WORKING PAPER NO. 08-30
ON THE IMPLEMENTATION OF MARKOV-PERFECT INTEREST RATE AND MONEY SUPPLY RULES: GLOBAL AND LOCAL UNIQUENESS

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December 5, 2008

Supersedes Working Paper 07-27

On the Implementation of Markov-Perfect Interest Rate and Money Supply Rules: Global and Local Uniqueness *

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Abstract

Currently there is a growing literature exploring the features of optimal monetary policy in New Keynesian models under both commitment and discretion. This literature usually solves for the optimal allocations that are consistent with a rational expectations market equilibrium, but it does not study how the policy can be implemented given the available policy instruments. Recently, however, King and Wolman (2004) have shown that a time-consistent policy cannot be implemented through the control of nominal money balances. In particular, they find that equilibria are not unique under a money stock regime. We find that their conclusion of non-uniqueness of Markov-perfect equilibria is sensitive to the instrument of choice. *Surprisingly*, if, instead, the monetary authority chooses the nominal interest rate there exists a unique Markov-perfect equilibrium. We then investigate under what conditions a time-consistent planner can implement the optimal allocation by just announcing his policy rule in a decentralized setting.

* This is a substantially revised version of a paper previously circulating under the title "Interest Rate versus Money Supply Instruments: On the Implementation of Markov-Perfect Policy." We would like to thank Alex Wolman, Per Krusell, Bob King, and Jesus Fernandez Villeverde for comments. The views expressed in this paper are those of the authors and do not necessarily represent those of the Federal Reserve Bank of Philadelphia, the Federal Reserve Bank of Richmond, or the Federal Reserve System. This paper is available free of charge at www.philadephiafed.org/research-and-data/

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1 Introduction

Currently there is a growing literature exploring the features of optimal monetary policy in New Keynesian models under both commitment and discretion. This work usually assumes that the optimal policy solves a constrained planning problem where the policymaker chooses among all allocations that are consistent with a market equilibrium. Recently, however, attention has been paid to how to implement the optimal policy through instrument rules. We believe that this is an important area of inquiry, because the institutions responsible for setting policies rarely have direct control over allocations. It is therefore important to understand whether or not a planner's allocations are obtainable with a given institutional structure.

For the case of time-consistent policies that are Markov-perfect, King and Wolman (2004) have examined implementation issues when the monetary authority uses nominal money balances as the policy instrument in a sticky price environment. Surprisingly, they find that equilibria are no longer unique under a money-supply regime. Conditional on a given continuation allocation determined by the future policymaker, the current policymaker cannot implement a unique point-in-time equilibrium. These multiple equilibria are supported by strategic complementarities in the price-setting process. In particular, if a price-setting firm believes that all other price-adjusting firms will set relatively high (low) prices, then it will be optimal for the individual firm to set a relatively high (low) price. We clarify how strategic complementarities interact with the money-supply rule. In particular, we show that multiple equilibria arise because the money-supply rule weakens the existing strategic complementarities in the price-setting process for low inflation outcomes. Each one of the multiple point-in-time equilibria for a money-supply rule is associated with a different inflation rate and nominal interest rate.

We then study the implementability of a Markov-perfect nominal interest rate policy, since actual monetary policy is usually implemented through interest rate policies. We find that a policy that uses the nominal interest rate as the policy instrument implements a unique point-in-time equilibrium. Given the well-understood problems involving interest rate instruments in other settings, this may be an unexpected result. We obtain this result because contrary to the money-supply rule, the nominal interest rate instrument uniformly strengthens strategic complementarities and thereby eliminates multiple equilibria.

Once we have established that a nominal interest rate rule can implement the unique Markov-perfect equilibrium of the planning problem, we ask if this policy rule can also implement a unique rational expectations equilibrium. Here we are faced with a long-established literature, starting with Sargent and Wallace (1975), that shows that interest rate policies tend to yield indeterminate equilibria unless the interest rate is conditioned on other variables. However, because we are considering only Markov-perfect equilibria, the planner is prohibited from making the interest rate conditional on other endogenous variables. The restriction of analyzing only unconditional nominal interest rate rules introduces a real indeterminacy into the local dynamics of the rational expectations equilibrium in our environment.[1]

On the one hand, our analysis of Markov-perfect equilibria sidesteps this issue because it essentially picks McCallum's (1983) minimal state variable solution as the rational expectations equilibrium. Thus, we find that the minimum state variable solution for the fully decentralized environment is locally unique with sticky prices, a result that does not hold when prices are flexible.[2] For those who find this equilibrium restriction compelling, one may interpret this result as allowing the planner to achieve the optimal time-consistent allocation in a fully decentralized environment. If not, then one can also employ a technique that has been used to make interest rate policies yield determinate equilibria; see, e.g., Carlstrom and Fuerst (2001) and Adão, Correia, and Teles (2003). We can maintain the assumption that the monetary policy rule is Markov-perfect and yet eliminate the real indeterminacy from the rational expectations equilibrium by assuming that the rule simultaneously sets

[1] The presence of a real indeterminacy for fixed nominal interest rate policies in sticky price models has been pointed out before in Carlstrom and Fuerst (1998).

[2] The indeterminancy of nominal interest rate rules in a flexible price environment has been studied extensively; for example, see McCallum (1986) or Boyd and Dotsey (1994).

the nominal interest rate and the nominal money supply. This approach selects the optimal allocation out of the multiplicity of possible equilibria.

The paper proceeds as follows. First, we briefly describe the problem of implementing the allocations from Markov-perfect optimal policies through instrument rules. Then we study the problem in a standard New Keynesian economy, which is identical to the one used by King and Wolman (2004). We first review the King and Wolman (2004) result that using a money-supply instrument generates multiple equilibria. We then show that using an interest rate instrument uniquely implements the Markov-perfect allocation. Finally, we discuss how a synthesis of the two instruments, the money supply and the nominal interest rate, uniquely implements the Markov-perfect allocation as a rational expectations equilibrium. A brief summary concludes.

2 Implementation of Markov-Perfect Policies

Consider a policymaker that chooses a sequence of allocations, $\{x_t, s_t\}$, in order to maximize the welfare of a representative agent

$$\sum_{t=0}^{\infty} \beta^t U\left(x_t, s_t\right) \tag{1}$$

subject to the constraint that the allocations are consistent with a competitive equilibrium[3]

$$F\left(x_t, s_t; x_{t+1}, s_{t+1}\right) = 0. \tag{2}$$

The vector of allocations (x_t, s_t) may contain prices and quantities. Implicit in the statement of the planning problem is the assumption that the policymaker directly controls prices and quantities, subject to the constraint that they are consistent with the optimizing behavior of agents in the competitive equilibrium (CE).

[3] Even though in the particular economy we will study below firms that set their prices are assumed to behave as monopolistic competitors, we will use the terminology 'competitive equilibrium.'

We distinguish between pre-determined state variables, s_t, and other non-predetermined variables, x_t. From the CE constraint, it is apparent that the policymaker's current choices are constrained by the expectations about future outcomes. This feature can give rise to a time-consistency problem, in that a policymaker that plans an optimal time path for all future choices has an incentive to deviate from that plan in the future if he has the option to reoptimize (Kydland and Prescott (1977)). Solutions to the planning problem that are derived under the assumption that the policymaker will never deviate from the plan devised at time zero are called full-commitment solutions.

We study Markov-perfect policies, a class of policies that avoids the time-consistency problem. Markov-perfect policies restrict choices to be contingent only on payoff-relevant state variables. At any point in time the policymaker is assumed to decide only on the current non-predetermined variables, x_t, and next period's state variables, s_{t+1}, taking the current state, s_t, and the decisions of future policymakers as given. In particular, it is assumed that future policy decisions are characterized by a policy rule

$$(x_\tau, s_{\tau+1}) = (G_x(s_\tau),\ G_s(s_\tau)) \text{ for } \tau > t,$$

that is consistent with the CE constraints. Conditional on the policy rule, a decision on next period's state generates a sequence of allocations $\{(x_\tau, s_{\tau+1}) : \tau > t\}$ and thereby a continuation value implied by future policies and the choice of next period's state

$$V(s_{t+1}; G) = \sum_{\tau=0}^{\infty} \beta^\tau U\left[x_{t+1+\tau}(s_{t+1}; G), s_{t+1+\tau}(s_{t+1}; G)\right] \tag{3}$$

Today's optimal policy choice then solves the problem

$$\begin{aligned} g(s_t; G) &= \arg\max_{x_t, s_{t+1}} \{U(x_t, s_t) + \beta V(s_{t+1}; G)\} \\ &\quad \text{s.t. } F\left[x_t, s_t, G_x(s_{t+1}), s_{t+1}\right] = 0. \end{aligned} \tag{4}$$

A Markov-perfect equilibrium is a fixed point for the policy rules g and G,

$$g(s; G) = G(s) \text{ for all } s. \tag{5}$$

In general, it is difficult to characterize Markov-perfect equilibria since they involve the search for a fixed point in function spaces. Most of the literature studying Markov-perfect equilibria is restricted to linear-quadratic or to higher order polynominal local approximations of the original problem. One of our objectives is to see if Markov-perfect equilibria can be globally implemented, and thus local approximation methods are not useful.[4] Global nonlinear computational methods tend to be limited to a small number of state variables. Furthermore, the computational procedure usually works on the assumption that the policy rule is unique. Thus again this approach is not helpful for our question.

The study of Markov-perfect equilibria simplifies considerably if there are no payoff-relevant state variables. In this case the optimal Markov-perfect policy will be a constant allocation, x^*, that solves the problem

$$x^* = \arg\max_x u(x) \text{ s.t. } f(x, x^*) = 0. \tag{6}$$

We are concerned with the implementation of Markov-perfect policies through a policy rule where the policymaker does not choose all elements of the allocation. For this purpose we assume that the allocation vector, x, can be partitioned into two subsets, variables determined by the private sector, y, and policy instruments, z. A policy rule is then a constant vector z^*. A Markov-perfect policy is implementable if conditional on the policy rule z^*, the market constraints (2) define a unique rational expectations equilibrium. In order for the Markov-perfect policy to be implementable, *three* conditions have to be satisfied.

First, there needs to be a unique steady state, since in the absence of state variables

[4] Local approximations are also often problematic, since the steady state of the Markov-perfect equilibrium around which we want to approximate the economy usually depends on the shape of the policy rule, which is not known a priori.

admissable continuation values will be steady-state values. Second, there has to be a unique point-in-time equilibrium. By this we mean that conditional on the current policy and the Markov-perfect outcome for the next period there exists a unique allocation and prices in the current period such that we have a market equilibrium

$$\exists! y \text{ s.t. } f\left(y, z^*, y^*, z^*\right) = 0. \tag{7}$$

King and Wolman (2004) demonstrate for a simple model with sticky prices that a money stock rule does not give rise to a unique point-in-time equilibrium. This result is due to non-linearities induced by the money stock rule. Below we will argue that in the same environment an interest rate rule implements a unique point-in-time equilibrium.

Third, there has to be a unique dynamic equilibrium. By this we mean that conditional on the policy rule today and in all future periods there exists a unique rational expectations equilibrium

$$\exists! \left\{y_t\right\} \text{ s.t. } f\left(y_t, z^*, y_{t+1}, z^*\right) = 0 \text{ for all } t. \tag{8}$$

This is just the usual condition for the existence of a unique rational expectations equilibrium conditional on some policy rule. We will argue that even though an interest rate policy has a unique point-in-time equilibrium, it is not dynamically unique *unless one is willing to limit consideration to minimum state variable solutions* Although this result has the flavor of the Sargent and Wallace (1975) result on the indeterminacy of interest rate policies, it differs from their result in that, for our example, the real allocation is indeterminate rather than the price level alone being indeterminate. We will also argue that a more general concept of the Markov-perfect policy rule eliminates this indeterminacy. In particular, if the policymaker jointly determines the money growth rule and the interest rate, albeit consistent with the CE conditions, the policy rule will uniquely implement the Markov-perfect allocation and prices.

3 The Economy

There is an infinitely lived representative household with preferences over consumption and leisure. The consumption good is produced using a constant-returns-to-scale technology with a continuum of differentiated intermediate goods. Each intermediate good is produced by a monopolistically competitive firm with labor as the only input. Intermediate goods firms set the nominal price for their products for two periods, and an equal share of intermediate firms adjusts their nominal price in any particular period. Also, in what follows we restrict our analysis to perfect foresight economies.

3.1 The representative household

The representative household's utility is a function of consumption c_t, and the fraction of time spent working n_t,

$$\sum_{t=0}^{\infty} \beta^t \left[\ln c_t - \chi n_t \right], \tag{9}$$

where $\chi \geq 0$, and $0 < \beta < 1$. The household is assumed to hold money in order to pay for consumption purchases

$$P_t c_t \leq M_t. \tag{10}$$

The household's period budget constraint is

$$P_t c_t + B_t + M_t \leq W_t n_t + R_{t-1} B_{t-1} + M_{t-1} - P_{t-1} c_{t-1} + V_t + T_t, \tag{11}$$

where P_t is the nominal price level, W_t is the nominal wage rate, B_t are the end-of-period holdings of nominal bonds, T_t are lump-sum transfers, and R_{t-1} is the gross nominal interest rate on bonds. The agent owns all firms in the economy, and V_t is nominal profit income from firms. The household can adjust money holdings M_t at the beginning of the period. We will use the term "real" to denote nominal variables deflated by the nominal price level, which is the price of the aggregate consumption good, and we use lower case letters to denote real variables. For example, real balances are $m_t \equiv M_t/P_t$.

The relevant first order conditions of the representative household's problem are[5]

$$
\begin{aligned}
1/c_t &= \lambda_t & (12)\\
w_t/c_t &= \chi, & (13)\\
\lambda_t &= \beta\lambda_{t+1}\frac{R_t}{P_{t+1}/P_t}, & (14)
\end{aligned}
$$

Equation (12) equates the multiplier on the household's budget constraint, λ, with the marginal utility of consumption. Equation (13) states that the marginal utility derived from the real wage equals the marginal disutility from work. Equation (14) is the Euler equation, which states that the marginal utility loss from saving one more unit today is equated with the discounted marginal utility gain from the real rate of return on savings tomorrow.

3.2 Firms

The consumption good is produced using a continuum of differentiated intermediate goods as inputs to a constant-returns-to-scale technology. Producers of the consumption good behave competitively in their markets. There is a measure one of intermediate goods, indexed $j \in [0,1]$. Production of the consumption good c as a function of intermediate goods $y(j)$ is

$$
c_t = \left[\int_0^1 y_t(j)^{(\varepsilon-1)/\varepsilon} dj\right]^{\varepsilon/(\varepsilon-1)} \tag{15}
$$

where $\varepsilon > 1$. Given nominal prices $P(j)$ for the intermediate goods, the nominal unit cost and price of the consumption good is

$$
P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon} dj\right]^{1/(1-\varepsilon)}. \tag{16}
$$

[5] The cash constraint (10) is binding for positive interest rates, and we have already substituted out for the Lagrange multiplier on the cash constraint.

For a given level of production, the cost-minimizing demand for intermediate good j depends on the good's relative price, $p(j) \equiv P(j)/P$,

$$y_t(j) = p_t(j)^{-\varepsilon} c_t. \tag{17}$$

Each intermediate good is produced by a single firm, and j indexes both the firm and good. Firm j produces $y(j)$ units of its good using a constant-returns technology with labor as the only input,

$$y_t(j) = n_t(j). \tag{18}$$

Each firm behaves competitively in the labor market and takes wages as given. Real marginal cost in terms of consumption goods is

$$\psi_t = w_t. \tag{19}$$

Since each intermediate good is unique, intermediate goods producers have some monopoly power, and they face downward sloping demand curves (17). Intermediate goods producers set their nominal price for two periods, and they maximize the discounted expected present value of current and future profits:

$$\max_{P_t(j)} \left[\frac{P_t(j)}{P_t} - \psi_t\right] y_t(j) + \beta \frac{\lambda_{t+1}}{\lambda_t} \left[\frac{P_t(j)}{P_{t+1}} - \psi_{t+1}\right] y_{t+1}(j). \tag{20}$$

Since the firm is owned by the representative household, the household's intertemporal marginal rate of substitution is used to discount future profits. Using the definition of the firm's demand function (17) and the household's intertemporal rate of substitution, the first order condition for profit maximization can be written as

$$\left[\frac{P_t(j)}{P_t}\right]^{-\varepsilon-1} \left[\mu\psi_t - \frac{P_t(j)}{P_t}\right] + \beta \left[\frac{P_t(j)}{P_{t+1}}\right]^{-\varepsilon-1} \left[\mu\psi_{t+1} - \frac{P_t(j)}{P_{t+1}}\right] = 0 \tag{21}$$

where $\mu = \varepsilon / (\varepsilon - 1)$ is the static markup with flexible prices.

3.3 A symmetric equilibrium

We will study a symmetric equilibrium where all firms that face the same constraints behave the same. This means that in every period there will be two firm types: the firms that adjust their nominal price in the current period, type 0 firms with relative price p_0, and the firms that adjusted their price in the previous period, type 1 firms with current relative price p_1. Each period half of all firms have the option to adjust their nominal price. The equilibrium of the economy is completely described by the sequence of marginal cost, relative prices, inflation rates, nominal interest rates, and real balances, $\{\psi_t, p_{0,t}, p_{1,t}, \pi_{t+1}, R_t, m_t\}$, such that

$$
\begin{aligned}
0 &= (p_{0,t})^{-\varepsilon-1} (\mu\psi_t - p_{0,t}) + \beta \left(\frac{p_{0,t}}{\pi_{t+1}}\right)^{-\varepsilon-1} \left(\mu\psi_{t+1} - \frac{p_{0,t}}{\pi_{t+1}}\right) \frac{1}{\pi_{t+1}} && (22) \\
1 &= 0.5 \left[p_{0,t}^{1-\varepsilon} + p_{1,t}^{1-\varepsilon}\right] && (23) \\
\pi_{t+1} &= \frac{p_{0,t}}{p_{1,t+1}} && (24) \\
m_t &= \psi_t / \chi && (25) \\
\psi_t &= \frac{\pi_{t+1}}{\beta R_t} \psi_{t+1} && (26)
\end{aligned}
$$

Equation (22) restates the optimal pricing equation, (21), for a firm that can adjust its price in the current period. Equation (23) is the price index equation (16) for relative prices. Equation (24) relates the inflation rate $\pi_t \equiv P_t / P_{t-1}$ to the ratio of a price-adjusting firm's optimal current relative price and next period's preset relative price. Equation (25) relates real balances to marginal cost, using the household's optimal labor supply condition, (13), together with the fact that real balances are equal to consumption. Equation (26) is the household Euler equation, (14), after substituting for the marginal utility of income from (12) and (13). For ease of exposition we will drop time subscripts from now on and denote next period's values by a prime.

Allocations in this economy are suboptimal because of two distortions. The first distortion results from the monopolistically competitive structure of intermediate goods production: the price of an intermediate good is not equal to its marginal cost. The average markup in the economy is the inverse of the real wage, P/W, which is, according to equation (19), the inverse marginal cost $1/\psi$. The second distortion reflects inefficient production when relative prices are different from one. Using the firm's demand function (17) and aggregate production (15) we can obtain the total demand for labor as a function of relative prices and aggregate output. Solving aggregate labor demand for aggregate output we obtain an 'aggregate' production function

$$c = an \text{ with } a \equiv 2/\left[p_0^{-\varepsilon} + p_1^{-\varepsilon}\right], \tag{27}$$

Given the symmetric production structure, equations (15) and (18), efficient production requires that equal quantities of each intermediate good be produced. The degree of allocational inefficiency is reflected in the term $a \leq 1$. The allocation is efficient if $a = 1$, implying that $p_0 = p_1 = 1$.

The policymaker is assumed to maximize lifetime utility of the representative agent, taking the competitive equilibrium conditions (22)-(26) as constraints. For a time-consistent Markov-perfect policy the policymaker takes future policy choices as given and policy choices are functions of payoff-relevant state variables only. Because there are no state variables in our example, this amounts to the planner maximizing the current period utility function of a representative agent and choosing an unconditional value for the policy instrument. Taking future policy as given means that the planner has no control over future outcomes, such as future relative prices or allocations.

One usually states the problem in terms of the planner choosing the market allocation. In this case we can view the planner choosing a vector $y = (p_0, p_1, \pi', \psi)$ subject to constraints (22)-(24), and conditional on the choices of next period's policymaker, y'. The planner's choices determine the representative household's utility through their impact on allocational efficiency and the markup. In this model, with $\varepsilon = 11$, implying a markup of approximately 10 percent, and $\chi = 1/1.1$ the optimal allocation of consumption and labor is .9996 and 1.0 respectively. Thus, there is very little allocational inefficiency. This allocation implies an annual inflation rate of 1.82 percent and a nominal interest rate of 2.84 percent. We will use this parameterization in the following examples.

Note that the statement of the planner's problem in terms of the market allocation does not involve any reference to the policy instrument, z, be it real balances or the nominal interest rate. To determine whether the Markov-perfect equilibrium can be implemented as described in equations (7) or (8), we have to characterize the feasible set for market outcomes y conditional on the policy instrument.

4 Implementation of Point-in-Time Equilibria

In most models of monetary economies money-supply policies lead to a unique equilibrium with a determinate price level, whereas interest rate policies imply equilibrium indeterminacy. Exactly the opposite is true for the simple economy we have just described. As King and Wolman (2004) have shown, a Markov-perfect money-supply rule will imply non-uniqueness for the point-in-time equilibrium, and as we will show, a Markov-perfect interest rate policy will imply a unique point-in-time equilibrium. It turns out that (non)uniqueness of the equilibrium is related to the presence of strategic complementarities in the price-setting process and how the policy rule amplifies or weakens these complementarities.

Before we discuss the two policy rules, we want to demonstrate that strategic complementarities are inherent to the firms' price-setting problem. In the context of the model's monopolistic-competition framework, strategic complementarities are said to be present if it is optimal for an individual price-adjusting firm to increase its own relative price, p_0, if all other price-adjusting firms increase their relative price, $\bar{p}_0$. To study this issue we use a graphic representation of the individual firm's FOC for profit maximization, (22), which states that the sum of today's marginal profit, $MP(p_0, \psi)$, and tomorrow's discounted marginal profit, $\beta MP(p_0/\pi', \psi')/\pi'$, has to be zero. For the profit maximization problem to be well-defined we need the profit function to be concave; that is, the marginal profit function MP is decreasing in the relative price. In the Appendix we also show that

Proposition 1 *With constant marginal cost, $\psi = \psi'$, tomorrow's marginal profit, $MP(p_0/\pi', \psi')/\pi'$, is increasing in the inflation rate π' for a neighborhood around zero inflation, $\pi' = 1$.*

In Figure 1 we graph today's (red line) and tomorrow's marginal profit (blue line) for an individual firm conditional on all other firms' relative price, $\bar{p}_0$, and a positive inflation rate. The positive inflation rate erodes the firm's relative price tomorrow and therefore the firm will set its optimal price, p_0, above the static profit-maximizing relative price, $\mu\psi$, such that it balances today's negative marginal profit against tomorrow's positive marginal profit. Now suppose that all other firms increase their relative price. It follows from expression (24) that the inflation rate will increase, $\pi' = \bar{p}_0/p_1'$, and this will shift tomorrow's marginal profit curve up, leaving today's marginal profit curve unchanged. It is then optimal for the individual firm to also increase its own relative price. Thus, there is a source of strategic complementarities, independent of monetary policy. The choice of monetary policy instrument will modify strategic complementarities through its general equilibrium feedback effect on marginal cost.

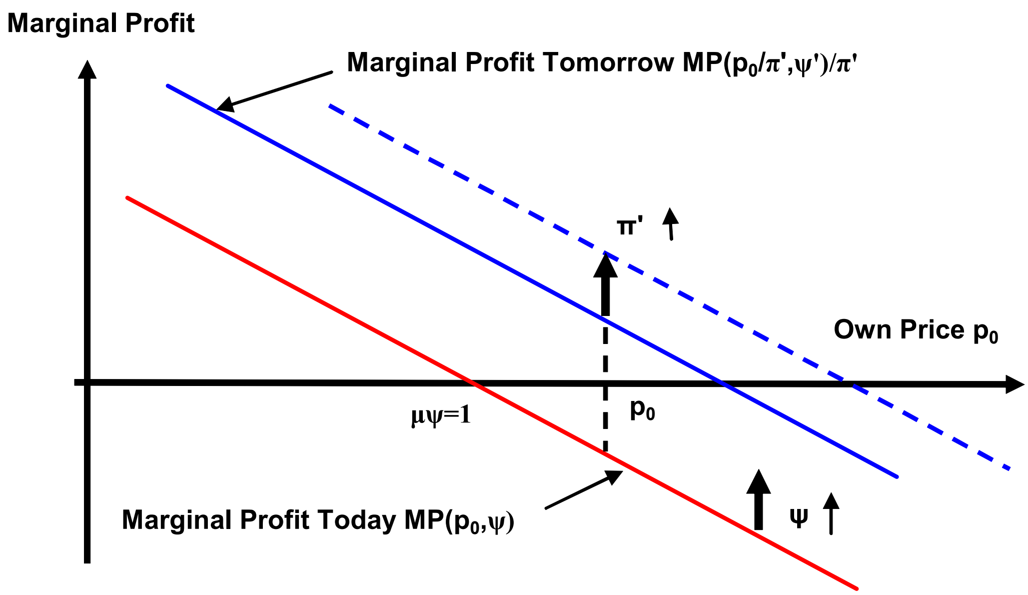

Figure 1. Strategic Complementarities

4.1 A money supply policy

We now review King and Wolman's (2004) analysis of a Markov-perfect nominal money stock rule. We first show that such a policy will in general imply the existence of multiple steady states. Furthermore, we show that the point-in-time equilibrium is not unique even conditional on a particular steady state allocation for the continuation of the economy. The equilibrium is not unique because for low inflation rates the money stock rule weakens the existing strategic complementarities in the price-setting process.

King and Wolman (2004) assume a homogeneous monetary policy rule that sets the nominal money stock in proportion to the preset nominal price from the last period

$$M = \tilde{m} P_1. \tag{28}$$

In terms of the Markov problem (7) the policy instrument is $z = \tilde{m}$. Combining the policy rule (28) with the money demand equation (10) yields the modified policy rule in real terms

$$c = \tilde{m} p_1. \tag{29}$$

Finally, combining (29) with the optimal labor supply condition (13) yields the equilibrium condition for marginal cost

$$\psi = \chi \tilde{m} p_1. \tag{30}$$

4.1.1 (Non)uniqueness of the steady state

We now show that for most values of the money-supply policy parameter, $\tilde{m}$, the steady-state of the economy will not be unique. Since in a Markov-perfect equilibrium without state variables the expected future policy has to be a steady-state, non-uniqueness of the steady-state alone suggests that the monetary policy rule may result in indeterminacy of the point-in-time equilibrium.

Proposition 2 *There exist values $\tilde{m}_{\min} < \tilde{m}_1 = 1/(\chi\mu) \leq \tilde{m}_2$ such that (1) if $\tilde{m} \in (\tilde{m}_{\min}, \tilde{m}_1]$ then there exists a unique non-inflationary steady-state; (2) if $\tilde{m} \in (\tilde{m}_1, \tilde{m}_2)$, then there exist two inflationary steady-states; (3) if $\tilde{m} = \tilde{m}_2$ then there exists a unique inflationary steady-state; and (4) if $\tilde{m} > \tilde{m}_2$ then no steady-state exists.*

Proof. Substitute (30) for marginal cost in (22) and obtain the following steady-state mapping from the inflation rate to the policy parameter

$$\tilde{m} = \frac{1}{\chi\mu} h(\pi^*) \text{ and } h(\pi^*) = \frac{\pi^* + \beta \pi^{*\varepsilon}}{1 + \beta \pi^{*\varepsilon}}.$$

In steady-state, the nominal interest rate, $R \geqslant 1$, and because $\beta R = \pi^*$, $\pi^* \geqslant \beta$. For $\pi^* \in (\beta, 1]$, $h(\pi^*)$ is strictly increasing and less than one. For positive inflation, $\pi^* > 1$, the function h satisfies (1) $h(\pi^*) > h(1) = 1$ and (2) $h(\infty) = 1$. Since h is continuous the function must eventually be decreasing if it is to approach 1 as $\pi^* \to \infty$. So there must exist an inflation rate π_2 such that $h(\pi^*) \leq h_2 = h(\pi_2)$. Furthermore, h is monotonically increasing (decreasing) for $\pi^* < \pi_2$ ($\pi^* > \pi_2$). Let $\tilde{m}_1 = 1/(\chi\mu)$ and $\tilde{m}_2 = \tilde{m}_1 h_2$. The proposition follows immediately from the properties of the h function.■

Figure 2 displays the steady-state inflation rates π^* consistent with the money rule $\tilde{m}$ for the parameter values used in section 3.3, in particular, $\chi\mu = 1$. Note that $\tilde{m}_1$ is the money-supply policy parameter associated with a zero steady-state inflation rate, and that the range of policy parameters associated with multiple steady-states is relevant, since the optimal Markov-perfect policy is inflationary.

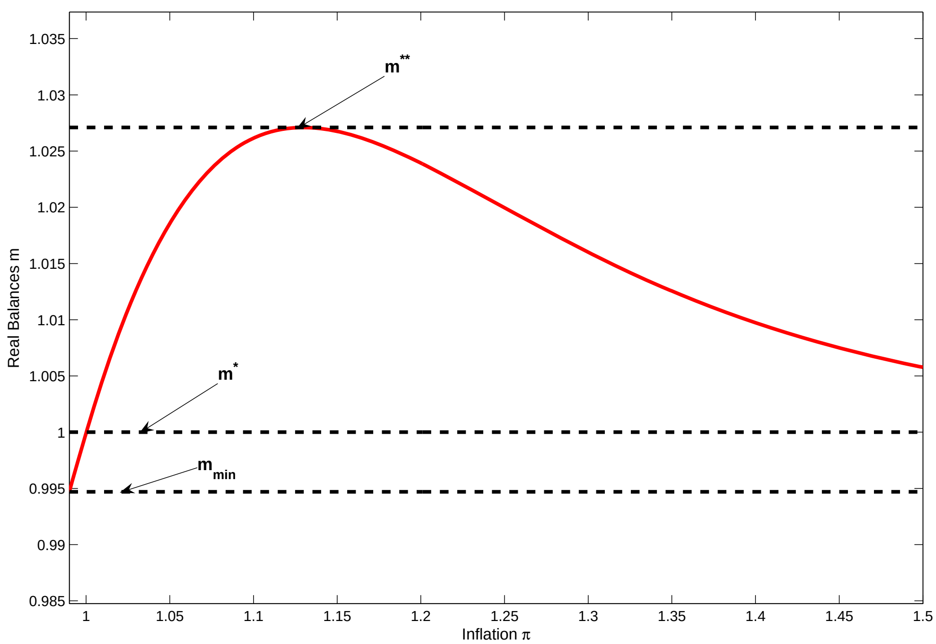

Figure 2. Steady State Multiplicity with a Money Rule

4.1.2 Non-uniqueness of the point-in-time equilibrium

Suppose that we choose one of the possible steady-states as a continuation of the economy in the next period. We now show that the choice of a money-supply instrument weakens strategic complementarities when the average firm chooses a low relative price, and that the complementarities persist when the average firm chooses a high relative price. The resulting shape change of the optimal reaction function, that is, the mapping from the average firm's relative price to an individual firm's optimal relative price response, gives rise to multiple point-in-time equilibria.

Consider again the response of an individual firm to an increase in the relative price set by all other firms, but now allow for the feedback coming through the money stock policy. When all other price-adjusting firms increase their relative price, it follows from the price index equation, (23), that the preset relative price, $\bar{p}_1$, declines. From equation (30) it then follows that today's marginal cost declines, which in turn shifts down today's marginal profit curve in Figure 1. Thus the policy-induced feedback effect reduces the individual firm's need to increase its own relative price in response to the general price increase; that is, it weakens the strategic complementarities.

It is easily shown that the impact of $\bar{p}_0$ on $\bar{p}_1$ declines with $\bar{p}_0$. Thus, strategic complementarities are weakened the most when the relative price of price-adjusting firms is the lowest. The resulting shape of a firm's optimal response function is depicted in Figure 3 for the parameter values used in section 3.3, and assuming that next period's policy generates a steady-state inflation rate $\pi = 1.05$. We can see that for low values of other firms' relative price choice, there are no strategic complementarities, and the reaction function is quite flat. If other firms start setting higher relative prices, then an individual firm's own optimal relative price starts to increase and the rate at which it responds also increases. Thus the reaction function becomes steeper than the 45 degree line and multiple equilibria due to self-fulfilling expectations are possible. In the Appendix we prove the following Proposition.

Proposition 3 *Suppose the current and future policymakers use the same money stock rule $\tilde{m}$. If $\tilde{m} \in (\tilde{m}_1, \tilde{m}_2)$, then, in general, at least two point-in-time equilibria exist. If $\tilde{m} = \tilde{m}_1$ then the point-in-time equilibrium is unique.*

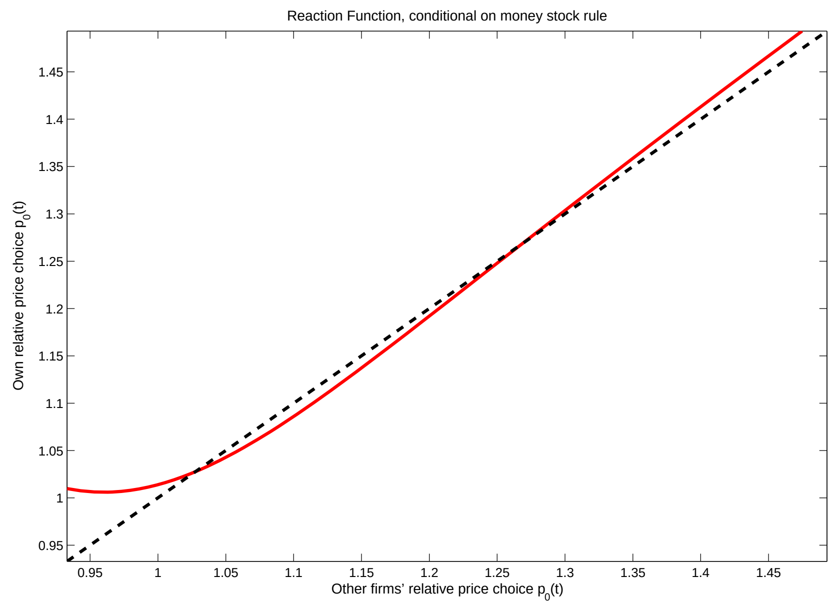

Figure 3. Optimal Response Function for Money Rule

4.2 An interest rate policy

In this section we evaluate the benefits of using an interest rate instrument to implement Markov-perfect policies. We find that steady-states and point-in-time equilibria are unique, despite the fact that the reaction function remains characterized by strategic complementarities. In what follows, we solve for the current equilibrium, y, conditional on current policy $z = R$ and future equilibrium outcomes y'. With a fixed nominal interest rate, policy affects marginal cost through the Euler equation,

$$\psi = \frac{\psi'}{\beta R p_1'} p_0 \tag{31}$$

which combines (24) and (26). We first show existence and uniqueness of the steady-state and the point-in-time equilibrium. We then show that uniqueness occurs despite the continued presence of strategic complementarities. Indeed, the interest rate rule strengthens existing strategic complementarities. Finally, we discuss the existence of a unique rational expectations equilibrium for the policy rule.

Proposition 4 *Conditional on the nominal interest rate $R \geqslant 1$, there exists a unique steady-state (p_0^*, p_1^*, ψ^*).*

Proof. Equation (31) and (24) determine the unique steady-state inflation rate

$$\pi^* = \beta R. \tag{32}$$

Equations (23), (24), and (32) uniquely determine the steady-state relative prices

$$p_0^{*\varepsilon-1} = 0.5\left(1 + \pi^{*\varepsilon-1}\right) \text{ and } p_1^* = p_0^*/\pi^*. \tag{33}$$

From equation (22) we obtain the steady-state marginal cost

$$\psi^* = \frac{1}{\mu}\frac{1 + \beta\pi^{*\varepsilon-1}}{1 + \beta\pi^{*\varepsilon}} p_0^*. \tag{34}$$

■

Uniqueness of the point-in-time equilibrium with an inflationary interest rate rule is not due to the elimination of strategic complementarities but to a strengthening of the strategic complementarities. Consider again the response of an individual firm to an increase in the relative price set by all other firms, but now allow for the feedback coming through the interest rate policy. From equation (31) it now follows that today's marginal cost increases, which in turn shifts up today's marginal profit curve in Figure 1. Thus the policy-induced feedback effect increases the individual firm's need to increase its own relative price in response to the general price increase; that is, it strengthens the strategic complementarities.

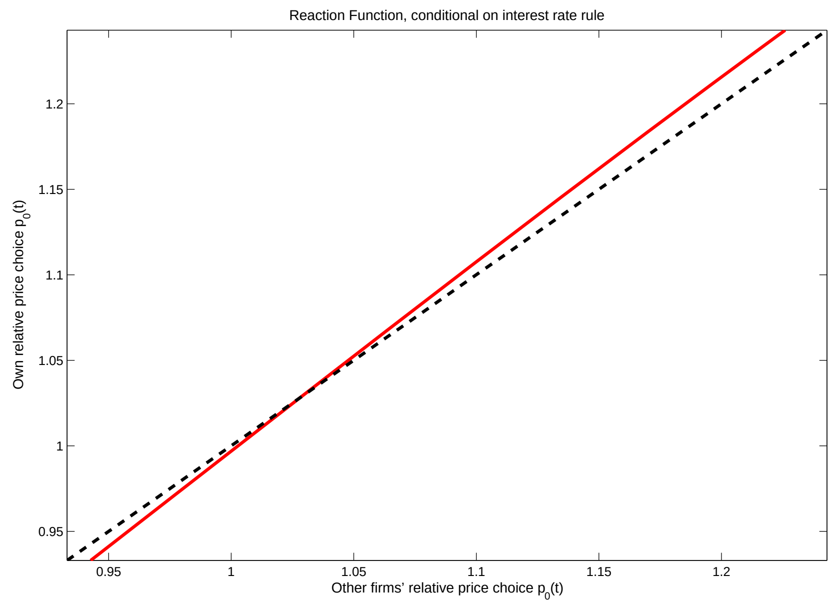

Figure 4. Optimal Response Function for Nominal Interest Rate Rule

Figure 4 displays the reaction function for the interest rate policy conditional on the parameterization used in Section 3.3 and assuming that next period's policy generates a steady-state inflation rate $\pi = 1.05$. In the following proposition we state that as long as tomorrow's policy does not try to implement price stability, there will always exist a unique point-in-time equilibrium for the current period.

Proposition 5 *(A) If next period's policy choice attains an inflationary or deflationary steady-state outcome, then (1) for any nominal interest rate for which a current period equilibrium exists it is unique, and (2) there always exists a nominal interest rate for which an equilibrium exists. (B) If next period's policy choice attains a steady-state outcome with stable prices, then (1) the current period equilibrium is indeterminate if current policy also tries to attain the stable-price steady-state* $\beta R = 1$*; (2) no current period equilibrium exists if* $\beta R \neq 1$.

4.3 Dynamic (in)determinacy

The dynamics of the rational expectations equilibrium conditional on the interest rate policy are characterized by the first order vector difference equation in (p_{0t}, ψ_t) determined by the Euler equation (31) for marginal cost and the rewritten first order condition for optimal pricing (22),

$$\frac{\psi}{p_0} = \frac{\psi'}{\beta R p_1'}. \tag{35}$$

$$(p_0)^{1-\varepsilon}\left(1-\mu\frac{\psi}{p_0}\right) = -\beta\left(p_1'\right)^{1-\varepsilon}\left(1-\mu\frac{\psi'}{p_1'}\right). \tag{36}$$

Recall that the preset relative price p_1 is determined by p_0 through the price index equation (23). For a locally unique rational expectations equilibrium to exist, both eigenvalues of the linearized difference equation system (35) and (36) have to be greater than one. For the parameterization that we have used in section 3.3, only one of the eigenvalues is greater than one, independent of the steady-state inflation rate around which we approximate the equation system. In the Appendix we also show that for inflation rates close to one and for very large inflation rates, only one of the eigenvalues is greater than one, independent of the parameter values. Thus the solution to the linearized difference equation system tends to be indeterminate, and the rational expectations equilibrium is not locally unique.[6] Furthermore, since the dynamics of the economy are characterized in terms of real variables, the real allocation is indeterminate. This contrasts with Sargent and Wallace's (1975) study in which nominal interest rate policies imply an indeterminate price level but a determinate real allocation. In our environment, the price level is determinate conditional on a given real allocation,

$$P_t = P_{0,t-1}/p_{1,t},\ P_{0,t} = p_{0t}P_t, \text{ and } P_{1t} = p_{1,t}P_t. \tag{37}$$

However, if we restrict the solution of the local dynamics to be in accord with McCallum's

[6] In the Appendix we also show that the money supply rule supports a locally unique rational expectations equilibrum.

(1983) minimum state variable solution, there is only one such solution, namely $p_{0,t} = p_0^*$ and $\psi_t = \psi^*$, and the nominal prices are determined conditional on the law of motion, (37), and the initial price, $P_{0,-1}$. This is easy to see, since there are no state variables and the minimum state variable solution must be the steady-state, which is unique. We also note that in an economy like ours with flexible prices, it is well known that the minimum state variable solution still displays nominal indeterminacy. This difference indicates another important distinction between flexible and sticky price environments.

The policymaker can uniquely implement the Markov-perfect equilibrium through a policy that jointly determines the nominal interest rate and the money stock. The choice of a nomial interest rate eliminates the potential for multiple point-in-time equilibria, whereas the money rule picks the Markov-perfect equilibrium allocation among the possible solutions to the system of dynamic equations.[7]

The usual procedure to eliminate dynamic indeterminacies arising from a fixed nominal interest rate policy – making the interest rate decision contingent on other endogenous variables; see, e.g., McCallum (1986), Boyd and Dotsey (1994), or Carlstrom and Fuerst (1998) – cannot be used to implement Markov-perfect equilibria. This approach is not applicable, since, by definition, decisions in Markov-perfect equilibria can depend only on payoff-relevant state variables and not other endogenous variables, be they lagged or contemporaneous.[8] Reputation-based time-consistent policies can eliminate dynamic indeterminacies in a way similar to our approach. For example, Atkeson, Chari, and Kehoe (2007) implement unique equilibria based on nominal interest rate rules through the specification of the off-equilibrium behavior of the policymaker's history-contingent decision rules.

[7] We note, however, that implementing this combination policy requires the monetary instrument to be state contingent. In order to replicate the optimal allocations, the policy-maker would need complete information. In an environment with incomplete information, it would be interesting to explore the properties of this type of combination policy.

[8] This argument does not apply to the implementation of optimal policies with full-commitment; see, e.g., by Giannoni and Woodford (2002).

5 Conclusion

This paper has analyzed the importance of the monetary policy instrument in decentralizing a time-consistent planner's optimal policy. In that regard, it is part of a growing literature investigating the implementation of optimal plans. We have shown that whether a planner uses a money instrument or an interest rate instrument is crucial for determining if optimal Markov-perfect allocations can be attained via the appropriate setting of the instrument. King and Wolman (2004) were the first to alert us to the non-trivial ramifications of decentralization. They produced a surprising result of significant impact, namely, that decentralization is a non-trivial problem. With regard to using a money instrument, implementation of the optimal allocation is unattainable. A time-consistent planner using a money instrument could not achieve the allocations chosen by a planner who was able to directly pick allocations. In fact, they showed that steady-states and equilibria were not unique at the optimal inflation rate. Since, in reality, no central bank picks allocations, this result presents a challenge for understanding just how a time-consistent central bank might operate.

Intuition gained from the early rational expectations literature on monetary policy as depicted in Sargent and Wallace (1975) would suggest that an interest rate instrument would have similar problems. Here we have shown that it does not. A planner using an interest rate instrument can achieve the Markov-perfect allocations of the standard time-consistent planning problem. The result occurs for two key reasons. The interest rate instrument pins down future inflation in ways unobtainable using a money instrument and, in so doing, increases the degree of strategic complementarity that arises from the monopolistically competitive price-setting problem itself.

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Appendix

A Proof of Proposition 1. Strategic Complementarities

The optimal relative price of a price-setting firm satisfies the FOC for profit maximization (22). With constant marginal cost, $\psi = \psi'$, and positive inflation this implies

$$p_0 \geq \mu\psi \geq 1 \geq p_0/\pi' \tag{A.1}$$

since the marginal profit function is decreasing in p_0. The derivative of the firm's marginal profit tomorrow with respect to inflation is

$$\frac{\partial MP\left(p_0/\pi', \psi\right)/\pi'}{\partial \pi'} = (\varepsilon - 1)\left(\frac{p_0}{\pi'}\right)^{-\varepsilon-1}\left(\mu^2\psi - \frac{p_0}{\pi'}\right)\frac{1}{\pi'^2} \tag{A.2}$$

Thus tomorrow's marginal profit is increasing in inflation iff

$$\mu^2\psi > \frac{p_0}{\pi'}. \tag{A.3}$$

Note that with zero inflation the optimal relative price satisfies $p_0 = \mu\psi$. Since we have a positive markup, $\mu > 1$, we get

$$\mu^2\psi > \mu\psi = p_0. \tag{A.4}$$

By continuity condition (A.3) is satisfied for a neighborhood around zero inflation.■

B Proof of Proposition 3. Nonuniqueness of PITE with Money Rule

Suppose that today's and tomorrow's policymakers choose the same policy rule, $\tilde{m} = \tilde{m}' \in (\tilde{m}_1, \tilde{m}_2)$. From Proposition 2 this policy is consistent with the existence of two steady-state equilibria. We now show that even conditional on choosing future behavior to be in accord with one of the two possible steady-states, $p_1' = p_1^*$ and $\psi' = \psi^*$, there exist two point-in-time equilibria in the current period. An individual firm's optimal relative price is determined by the profit maximization condition, (22),

$$p_0 = \mu\frac{\psi + \beta\psi^*\pi'^{\varepsilon}}{1 + \beta\pi'^{\varepsilon-1}}, \tag{B.1}$$

conditional on today's marginal cost and tomorrow's marginal cost and inflation rate. Together with the policy rule (30) and the definition of the inflation rate (24), the reaction function simplifies to

$$\frac{1}{\mu\chi\tilde{m}}p_0 = p_0\frac{(p_1/p_0) + \beta\left(p_0/p_1^*\right)^{\varepsilon-1}}{1 + \beta\left(p_0/p_1^*\right)^{\varepsilon-1}} = g\left(p_0, p_1^*\right). \tag{B.2}$$

In equation (B.2) the left-hand-side price p_0 is interpreted as an individual firm's optimal relative price in response to the expected aggregate relative prices, p_0 and p_1^*, on the right-hand-side. Note that the price index equation (23) implies that p_1 is a decreasing function of p_0. For parameter values and policy choice such that $\mu\chi\tilde{m} = 1$, we can interpret g as the reaction function and Figure 3 can be used to visualize the argument below.

One can show that the 'reaction' function g in terms of the relative price p_0 intersects the 45-degree line at $p_0 = 1$, and is above (below) the 45-degree line when p_0 is less than (greater than) one,

$$g\left(p_0, p_1^*\right) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} p_0 \text{ for } p_0 \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 1 \tag{B.3}$$

As p_0 becomes large, $g(p_0, p_1^*)$ converges to the 45-degree line from below,

$$\lim_{p_0 \to \infty} g\left(p_0, p_1^*\right) = p_0. \tag{B.4}$$

With some some additional algebra one can show that the derivative of the g function at $p_0 = 1$ is

$$\frac{\partial g\left(p_0, p_1^*\right)}{\partial p_0}|_{p_0=1} = -\frac{1 - \beta\left(p_1^*\right)^{1-\varepsilon}}{1 + \beta\left(p_1^*\right)^{1-\varepsilon}}. \tag{B.5}$$

We can now show that for $\tilde{m} \in (\tilde{m}_1, \tilde{m}_2)$ the LHS and the RHS of expression (B.2) will in general intersect twice. On the one hand, from Proposition 2 it follows that since $\tilde{m} > \tilde{m}_1$, that is, $\mu\chi\tilde{m} > 1$, the slope coefficient of the LHS linear expression in p_0 is less than one. Thus the LHS defines a line through the origin below the 45-degree line. On the other hand, the RHS of (B.2) intersects the 45-degree line at $p_0 = 1$, and stays above (below) the 45-degree line whenever p_0 is less than (greater than) one. Furthermore, as p_0 becomes arbitrarily large the RHS of (B.2) converges to the 45-degree line from below.

Since the LHS is strictly below the RHS for $p_0 \leq 1$, the two curves do not intersect in this range. We know that at least one intersection point exists, since we consider policy rules that are consistent with the existence of a steady-state, and the steady-state price is a solution to the reaction function (B.2). Thus there must be an intersection point for $p_0 > 1$.

If $\tilde{m} = \tilde{m}_1$, then we know that a unique non-inflationary steady-state with $p_0 = 1$ exists, and this steady-state also satisfies (B.2). For this case, the LHS is the 45-degree line and the RHS has a unique intersection with the 45-degree line at $p_0 = 1$. Furthermore, from (B.5) it follows that the slope of the RHS at $p_0 = 1$ is negative. With a marginally larger value of $\tilde{m}$, the slope of the LHS becomes less than one, and there will be at least two intersections with the RHS to the right of $p_0 = 1$.■

C Proof of Proposition 4. (Non)uniqueness of PITE with Interest Rate Rule

The current equilibrium is defined by the two equations (31) and (22), which map the current period relative price p_0 to current period marginal cost ψ. Rewriting (22), we have

$$\psi = f_1(p_0) = \left(\frac{1}{\beta R}\frac{\psi'}{p_1'}\right) p_0 \tag{C.1}$$

$$\psi = f_2(p_0) = \frac{1}{\mu}\left(p_0 + \beta A' p_0^{\varepsilon}\right). \tag{C.2}$$

where, $A' = (p_1')^{1-\varepsilon}\left(1 - \mu\frac{\psi'}{p_1'}\right)$, and next period's variables are evaluated at their steady-state values, p_1^* and ψ^* as determined by (32), (33) and (34). An intersection of the two functions represents a potential equilibrium.

The two functions always intersect at $p_0 = 0$, but $p_0 = 0$ is not a feasible outcome, since the price index equation (23) together with p_1 being positive implies a lower bound, $\underline{p}_0$, for the optimal relative price. Both functions are strictly increasing at $p_0 = 0$,

$$\frac{\partial f_1}{\partial p_0} = \frac{1}{\mu\beta R}\frac{\pi^* + \beta(\pi^*)^{\varepsilon}}{1 + \beta(\pi^*)^{\varepsilon}} \tag{C.3}$$

$$\frac{\partial f_2}{\partial p_0} = \frac{1}{\mu}\left(1 + \beta A' \varepsilon p_0^{\varepsilon - 1}\right). \tag{C.4}$$

The function f_2 is strictly concave (linear, strictly convex) if $A' < 0$ ($A' = 0$, $A' > 0$),

$$\frac{\partial^2 f_2}{\partial p_0^2} = \frac{1}{\mu}\beta A' \varepsilon(\varepsilon - 1) p_0^{\varepsilon - 2}. \tag{C.5}$$

The sign of the term A' depends on the inflationary stance of next period's steady-state policy. From (22) we get

$$\begin{aligned}
\beta A' &= \beta(p_1^*)^{1-\varepsilon}\left\{1 - \mu\left[\frac{1}{\mu}\frac{1 + \beta(\pi^*)^{\varepsilon - 1}}{1 + \beta(\pi^*)^{\varepsilon}} p_0^*\right]\frac{1}{p_1^*}\right\} \\
&= \beta(p_1^*)^{1-\varepsilon}\left\{1 - \pi'\frac{1 + \beta(\pi^*)^{\varepsilon - 1}}{1 + \beta(\pi^*)^{\varepsilon}}\right\} \\
&= \beta(p_1^*)^{1-\varepsilon}\left\{\frac{1 - \pi^*}{1 + \beta(\pi^*)^{\varepsilon}}\right\}.
\end{aligned} \tag{C.6}$$

The first equality uses the steady-state expression for next period's marginal cost (34), and the second equality uses the steady-state expression for next period's inflation rate (33). Thus A' is negative (positive) if next period's policy is inflationary, $\pi^* > 1$ (deflationary, $\pi^* < 1$), and $A' = 0$ if next period's policy implements price stability, $\pi^* = 1$.

If next period's policy is inflationary and an intersection between f_1 and f_2 exists for positive values of p_0, the intersection point is unique since the function f_1 is linear and the function f_2 is strictly concave. The two functions intersect for positive p_0 if at $p_0 = 0$ the function f_2 is steeper than f_1,

$$\frac{\partial f_1}{\partial p_0} = \frac{1}{\mu} \frac{1}{\beta R} \frac{\pi^* + \beta (\pi^*)^\varepsilon}{1 + \beta (\pi^*)^\varepsilon} < \frac{1}{\mu} = \left. \frac{\partial f_2}{\partial p_0} \right|_{p_0 = 0} \tag{C.7}$$

This condition can always be satisfied for a sufficiently large nominal interest rate $R \geq 1$. In other words the policymaker can always find an interest rate for which the functions intersect. Recall that there is a lower bound for feasible relative prices $\underline{\mathrm{p}}_0$, so the policymaker has to choose an interest rate that implies a sufficiently large value for the relative price p_0. A policymaker can always find such an interest rate, since he can always replicate the steady-state by choosing $R = R^*$. Thus there exists a choice for R such that an equilibrium exists and it is unique. An analogous argument applies if next period's policy is deflationary.

If next period's policy implements price stability, that is, $\psi^* = 1/\mu$ and $p_1^* = 1$, then the only policy for today that is consistent with the existence of an equilibrium is a nominal interest rate such that $\beta R = 1$. But then equations (C.1) and (C.2) are satisfied for any feasible combination of (p_0, ψ) such that

$$p_0 > \underline{\mathrm{p}}_0 \text{ and } \psi = p_0/\mu.$$

If current policy is inflationary or deflationary, $\beta R \neq 1$, then the only solution to equations (C.1) and (C.2) is $p_0 = 0$. But $p_0 = 0$ is not a feasible outcome, so no equilibrium exists. ■

D Local (in)determinacy of the rational expectations equilibrium

D.1 The money stock rule

If the policymaker follows a money-supply rule, we can substitute for marginal cost from expression (30) in equation (36) and get a difference equation in the optimal relative price only

$$(p_0)^{1-\varepsilon} \left(1 - \mu \chi \tilde{m} \frac{p_1}{p_0}\right) = -\beta (p_1')^{1-\varepsilon} (1 - \mu \chi \tilde{m}).$$

The use of the price index equation (23) for the preset relative price is implicit. Log-linearzing this equation at a steady-state yields

$$\hat{p}_0 = \beta \frac{1 - \mu \chi \tilde{m}}{1 - \mu \chi \tilde{m} \left[(\varepsilon \pi + \pi^\varepsilon) / (\varepsilon - 1) \right]} \hat{p}_0'$$

and the equilibrium is locally unique if the absolute value of the scale coefficient on the right-hand-side is less than one. We now use the expression for the steady-state with a money rule and get

$$\hat{p}_0 = \beta \frac{h(\pi) - 1}{h(\pi) \left(\frac{\varepsilon \pi + \pi^\varepsilon}{\varepsilon - 1} \right) - 1} \hat{p}_0'$$

For positive inflation rates the function $h(\pi) > 1$ and the expression in parentheses in the denominator is greater than one. Therefore the numerator is smaller than the denominator, and the scale coefficient is less than one. Thus the equilibrium is locally stable for any positive steady-state inflation rate.

D.2 The nominal interest rate rule

We study the difference equation (35) and (36) in the transformed marginal cost $z = \psi/p_0$

$$\begin{aligned} \beta R z &= z' q' \\ (p_0)^{1-\varepsilon}(1-\mu z) &= -\beta (p_1')^{1-\varepsilon}(1-\mu z' q') \end{aligned}$$

where $q \equiv p_0/p_1$. Log-linearizing the two equations around the steady-state yields

$$\hat{z} = \hat{z}' + \left(1 + q^{1-\varepsilon}\right)\hat{p}_0' \tag{D.8}$$

$$(\varepsilon - 1)\hat{p}_0 + \frac{\mu z}{1-\mu z}\hat{z} = \frac{\mu q' z'}{1-\mu q' z'}\hat{z}' + a\hat{p}_0' \tag{D.9}$$

$$\text{and } a = (1-\varepsilon)(q')^{1-\varepsilon} + \frac{\mu q' z'}{1-\mu q' z'}\left(1 + (q')^{1-\varepsilon}\right)$$

Note that in the steady-state $q = q' = \pi$. Solving for future p_0 and z yields

$$\begin{bmatrix} \hat{z}' \\ \hat{p}_0' \end{bmatrix} = C \begin{bmatrix} \hat{z} \\ \hat{p}_0 \end{bmatrix} \tag{D.10}$$

$$\begin{aligned} \text{with } C &= \Delta \begin{bmatrix} (1-\varepsilon)\pi^{1-\varepsilon} + (1+\pi^{1-\varepsilon})b & -(\varepsilon-1)(1+\pi^{1-\varepsilon}) \\ -b & \varepsilon - 1 \end{bmatrix}, \\ \Delta &= -\pi^{\varepsilon-1}/(\varepsilon-1), \text{ and} \\ b &= \frac{\pi + \beta\pi^{\varepsilon}}{\beta\pi^{\varepsilon}}\frac{1+\beta\pi^{\varepsilon}}{1-\pi}. \end{aligned}$$

Let tr denote the trace of C and det denote the determinant of C,

$$\det = -\pi^{\varepsilon-1} \tag{D.11}$$

$$tr = 1 - \pi^{\varepsilon-1} + \frac{1}{\varepsilon-1}\frac{\pi}{\pi-1}\underbrace{\frac{(1+\beta\pi^{\varepsilon})}{\beta\pi^{\varepsilon}}\left(1+\beta\pi^{\varepsilon-1}\right)\left(1+\pi^{\varepsilon-1}\right)}_{f(\pi)} \tag{D.12}$$

From the Schur-Cohn Criterion and some algebra we get that the roots of the characteristic polynomial of C satisfy the following conditions:

- both roots are inside the unit circle if

$$|\det| < 1 \text{ and } |tr| < 1 + \det; \tag{D.13}$$

- both roots are outside the unit circle if

$$\begin{aligned} \text{either det} &> 1 \text{ and } |tr| < 1 + \det \\ \text{or det} &< -1 \text{ and } |tr| < -(1 + \det); \end{aligned} \tag{D.14}$$

- one root is inside and one root is outside the unit circle if

$$|tr| > |1 + \det|; \tag{D.15}$$

For price stability, $\pi = 1$, the rational expectations equilibrium is locally indeterminate since $1 + \det = 0$ and $tr = \infty$ and condition (D.15) is satisfied. Therefore the rational expectations equilibrium is locally indeterminate for a neighborhood of price stability. For large values of the inflation rate, condition (D.15) is also satisfied, since the polynominal in the inflation rate defined by the function f is of higher order than $\varepsilon - 1$. We are unable to prove that condition (D.15) is satisfied for all intermediate values of the inflation rate but, given our baseline calibration in section 3.3 condition, (D.15) is satisfied for all values of the inflation rate.