every $\bar{\mu}$. Define $\hat{\omega}_t \equiv \omega_t - \omega_N$, $s_N \equiv \varphi'(\omega_N)$, and $q_N \equiv F_m(\omega_N, 0)$. It follows that $H'(\omega_N) = \frac{1-s_N}{q_N}$. At the critical parameter value, the Jacobian is

$$J_N(0) = \begin{pmatrix} s_N & q_N \\ 0 & 1 \end{pmatrix}.$$

Since $|s_N| < 1$, the wealth direction is stable, while the unit eigenvalue generates a one-dimensional center manifold. Its tangent satisfies $\hat{\omega} = \frac{m}{H'(\omega_N)}$. Precisely, the parameter-dependent center manifold can be written as

$$\hat{\omega} = \frac{m}{H'(\omega_N)} + \mathcal{O}(m^2 + |\delta|m).$$

Locally, the first dynamic equation can be written as

$$\hat{\omega}_{t+1} = s_N \hat{\omega}_t + q_N m_t + \mathcal{O}(||(\hat{\omega}_t, m_t)||^2).$$

The second dynamic equation is exactly

$$m_{t+1} = \frac{\mu_N(1+\delta) + a\hat{\omega}_t + bm_t}{A}(\omega_N + \hat{\omega}_{t+1})m_t.$$

At $\delta = 0$, we have

$$\hat{\omega}_t = \frac{m_t}{H'(\omega_N)} + \mathcal{O}(m_t^2).$$

Substituting this into the first dynamic equation, we find that $\hat{\omega}_{t+1} = \mathcal{O}(m_t)$. Now set $\delta = 0$ in the second dynamic equation. Since $\hat{\omega}_{t+1} = \mathcal{O}(m_t)$, we find that $\hat{\omega}_{t+1} m_t = \mathcal{O}(m_t^2)$. Therefore, $m_{t+1} = m_t + \mathcal{O}(m_t^2)$. In other words, the difference $m_{t+1} - m_t$ has no linear term. Then, we have

$$\hat{\omega}_{t+1} = \frac{m_t}{H'(\omega_N)} + \mathcal{O}(m_t^2).$$

On the center manifold, we have

$$m_{t+1} = m_t + \left[\frac{1}{\omega_N H'(\omega_N)} + \frac{a}{\mu_N H'(\omega_N)} + \frac{b}{\mu_N}\right] m_t^2 + \mathcal{O}(m_t^3).$$

Therefore, allowing $\delta \neq 0$ gives

$$m_{t+1} = (1+\delta)m_t + Cm_t^2 + \mathcal{O}(m_t^3 + |\delta| m_t^2),$$

where

$$C \equiv \frac{a + bH'(\omega_N) + \frac{A}{\omega_N^2}}{\mu_N H'(\omega_N)}.$$

The two nearby steady states of this reduced equation satisfy

$$m\left[\delta + Cm + ...\right] = 0.$$

One solution is $m = 0$. If $C \neq 0$, the other solution is

$$m_M(\delta) = -\frac{\delta}{C} + \mathcal{O}(\delta^2).$$

Since $H'(\omega_N) > 0$, $m_M > 0 \Leftrightarrow \omega_M > \omega_N \Leftrightarrow \delta C < 0$.

At the boundary steady state, $\lambda_N^s = s_N$ and $\lambda_N^C = 1 + \delta$. At the dual-money steady state, $\lambda_M^s = s_N + \mathcal{O}(\delta)$ and $\lambda_M^C = 1 - \delta + \mathcal{O}(\delta^2)$. Since $\delta C < 0$, the results stated in the proposition follow directly. ■

Proof of Proposition 13. Condition (29) implies that the Friedman rule is in the return regime, according to Proposition 8. Therefore, the first-best allocation is available, given that no entrepreneur with a socially productive project is collateral-constrained and the opportunity cost of holding monetary assets is zero. ■

Proof of Proposition 14. If $\beta > 2A$, a small increase in μ lowers the project cutoff, relaxes under-investment, and produces a first-order CM investment gain, while the DM welfare loss is only second order at $\mu \downarrow \beta$. On the other hand, if $\beta < 2A$, a small increase in μ raises the project cutoff and worsens both distortions, so the Friedman rule is locally optimal. ■

Proof of Proposition 15. The condition $\bar{\gamma} > 1/(4\theta A)$ ensures that $\gamma(\mu) < \bar{\gamma}$. For all $\mu > \mu_E$, we have $\gamma(\mu) < \beta^{-1}$. On the descending collateral branch, it follows that $\gamma'(\mu) < 0$. Therefore, $\mathcal{W}_I'(\mu) < 0$. Also, we have $\mathcal{W}_D'(\mu) < 0$. Hence, $\mathcal{W}'(\mu) < 0$ for every $\mu > \mu_E$.

If $\beta > 2A$ and $\mu_E < \mu_N$, then $\mathcal{W}'(\beta^+) > 0$ and $\mathcal{W}'(\mu_E) < 0$. Moreover, $\mathcal{W}$ is strictly decreasing for $\mu > \mu_E$. Thus, the global maximizer lies in the compact interval $[\beta, \mu_E]$. It cannot be β because the right derivative is positive, and it cannot be μ_E because the left derivative is negative. Therefore, a global optimum exists in $\mu^* \in (\beta, \mu_E)$. ■

Proof of Proposition 16. The stated conditions imply

$$h'(\omega) = -\frac{\Lambda_{12}}{Q_{zz}},$$

so h is a contraction and has a unique fixed point. On the collateral branch, $\gamma \geqslant \frac{1}{2\beta}$ implies $\Lambda_{12} \geqslant 0$. Since $Q_{zz} \leqslant -c < 0$, it follows that $0 \leqslant h' \leqslant \frac{d}{c}$. Also, we have

$$r'(\omega) = \Psi_{\omega'}(h(\omega), \omega)h'(\omega) + \Psi_{\omega}(h(\omega), \omega).$$

On the collateral branch, $h(\omega) < \theta\Gamma$ ensures $\Psi_{\omega} > 0$, and hence $r' > 0$.

Because h is increasing and has a unique fixed point, $\omega < \omega^{**} \Rightarrow \omega < h(\omega) \leqslant \omega^{**}$. Since

r is increasing, $r(h(\omega)) > r(\omega)$. Then,

$$k(\omega) = \frac{A}{h(\omega)} \frac{r(h(\omega))}{r(\omega)} > \frac{A}{h(\omega)} \geqslant \frac{A}{\omega^{**}} = \mu^{**}.$$

Similarly, if $\omega > \omega^{**}$, we have $\omega^{**} \leqslant h(\omega) < \omega$, so $r(h(\omega)) < r(\omega)$, and therefore

$$k(\omega) < \frac{A}{h(\omega)} \leqslant \frac{A}{\omega^{**}} = \mu^{**}.$$

■

B Feedback Policy Rule

Here, we derive some properties of the dynamic system under a feedback rule of the form: $\mu = \mu(\omega, m)$. This rule does not change the one-period feasible set. However, it changes the viability kernel. For $m > 0$, one-step feasibility requires

$$\underline{m}(F(\omega, m)) \leqslant \frac{\mu(\omega, m) F(\omega, m) m}{A} \leqslant \overline{m}(F(\omega, m)).$$

The locus $\omega_{t+1} = \omega_t$ remains unchanged. For $m > 0$,

$$\frac{m_{t+1}}{m_t} = \frac{\mu(\omega_t, m_t) F(\omega_t, m_t)}{A}.$$

Thus, $m_{t+1} > m_t \Leftrightarrow \mu(\omega_t, m_t) F(\omega_t, m_t) > A$. The locus $m_{t+1} = m_t$ is given by

$$N(\omega, m) \equiv \mu(\omega, m) F(\omega, m) - A = 0.$$

Because $m = 0$ implies $m' = 0$, the private-money-only boundary is invariant under every finite affine rule: $\mu(\omega, m) = a\omega + bm + \bar{\mu}$.

A dual-money steady state (ω_M, m_M) satisfies $m_M = H(\omega_M)$ and $\mu(\omega_M, H(\omega_M)) = \frac{A}{\omega_M}$. Define

$$R(\omega) \equiv \bar{\mu} + a\omega + bH(\omega) - \frac{A}{\omega}.$$

A sufficient condition for the uniqueness of the dual-money steady state is $R'(\omega) > 0$ throughout the relevant interval.

B.1 Stability of the single-money steady state

Suppose that the feedback rule is locally affine near the bifurcation point. The Jacobian at $(\omega_N, 0)$ is

$$D\hat{T}(\omega_N, 0) = \begin{pmatrix} \varphi'(\omega_N) & F_m(\omega_N, 0) \\ 0 & \lambda_N \end{pmatrix},$$

where $\lambda_N = \frac{a\omega_N + \bar{\mu}}{\mu_N}$. Therefore,

$$a\omega_N + \bar{\mu} < \mu_N \Rightarrow \lambda_N < 1,$$

so the boundary steady state attracts positive real balances locally;

$$a\omega_N + \bar{\mu} > \mu_N \Rightarrow \lambda_N > 1,$$

so the boundary steady-state repels positive real balances locally. If $|\varphi'(\omega_N)| < 1$ and $0 < \lambda_N < 1$, the trapping-rectangle argument continues to apply: an open set of trajectories with $m_0 > 0$ converges to $(\omega_N, 0)$.

B.2 Local dynamics at a dual-money steady state

Let (ω_M, m_M) denote an interior dual-money steady state away from the kink. The Jacobian at (ω_M, m_M) is

$$D\hat{T}(\omega_M, m_M) = \begin{pmatrix} p & q \\ \frac{m_M}{\omega_M}(p+u) & 1+\kappa+v \end{pmatrix},$$

where $p \equiv F_\omega(\omega_M, m_M)$, $q \equiv F_m(\omega_M, m_M) > 0$, $u \equiv \frac{a\omega_M^2}{A}$, $v \equiv \frac{b\omega_M m_M}{A}$, and $\kappa \equiv \frac{m_M q}{\omega_M} > 0$. Let τ denote the trace of the Jacobian matrix, and let Δ denote its determinant. The characteristic polynomial is $P(\lambda) = \lambda^2 - \tau\lambda + \Delta$. It can be shown that $sgn[P(1)] = -sgn[R'(\omega_M)]$.

A saddle (and therefore local determinacy of equilibrium with one predetermined variable and one jump variable) exists when $P(1)P(-1) < 0$. If $R'(\omega_N) > 0$, then $P(1) < 0$; thus, the roots are necessarily real, and one of them exceeds one. The dual-money steady state is a saddle if and only if $P(-1) > 0$. In that case, we have $\Delta > 0 \Rightarrow 0 < \lambda_s < 1$, so convergence is monotonic, where λ_s denotes the stable root. Alternatively, we have $\Delta < 0 \Rightarrow -1 < \lambda_s < 0$, so convergence exhibits damped oscillations.