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ABSTRACT

We study, theoretically and quantitatively, the general equilibrium of an economy in which households smooth consumption by means of both a riskless asset and unsecured loans with the option to default. The default option resembles a bankruptcy filing under Chapter 7 of the U.S. Bankruptcy Code. Competitive financial intermediaries offer a menu of loan sizes and interest rates wherein each loan makes zero profits. We prove existence of a steady state equilibrium and characterize the circumstances under which a household defaults on its loans. We show that our model accounts for the main statistics regarding bankruptcy and unsecured credit while matching key macroeconomic aggregates and the earnings and wealth distributions. We use this model to address the implications of a recent policy change that introduces a form of ``means-testing" for households contemplating a Chapter 7 bankruptcy filing. We find that this policy change yields large welfare gains.

1 Introduction

In this paper we analyze a general equilibrium model with unsecured consumer credit that incorporates the main characteristics of U.S. consumer bankruptcy law and replicates the key empirical characteristics of unsecured consumer borrowing in the U.S. Specifically, we construct a model consistent with the following facts:

- Borrowers can default on their loans by filing for bankruptcy under the rules laid down in Chapter 7 of the U.S. bankruptcy code. In most cases, filing for bankruptcy results in seizure of all (non-exempt) assets and a full discharge of household debt. Importantly, filing for bankruptcy protects a household's current and future earnings from any collection actions by those to whom the debts were owed.
- Post-bankruptcy, a household's credit rating deteriorates and it has serious difficulty getting new (unsecured) loans for a period of about 10 years.[1]
- Households that default are typically in poor financial shape. [2]
- There is free entry into the consumer loan industry and the industry behaves competitively.[3]
- There is a large amount of unsecured consumer credit.[4]
- A large number of people who take out unsecured loans default each year.[5]

A key contribution of our paper is to establish a connection between the recent facts on household debt and the bankruptcy filing rate and the theory of consumer behavior that macroeconomists routinely use to address micro and macro observations on household consumption. This connection is established by modifying the equilibrium models of Imrohoroğlu (1989), Huggett (1993), and Aiyagari (1994) to include default and by organizing the facts on consumer debt and bankruptcy in light of the model.

[1]This is documented in Musto (1999).

[2]This is documented in, for example, Flynn (1999).

[3]See Evans and Schmalensee (2000), Ch.10, for a compelling defense of the view that the unsecured consumer credit industry in the U.S. is competitive.

[4]The Board of Governors of the Federal Reserve System constructs a measure of revolving consumer debt that excludes debt secured by real estate, as well as automobile loans, loans for mobile homes, trailers, or vacations. This measure is probably a subset of unsecured consumer debt and it amounted to $692 billion in 2001, or almost 7 percent of the $10.2 trillion that constitutes U.S. GDP.

[5]In 2001, 1.45 million people filed for bankruptcy in the U.S., of which just over 1 million were under Chapter 7 (as reported by the American Bankruptcy Institute).

Turning first to the theory, we analyze an environment where households with infinitely long planning horizons choose how much to consume and how much to save or borrow. Households face uninsured idiosyncratic shocks to income, preferences, and wealth and therefore have a motive to accumulate assets and to sometimes borrow in order to smooth consumption. We permit households to default on their loans. This default option resembles a Chapter 7 bankruptcy filing in which debts are discharged. We abstract from the out-of-pocket expenses of declaring bankruptcy but assume that a bankruptcy remains on a household's credit record for some (random) length of time which, on average, is compatible with the length of time mandated by law. Consistent with available evidence, we assume that a household with a bankruptcy on its credit record is shut out of the credit market and experiences inconveniences which we model as a small reduction in the household's earning capability.

It should be clear from this basic setup that an indebted household will weigh the benefit of maintaining access to the unsecured credit market against the benefit of declaring default and having its debt discharged. Accordingly, credit suppliers who make unsecured loans will have to price their loans taking into account the likelihood of default. We assume a market arrangement where credit suppliers can link the price of their loans to the observable total debt position of a household and to a household's type. The first theoretical contribution of the paper is to prove the existence of a general equilibrium in which the price charged on a loan of a given size made to a household of a given type exactly compensates lenders for the objective default frequency on loans of that size made to households of that type. This demonstration is made challenging by the fact that the default option could result in discontinuities (with respect to price and other parameters) in the steady-state distribution of households.

A second theoretical contribution of the paper is a characterization of default behavior. Specifically, we demonstrate that for each level of debt and for each household type, the set of earnings that trigger default is an interval: an income-rich household is better off repaying its debt and saving, and an income-poor household is better off repaying its debt and borrowing. This result is important for computation because it makes the task of calculating equilibrium default probabilities manageable.

A third theoretical contribution is to show that our equilibrium loan price schedules determine, endogenously, the borrowing limit facing each type of household. This is theoretically significant since borrowing constraints often play a key role in empirical work regarding consumer spending. Thus, we believe it is important to provide a theory of borrowing constraints that derives from the institutional and legal features of the U.S. unsecured consumer credit market.

Turning to our quantitative work, we first organize facts on consumer earnings, wealth, and indebtedness from the 2001 Survey of Consumer Finances (SCF) in light of the reasons cited for bankruptcy by Panel Study of Income Dynamics survey participants between 1984 and 1995. Our model successfully generates statistics that closely resemble these facts. To accomplish this, we model shocks that correspond to the reasons people give for filing for bankruptcy and which replicate the importance (for the filing frequency and debt) of each reason given. One of the three shocks is a standard earnings shock that captures the job-loss and credit-misuse reasons. A second shock is a preference shock that captures the effects of marital disruptions. The third shock is a liability shock that captures motives related to unpaid health-care bills and lawsuits. This last shock is important because it captures events that create liabilities without a person having actually borrowed from a financial intermediary – a fact that turns out to be important for simultaneously generating large amounts of debt and default. To incorporate this liability shock the model had to be expanded to incorporate a hospital sector.

We use our calibrated model to study the effects of a recent change in bankruptcy law that discourages above-median-income households from filing under Chapter 7. We find that the policy change has a substantial impact. There is a roughly three-fold increase in the level of debt extended without a significant increase in the total amount defaulted. We also find a significant welfare gain from this policy: households are willing (on average) to pay around 1 percent of annual consumption to implement this policy.

Our paper is related to several recent strands of literature on unsecured debt. One strand studies optimal contractual arrangements in the presence of commitment problems. For instance, Kocherlakota (1996) designs state (earnings) contingent bilateral contracts where the threat of punishment to autarky is sufficient to ensure that a given household does not default. Similarly, Zhang (1997) calculates borrowing limits to preclude the existence of an incentive to default. Kehoe and Levine (2001) embed this idea in a general equilibrium framework. These papers have the implication that it is in the state where earnings are high that households want to default but the binding individual rationality constraint prevents equilibrium default. To model equilibrium default we depart from this literature in an important way. In our framework a loan contract between the lending institution and a household specifies the household's next-period obligation independent of any future shock but gives the household the option to default. The interest rate on the contract can, however, depend on such things as the household's current total debt, credit rating, and demographic characteristics that provide partial information on a household's earnings prospects (such as its zip code). This assumption is motivated by the typical credit card arrangement.[6] Because of the limited dependence of the loan contract on future shocks, our framework is closer to the literature on default with incomplete markets as in Dubey, Geanokoplos, and Shubik (2000) and Zame (1994). Zame's work is particularly relevant because he shows that with incomplete markets, it may be efficient to allow a bankruptcy option to debtors.

In innovative work, Athreya (2002) analyzes a model that includes a default option with stochastic punishment spells. But in his model financial intermediaries charge the *same* interest rate on loans of different sizes even though a large loan induces a higher probability of default than a small loan. As a result, small borrowers end up subsidizing

[6]For more detail on the form of the standard credit card "contract" see Section III of Gross and Souleles (2002).

large borrowers, a form of cross-subsidization that is not sustainable with free entry of intermediaries.[7] Enforcing zero profits on loans of varying sizes complicates our equilibrium analysis because there is now a *schedule* of loan prices to solve for rather than a single interest rate on loans.[8]

The paper is organized as follows. In Section 2 we give a brief backgroud to the U.S. bankruptcy code as it applies to consumer debt. In Section 3 we describe our model economy and characterize the behavior of agents. We prove existence of equilibrium in Section 4 and characterize properties of the equilibrium loan schedules. We describe and discuss our calibration targets in Section 5. We discuss the properties of the calibrated economy in Section 6. In Section 7, we pose and answer our policy question. All proofs are given in the Appendix.

2 Bankruptcy in the U.S: Process and Consequences

In the United States, the right to petition for relief from the burden of debt has existed since colonial times. Article I, section 8, of the U.S. constitution authorizes Congress to "enact uniform Laws on the subject of Bankruptcies." Under this authority various bankruptcy laws have been enacted over the years, and, at present, the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005 provides federal guidelines for debt relief.[9]

There has been a large and growing number of individuals who filed for bankruptcy each year since 1978. In 2004 alone, more than 1 million individuals, or about 1 percent of U.S. households, filed for bankruptcy under Chapter 7, and an additional 456 thousand filed under other chapters. Since the late 1960s, Chapter 7 filings have annually averaged around 70 percent of all individual filings.

The procedure for completing a filing under Chapter 7 is as follows. An individual debtor seeking relief fills out a set of standardized forms that collect information on his or her existing debts, income, property, and monthly living expenses. The individual then files for bankruptcy in a special bankruptcy court, and the court informs the creditors listed by the individual in the filing of that fact. Once the creditors learn of the filing they are required by law to cease all actions to collect their debts. In about a month's time, the creditors meet with the debtor to determine whether there are any non-exempt assets that can be liquidated

[7]Lehnert and Maki (2000) have a model with competitive financial intermediaries who can precommit to long-term credit contracts in which a similar type of cross-subsidization is also permitted.

[8]Livshits, MacGee, and Tertilt (2003) follow our approach where the zero profit condition is applied to loans of varying size. However, they assume that creditors can garnish wages of a bankrupt person in the period in which that person files for bankruptcy and that a person has unrestricted access to unsecured credit in the period immediately following default.

[9]This act will take effect on October 17, 2005. It is expected to reduce the number of people who qualify for a Chapter 7 filing and make it harder to avoid collection efforts. See `http://bankruptcy.findlaw.com/new-bankruptcy-law/`.

to pay off unsecured debts.[10] Although the meeting also affords an occasion for creditors to verify the income and expense information on the debtor's filing forms, they rarely do so in practice.

While some unsecured debts (such as student loans) are not dischargeable, unsecured consumer loans such as credit card debt are dischargeable. A discharge releases the debtor from personal liability for discharged debts and prevents the creditors owed those debts from taking any future action against the debtor or his property to collect the debts. From start to finish, the process takes, on average, about four months and costs filers about $200, not including attorney's fees. After filing, the individual loses the right to file for another bankruptcy under Chapter 7 for six years.

In contrast to Chapter 7, filing under Chapter 13 leads to a rescheduling of debt rather than immediate discharge. The rescheduling generally results in a situation where the debtor promises to make additional payments on existing debts over a period of three to five years, followed by a discharge of any remaining debt. By and large, a Chapter 13 filing is not as beneficial to an individual as a Chapter 7 filing and hence is not the preferred Chapter for most individuals seeking debt relief.[11] In any case, for many Chapter 13 filings the resulting rescheduling does not succeed and leads ultimately to a filing under Chapter 7.

This brief description of the bankruptcy procedure should make clear that defaulting on unsecured consumer loans does not take much time or money. But while easy to do, filing for bankruptcy does have some adverse consequences. These stem from the fact that a bankruptcy filing remains on record on an individual's credit history for a period of 10 years from the date of filing. After the 10-year period is over, federal law mandates that the record of the filing be deleted from the household's credit history. During those 10 years, the individual's access to unsecured consumer loans is demonstrably impaired. As documented carefully in Musto (1999), individuals who filed for bankruptcy enjoy better access to unsecured consumer credit when the record of their filing disappears from the view of potential creditors after 10 years. This is apparent in the improvement in their overall credit score, in the number and borrowing capacity of their credit cards, and in their credit relationships more generally. Provided their credit history is not poor for other reasons, individuals with a record of bankruptcy typically see their total credit card balance rise from well under $1000 in the first six post-filing years to well over $2000 in the 11th post-filing year. In addition, credit-card credit granted during the early post-filing years may be secured against a deposit.

The bankruptcy flag in an individual credit history has other consequences as well. These arise because credit histories are also accessed by entities other than credit-granting agencies.

[10] Some assets are exempt from liquidation (allowable exemptions vary by state: in Texas and Florida an individual's home equity is exempt, while in Iowa the total value of exemptions permitted is just $500).

[11] In some cases an individual may be denied a petition to file under Chapter 7 if the bankruptcy court feels that use of Chapter 7 constitutes an abuse of the law. In such cases, the individual has the option to file under Chapter 13.

For instance, a landlord may request a credit report on a prospective renter, and a potential employer may wish to see a job applicant's credit history. In such cases, an adverse credit history is likely to impose costs.

To summarize, filing for a discharge of unsecured consumer debt under Chapter 7 is not very costly in terms of time or money. Doing so discharges existing debts but makes it difficult for the individual to get new unsecured loans for the next 10 years. In addition, the individual suffers costs that result from having an impaired credit history. These are the key institutional features we incorporate in the model presented in the next section.

3 The Model Economy

We begin by specifying the legal and physical environment of our model economy. Then we describe a market arrangement for the economy. This is followed by a statement of the decision problems of households, firms, financial intermediaries, and the hospital sector.

3.1 Legal Environment

We model the default option to resemble, in procedure and consequences, a Chapter 7 bankruptcy filing. Consider a household that starts the period with some unsecured debt. If the household files for bankruptcy (and we permit a household to do so irrespective of its current income or past consumption level) then the following things happen:

1. The household's beginnning of period liabilities are set to zero (i.e., its debts are discharged) and the household is not permitted to save in the current period. The latter assumption is a simple way to recognize that a household's attempt to accumulate assets during the filing period will result in those assets being seized by creditors.

2. The household begins next period with a record of bankruptcy. Let $h_t \in \{0, 1\}$ denote the "bankruptcy flag" for a household in period t, where $h_t = 1$ indicates in period t a record of a bankruptcy filing in the past and $h_t = 0$ denotes the absence of any such record. In what follows, we will refer to h as simply the household's credit record, with the record being either clean ($h = 0$) or tarnished ($h = 1$). Thus, a household that declares bankruptcy in period t, starts period $t + 1$ with $h_{t+1} = 1$.

3. A household that begins a period with a record of bankruptcy cannot get new loans, an assumption that is broadly consistent with the experience of bankrupt individuals in Musto (1999).[12] Also, a household with a record of bankruptcy experiences a loss

[12] We interpret the assumption that firms do not lend to households with a record of a bankruptcy filing in their credit history as a legal restraint on firm behavior. The central banking authority restricts the type of assets that can be held by credit firms. In addition, note that this restriction has the full support of the incumbent firms in the unsecured credit industry. Lifting this restriction will reduce the costs of filing for bankruptcy for consumers, resulting in more defaults than expected and losses for the incumbent credit firms.

equal to a fraction $0 < \gamma < 1$ of earnings, a loss intended to capture the pecuniary costs of a bad credit record.[13]

4. There is an exogenous positive probability λ that a household with a record of bankruptcy will have its record expunged in the following period. That is, a household that starts period t with $h_t = 1$ will start period $t+1$ with $h_{t+1} = 0$ with probability λ. This is a simple, albeit idealized, way of modeling the fact that a bankruptcy flag remains on an individual's credit history for only a finite number of years.

3.2 Preferences and Technologies

At any given time there is a unit mass of households. Each household is endowed with one unit of time. Households differ in their labor efficiency $e_t \in E = [e_{\min}, e_{\max}] \subset I\!R_{++}$ and in certain characteristics $s_t \in S$, where S is a finite set. There is a constant probability $(1-\rho)$ that any household will die at the end of each period. Households that do not survive are replaced by newborns who have a good credit rating ($h_t = 0$), zero assets ($\ell_t = 0$), and with labor efficiency and characteristics drawn independently from a joint probability measure $(S \times E, \mathcal{B}(S \times E), \psi)$ where and $\mathcal{B}(\cdot)$ denotes the Borel sigma algebra. Surviving households independently draw their labor efficiency and characteristics at time t from a Markov process defined on the measurable space $(S \times E, \mathcal{B}(S \times E))$ with transition function $\Phi(e_t|s_t)\Gamma(s_{t-1}, s_t)$ where $\Phi(e_t|s_t)$ is a conditional density function. We assume that for all s_t, the probability measure defined by $\Phi(e_t|s_t)$ is atomless.

There is one composite good produced according to an aggregate production function $F(K_t, N_t)$ where K_t is the aggregate capital stock that depreciates at rate δ and N_t is aggregate labor in efficiency units in period t. We make the following assumptions about technology:

Assumption 1. For all $K_t, N_t \geq 0$, F satisfies: (i) constant returns to scale; (ii) diminishing marginal returns with respect to the two factors; (iii) $\partial^2 F/\partial K_t \partial N_t > 0$; (iv) Inada conditions with respect to K_t, namely, $\lim_{K_t \to 0} \partial F/\partial K_t = \infty$ and $\lim_{K_t \to \infty} \partial F/\partial K_t = 0$; and (v) $\partial F/\partial N_t \geq b > 0$.

The composite good can be transformed one-for-one into consumption, investment, and medical services. As described in detail later, unforeseen medical expenditure is an oft-cited reason for Chapter 7 bankruptcy filing.

Taking into account the possibility of death, the preferences of a household are given by the expected value of a discounted sum of momentary utility functions:

$$E_0 \left\{ \sum_{t=0}^{\infty} (\beta\rho)^t \, u(c_t, s_t) \right\}, \tag{1}$$

[13] For instance, there are substantial annual fees associated with secured credit cards.

where $0 < \beta < 1$ is the discount factor and c_t is consumption in period t. We make the following assumptions on preferences.

Assumption 2. For any given s, $u(\cdot, s)$ is strictly increasing, concave, and differentiable. Furthermore, there exist $\underline{s}$ and $\overline{s}$ in S such that for all c and s, $u(c, \underline{s}) \leq u(c, s) \leq u(c, \overline{s})$.

Consumption of medical services does not appear in the utility function because we treat this consumption as non-discretionary.[14] When they occur, the household is presented with a hospital bill $\zeta(s_t)$. We assume that each surviving household has a strictly positive probability of experiencing a medical expense. Specifically, there exists $\widehat{s} \in S$ for which $\zeta(\widehat{s}) > 0$ and $\Gamma(s_{t-1}, \widehat{s}) > 0$ for all s_{t-1}.

3.3 Market Arrangements

We assume competitive factor markets. The real wage per efficiency unit is given by $w_t((\in W = [w_{\min}, w_{\max}]$ with $w_{\min} > 0))$. The rental rate on capital is given by r_t.

The addition of a default option necessitates a departure from the conventional modeling of borrowing and lending opportunities. In particular, we posit a market arrangement where unsecured loans of different sizes for different types of households are treated as distinct financial assets. This expansion of the "asset space" is required to correctly handle the competitive pricing of default risk, a risk that will vary with the size of the loan and household characteristics. In our model a household with characteristics s_t can borrow or save by purchasing a single one-period pure discount bond with a face value in a finite set $L \subset \mathbb{R}$. The set L contains 0 and positive and negative elements. We will denote the largest and smallest elements of L by $\ell_{\max} > 0$ and $\ell_{\min} < 0$, respectively. We will assume that $F_K(\ell_{\max}, e_{\min}) - \delta > 0$.

A purchase of a discount bond in period t with a non-negative face value ℓ_{t+1} means that the household has entered into a contract where it will receive $\ell_{t+1} \geq 0$ units of the consumption good in period $t+1$. A purchase of a discount bond with a negative face value ℓ_{t+1} means that the household has entered into a contract where it promises to deliver, conditional on not declaring bankruptcy, $-\ell_{t+1} > 0$ units of the consumption good in period $t+1$; if it declares bankruptcy, the household delivers nothing. A purchase of a discount bond with a negative face value ℓ_{t+1} "costs" a household with characteristics s_t the amount $q_{\ell_{t+1},s_t} \cdot \ell_{t+1}$ in period-t consumption goods (i.e., the household receives $q_{\ell_{t+1},s_t} \cdot (-\ell_{t+1})$ units of the period-t consumption good). Thus, the total number of financial assets available to be traded is $N_L \cdot N_S$, where N_X denotes the cardinality of the set X. Let the entire set of $N_L \cdot N_S$ prices in period t be denoted by the vector $q_t \in \mathbb{R}_+^{N_L \cdot N_S}$. We restrict q_t to lie in a compact set $Q \equiv [0, q_{\max}]^{N_L \cdot N_S}$ where $1 \geq q_{\max} \geq 0$. In the section on steady state equilibrium the upper bound on q will follow from assumptions on fundamentals.

[14] Alternatively, we could assume that unless the medical expenditure is incurred the household receives $-\infty$ utility.

Households purchase these bonds from financial intermediaries. We assume that both losses and gains resulting from death are absorbed by financial intermediaries. That is, a household that purchases a negative face value bond honors its obligation only if it survives and does not declare bankruptcy, and, symmetrically, an intermediary that sells a positive face value discount bond is released from its obligation if the household to which the contract was sold is not around to collect. We assume that there is a market where intermediaries can borrow or lend at the risk-free rate i_t. Also, without loss of generality, we assume that physical capital is owned by intermediaries who rent it to composite goods producers. There is free entry into financial intermediation and intermediaries can exit costlessly by selling all their capital.

The hospital sector takes in the composite good as an intermediate input and transforms it one-for-one into medical services. In our model, as in the real world, some households may default and not pay their medical bills $\zeta(s_t)$. We assume that if some proportion of aggregate medical bills is not paid back due to default, then hospitals supply medical services in the amount $\zeta(s_t)/m_t$ to households with characteristic s_t where $m_t \geq 1$ is chosen to ensure zero profits.

3.4 Decision Problems

The timing of events in any period are: (i) idiosyncratic shocks s_t and e_t are drawn for survivors and newborns; (ii) capital and labor are rented and production of the composite good takes place; (iii) household default and borrowing/saving decisions are made, and consumption of goods and services takes place. In what follows, we will focus on steady state equilibria where $w_t = w$, $r_t = r$, $i_t = i$, and $q_t = q$.

3.4.1 Households

We now turn to a recursive formulation of a household's decision problem. We denote any period t variable x_t by x and its period $t+1$ value by x'.

In addition to prices, the household's current period budget correspondence $B_{\ell,h,s,d}(e;q,w)$ depends on its exogenous state variables s and e, its beginning of period asset position ℓ, and its credit record h. It will also depend on the household's default decision $d \in \{0,1\}$, where $d = 1$ indicates that the household is exercising its default option and $d = 0$ indicates that it's not. Then $B_{\ell,h,s,d}(e;q,w)$ has the following form:

1. If a household with characteristics s has a good credit record ($h = 0$) and does not exercise its default option ($d = 0$) then

$$B_{\ell,0,s,0}(e;q,w) = \{c \in I\!\!R_+, \ell' \in L : c + q_{\ell',s}\, \ell' \leq e \cdot w + \ell - \zeta(s)\}. \tag{2}$$

We take into account the possibility that the budget correspondence may be empty in this case. In particular, if the household is deep in debt, earnings are low, new loans are expensive, and/or medical bills are high, then the household may not be able to afford non-negative consumption. As discussed below, allowing the budget correspondence to be empty permits us to analyze both voluntary and "involuntary" default.

2. If a household with characteristics s has a good credit record ($h = 0$) and net liabilities ($\ell - \zeta < 0$) and exercises its default option ($d = 1$), then

$$B_{\ell,0,s,1}(e; q, w) = \{c \in I\!R_+, \ell' = 0 : c \leq e \cdot w\}. \tag{3}$$

In this case, net liabilities disappear from the budget constraint and no saving is possible in the default period. That is we assume that during a bankruptcy proceeding a household cannot hide or divert funds owed to creditors.

3. If a household with characteristics s has a bad credit record ($h = 1$) and net liabilities are non-negative ($\ell - \zeta \geq 0$) then

$$B_{\ell,1,s,0}(e; q, w) = \{c \in I\!R_+, \ell' \in L^+ : c + q_{\ell',s}\, \ell' \leq (1 - \gamma)e \cdot w + \ell - \zeta(s)\}, \tag{4}$$

where $L^+ = L \cap I\!R_+$. With a bad credit record, the household is not permitted to borrow and is subject to pecuniary costs of a bad credit record.

4. If a household with characteristics s has a bad credit record ($h = 1$) and ($\ell - \zeta < 0$) then

$$B_{\ell,1,s,1}(e; q, w) = \{c \in I\!R_+, \ell' = 0 : c \leq (1 - \gamma)\, e \cdot w\}. \tag{5}$$

A household with bad credit record and a net medical liability pays only up to its assets. We further assume that a household whose medical liabilities are being "discharged" in this way cannot save in that period and begins next period with a bad credit record. For this reason we denote this case by setting $d = 1$.[15]

To set up the household's decision problem, define $\mathcal{L}$ to be all possible (ℓ, h, s)-tuples, given that only households with a good credit record can have debt and let $N_{\mathcal{L}}$ be the cardinality of $\mathcal{L}$. Then, $\mathcal{L} \equiv \{L^{--} \times \{0\} \times S\} \cup \{L^+ \times \{0, 1\} \times S\}$, where $L^{--} = L \backslash L^+$. Let $v_{\ell,h,s}(e; q, w)$ denote the expected lifetime utility of a household that starts with (ℓ, h, s) and e and faces the prices q and w and let $v(e; q, w)$ be the vector $\{v_{\ell,h,s}(e; q, w) : \{\ell, h, s\} \in \mathcal{L}\}$ in the set $\mathcal{V}$ of all continuous (vector-valued) functions $v : E \times Q \times W \to I\!R^{N_{\mathcal{L}}}$.

The household's optimization problem can be described in terms of a vector-valued operator $(\mathcal{T}v)(e; q, w) = \{(Tv)(\ell, h, s, e; q, w) : (\ell, h, s) \in \mathcal{L}\}$ which yields the maximum lifetime utility achievable if the household's future lifetime utility is assessed according to a given function $v(e; q, w)$.

[15]Since our model abstracts from Chapter 13 bankruptcy, we cannot let households with a Chapter 7 bankruptcy on record file again for bankruptcy; in reality such a household's bankruptcy option is limited to Chapter 13 only. We make the above assumption to guarantee that such a household's budget set is always non-empty.

Definition 1. For $v \in \mathcal{V}$, let $(Tv)(\ell, h, s, e; q, w)$ be defined as follows:

1. For $h = 0$ and $B_{\ell,0,s,0}(e; q, w) = \varnothing$:

$$(Tv)(\ell, 0, s, e; q, w) = u(e \cdot w, s) + \beta\rho \int v_{0,1,s'}(e'; q, w)\, \Phi(e'|s')\, \Gamma(s, ds')\, de'.$$

2. For $h = 0$, $B_{\ell,0,s,0}(e; q, w) \neq \varnothing$, and $\ell - \zeta(s) < 0$:

$$(Tv)(\ell, 0, s, e; q, w) = \max \left\{ \begin{array}{c} \max_{c,\ell' \in B_{\ell,0,s,0}} u(c, s) + \beta\rho \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de', \\ u(e \cdot w, s) + \beta\rho \int v_{0,1,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right\}.$$

3. For $h = 0$, $B_{\ell,0,s,0}(e; q, w) \neq \varnothing$, and $\ell - \zeta(s) \geq 0$:

$$T(v)(\ell, 0, s, e; q, w) = \max_{c,\ell' \in B_{\ell,0,s,0}} u(c, s) + \beta\rho \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de'.$$

4. For $h = 1$ and $\ell - \zeta(s) < 0$:

$$(Tv)(\ell, 1, s, e; q, w) = u(e \cdot w(1 - \gamma), s) + \beta\rho \int v_{0,1,s'}(e'; q, w)\, \Phi(e'|s')\Gamma(s, ds')de'.$$

5. For $h = 1$ and $\ell - \zeta(s) \geq 0$:

$$(Tv)(\ell, 1, s, e; q, w) = \max_{c,\ell' \in B_{\ell,1,s,0}} u(c, s) + \beta\rho \left[\begin{array}{c} \lambda \int v_{\ell',1,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \\ +(1 - \lambda) \int v_{\ell',0,s'}(e'; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right].$$

The first part of this definition says that if the household has debt and the budget set conditional on not defaulting is empty, the household must default. In this case, the expected lifetime utility of the household is simply the sum of the utility from consuming its current earnings and the discounted expected utility of starting next period with no assets and a bad credit record. The second part says that if the household has net liabilities and the budget set conditional on not defaulting is not empty, the household chooses whichever default option yields higher lifetime utility. In the case where both options yield the same utility the household may choose either. The difference between default under part 1 and default under part 2 is the distinction between "involuntary" and "voluntary" default. In the first case, default is the *only* option, while in the second case it's the *best* option. The third part applies when a household with good credit record has no net liabilities. In this case, the household does not have the default option and simply chooses how much to borrow/save.[16]

[16] We don't permit households to default on liabilities $\zeta$when $\ell - \zeta \geq 0$.This is without loss of generality since all assets of a household can be seized during a bankruptcy filing (no exempt assets).

The final two parts apply when the household has a bad credit record and hence no debt. It distinguishes between the case where it has some net liability (which arises from a large enough liability shock) and the case where it does not. In the first case, the household is permitted to partially default on its liabilities as described earlier. In the second case the household simply chooses how much to save.

Theorem 1 (Existence of a Recursive Solution to the Household Problem). There exists a unique $v^* \in \mathcal{V}$ such that $v^* = \mathcal{T}(v^*)$. Furthermore: (i) v^* is bounded and increasing in ℓ and e; (ii) a bad credit record reduces v^*; (iii), the optimal policy correspondence implied by $\mathcal{T}(v^*)$ is compact-valued and upper hemi-continuous; and (iv) provided $u(0, s)$ is sufficiently low, default is strictly preferable to zero consumption and optimal consumption is always strictly positive.

Because certain actions involve discrete choice, $\mathcal{T}(v^*)$ generally delivers an optimal policy *correspondence* instead of a function. Given property (iii) of Theorem 1, the Measurable Selection Theorem (Theorem 7.6 of Stokey and Lucas) guarantees the existence of measurable policy functions for consumption $c^*_{\ell,h,s}(e; q, w)$, asset holdings $\ell'^*_{\ell,h,s}(e; q, w)$, and default decision $d^*_{\ell,h,s}(e; q, w)$.

The default decision rule along with the probabilistic erasure of a bankruptcy flag on the household's credit record implies a mapping $H^*_{(q,w)} : (\mathcal{L} \times E) \times \{0, 1\} \to [0, 1]$ which gives the probability that the household's credit record next period is h'. The mapping H^* is given by:

$$H^*_{(q,w)}(\ell, h, s, e, h' = 1) = \begin{cases} 1 & \text{if } d^*_{\ell,h,s}(e; q, w) = 1 \\ \lambda & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 1 \\ 0 & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 0, \end{cases}$$

$$H^*_{(q,w)}(\ell, h, s, e, h' = 0) = \begin{cases} 0 & \text{if } d^*_{\ell,h,s}(e; q, w) = 1 \\ 1-\lambda & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 1 \\ 1 & \text{if } d^*_{\ell,h,s}(e; q, w) = 0 \text{ and } h = 0. \end{cases}$$

Then we can define a transition function $GS^*_{(q,w)} : (\mathcal{L} \times E) \times (2^{\mathcal{L}} \times \mathcal{B}(E)) \to [0, 1]$ for a surviving household's state variables given by

$$GS^*_{(q,w)}((\ell, h, s, e), Z) \tag{6}$$
$$= \int_{Z_h \times Z_s \times Z_e} \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q,w) \in Z_\ell\}} \, H^*_{(q,w)}(\ell, h, s, e, dh') \; \Phi(e'|s') de' \; \Gamma(s, ds')$$

where $Z \in 2^{\mathcal{L}} \times \mathcal{B}(E)$ and Z_j denotes the projection of Z on $j \in \{\ell, h, s, e\}$ [[and where $\mathbf{1}_{.}$ is the indicator function]]. Since a household in state (ℓ, h, s, e) could die and be replaced with a newborn, we can define a transition function $GN : (\mathcal{L} \times E) \times (2^{\mathcal{L}} \times \mathcal{B}(E)) \to [0, 1]$ to a newborn's initial conditions given by

$$GN((\ell, h, s, e), Z) = \int_{Z_s \times Z_e} \mathbf{1}_{\{(\ell', h')=(0,0)\}} \psi(ds', de'). \tag{7}$$

Combining these, we can define the transition function $G^*_{(q,w)} : (\mathcal{L} \times E) \times (2^{\mathcal{L}} \times \mathcal{B}(E)) \to [0, 1]$ for the economy as a whole by

$$G^*_{(q,w)}((\ell, h, s, e), Z) = \rho GS^*_{(q,w)}((\ell, h, s, e), Z) + (1 - \rho)GN((\ell, h, s, e), Z). \tag{8}$$

Finally, given the transition function G^*, we can describe the evolution of the distribution of households μ across their state variables (ℓ, h, s, e) for any given prices (q, w) by use of an operator Υ. Specifically, let $\mathcal{M}(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$ denote the space of probability measures. For any probability measure $\mu \in \mathcal{M}(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$ and any $Z \in 2^{\mathcal{L}} \times \mathcal{B}(E)$, define $(\Upsilon_{(q,w)}\mu)(Z)$ by

$$(\Upsilon_{(q,w)}\mu)(Z) = \int G^*_{(q,w)}(\ell, h, s, e, Z) d\mu. \tag{9}$$

Theorem 2 (Existence of a Unique Invariant Distribution). For any $(q, w) \in Q \times W$ and any measurable selection from the optimal policy correspondence, there exists a unique $\mu_{(q,w)} \in \mathcal{M}(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$ such that $\mu_{(q,w)} = \Upsilon_{(q,w)}\mu_{(q,w)}$.

3.4.2 Characterization of the Default Decision

Since the option to default is the novel feature of this paper, it's useful to establish some results on the manner in which the decision to default varies with a household's level of earnings and with its level of debt. We will characterize the default decision in terms of the the *maximal default set* $\overline{D}^*_{\ell,h,s}(q, w)$. This set is defined as follows: for $h = 0$ and $\ell - \zeta(s) < 0$ it consists of the set of e's for which either the budget set $B_{\ell,0,s,0}(e; q, w)$ is empty or the value from not defaulting does not exceed the value from defaulting; for $h = 1$ and $\ell - \zeta(s) < 0$ it consists of the entire set E. The maximal default set will coincide with the set of e for which $d^*_{\ell,h,s}(e; q, w) = 1$ if households that are indifferent between defaulting and not defaulting choose always to default.

Theorem 3 (The Maximal Default Set Is a Closed Interval). If $\overline{D}^*_{\ell,h,s}(q, w)$ is non-empty, it is a closed interval.

The intuition for this result can be seen in the following way. Suppose that there are two efficiency levels, say e_1 and e_2 with $e_1 < e_2$, for which it is optimal for the household to default on its debt. Now consider an efficiency level $\widehat{e}$ that's intermediate between e_1 and e_2. Suppose that the household prefers to maintain access to the credit market at $\widehat{e}$ even though it defaults at a higher earnings level e_2. It seems intuitive then that the reason for not defaulting at the lower earnings level associated with $\widehat{e}$ must be that the household finds it optimal to consume more than its earnings and incur even more debt. On the other hand, the fact that the household defaults at the efficiency level e_1 but maintains access to the credit market at the higher efficiency level $\widehat{e}$ suggests that the reason for not defaulting at the earnings level associated $\widehat{e}$ must be that the household finds it optimal to consume less than its earnings and reduce its level of indebtedness. Since the household cannot simultaneously be consuming more and less than the earnings level associated with $\widehat{e}$, it follows that the household must default at the efficiency level $\widehat{e}$ as well.

Theorem 4 (Maximal Default Set Expands with Liabilities). If $\ell^0 > \ell^1$, then $D^*_{\ell^0,h,s}(q,w) \subseteq D^*_{\ell^1,h,s}(q,w)$.

The result follows from the property that $v^*_{\ell,0,s}(e;q,w)$ is increasing in ℓ and the utility from default is independent of the level of net liabilities. Figure 1 helps to visualize this.

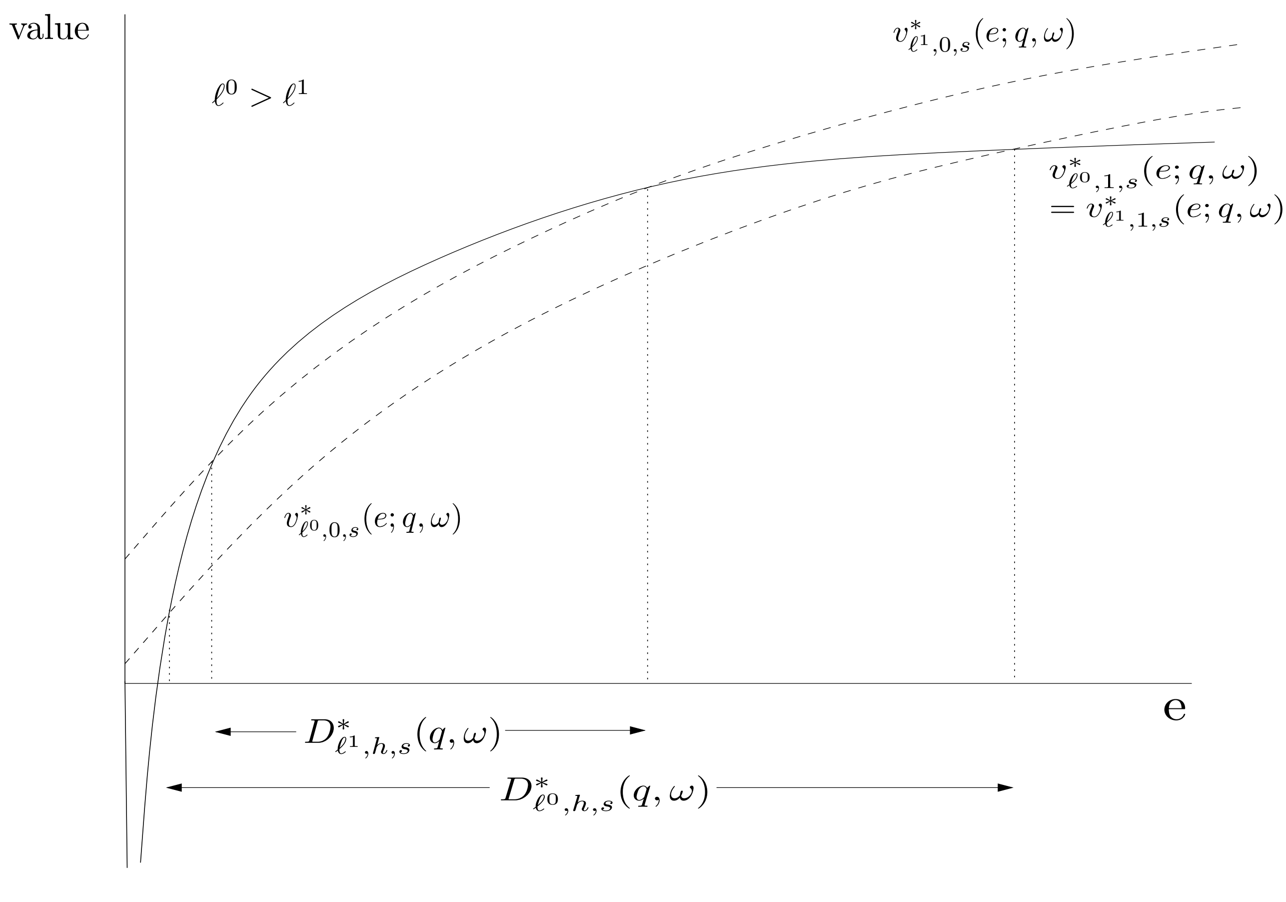

Figure 1: Typical Default Sets Conditional on Household Type

3.4.3 Firms

Firms producing the composite good face a static optimization problem of choosing non-negative quantities of labor and capital to maximize $F(K_t, N_t) - w \cdot N_t - r \cdot K_t$. The necessary conditions for profit maximization imply (with equality if the optimal N_t and K_t are strictly positive) that

$$w \geq \frac{\partial F(K_t, N_t)}{\partial N_t} \text{ and } r \geq \frac{\partial F(K_t, N_t)}{\partial K_t}. \tag{10}$$

3.4.4 Financial Intermediaries

The intermediary chooses the number $a_{\ell_{t+1},s_t} \geq 0$ of type (ℓ_{t+1}, s_t) contracts to sell and the quantity $K_{t+1} \geq 0$ of capital to own for each t to maximize the present discounted value of current and future cash flows

$$\sum_{t=0}^{\infty}(1+i)^{-t}\pi_t, \tag{11}$$

given K_0 and $a_{\ell_0,s_{-1}} = 0$. The period t cash flow is given by

$$\pi_t = \left[\begin{array}{c} (1-\delta+r)K_t - K_{t+1} \\ +\sum_{(\ell_t,s_{t-1})\in L\times S} \rho(1-p_{\ell_t,s_{t-1}})a_{\ell_t,s_{t-1}}(-\ell_t) \\ -\sum_{(\ell_{t+1},s_t)\in L\times S} q_{\ell_{t+1},s_t}a_{\ell_{t+1},s_t}(-\ell_{t+1}) \end{array} \right] \tag{12}$$

where p_{ℓ_{t+1},s_t} is the probability that a contract of type (ℓ_{t+1}, s_t), where $\ell_{t+1} < 0$, experiences default and it is understood that $p_{\ell_{t+1},s_t} = 0$ for $\ell_{t+1} \geq 0$.[17] Note that the calculation of cash flow takes into account that some borrowers will not survive to repay their loans and some depositors will not survive to collect on their deposits.[18]

[17] Note that households with $\ell_{t+1} \geq 0$ may still default on their medical liabilities if those liabilities are sufficiently high.

[18] Here, and in the household's decision problem, we assumed that a household enters into a single contract with some firm. This simplifies the situation in that a household's end-of-period asset holding is the same as ℓ', the size of the single contract entered into by the household. However, this is without loss of generality in the following sense. Let households write any collection of contracts $\{\ell'^k \in L\}$ as long as $\ell' = \sum_k \ell'^k \in L$. Consistent with the procedures of a Chapter 7 bankruptcy filing, assume that a household has the option to either (i) default on all negative face value subcontracts (i.e., loans) or (ii) not default on any of them. In the case of default, assume that creditor-firms can liquidate any positive face value subcontracts held by the household and use the proceeds to recover their loans in proportion to the size of each loan. With these bankruptcy rules in place, the price charged on any subcontract in the collection $\{\ell'^k \in L\}$ must be the price that applies to the single contract of size ℓ'. Consequently, as long as credit suppliers can condition their loan price on total end-of-period debt position of a household, there is a market arrangement in which the household is indifferent between writing a single contract or a collection of subcontracts with the same total value. Parlour and Rajan (2001) analyze equilibrium in a two-period model of unsecured consumer debt when such conditioning is not possible.

If a solution to the intermediary's problem exists, then optimization implies

$$i \geq r - \delta, \tag{13}$$

$$q_{\ell_{t+1},s_t} \leq \frac{\rho}{1+i} \quad \text{if } \ell_{t+1} \geq 0, \tag{14}$$

and

$$q_{\ell_{t+1},s_t} \geq \frac{\rho}{1+i}(1 - p_{\ell_{t+1},s_t}) \quad \text{if } \ell_{t+1} < 0. \tag{15}$$

If the optimal K_{t+1} is strictly positive, then (13) holds with equality. Similarly, if any optimal a_{ℓ_{t+1},s_t} is strictly positive, the associated condition (14) or (15) holds with equality. Furthermore, any non-negative sequence $\left\{K_{t+1}, a_{\ell_{t+1},s_t}\right\}_{t=0}^{\infty}$ implies a sequence of risk-free bond holdings $\{B_{t+1}\}_{t=0}^{\infty}$ by the intermediary given by the recursion

$$B_{t+1} = (1+i)B_t + \pi_t, \tag{16}$$

where $B_0 = 0$.

3.4.5 Hospital Sector

Hospital revenue received from a household in state ℓ, h, s,and e, is given by

$$\left[(1 - d^*_{\ell,h,s}(e;q,w))\zeta(s) + d^*_{\ell,h,s}(e;q,w)\max\{\ell, 0\}\right].$$

Observe that if a household has positive assets but negative net (after medical shock) liabilities and defaults, the hospital receives ℓ. If the household's assets are negative and it defaults, the hospital receives nothing. As noted before, for a bill of ζ, the hospital's resource cost is given by ζ/m. Thus, hospital profits in period t are given by

$$\int \left[(1 - d^*_{\ell,h,s}(e;q,w))\zeta(s) + d^*_{\ell,h,s}(e;q,w)\max\{\ell, 0\} - \zeta(s)/m\right] d\mu_t \tag{17}$$

where μ_t is the distribution of households over $\mathcal{L} \times E$ at time t. In steady state, m must be consistent with zero profits for the hospital sector.

4 Steady-State Equilibrium

In this section we define and establish the existence of a steady-state equilibrium and characterize some properties of the equilibrium loan price schedule. The proof of existence uses Brouwer's FPT for a continuous function on a compact domain. Nevertheless, the proof is not straightforward. The nature of the difficulty – which is related to the possibility of default – is discussed later in this section.

Definition 2. A steady-state competitive equilibrium is a set of strictly positive prices w^*, r^*, i^*, a non-negative loan-price vector q^*, a non-negative default frequency vector $p^* = (p^*_{\ell',s})_{\ell' \in L, s \in S}$, a non-negative hospital mark-up m^*, strictly positive quantities of aggregate labor and capital N^*, K^*, a non-negative vector of quantities of contracts $a^* = (a^*_{\ell',s})_{\ell' \in L, s \in S}$, bond holdings by the intermediary B^*, decision rules $\ell'^*_{\ell,h,s}(e; q^*, w^*), d^*_{\ell,h,s}(e; q^*, w^*), c^*_{\ell,h,s}(e; q^*, w^*)$ and a probability measure μ^* such that:

(i) $\ell'^*_{\ell,h,s}(e; q^*, w^*)$, $d^*_{\ell,h,s}(e; q^*, w^*)$ and $c^*_{\ell,h,s}(e; q^*, w^*)$ solve the household's optimization problem;

(ii) N^*, K^* solve the firm's static optimization problem;

(iii) K^*, a^* solve the intermediary's optimization problem;

(iv) $p^*_{\ell',s} = \int d^*_{\ell',0,s'}(e'; q^*, w^*)\Phi(e'|s')\Gamma(s; ds')de'$ for $\ell' < 0$ and $p^*_{\ell',s} = 0$ for $\ell' \geq 0$ (intermediary consistency);

(v) $\int \left[(1 - d^*_{\ell,h,s}(e; q^*, w^*))\zeta(s) + \; d^*_{\ell,h,s}(e; q^*, w^*) \max\{\ell, 0\} - \zeta(s)/m^*\right] d\mu^* = 0$ (zero profits for the hospital sector);

(vi) $\int e \;\; d\mu^* = N^*$ (the labor market clears);

(vii) $\int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}} \mu^*(d\ell, dh, s, de) = a^*_{\ell',s}$, $\forall(\ell', s) \in L \times S$ (each loan market clears);

(viii) $B^* = 0$ (the bond market clears);

(ix) $\int c^*_{\ell,h,s}(e; q^*, w^*)d\mu^* + \int \frac{\zeta(s)}{m^*} d\mu^* + \delta K^* = F(K^*, N^*) - \gamma w^* \int e\mu^*(d\ell, 1, ds, de)$ (the goods market clears);

(x) $\mu^* = \mu_{(q^*,w^*)}$ where $\mu_{(q^*,w^*)} = \Upsilon_{(q^*,w^*)}\mu_{(q^*,w^*)}$ (μ^* is an invariant probability measure).

We will use the above definition to derive a set of price equations whose solution implies the existence of a steady state. Conditions (ii) and (iii) in the definition imply the following equations. Since N^* and K^* are strictly positive, the first order conditions for the firm (10) and the intermediary (13) imply:

$$w^* = \frac{\partial F(K^*, N^*)}{\partial N^*}, \quad r^* = \frac{\partial F(K^*, N^*)}{\partial K^*}, \quad i^* = r^* - \delta. \tag{18}$$

For $a^*_{\ell',s} > 0$, the intermediary first order conditions (14) or (15) imply

$$q^*_{\ell',s} = \frac{\rho(1 - p^*_{\ell',s})}{1 + i^*}. \tag{19}$$

For $a^*_{\ell',s} = 0$ we will look for an equilibrium where the intermediary is indifferent between selling and not selling the associated (ℓ', s) contract. Then (19) holds for these contracts as well.

Condition (viii) implies the following equation. From the recursion (16), bond market clearing (viii) implies cash flow (12) can be written

$$\left[(1-\delta+r^*)K^* - \sum_{(\ell,\sigma)\in L\times S} \rho(1-p^*_{\ell,\sigma})a^*_{\ell,\sigma}\ell\right] - \left[K^* - \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'\right] = 0$$

or using (18) and (19)

$$(1-\delta+r^*)\left[K^* - \sum_{(\ell,\sigma)\in L\times S} q^*_{\ell,\sigma}a^*_{\ell,\sigma}\ell\right] - \left[K^* - \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'\right] = 0.$$

Therefore, bond market clearing in steady state implies

$$K^* = \sum_{(\ell',s)\in L\times S} q^*_{\ell',s}a^*_{\ell',s}\ell'. \tag{20}$$

It can be shown that the goods market clearing condition (ix) is implied by the other conditions for an equilibrium.[19] Thus, we can summarize an equilibrium by the following set of four equations. The first two are price equations that incorporate household optimization (i), intermediary consistency (iv), labor market clearing (vi), loan market clearing (vii), and (20) into (18) and (19) to yield:

$$w^* = F_N\left(\sum_{(\ell',s)\in L\times S} \ell' q^*_{\ell',s}\int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de), \int e\, d\mu^*\right) \tag{21}$$

$$q^*_{\ell',s} = \begin{cases} \dfrac{\rho}{1+F_K\left(\sum_{(\ell',s)\in L\times S}\ell' q^*_{\ell',s}\int \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de),\int e\, d\mu^*\right)-\delta} & \text{for } \ell'\geq 0 \\[2ex] \dfrac{\rho\left(1-\int d^*_{\ell',0,s'}(e';q^*,w^*)\Phi(e'|s')\Gamma(s;ds')de'\right)}{1+F_K\left(\sum_{(\ell',s)\in L\times S}\ell' q^*_{\ell',s}\int \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;q^*,w^*)=\ell'\}}\mu^*(d\ell,dh,s,de),\int e d\mu^*\right)-\delta} & \text{for } \ell' < 0 \end{cases} \tag{22}$$

The other two equations are given by (v) and (x).

Proving the existence of a steady-state equilibrium reduces to proving that there is a fixed point to equations (21)- (22) where the invariant distribution μ^* is itself a fixed point of a Markov process whose transition probabilities depend on the price vector. Provided the aggregate production function has continuous first derivatives (and these derivatives satisfy certain boundary conditions) a solution to this nested fixed point problem will exist (as a

[19]This is a nontrivial accounting exercise given that our environment admits default on loans and medical bills. For reasons of space we omit a proof here. The proof is given in the appendix of the working paper version of this paper available on our web sites.

simple consequence of the Brouwer's FPT) if μ^* is continuous with respect to the price vector. Given a continuum of households, a sufficient condition for the continuity of μ^* is that the set of households that are indifferent between any two courses of action be of (probability) measure zero. The assumption that the efficiency shock e is drawn from a distribution with a continuous cdf goes a long way toward ensuring this but, surprisingly, not *all* the way. Even with this assumption we cannot rule out that a continuum of households may be indifferent between defaulting and paying back.[20] To work around this problem we first establish the existence of a steady-state equilibrium for an environment in which there is an additional bankruptcy cost that is paid in the filing period. The nature of this cost ensures that the set of households that are indifferent between defaulting and paying back is finite and thereby restores the continuity of the invariant distribution with respect to the price vector. We then take a sequence of steady-state equilibria in which the filing-period bankruptcy cost converges to zero and establish that the limit of this sequence is a steady-state equilibrium for the environment of this paper.

The equilibrium loan price vector has the property that all positive face-value loans (household deposits) bear the risk-free rate and negative face-value loans (household borrowings) bear a rate that reflects the risk-free rate and a premium for the objective default probability on the loan. Given the risk-free rate, which in equilibrium will depend on μ^*, default probabilities (and hence loan prices) do not depend on μ^*. This is because free entry into financial intermediation implies that cross-subsidization across loans of different sizes is not possible; i.e. it's not possible for intermediaries to charge more than the cost of funds on small low-risk loans in order to offset losses on large higher-risk loans. For if there were positive profits in some contracts that were offsetting the losses in others, intermediaries could enter the market for those profitable loans. In contrast, in the environment of Athreya (2002) and Lehnert and Maki (2000) such cross-subsidization does occur and the calculation of intermediary profits requires knowledge of the distribution of customers over various loan sizes.

Theorem 5 (Existence) A steady-state competitive equilibrium exists.

For a finite r^*, it is possible that there are contracts $(\ell' < 0, s)$ whose equilibrium price $q^*_{\ell',s} = 0$. Even in this case, intermediaries are indifferent as to how many loans of type (ℓ', s) they "sell"; "selling" these loans doesn't cost the intermediary anything (since the price is zero) and it (rationally) expects the loans to generate no payoff in the following period. From the perspective of a household, taking out one of these free loans buys nothing in the current period but saddles the household with a liability. Since the household can do better by choosing $\ell' = 0$ in the current period, there is no demand for such loans either.

[20] This case will occur if a household that is indifferent between defaulting and paying back finds it optimal to consume its endowment when paying back. Then, *ceteris paribus*, households with slightly higher or slightly lower es will also be indifferent between defaulting and paying back.

We now deal with the limits of the set L, for a given s. Models of precautionary savings have the property that when $\beta\rho(1+r^*-\delta) < 1$ there is an upper bound on the amount of assets a household will accumulate. This upper bound arises because as wealth gets larger, the coefficient of variation of income goes to zero, and hence the role of consumption smoothing vanishes.[21] Since ours is also a model of precautionary saving, the same argument applies and $\ell_{\max}$ exists. With respect to the debt limit, $\ell_{\min}$, it can be set to any value less than or equal to $[e_{\max} \cdot w_{\max}]\,[(1+r^*-\delta)/(1-\rho+r^*-\delta)]$. This expression is the largest debt level that could be paid back by the luckiest household facing the lowest possible interest rate and is the polar opposite of the one in Huggett (1993), Aiyagari (1994), and Athreya (2002). As we show in the next theorem, for any s, a loan of this size or larger would have a price of zero in any equilibrium. Hence, as long as the lower limit is at least as low as $-[e_{\max} \cdot w_{\max}]\,[(1+r^*-\delta)/(1-\rho+r^*-\delta)]$, it will not have any effect on the equilibrium price schedule. We now turn to characterizing the equilibrium price schedule.

Theorem 6 (Characterization of Equilibrium Prices) In any steady-state competitive equilibrium: (i) $q^*_{\ell',s} = \rho(1+r^*-\delta)^{-1}$ for $\ell' \geq 0$; (ii) if the grid for L is sufficiently fine, there exists $\ell^0 < 0$ such that $q^*_{\ell^0,s} = \rho(1+r^*-\delta)^{-1}$; (iii) if the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, $0 > \ell^1 > \ell^2$ implies $q^*_{\ell^1,s} \geq q^*_{\ell^2,s}$; (iv) when $\ell_{\min} \leq -[e_{\max} \cdot w_{\max}]\,[(1+r^*-\delta)/(1-\rho+r^*-\delta)]$, $q^*_{\ell_{\min},s} = 0$.

The first property simply says that firms charge the risk-free rate on deposits. The second property says that if the grid is taken to be fine enough, there is always a level of debt for which it is never optimal for any household to default. As a result, competition leads firms to charge the risk-free rate on these loans as well. The third property says that the price on loans falls with the size of loans, i.e., the implied interest rate on loans rises with the size of the loan. The final property says that the price on loans eventually become zero; i.e., for any household the price on a loan of size $[e_{\max} \cdot w_{\max}]\,[(1+r^*-\delta)/(1-\rho+r^*-\delta)]$ or larger is always zero in every equilibrium. In other words, the equilibrium delivers an endogenous credit limit for each household with characteristics s.

5 Mapping the Model to U.S. Data

The objective of the quantitative work reported in this and the next section is to establish that our framework can give a plausible account of the overall facts on bankruptcy and credit. The challenging part is to simultaneously account for a high frequency of default *and* significant levels of unsecured debt – the reason being that a high default frequency makes unsecured debt very expensive and therefore rare in the model. We have found two features of the real world to be key in getting the model to plausibly account for aggregate default and credit statistics. First, not all unsecured consumer debt is a result of borrowing from

[21]See Huggett (1993) and Aiyagari (1994) for a detailed argument.

financial intermediaries – some of it is in the nature of an "involuntary" loan resulting from reasons such as large medical bills. Second, marital disruption is often cited as a reason for filing – something that is not necessarily related to earnings shocks. In our model we take into account the possibility of "involuntary" loans through our modeling of non-discretionary medical expenses and we take into account private life-events (such as divorce) as a possible trigger for default through the preference shock. There is a third feature of the real world that we believe to be important as well but have not taken into account in the model. In the real world, many households hold both unsecured debt *and* (non-exempt) assets – a fact that no doubt lowers the default premium on unsecured loans and makes them less expensive. We skirt this issue by focusing on the net asset positions of households but (as explained below) this impairs our ability to explain some aspects of the data.

We map four different versions of our model to the data. The versions differ by which idiosyncratic shocks are included – namely in the specification of the set S. We use the reasons for bankruptcy cited by PSID survey participants to determine targets for the fraction of consumer debt, the fraction of indebted households, and the fraction of people filing for bankruptcy that should be accounted for by each version of the model. By proceeding in this fashion we are able to obtain a more refined understanding of how high levels of debt can be reconciled with high default frequency. As a by-product we are able to show that our model can plausibly replicate the survey statistics on reasons for filing. We should note that plausibility in this context means that the model should explain the debt and default statistics without generating counterfactual predictions for macroeconomic aggregates and for earnings and wealth distributions.

5.1 Model Specification

We start by specifying the model with only earnings shocks (which we call the baseline) as is traditional in the macroeconomics literature. Next we move to a model with earnings and preference shocks and then to a model with earnings and medical expense shocks. Finally we consider the model with all three shocks.

5.1.1 The Model Economy with Earnings Shocks (Baseline)

The baseline model economy has only idiosyncratic earnings shocks. There are 17 parameters to be specified. These parameters are listed below in separate categories with the number of parameters in each category appearing in parentheses.

Demographics (1) The probability of survival is ρ (which implies that the mass of new entrants is $1 - \rho$).

Preferences (2) We assume standard time–separable constant relative risk aversion preferences that are characterized by two parameters, the discount rate, β, and the risk aversion coefficient, σ.

Technology (3) There are two parameters that determine the properties of the production function: the exponent on labor in the Cobb-Douglas production function (θ) and the depreciation rate (δ). We also place in this category the fraction of lost earnings while a household has a bankruptcy on its credit record, γ.

Legal system (1) The legal system is characterized by the average length of the exclusion from access to credit, (λ).

Earnings process (10) The process for earnings requires the specification of a Markov chain for s and of the distribution of e conditional on s. We use a three-state Markov chain Γ that we loosely identify with "super-rich" (s_1), "white-collar" (s_2), and "blue-collar" (s_3). The persistence of the latter two states ensures that earnings display a sizable positive autocorrelation. The first state provides the opportunity and incentive for a high concentration of earnings and wealth (see Castañeda, Díaz-Giménez, and Ríos-Rull (2003)). This specification requires 6 parameters in the Markov transition matrix but we reduce it to 4 by setting the probability of moving from blue-collar to super-rich and *vice versa* to zero. For the distribution of earnings we need 6 more parameters, 5 of which pertain to the upper and lower limits of the range of earnings for each type (units do not matter and that frees up one parameter) and one additional parameter to specify the shape of the cdf of earnings. We assume the following one-parameter functional form for the distribution function:

$$\int_{e^s_{\min}}^{y} \Phi(e|s) = P[e \leq y|s] = \left[\frac{y - e^s_{\min}}{e^s_{\max} - e^s_{\min}}\right]^{\varphi}. \tag{23}$$

5.1.2 The Model Economy with Earnings and Preference Shocks

In this economy, we keep the shocks to earnings and add a multiplicative shock to the utility function. The preference shock follows a two-state Markov process independent of earnings. This version has three additional parameters — one for the relative magnitude of the shock (again, units do not matter and that frees up one parameter) and two for the transition probabilities. For reasons explained below we set the probability of remaining in the high state to zero. Therefore, this version has only two additional parameters. With preference shocks, the overall Markov process Γ contains $3 \times 2 = 6$ states.

5.1.3 The Model Economy with Earnings and Liability Shocks

In this economy, we keep the shocks to earnings and add the liability shock ζ. We assume that ζ can take on only two values, zero and some positive number. The shock is assumed to be independent of earnings and $i.i.d$ over time. Therefore, this version has two additional parameters. With liability shocks, the Markov process Γ contains $3 \times 2 = 6$ states.

5.1.4 The Model Economy with All Shocks

This economy includes all three shocks. Accordingly, the economy has 4 additional parameters (2 for preference shocks and 2 for the liability shock) compared to the baseline model economy. With all shocks, the overall Markov process Γ has $3 \times 2 \times 2 = 12$ states.

5.2 Data Targets

We select model parameters to match three sets of statistics: aggregate statistics, earnings and wealth distribution-related statistics, and statistics on debt and bankruptcy. The targets – which appear in Table 3 – contain relatively standard targets for macroeconomic variables such as capital-output ratio, labor share, and so on. They also include statistics of the earnings and wealth distribution obtained from the 2001 SCF. These statistics are selected points of Lorenz curves as well as the Gini indices and the mean to median ratios. An important additional target is the autocorrelation of earnings – set at 0.5. This target is a compromise between the high persistence of earnings of most households present in U.S. panel data sets and the possible lower volatility of very high earners on whom there is no direct evidence (Castañeda, Díaz-Giménez, and Ríos-Rull (2003)).

We now turn to the debt and bankruptcy targets and discuss them in more detail since they are novel. First, according to the Fair Credit Reporting Act, a bankruptcy filing stays on a household's credit record for 10 years. This fact is used in our model to calibrate the length of exclusion from the credit market.

Second, according to the Administrative Office of the U.S. Courts, the total number of filers for personal bankruptcy under Chapter 7 was 1.087 million in 2002. According to the Census Bureau, the total population above age 20 in 2002 was 201 million. Therefore, the ratio of people who file to total population over age 20 is 0.54%.

Third, since in our model households can only save or borrow, we use the 2001 Survey of Consumer Finances to obtain the consolidated asset position of households. Only people with negative net worth are considered to be debtors. We exclude households with negative net worth larger than 120% of average income since the debts are likely to be due to entrepreneurial activity that our model abstracts from. These households are a very small fraction of the SCF (comprising only 0.13% of the original sample of SCF 2001).[22] The average net negative wealth of the remaining households is \$631.46, which divided by per household GDP of \$94,077 is 0.0067. Thus, we take the target debt-to-income ratio to be 0.67%.

Fourth, after excluding the few households with debt more than 120% of average income, 6.7% of the households in the 2001 SCF had negative net worth.[23]

[22] The average amount of debt for this group is \$100,817, or 145% of the average income, and their income is relatively high.

[23] We also note that 2.6% of the households had zero wealth in the 2001 SCF.

Table 1: Reasons for Filing for Bankruptcy (PSID, 1984-95)

1	Job loss	12.2%
2	Credit misuse	41.3%
3	Marital disruption	14.3%
4	Health-care bills	16.4%
5	Lawsuit/Harassment	15.9%

Source: Chakravarty and Rhee (1999)

The last three statistics are the relevant bankruptcy and debt targets for a model that includes all important motives for bankruptcy. If a model does not include all motives, the appropriate targets would be some fraction of these statistics. According to Chakravarty and Rhee (1999) the Panel Study of Income Dynamics (PSID) classified the reasons for bankruptcy filings into five categories and we report their findings in Table 1. Among the five reasons listed, we associate the first two (job loss and credit misuse) with earnings shocks; marital disruption with preference shocks; and the final two (health-care bills and lawsuits/harassment) with liability shocks. Given these associations, we allocate the total volume of debt, the fraction of households in debt, and the fraction of filings according to the fraction of people citing the above reasons. For instance, given that 53.5% of households cited reasons we associate with earnings shocks, we assume that the fraction of filings corresponding to this reason is 0.29% (i.e., $0.00535 \times 0.0054 = 0.0029$). We report these targets specific to each of the model economies in Table 2, where E denotes our baseline model with only earnings shocks, EP denotes the model with earnings and preference shocks, EL denotes the model with earnings and liability shocks, and EPL denotes the model with all three shocks.

5.3 Moments Matching Procedure

The baseline model economy has 17 parameters, which we classify into two groups. The first group consists of 5 parameters, each of which can be pinned down independently of all other parameters by one target. The survival rate ρ is determined to match the average length of adult life, which we take to be 40 years, a good compromise for an economy without population growth or changes in marital status. The labor share of income is taken to be 0.64, which determines θ. The depreciation rate δ is taken to be 0.10, which is consistent with a wealth to output ratio of 3.08 (its value according to the 2001 Survey of Consumer Finances) and the consumption to output ratio of 0.70. The transition probability λ, which governs the average length of time that a bankruptcy remains on a person's credit record is set to 0.1, consistent with the Fair Credit Reporting Act. The coefficient of relative risk aversion σ is fixed at 2.

Table 2: Debt and Bankruptcy Targets for Each Model Economy

Economy	U.S.	E	EP	EL	EPL
Reasons covered	1,2,3,4,5	1,2	1,2,3	1,2,4,5	1,2,3,4,5
Percent of all bankruptcies	100%	53.5%	67.8%	85.7%	100%
Percent of households filing	0.54%	0.29%	0.37%	0.46%	0.54%
Debt-to-Income ratio	0.67%	0.36%	0.46%	0.58%	0.67%
Percent of households in debt	6.7%	3.6%	4.6%	5.8%	6.7%

Note: The numbers in the "Reasons covered" row are associated with the number in the previous table.

The 12 parameters in the second group — including the discount rate β, the cost of having a bad credit record γ, 4 parameters governing the transition of type characteristics Γ, 5 parameters defining the bounds of efficiency levels for the type characteristics, and 1 parameter characterizing the shape of the distribution function of the efficiency shocks in (23) φ — are determined jointly by minimizing the weighted sum of squared errors between the 17 remaining targets and the corresponding statistics generated by the model. Our weighting matrix puts more emphasis on matching the debt and bankruptcy filing targets than on earnings and wealth distribution targets.

Since the computational task of simulating equilibrium model moments was extremely burdensome (each equilibrium requires computing thousands of equilibrium loan prices – recall that we have to solve for equilibrium loan price *schedules* – and it took thousands of computed equilibria to find satisfactory configurations of parameter values), the selection of parameter values for the three additional model economies is not done from scratch. Specifically, we maintained the 10 parameter values for earnings shocks estimated for the baseline model economy in all other model economies. In addition, we maintain the 5 parameters in the first group for all the model economies. For the economy with preference shocks (EP economy), we re-estimate 2 parameters from the baseline model (β and γ) as well as estimate the 2 parameters that characterize the preference shock process. The 4 target statistics we match for the EP economy are: *(i)* the fraction of households in debt; *(ii)* the fraction of bankruptcies; *(iii)* the capital-output ratio; and *(iv)* the capital-debt ratio.[24] For the economy with liability shocks, we re-estimate β and γ as well as estimate

[24] In an earlier version of this paper we specified a more general process for the preference shock but this did not help in matching the data. In particular, it is impossible to generate a large amount of debt when preference shocks are persistent or even iid. When the process is persistent or iid, financial intermediaries know that a household borrowing to accommodate a preference shock may get the same shock next period and file for bankruptcy. So they limit loans to these households. With observable shocks, generating a high level of equilibrium debt in response to a preference shock requires that the probability of receiving two of

the two parameters that characterize the *i.i.d.* liability shocks to match appropriate values of the above 4 target statistics. Finally, for the economy with all three shocks we maintain the parameters for the earnings shocks from the baseline model economy, the parameters for preference shocks from the EP economy, and the parameters for liability shocks from the EL economy. This left us with only β and γ to estimate. In this final step we put more weight on matching the fraction of bankruptcies and the debt-output ratio relative to matching the capital-output ratio and fraction of households in debt.

5.4 Computation of the Steady State

The computation of the equilibrium requires three steps: the inner loop, where the decision problem of households given parameter values and prices (including loan prices) is solved; the middle loop, where market-clearing prices are obtained; and the outside loop – or estimation loop – where parameter values that yield equilibrium allocations with the desired (target) properties are determined. We use a variety of (almost) off-the-shelf techniques, powerful software (Fortran 90) and hardware (26 processors Beowulf cluster) to accomplish our task. Space considerations precludes a more detailed discussion of the computational "tricks" employed to improve the speed and accuracy of calculations.

5.5 Results

Table 3 reports the target statistics and their counterparts in the baseline model economy as well as values selected for each parameter. Table 4 reports the target statistics and their model counterparts for the three other economies (and the baseline for comparison).[25] Table 5 displays the estimated parameter values other than those estimated in the baseline model.

The baseline model economy successfully replicates the macro and distributional targets. The capital-output ratio is exactly as targeted and so is the earnings Gini. The wealth Gini is somewhat lower than in the data, but as Figures 2 and 3 show, the overall fit of the model along these dimensions is quite good. The EP and EL economies replicate macro targets successfully, but in the EPL economy, the capital-output ratio is slightly below the U.S. level. This occurred because we maintained the same parameter values for the shock processes as obtained from other economies and were thus left with only two free parameters to estimate. These two parameters were used to match the fraction of defaulters and the debt-to-income ratio since we consider these to be more important for our study.

Turning to the debt and bankruptcy targets, all the models successfully replicate the percentage of defaulters and the debt-to-income ratio. On the other hand, the percentage of

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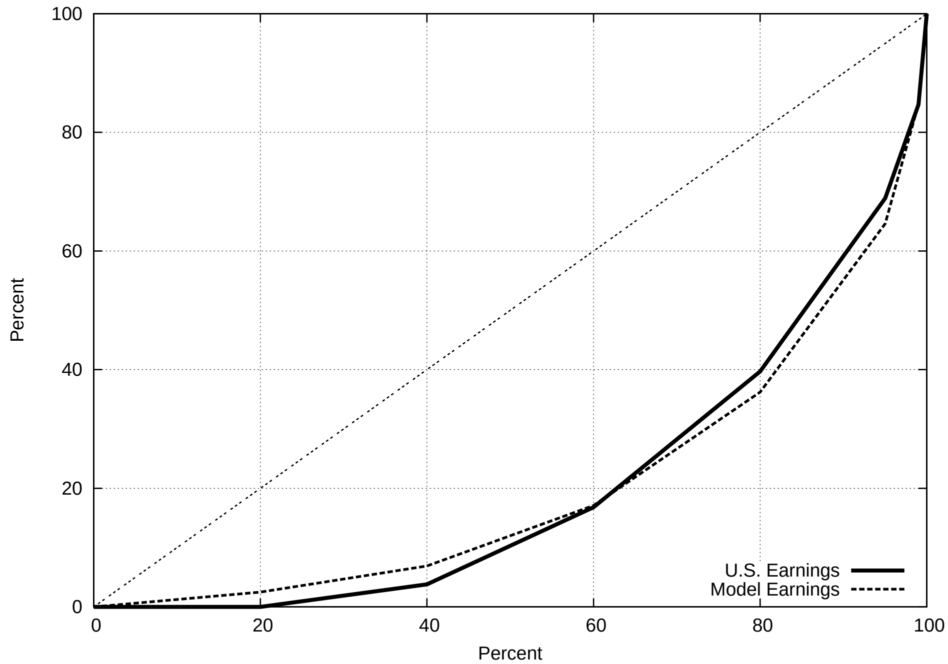
these shocks in a row be as small as possible (i.e. zero).

[25]These economies have the same statistics with regard to the 5 aggregate statistics determined independently and with respect to the earnings distribution. The wealth distributions do differ relative to the baseline but the differences are relatively minor

Figure 2: Earnings Distributions for the U.S. and Baseline Model



households in debt is measurably higher in the baseline model relative to target. Discrepancies between model statistics on the percentage of households in debt and the targets also arise in the other economies. These discrepancies are hard to overcome for the following reason. In the model, the households who borrow are those with a negative net asset position. Because indebted households do not have any assets to lose they are more prone to default. Consequently the default premium on loans is very high, which works to reduce the participation of households in the credit market. Given the structure of the model, it is very difficult to attain a combination of a small number of indebted households and a relatively large number of bankruptcy filings. The difference between the model and the U.S. economy lies in the fact that a typical indebted American household has both liabilities *and* assets and the default rate on each loan is much lower than in this model.[26]

6 Properties of the Model Economies

6.1 Distributional Properties

Figure 4 shows the histogram of the wealth distribution in the baseline model, excluding the long right tail which comprises about 15% of the population. The asset holdings of households with a good credit record and with a bad credit record are plotted separately. For households with a good credit record, the model generates a pattern of the wealth

[26]This difference is also the reason we do not target interest rate statistics. In the model all borrowers have negative net worth and therefore pay very high interest rates. In the real world many borrowers also own non-exempt assets and the interest rate they pay presumably reflects this fact. To target interest rates in a meaningful way would require a model in which households hold both assets and liabilities.

Table 3: Baseline Model Statistics and Parameter Values

Statistic	Target	Model	Parameter	Value
Targets determined independently				
Average length of life	40 years	40 years	ρ	0.975
Coefficient of risk aversion	2.0	2.0	σ	2.000
Labor share of income	0.64	0.64	θ	0.640
Depreciation rate of capital	0.10	0.10	δ	0.100
Average length of punishment	10 years	10 years	λ	0.100
Targets determined jointly				
Percent of defaulters	0.29%	0.29%	β	0.919
Percent in debt	3.6%	4.9%	γ	0.025
Capital-output ratio	3.08	3.08	$\Gamma_{1,1}$	0.019
Debt-to-output ratio	0.0036	0.0036	$\Gamma_{2,1}$	0.001
Autocorrelation of earnings	0.50	0.49	$\Gamma_{2,3}$	0.282
Earnings Gini	0.61	0.61	$\Gamma_{3,3}$	0.966
Earnings mean/median	1.57	1.98	$e^1_{\max}/e^3_{\min}$	19573.0
Earnings earned by 2nd quintile	4.0%	4.4%	$e^1_{\min}/e^3_{\min}$	11608.1
Earnings earned by 3rd quintile	13.0%	10.2%	$e^2_{\max}/e^3_{\min}$	117.8
Earnings earned by 4th quintile	22.9%	19.1%	$e^2_{\min}/e^3_{\min}$	34.3
Earnings earned by 5th quintile	60.2%	63.9%	$e^3_{\max}/e^3_{\min}$	23.5
Earnings earned by top 2-5%	15.8%	20.2%	φ	0.479
Earnings earned by top 1%	15.3%	15.2%		
Wealth Gini	0.80	0.73		
Wealth mean/median	4.03	3.25		
Wealth earned by 2nd quintile	1.3%	2.9%		
Wealth earned by 3rd quintile	5.0%	5.9%		
Wealth earned by 4th quintile	12.2%	15.1%		
Wealth earned by 5th quintile	81.7%	75.7%		
Wealth earned by top 2-5%	23.1%	15.3%		
Wealth earned by top 1%	34.7%	31.3%		

Table 4: **Key Statistics and Targets for All Model Economies**

Target statistics	U.S. Economy	Model Economy
E (Baseline) Economy		
Percent of defaulters	0.29%	0.29%
Percent in debt	3.6%	6.2%
Capital-output ratio(%)	308	308
Debt-to-output ratio(%)	0.36	0.36
EP Economy		
Percent of defaulters	0.37%	0.37%
Percent in debt	4.6%	4.8%
Capital-output ratio(%)	308	308
Debt-to-output ratio(%)	0.46	0.46
EL Economy		
Percent of defaulters	0.46%	0.46%
Percent in debt	5.8%	5.7%
Capital-output ratio(%)	308	308
Debt-to-output ratio(%)	0.58	0.58
EPL Economy		
Percent of defaulters	0.54%	0.54%
Percent in debt	6.7%	5.4%
Capital-output ratio(%)	308	297
Debt-to-output ratio(%)	0.67	0.67

Table 5: Parameter Values in Economies with Preference and/or Liability Shock

Parameter	E	EP	EL	EPL
Discount factor	0.919	0.898	0.918	0.887
Fraction of earnings lost due to bankruptcy	0.025	0.002	0.068	0.032
Magnitude of high preference shocks	–	16.8	–	16.8
Persistence of normal preference shocks	–	0.809	–	0.809
Magnitude of liability shock (over average income)	–	–	1.72	1.73
i.i.d. probability of liability shock	–	–	0.012	0.012

Figure 3: Wealth Distributions for the U.S. and Baseline Model

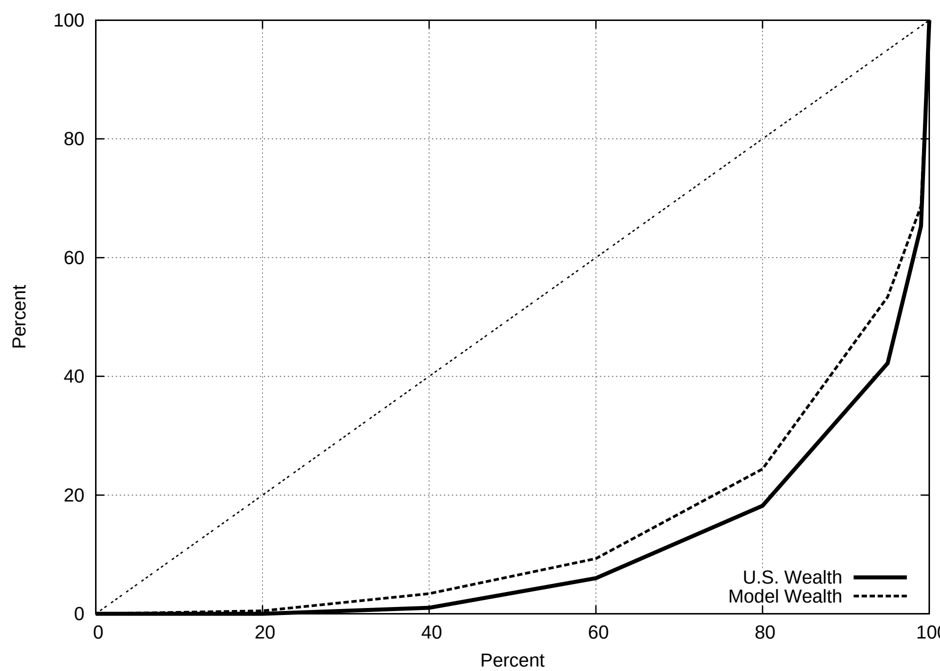
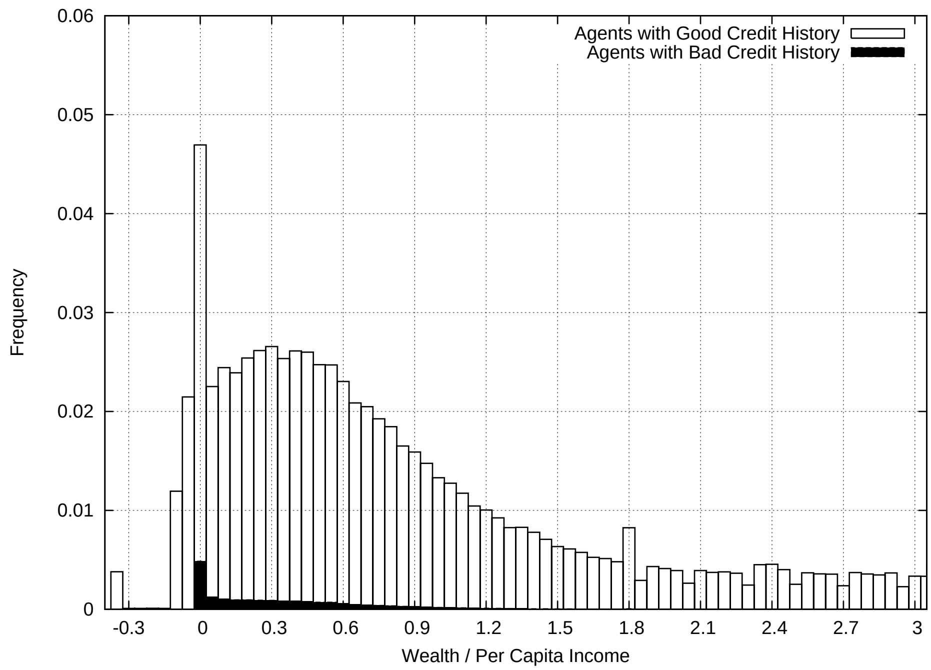


Figure 4: Wealth Histogram in the Baseline Model



distribution that is typical of overlapping generations model. There is a significant fraction of households with zero wealth, many of whom are newborns. Because most households accumulate some savings there is another small peak in the histogram around 30% of average income. There is also a relatively large fraction of households with small amounts of debt relative to average income and there are some households with debt in the neighborhood of 35% of average income. There are no households borrowing more than 35% because the amount of current consumption derived from borrowing declines beyond this level of debt due to steeply rising default premia.[27]

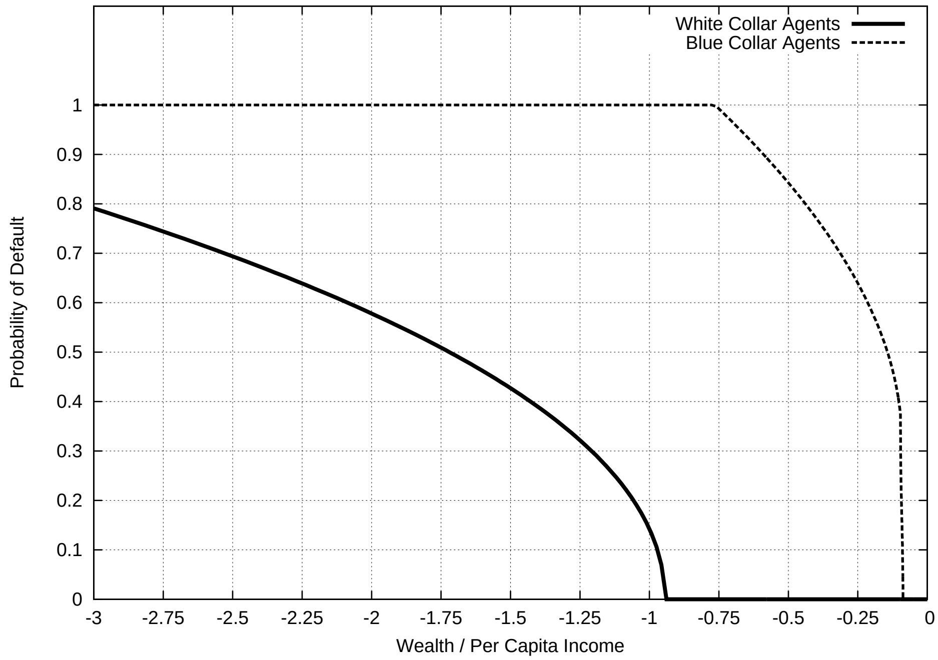
Households with a bad credit record consist mostly of households with very few assets. No one in this group has debt because these households are precluded from borrowing. The right tail of this distribution is relatively long, indicating that some households remain with a bad credit record for many periods and have relatively high earnings realizations.

6.2 Bankruptcy Filing Properties

Figure 5 shows default probabilities in the baseline model, conditional on whether households are blue-collar or white-collar in the *current* period, on loans taken out in the previous period. We wish to make three points. First, the probability of filing for bankruptcy is higher for blue-collar than white-collar households for every level of debt. This is a natural consequence of white-collar households receiving higher earnings on average than blue-collar households. For instance, at a debt level of average income no white-collar worker is expected to default while *all* blue-collar workers are expected to default. Second, the default probabilities for both types of households are rising in the level of debt, which is consistent with Theorem

[27]This point is discussed in more detail in section 6.5

Figure 5: Baseline Default Probabilities for Blue- and White-Collar Households



4. Third, no one is expected to file for bankruptcy with a level of debt near zero, which is consistent with Theorem 6.ii. In particular, even the blue-collar households are not expected to default if their debt is less than 9% of average income. The threshold debt level below which there is no default for white-collar households is about 94% of average annual income.

Table 6 shows the number of people filing for bankruptcy by earning quintiles as a fraction of the entire population and as a fraction of those in debt. Across the four economies, the conditional probability of bankruptcy for households in the first and second earnings quintiles is very similar but declines rapidly for the third and fourth quintiles. And there are few defaulters in the top quintile. For economies with liability shocks, there are two differences.[28]. First, defaulters as a percent of households in debt is lower than in the other two economies because many households without debt file for bankruptcy when they experience a liability shock. Second, even households in the highest earnings quintile might file for bankruptcy since the magnitude of the liability shock is so large. Notice how this table highlights the different role played by earnings and by wealth in shaping the bankruptcy

[28]For the EL economy, aggregate medical services are 0.79% of output, while medical expenditure is 0.62% of output. This implies that the markup m is 27.5%. For the EPL economy, aggregate medical services are 0.83% of output, while medical expenditure is 0.68% of output. This implies that the markup in the EPL economy is 20.7%.

Table 6: **Earnings and Bankruptcies.**

Economy	E	EP	EL	EPL
Over the total population				
1st quintile	0.46%	0.57%	0.61%	0.68%
2nd quintile	0.46%	0.56%	0.60%	0.67%
3rd quintile	0.44%	0.46%	0.52%	0.57%
4th quintile	0.10%	0.05%	0.13%	0.14%
5th quintile	0.00%	0.00%	0.01%	0.02%
total	0.29%	0.37%	0.46%	0.54%
Over the population in debt				
1st quintile	8.70%	10.86%	3.36%	2.96%
2nd quintile	8.70%	10.80%	3.35%	2.93%
3rd quintile	8.46%	10.24%	3.15%	2.72%
4th quintile	4.00%	2.46%	1.18%	0.96%
5th quintile	0.00%	0.00%	0.19%	0.19%
total	5.97%	7.79%	1.14%	1.15%

decision.[29]

6.3 Loan Price Properties

Since household type is quite persistent, the lower probabilities of bankruptcy of white-collar households translate into their having a lower default premium (higher q) than blue-collar households. Figure 6 shows the price schedule of loans conditional on the amount of debt, for white- and blue-collar households. For a debt level of less than 9% of average income, the price schedule is flat since there is no default premium; even if a borrowing household turns out to be blue-collar next period, the probability that this household defaults is zero. For a higher level of debt the loan price schedule for a white-collar household lies above that of blue-collar households. This is because type shocks are persistent and current white-collar workers are less likely to default next period. White-collar households whose debt is smaller than 94% of average income have to pay positive default premiums even though we can see in Figure 5 that white-collar households with smaller than that amount of debt have zero

[29]One aspect of default behavior that is not evident in these tables is that in every case households below some earnings threshold default. Although the theory allows for a second (lower) threshold below which people pay back, that does not happen in equilibrium.

Figure 6: Baseline Loan Prices for Blue- and White-Collar Households

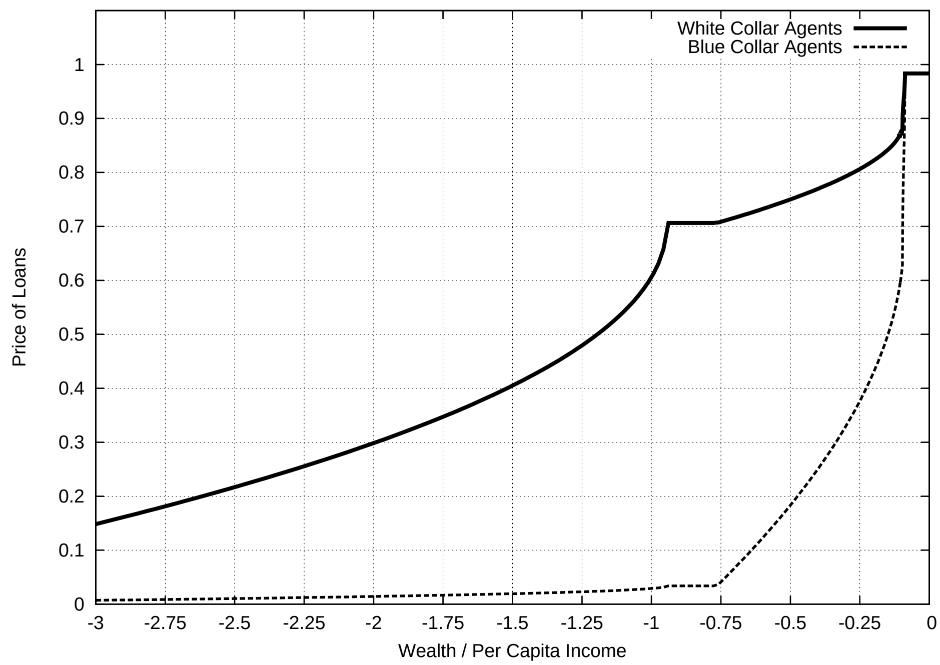


Table 7: Interest Rates in the Baseline Model

Rate of return of capital	1.69%
Risk-free interest rate	1.69%
Average loan interest rate (weighted by persons)	21.50%
Implied average default premium	19.82%

probability of default. This is because some of the white-collar households in the current period become blue-collar households next period (and they have a positive probability of bankruptcy). The kink in the loan price schedule for white-collar households at 94% of average income reflects the property of the default probability of white-collar households which becomes flat at debt levels below that level.

Table 7 summarizes interest rates in the baseline model economy. The annual equilibrium rate of return of capital is 1.69%. Since the competitive loan industry cannot charge a positive premium for loans without any risk, the risk-free interest rate is the same as the rate of return of capital. The average interest rate on loans (weighted by the number of households in debt) is 21.50%, implying an average default premium of 19.82%.[30]

[30] If we weight by the amount of debt for each debtor, the average loan interest rate is 91.69% with the average default premium of 90.00%. The latter average interest rate is substantially higher than the average rate paid per household because there are a small number of households who borrow a large amount at very

6.4 Accounting for Debt and Default

These properties of default and loan price schedules indicate different roles of blue-collar and white-collar households in accounting for aggregate filing frequency and consumer debt. Blue-collar households receive (on average) lower earnings every period and therefore frequently face a need to borrow in order to smooth consumption. On the other hand if they receive a sequence of bad earnings shocks they find it beneficial to file for bankruptcy and erase their debt. Since they are more likely to default, blue-collar households have to pay a relatively high default premium and the premium soars as the size of the loan increases. As a result blue-collar households borrow relatively frequently in small amounts and constitute the majority of those who go bankrupt. But because they borrow small amounts they account for only a small portion of aggregate consumer debt. In contrast, white-collar households face a lower default premium on their loans because they earn more on average. Therefore they borrow a lot more than blue-collar households when they suffer a series of bad earnings shocks. The households with large amounts of debt in our baseline model consist of these white-collar households. As long as these households remain white-collar they maintain access to credit markets. But they file for bankruptcy if their employment status changes to blue-collar because they then face an extremely high default premium on their debt. This story resembles the plight of some members of the American middle class who borrowed a lot because they were considered to be earning a sufficient amount but filed for bankruptcy following a big persistent adverse shock to their earning stream. To summarize, in our model blue-collar households account for a large fraction of bankruptcies and a large fraction of households in debt while white-collar households account for the large level of aggregate consumer debt.

6.5 A Comparison with Standard Exogenous Borrowing Limits

We conclude this section by comparing our results with the two extremes typically assumed in general equilibrium economies with heterogeneous agents: either agents are completely prevented from borrowing (the Bewley (1983) economy) or there is full commitment and hence agents can borrow up to the amount that they can repay with probability one (the Aiyagari (1994) economy). Table 8 compares the steady states of the Bewley and Aiyagari economies with our baseline (E) economy. As is apparent from the table, aggregate asset holdings in our economy are much closer to the Bewley model than the Aiyagari model.

A critical difference between these three models is the form of borrowing limit. The Bewley borrowing limit is exogenously set at zero. The Aiyagari borrowing limit is exogenously set at the level where a household can pay back across all possible realizations. In terms of our notation, it is given by:

$$\underline{\ell}^{Aiyagari} = \frac{\max\left\{w\underline{e} - \overline{\zeta}, 0\right\} \rho}{1 - \rho + (r^* - \delta)} \tag{24}$$

high interest rates. This is consistent with the histogram of household wealth shown in Figure 4.

Table 8: Comparison of Baseline Model with Bewley and Aiyagari Economies

Economy	Baseline	Bewley economy	Aiyagari economy
Availability of loans	Yes	No	Yes
Default premium	Yes	–	No
Output	100	100	100
Total asset	307.9	311.6	288.3
Total debt	0.360	–	9.4
Percentage of filers	0.29%	–	–
Percentage with bad credit record	2.31%	–	–
Percentage in debt	4.87%	–	29.10%
Rate of return of capital	1.69%	1.55%	2.48%
Avg loans rate (persons-weighted)	21.50%	–	2.48%

where $\overline{\zeta}$ is the upper bound of the liability shock. Note that Aiyagari's borrowing limit is zero for economies with liability shocks because the size of the shock is larger than the minimum potential earnings. Table 9 presents the endogenous borrowing limits across worker characteristics for each model economy we study as well as the Aiyagari model.

For each of the three earnings types (super-rich, white-collar, and blue-collar) two kinds of borrowing limits are presented. The first one (labeled as "q") is the smallest loan size for which the corresponding price q is zero, conditional on the type of household. The "q" borrowing limits for super-rich and white-collar households are the same (both types have a positive probability of being the highest type in the following period) while the borrowing limit is substantially lower for blue-collar households. The other borrowing limit (labeled as "ℓq") is the maximum amount of debt (optimally) chosen by households. The "ℓq" borrowing limit is substantially smaller than the corresponding "q" limit because of the sensitivity of the price of the loan to the size of the loan. If a household borrows an amount $\tilde{\ell}$ greater than the "ℓq," it actually receives *less* in the current period ($q\tilde{\ell}$) than it receives if it borrows at the "ℓq" borrowing limit and its future debt obligation is *larger*. Therefore, it is never optimal for a household to borrow more than the "ℓq" borrowing limit.

For our baseline economy, the "q" borrowing limits are larger across all household types than the Aiyagari limit. This might give the impression that our economy imposes a looser constraint than Aiyagari's economy. However, the "ℓq" borrowing limit for blue-collar households (0.36 of average income) is less than half of Aiyagari's limit (0.84 of average income). From the histogram of the wealth distribution presented earlier we know there is a mass of borrowers at this debt level. This mass of households is constrained by the "ℓq" borrowing limit. Since blue-collar households are the ones most likely in need of loans, our

Table 9: Comparison of Borrowing Limits

Earnings type	1 (Super-rich)		2 (White-collar)		3 (Blue-collar)		Aiyagari
	q	ℓq	q	ℓq	q	ℓq	
E economy	814.09	452.75	814.09	0.94	4.30	0.36	0.84
EP economy (low)	814.57	453.80	814.57	1.23	3.61	0.45	0.83
EP economy (high)	814.57	453.80	814.57	1.23	3.61	0.44	0.83
EL economy	814.39	453.70	814.39	1.01	4.32	0.37	0.00
EPL economy (low)	814.71	483.33	814.71	1.25	4.42	0.44	0.00
EPL economy (high)	814.71	483.33	814.71	1.25	4.42	0.44	0.00

[1] Unit is proportion to the average income of the respective economy.
[2] Since the liability shock is iid, the borrowing limit is independent of the value of the liability shock.

baseline economy imposes a stricter borrowing constraint for a subset of the population than Aiyagari's economy.

6.6 Borrowing Constraints and Consumption Inequality

Borrowing constraints have important implications for consumption inequality. Table 10 shows the earnings and consumption inequality of the baseline economy compared to the Bewley and Aiyagari economies.[31] In all the economies the degree of consumption inequality is substantially lower than the degree of earnings inequality since the households use savings to smooth consumption fluctuations. However, there is a slight difference in the consumption inequality across the three economies. The standard deviation of log consumption of the baseline economy is about 2.5% higher than for the Aiyagari economy. On the other hand, the standard deviation of log consumption in the baseline economy is 2.5% lower than in the Bewley economy.

7 Policy Experiment

Given that our model matches the relevant U.S. statistics on consumer debt and bankruptcy, it is possible to examine the consequences of a change in regulation that affects unsecured consumer credit. Here we evaluate a recent change to the bankruptcy law, which limits

[31] Since all three economies share the same stochastic process for earnings, earnings inequality is the same across all three economies.

Table 10: Consumption and Earnings Inequality

	Std Dev	Quintiles				
	of log	1st	2nd	3rd	4th	5th
Baseline model						
Earnings	1.198	2.5%	4.4%	10.2%	19.1%	63.9%
Consumption	0.663	6.8%	10.7%	13.5%	20.8%	48.1%
Aiyagari economy						
Earnings	1.198	2.5%	4.4%	10.2%	19.1%	63.9%
Consumption	0.647	7.2%	10.4%	13.6%	20.6%	48.7%
Bewley economy						
Earnings	1.198	2.5%	4.4%	10.2%	19.1%	63.9%
Consumption	0.680	6.7%	10.8%	13.6%	20.8%	48.1%

"above-median-income" households from filing under Chapter 7.[32] Since median income in the model economy is around 37% of average output, we assume that households in the model cannot file for bankruptcy if their current income is more than 37% of average output.[33] Table 11 reports the changes in the model statistics for each of the four economies with this policy with and without general equilibrium effects. We focus on the numbers with the general equilibrium effects but note that the general equilibrium effects are not crucial.

7.1 Effects on Allocations

As indicated in Figure 7, the filing restriction lowers default probabilities substantially. For blue-collar households the restriction reduces default probabilities for a large range of loan sizes and, for small set of loan sizes, reduces it to zero. The default probabilities drop for white-collar households as well, but the change is less pronounced. As a result, the

[32]The law is more complicated than our experiment. A person cannot file under Chapter 7 (and effectively would have to pursue Chapter 13) if all of the following three conditions are met: (1) The filer's income is at least 100 percent of the national median income for families of the same size up to four members; larger families use median income for a family of four plus an extra $583 for each additional member over four. (2) The minimum percentage of unsecured debt that could be repaid over 5 years is 25 percent or $5000, whichever is less. (3) The minimum dollar amount of unsecured debt that could be repaid over 5 years is $5000 or 25%, whichever is less. We summarize these criteria by restricting filing to those with lower than median earnings.

[33]But we assume that households are always allowed to file if not doing so results in negative consumption.

Table 11: Allocation Effects of Means-Testing in the Baseline Model

Economy	Baseline	Bankruptcy restriction	
Max earnings for filing	∞	Median income	
General equilibrium effect	–	No	Yes
Output	100	100	100
Total asset	307.9	302.9	304.2
Total debt	0.36	1.08	1.05
Percentage of filers	0.29	0.35	0.34
Percentage with bad credit record	2.31	2.79	2.73
Percentage in debt	4.87	8.73	8.61
Rate of return of capital	1.69	1.69	1.83
Avg loans rate (persons-weighted)	21.50	11.29	11.37

loan price schedules shift up (the default premium schedules shift down) for both types of households, as shown in Figure 8. Even though the change in *default probability* for the white-collar households is not substantial, the *default premia* on loans to both white- and blue-collar households drop substantially. This is because for both types of households there is a positive probability of being blue-collar in the next period.

Table 11 presents the changes in aggregate statistics resulting from this policy. Most interestingly, the number of bankruptcy filings increases by about 18% even though the default probability schedule conditional on type shifts down for each type of household. This occurs because the percentage of households in debt increases dramatically in response to lower interest rates on loans. Specifically, the percentage of households in debt almost doubles from 4.9% in the baseline to 8.6%. Total debt triples, implying that on average households take on bigger loans. Total asset holdings decline by more than 1%.

Table 12 shows how the policy change affects economies with preference and/or liability shocks. Only the results incorporating general equilibrium effects are shown.[34] The changes in the percentage of households in debt and the aggregate level of debt and assets show changes similar to that for the baseline economy.

The direction of the change in the percentage of bankruptcies depends on the relative strength of the negative effect due to means-testing and the positive effect due to the larger numbers of households in debt. For the EL and EPL economies, the positive effect dominates as in the baseline economy. But for the EP economy the negative effect dominates and

[34]Table 15 shows the results with and without the general equilibrium effects for all the economies. As evident, the general equilibrium effects do not play a significant role.

Figure 7: Default Probabilities in the Baseline Model With and Without Means-Testing

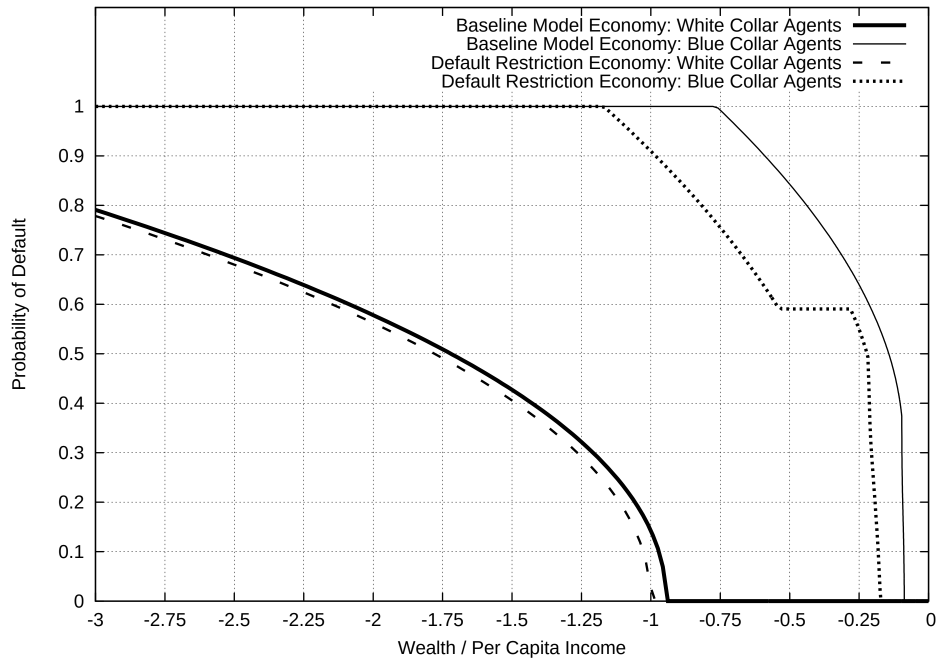


Figure 8: Loan Prices in the Baseline Model With and Without Means-Testing

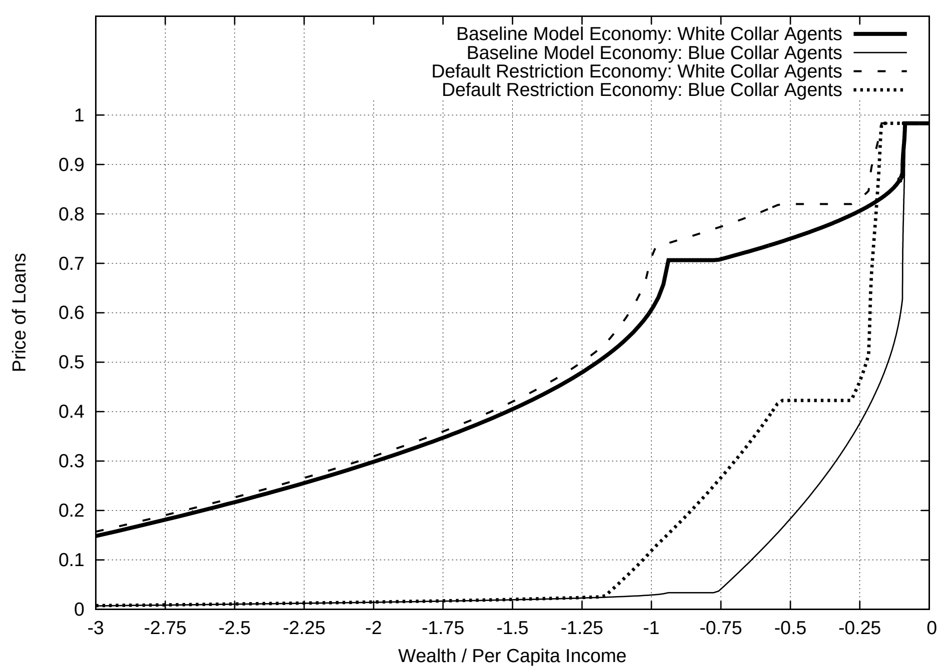


Table 12: Allocation Effects of Means-Testing in Other Models

Economy	EP economy		EL economy		EPL economy	
Max earnings for filing	∞	med(y)	∞	med(y)	∞	med(y)
Output	100	100	100	100	100	100
Total asset	308.3	305.7	308.1	305.0	296.6	294.7
Total debt	0.463	0.834	0.580	1.259	0.670	1.013
Percentage of filers	0.37	0.35	0.46	0.50	0.54	0.60
Percentage with bad credit record	2.95	2.82	3.69	4.04	4.33	4.78
Percentage in debt	4.75	6.75	5.72	8.96	5.36	6.81
Rate of return of capital	1.67	1.77	1.65	1.76	2.10	2.17
Avg loans rate (persons-weighted)	21.36	14.03	11.11	9.43	18.31	18.06

[1] The general equilibrium effect is taken into account for all the experiments.
[2] med(y) denotes median income.

the percentage of households filing for bankruptcy declines. The direction of the change in average interest rates also depends on these countervailing forces – for the EP and EL economies the average interest rate on loans drops as in the baseline economy, while the two effects virtually cancel each other out in the EPL economy. Finally, in all cases the percentage of households without access to credit (those with a bad credit record) moves in the same direction as the percentage of bankruptcies.

7.2 Effects on Welfare

We now turn to the welfare implications of this policy change. In assessing the welfare effects of any policy change one must take into account the transition path to the new steady state. However, there is a technical difficulty in computing an equilibrium transition path in our model. Since there are a very large number of prices (one for each household characteristic and each level of debt) the computation of an equilibrium transition path is computationally intractable. To work around this problem, we assume that the rate of return of capital and wages remain unchanged following the policy change. This assumption dramatically eases the computational burden because in our model loan prices depend only on the return to capital and the (household) characteristic-specific default probabilities. In particular, it does not depend on the distribution of households over the state space, which does evolve during the transition to the new steady state. One justification for our approach is that the positive results from the policy experiment confirm that the general equilibrium effects are not large. Finally, we should note that since our model abstracts from labor and family size decisions – which may be important margins for the choice of bankruptcy – our welfare estimates should be taken with a grain of salt.

Table 13: Welfare Comparisons for the Baseline Model

Average % gain in flow consumption	
With bad credit record	0.76
With good credit record and in debt	7.79
With good credit record and not in debt	1.38
Total	1.67
% of households in favor of reform	
With bad credit record	100.0
With good credit record and in debt	100.0
With good credit record and not in debt	100.0
Total	100.0

In addition, there is a second difficulty in conducting welfare analysis in our economy. In environments with multiple types of households there will generally not be agreement among different types as to the desirability of a policy change. Consequently, some form of aggregation is necessary. We use two aggregation criteria. The first criterion is the percentage of households that are made better off by the policy change and thus support it. The second criterion is the average gain as measured by the average of the percentage increase in consumption each household would be willing to pay in all future periods and contingencies so that the expected utility from the current period on in the initial steady state equals that of the equilibrium associated with the new policy. Because of our assumption on the functional form of the momentary utility function, the consumption equivalent welfare gain for a household of type (ℓ, h, s) can be computed as:

$$100\left(\left[\frac{\int \widetilde{v}_{\ell,h,s}(e;q,w)\Phi(e|s)de}{\int v_{\ell,h,s}(e;q,w)\Phi(e|s)de}\right]^{\frac{1}{1-\sigma}} - 1\right) \tag{25}$$

Table 13 reports the desirability of the policy change for the two aggregation criteria for the baseline model economy. Limiting bankruptcy filings to those with below-median income receives *unanimous* support. Every household gains from this policy reform, including households that are currently in debt. Indeed, currently indebted households gain the most as a group – an average of 7.8% of flow consumption. Evidently, the benefit of a lower default premium substantially exceeds the cost of losing the option to file for bankruptcy when household income happens to be higher than median earnings. The gain is large because following the policy change indebted households are induced to keep borrowing and therefore enjoy lower interest rates on loans for an extended period of time. Households with a good credit record and without any debt gain a modest 1.38% of flow consumption. Households

Table 14: Welfare Comparison For Other Models

Shock	Preference Shock		Liability Shock		Total
	Hit	Not	Hit	Not	
EP economy					
Proportion of households	0.160	0.840			1.000
Average % gain in flow consumption					
With bad credit record	0.41	0.52			0.50
With good credit record and debt	12.16	4.51			5.28
With good credit record and no debt	1.48	0.94			1.03
Total	1.77	1.11			1.22
% of households in favor of reform					
With bad credit record	100.0	100.0			100.0
With good credit record and debt	100.0	100.0			100.0
With good credit record and no debt	99.7	99.7			99.7
Total	99.8	99.8			99.8
EL economy					
Proportion of households			0.012	0.988	1.000
Average % gain in flow consumption					
With bad credit record			0.57	0.66	0.66
With good credit record and debt			-2.09	7.52	7.41
With good credit record and no debt			1.15	1.18	1.18
Total			0.96	1.50	1.49
% of households in favor of reform					
With bad credit record			100.0	100.0	100.0
With good credit record and debt			3.4	100.0	98.8
With good credit record and no debt			84.6	100.0	99.8
Total			80.8	100.0	99.8
EPL economy					
Proportion of households	0.160	0.840	0.012	0.988	1.000
Average % gain in flow consumption					
With bad credit record	0.28	0.36	0.29	0.35	0.35
With good credit record and debt	11.13	3.73	-1.78	4.57	4.49
With good credit record and no debt	1.07	0.68	0.71	0.75	0.75
Total	1.36	0.83	0.57	0.92	0.92
% of households in favor of reform					
With bad credit record	100.0	100.0	100.0	100.0	100.0
With good credit record and debt	98.8	98.9	9.5	100.0	98.9
With good credit record and no debt	99.7	99.9	86.3	100.0	99.8
Total	99.7	99.8	83.0	100.0	99.8

with a bad credit record gain the least – 0.76% – since they cannot benefit from lower loan prices until they are permitted to borrow again. Overall, the average gain from the reform is 1.67%. This is a large gain considering that it is a flow number.

Table 14 summarizes the welfare effects of restricting Chapter 7 filings to those with below-median earnings for the EP, EL, and EPL economies. Although support is no longer unanimous, it is still very widespread. The lack of unanimity comes from the fact that in these economies there are shocks, independent of earnings, that lead to onerous debt burdens. Therefore, restricting bankruptcy filings to when *earnings* are below median leaves some relatively high-income households open to low levels of consumption. Put differently, there are more households with a binding default restriction in these three economies relative to the baseline economy (where the fraction of such households is negligible). An example in the EL economy is the group that is currently hit with the liability shock: a majority of these households oppose the policy change because for them the default restriction is binding. The average welfare gain measured by changes in flow consumption is also lower than in the baseline economy. The average welfare gain is 1.2% in the EP economy, 1.5% in the EL economy, and 0.9% in the EPL economy.

8 Conclusions

In this paper we accomplished four goals. First, we developed a theory of default that is consistent with U.S. bankruptcy law. In the process we characterized some theoretical properties of the household's decision problem and proved the existence of a steady-state competitive equilibrium. A key feature of the model is that it treated different-sized consumer loans taken out by households with observably different characteristics as distinct financial assets with distinct prices. Second, we showed that the theory is quantitatively sound in that it is capable of accounting for the main facts regarding unsecured consumer debt and bankruptcy in the U.S. along with U.S. facts on macroeconomic aggregates and facts on inequality characteristics of U.S. earnings and wealth distributions. Third, we explored the implications of an important recent change in the bankruptcy law that limits the Chapter 7 bankruptcy option to households with below-median earnings. We showed that the likely outcome of this change will be a decrease in interest rates charged on unsecured loans, an increase in both the volume of debt and the number of borrowers and, potentially, an increase in the number of bankruptcies. Furthermore, our measurements indicated that the changes will be big – for instance, the volume of unsecured debt may increase 50% or more. Finally, we constructed measures of the welfare effects of the policy change. From the point of view of average consumption, our calculations indicate that the benefits of the change are large: on the order of 1% of average consumption. From the point of view of public support, we found that almost all households support the change.

In terms of future research, two issues seem important. First, analyzing environments in which households have some motive for simultaneously holding both assets and liabilities is likely to improve our understanding of the unsecured consumer credit market. Second,

Table 15: **Allocation Effects of Means-Testing in Other Models**

Max earnings for filing	∞	Median income	
General equilibrium effect	–	No	Yes
EP economy			
Output	100	100	100
Total asset	308.3	305.0	305.7
Total debt	0.463	0.846	0.834
Percentage of filers	0.37%	0.35%	0.35%
Percentage with bad credit record	2.95%	2.83%	2.82%
Percentage in debt	4.75%	6.82%	6.75%
Rate of return of capital	1.67%	1.67%	1.77%
Avg loans rate (persons-weighted)	21.36%	13.82%	14.03%
EL economy			
Output	100	100	100
Total asset	308.1	303.7	305.0
Total debt	0.580	1.280	1.259
Percentage of filers	0.46%	0.51%	0.50%
Percentage with bad credit record	3.69%	4.09%	4.04%
Percentage in debt	5.72%	9.05%	8.96%
Rate of return of capital	1.65%	1.65%	1.76%
Avg loans rate (persons-weighted)	11.11%	9.40%	9.43%
EPL economy			
Output	100	100	100
Total asset	296.6	294.2	294.7
Total debt	0.670	1.022	1.013
Percentage of filers	0.54%	0.60%	0.60%
Percentage with bad credit record	4.33%	4.80%	4.78%
Percentage in debt	5.36%	6.86%	6.81%
Rate of return of capital	2.10%	2.10%	2.17%
Avg loans rate (persons-weighted)	18.31%	17.93%	18.06%

incorporating unobserved differences among households with regard to willingness to default is also likely to improve our understanding of what happens to a household's credit opportunities after bankruptcy and, therefore, to the costs of default.

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Unabridged Appendix

A Proofs of Theorems 1 - 6

For reasons given in the text, the appendix generalizes the environment in the paper to include a bankruptcy cost $\alpha \cdot (e - e_{\min}) \cdot w$ with $\alpha \in [0, \overline{\alpha}] = A$ where $\overline{\alpha} < 1$. This requires us to expand the space on which the operator $\mathcal{T}$ is defined to include A and modify the operator $\mathcal{T}$ for case 2 (where the household chooses whether to default or not) in Definition 1 to be:

$$(Tv)(\ell, 0, s, e; \alpha, q, w) = \max \left\{ \begin{array}{c} \max_{c, \ell' \in B_{\ell,0,s,0}} u(c,s) + \beta\rho \int v_{\ell',0,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de', \\ u([e - \alpha \cdot (e - e_{\min})] \cdot w, s) + \beta\rho \int v_{0,1,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de' \end{array} \right\}.$$

A.1 Results for Theorems 1 and 2

The following restriction formalizes the assumption concerning $u(0, s)$ in part (iv) of Theorem 1.

Assumption A1. For every $s \in S$,

$$\begin{aligned} & u\left((1-\gamma)e_{\min} \cdot w_{\min}, s\right) - u(0, s) \\ & > \left(\frac{\beta\rho}{1-\beta\rho}\right) \left[u\left(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}\right) - u\left((1-\gamma)e_{\min} \cdot w_{\min}, \underline{s}\right)\right]. \end{aligned}$$

Definition A1. Let $\mathcal{V}$ be the set of all continuous (vector-valued) functions $v : E \times A \times Q \times W \to \mathbb{R}^{N_{\mathcal{L}}}$ such that $\forall q, w$:

$$v_{\ell,h,s}(e; \alpha, q, w) \in \left[\frac{u[e_{\min} \cdot w_{\min}(1-\gamma), \underline{s}]}{(1-\beta\rho)}, \frac{u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s})}{(1-\beta\rho)}\right], \tag{26}$$

$$\ell^0 \geq \ell^1 \Rightarrow v_{\ell^0,h,s}(e; \alpha, q, w) \geq v_{\ell^1,h,s}(e; \alpha, q, w), \tag{27}$$

$$e^0 \geq e^1 \Rightarrow v_{\ell,h,s}(e^0; \alpha, q, w) \geq v_{\ell,h,s}(e^1; \alpha, q, w) \tag{28}$$

$$v_{\ell,0,s}(e; \alpha, q, w) \geq v_{\ell,1,s}(e; \alpha, q, w), \tag{29}$$

$$\begin{aligned} & u(e_{\min} \cdot w_{\min}(1-\gamma), s) + \beta\rho \int v_{0,1,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de' \\ & > u(0, s) + \beta\rho \int v_{\ell_{\max},0,s'}(e', \alpha; q, w)\Phi(e'|s')\Gamma(s, ds')de'. \end{aligned} \tag{30}$$

Lemma A1. $\mathcal{V}$ is non-empty. With $\|v\| = \max_{\ell,h,s}\left\{\sup_{e,\alpha,q,w\in E\times A\times Q\times W}|v_{\ell,h,s}(e;\alpha,q,w)|\right\}$ as the norm, $(\mathcal{V},\|\cdot\|)$ is a complete metric space.

Proof. To prove $\mathcal{V}$ is non-empty, pick a constant (vector-valued) function whose value is in $\left[\frac{u(e_{\min}(1-\gamma)\cdot w_{\min},\underline{s})}{(1-\beta\rho)},\frac{u(e_{\max}\cdot w_{\max}+\ell_{\max}-\ell_{\min},\overline{s})}{(1-\beta\rho)}\right]^{N_{\mathcal{L}}}$. Such a function is continuous and obviously satisfies (26), (27), 28, and (29). Since the function is a constant, (30) is equivalent to $u(e_{\min}(1-\gamma)\cdot w_{\min},s)-u(0,s)>0$, which is satisfied by virtue of $e_{\min}(1-\gamma)\cdot w_{\min}>0$ and the strict monotonicity of $u(\cdot,s)$.

Next we prove $(\mathcal{V},\|\cdot\|)$ is complete. Let $\mathcal{C}$ be the set of all continuous (vector-valued) functions from $E\times A\times Q\times W\to R^{N_{\mathcal{L}}}$. Then, $(\mathcal{C},\|.\|)$ is a complete metric space. Since any closed subset of a complete metric space is also a complete metric space, it is sufficient to show that $\mathcal{V}\subset\mathcal{C}$ is closed in the norm $\|.\|$. Let $\{v_n\}$ be a sequence of functions in $\mathcal{V}$ converging to v, i.e., $\lim_{n\to\infty}\|v_n-v^*\|=0$. We need to show that $v^*\in\mathcal{V}$. If v^* violates any of the range and monotonicity properties of $\mathcal{V}$, there must be some v_n, for n sufficiently large, that violates those properties. But that would contradict the assertion that v_n belongs to $\mathcal{V}$ for all n. Hence, v^* must satisfy all the range and monotonicity properties (26)-(29). To prove that $v^*(e;\alpha,q,w)$ is continuous simply adapt the final part of the proof of Theorem 3.1 in Stokey-Lucas-Prescott to a vector-valued function. Specifically, let $\varepsilon>0$ and let $O_r(b)$ denote an open ball of radius r around b. Choose m such that $\|v_m(e;\alpha,q,w)-v^*(e;\alpha,q,w)\|<\varepsilon/3$. Fix e,α,q,w at $\widetilde{e},\widetilde{\alpha},\tilde{q},\widetilde{w}$. Choose $r>0$ such that for all $e,\alpha,q,w\in O_r(\widetilde{e},\widetilde{\alpha},\tilde{q},\widetilde{w})$, $v_m(e;\alpha,q,w)\in O_{\varepsilon/3}(v_m(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w}))$. This is possible because $v_m(e;\alpha,q,w)$ is continuous. Now consider the Euclidean distance between $v^*(e;\alpha,q,w)$ and $v^*(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})$, denoted $\|v^*(e;\alpha,q,w)-v^*(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E$, for $e,\alpha,q,x\in O_r(\widetilde{e},\widetilde{\alpha},\tilde{q},\widetilde{w})$. We have:

$$\begin{aligned}\|v^*(e;\alpha,q,w)-v^*(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E &\le \|v^*(e;\alpha,q,w)-v_m(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E\\ &+\|v_m(e;\alpha,q,w)-v_m(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E\\ &+\|v_m(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})-v^*(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E.\end{aligned}$$

Since $\|v_m(e;\alpha,q,w)-v^*(e;\alpha,q,w)\|<\varepsilon/3$, it follows that $\|v^*(e;\alpha,q,w)-v_m(e;\alpha,q,w)\|_E<\varepsilon/3$ for all e,α,q,x. Hence, the first and last terms on the r.h.s. of the above inequality are both less than $\varepsilon/3$. The middle term is less than $\varepsilon/3$ for all $e,q,x\in O_r(\widetilde{e},\widetilde{\alpha},\tilde{q},\widetilde{w})$ by the choice of r. Therefore, $\|v^*(e;\alpha,q,w)-v^*(\widetilde{e};\widetilde{\alpha},\tilde{q},\widetilde{w})\|_E<\varepsilon$ for all $e,\alpha,q,w\in O_r(\widetilde{e},\widetilde{\alpha},\tilde{q},\widetilde{w})$. Hence $v^*(e;\alpha,q,w)$ is continuous.

■

For any $v\in\mathcal{V}$, we first prove that $(Tv)(\ell,h,s,e;\alpha,q,w)$ is continuous in e,α,q, and w. To do so, we extend payoffs over infeasible actions associated with negative consumption in a continuous fashion. But first some preliminary definitions. Let

$$\begin{aligned}&c^{0,1}_{\ell,0,s}(e;\alpha,q,w)\equiv[e_{\min}+(1-\alpha)(e-e_{\min})]\cdot w>0,\\ &c^{0,1}_{\ell,1,s}(e;\alpha,q,w)\equiv e(1-\gamma)\cdot w>0,\\ &c^{\ell',0}_{\ell,h,s}(e;\alpha,q,w)\equiv w\cdot e(1-\gamma h)+\ell-\zeta(s)-q_{\ell',s}\ell'.\end{aligned}$$

Observe that $c^{\ell',0}_{\ell,h,s}(e;\alpha,q,w)$ may be negative. Also define

$$\omega_{\ell',h',s}(v) \equiv \int v_{\ell',h',s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s,ds')de'$$

where $\omega_{\ell',h',s}(v)$ is the expected life-time utility of household of characteristic s that will start next period with assets ℓ' and credit history h'. Since v depends on α,q,w, $\omega_{\ell',h',s}(v)$ also depends these variables. In what follows we will sometimes make this dependence explicit.

Now we can define payoffs for discrete actions $\{\ell',d\} \in L \times \{0,1\}$ as follows:

- For $h=0$ and $\ell-\zeta(s) \geq 0$,

$$\phi^{\ell',0}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,0,s}(e;\alpha,q,w),0\right\},s\right) + \beta\rho\omega_{\ell',0,s}(\alpha,q,w)$$

- For $h=0$ and $\ell-\zeta(s) < 0$,

$$\phi^{0,1}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(c^{0,1}_{\ell,0,s}(e;\alpha,q,w),s\right) + \beta\rho\omega_{0,1,s}(\alpha,q,w)$$
$$\phi^{\ell',0}_{\ell,0,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,0,s}(e;\alpha,q,w),0\right\},s\right) + \beta\rho\omega_{\ell',0,s}(\alpha,q,w)$$

- For $h=1$ and $\ell-\zeta(s) \geq 0$,

$$\phi^{\ell',0}_{\ell,1,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(\max\left\{c^{\ell',0}_{\ell,1,s}(e;\alpha,q,w),0\right\},s\right)+\beta\rho\left[\lambda\omega_{\ell',1,s}(\alpha,q,w)+(1-\lambda)\omega_{\ell',0,s}(\alpha,q,w)\right]$$

- For $h=1$ and $\ell-\zeta(s) < 0$,

$$\phi^{0,1}_{\ell,1,s}(e,\alpha,q,w;\omega(v)) \equiv u\left(c^{0,1}_{\ell,1,s}(e;\alpha,q,w),s\right) + \beta\rho\omega_{0,1,s}(\alpha,q,w).$$

Then we have:

Lemma A2. For any (ℓ',d), $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is continuous in e,α,q, and w.

Proof. Observe that $c^{\ell',d}_{\ell,h,s}(e;\alpha,q,w)$ are each continuous functions of e,α,q, and w and u is a continuous function in its first argument. Further, $\omega_{\ell',h',s}(v)$ is continuous in α,q and w because $v \in \mathcal{V}$ and integration preserves continuity.

■

Lemma A3. For $v \in \mathcal{V}$, $(\mathcal{T}v)(e;\alpha,q,w)$ is continuous in e,α,q, and w.

Proof. By Lemma A2, $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ is continuous. Hence, $\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega)$ is also continuous in e,α,q, and w. Then, it is sufficient to establish that $\forall \ell,h,s\in\mathcal{L}$,

$$(Tv)(\ell,h,s,e;\alpha,q,w)=\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v)).$$

To see this, first note that $(Tv)(\ell,h,s,e;\alpha,q,w)=\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega)$ provided the maximum is taken over feasible ℓ',d. Second, for infeasible ℓ',d the payoff $\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega)$ is assigned a value that is weakly dominated by some feasible ℓ',d. Specifically:

For $h=0$ and $\ell-\zeta(s)\geq 0$, we have

$$\begin{aligned}&u(w\cdot e+\ell-\zeta(s),s)+\beta\rho\omega_{0,0,s}(\alpha,q,w)\\&>u(w\cdot e_{\min}(1-\gamma),s)+\beta\rho\omega_{0,1,s}(\alpha,q,w)\\&>u(0,s)+\beta\rho\omega_{\ell',0,s}(\alpha,q,w),\forall\ell'\in L.\end{aligned}$$

The last inequality follows from (27) and (30) in the definition of the set $\mathcal{V}$. Hence, $\ell'=0$ and $d=0$ gives higher payoff than any infeasible ℓ',d.

For $h=0$, $\ell-\zeta(s)<0$, since $\overline{\alpha}<1$ we have

$$\begin{aligned}&u([e_{\min}+(1-\alpha)(e-e_{\min})]\cdot w,s)+\beta\rho\omega_{0,1,s}(\alpha,q,w)\\&>u(0,s)+\beta\rho w_{\ell',0,s}(\alpha,q,w),\end{aligned}$$

where again the last inequality follows from (27) and (30). Hence, $\ell'=0$ and $d=1$ gives higher payoff than any infeasible ℓ',d.

For $h=1$ and $\ell-\zeta(s)\geq 0$, we have

$$\begin{aligned}&u(e(1-\gamma)\cdot w+\ell-\zeta(s),s)+\beta\rho\left[\lambda\omega_{0,1,s}(\alpha,q,w)+(1-\lambda)\omega_{0,0,s}(\alpha,q,w)\right]\\&>u(e_{\min}(1-\gamma)\cdot w,s)+\beta\rho\omega_{0,1,s}(\alpha,q,w)\\&>u(0,s)+\beta\rho\omega_{\ell',0,s}(\alpha,q,w),\forall\ell'\in L\\&>u(0,s)+\beta\rho\left[\lambda\omega_{\ell',1,s}(\alpha,q,w)+(1-\lambda)\omega_{\ell',0,s}(\alpha,q,w)\right],\end{aligned}$$

where the first inequality follows from (29), the second inequality follows from (27) and (30), and the third inequality follows from (29) again. Hence, $\ell'=0$ and $d=0$ gives higher payoff than any infeasible ℓ',d.

For $h=1$ and $\ell-\zeta(s)<0$, $\phi^{\ell',1}_{\ell,1,s}(e,\alpha,q,w;v)=T(v)(\ell,1,s,e;\alpha,q,w)$ because there is only one action $\ell'=0$ and $d=1$.

Hence, $(Tv)(\ell,h,s,e;\alpha,q,w)=\max_{\ell',d}\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w;\omega(v))$ and therefore, $(\mathcal{T}v)(e;\alpha,q,w)$ is continuous.

■

Corollary to Lemma A3. For any $v\in\mathcal{V}$, the consumption implied by $(Tv)(\ell,h,s,e;\alpha,q,w)$ is strictly positive.

Proof. The exact same argument as in Lemma A3 can be used to establish that a *feasible* choice involving zero consumption is always strictly dominated by a feasible choice involving positive consumption.

■

Lemma A4. Given Assumption A1, $\mathcal{T}$ is a contraction mapping with modulus $\beta\rho$.

Proof. We first establish that $\mathcal{T}(\mathcal{V}) \subset \mathcal{V}$. For $v \in \mathcal{V}$, we have already established $\mathcal{T}$ is continuous by Lemma A3.

To establish that $\mathcal{T}$ preserves the boundedness property (26), note that since $q_{\ell_{\min},s} \in [0,1]$ consumption can never exceed $e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}$. Therefore,

$$\begin{aligned}
&(Tv)(\ell, h, s, e; \alpha, q, w) \\
&\leq u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}) + \beta\,\rho\,\omega_{\ell_{\max},0,s} \\
&\leq u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}) + \beta\,\rho\,\left\{\frac{1}{(1-\beta\rho)}\,u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s})\right\} \\
&= \frac{1}{(1-\beta\rho)}\,u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}),
\end{aligned}$$

where the first inequality follows from the strict monotonicity of $u(\cdot, s)$ and by properties (27) and (29).

Next, since $\alpha < 1$, $c = (1-\gamma)e_{\min} \cdot w_{\min}$ is a feasible choice for all ℓ, h, s, e, α, q, and w. Therefore,

$$(Tv)(\ell, h, s, e; \alpha, q, w) \geq \frac{1}{(1-\beta\rho)}\,u\left((1-\gamma)e_{\min} \cdot w_{\min}, \underline{s}\right).$$

Hence

$$\begin{aligned}
&(\mathcal{T}v)(e; \alpha, q, w) \\
&\in \left[\frac{1}{(1-\beta\rho)}\;u\left((1-\gamma)e_{\min} \cdot w_{\min}, \underline{s}\right),\; \frac{1}{(1-\beta\rho)}\;u\left(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, \overline{s}\right)\right]^{N_{\mathcal{L}}}.
\end{aligned}$$

To establish that $\mathcal{T}$ preserves the monotonicity property (27), consider a household with a given h, s and two different asset holdings $\ell^0 > \ell^1$. For any e, α, q, w : (i) if $0 > \ell^0$ then for $d \in \{0, 1\}$, $B_{\ell^1,0,s,d}(e; \alpha, q, w) \subseteq B_{\ell^0,0,s,d}(e; \alpha, q, w)$ and hence $(Tv)(\ell^1, 0, s, e; \alpha, q, w) < (Tv)(\ell^0, 0, s, e; \alpha, q, w))$; (ii) if $\ell^0 \geq 0 > \ell^1$ and $\ell^0 \geq \zeta(s)$, then $B_{\ell^1,0,s,d}(e; \alpha, q, w) \subseteq B_{\ell^0,0,s,0}(e; \alpha, q, w)$ and because v satisfies (29) it follows that $(Tv)(\ell^1, 0, s, e; \alpha, q, w) < (Tv)(\ell^0, 0, s, e; \alpha, q, w)$; (iii) if $\ell^0 \geq 0 > \ell^1$ and $\ell^0 < \zeta(s)$, then $B_{\ell^1,0,s,d}(e; \alpha, q, w) \subseteq B_{\ell^0,0,s,d}(e; \alpha, q, w)$ and hence $(Tv)(\ell^1, 0, s, e; \alpha, q, w) < (Tv)(\ell^0, 0, s, e; \alpha, q, w)$;(iv) if $\ell^1 \geq 0$ and $\ell^1 < \zeta(s) \leq \ell^0$ then $B_{\ell^1,h,s,d}(e; \alpha, q, w) \subseteq B_{\ell^0,h,s,0}(q)$ and because v satisfies (29) it follows that $(Tv)(\ell^1, 0, s, e; \alpha, q, w) < (Tv)(\ell^0, 0, s, e; \alpha, q, w)$; and (v) if $\ell^1 \geq 0$ and $\ell^1 \geq \zeta(s)$ then $B_{\ell^1,h,s,0}(e; \alpha, q, w) \subseteq B_{\ell^0,h,s,0}(q)$ and hence $(Tv)(\ell^1, 0, s, e; \alpha, q, w) < (Tv)(\ell^0, 0, s, e; \alpha, q, w)$.

To establish that $\mathcal{T}$ preserves the monotonicity property (28), consider a household with a given ℓ, h, s and two different efficiency levels $e^0 > e^1$. For any q, w, $B_{\ell,h,s,d}(e^1; q, w) \subseteq B_{\ell,h,s,d}(e^0; q, w)$ since $\alpha < 1$ and hence $(Tv)(\ell, h, s, e^1; q, w) < (Tv)(\ell, h, s, e^0; q, w)$.

To establish that $\mathcal{T}$ preserves the monotonicity property (29), consider a household with a given ℓ, s and two different credit records. For any e, q, w : (i) if $\ell - \zeta(s) < 0$, $B_{\ell,1,s,1}(e; \alpha, q, w) \subseteq B_{\ell,0,s,d}(e; \alpha, q, w)$ and hence $(Tv)(\ell, 1, s, e; \alpha, q, w) < (Tv)(\ell, 0, s, e; \alpha, q, w)$; (ii) if $\ell - \zeta(s) \geq 0$, $B_{\ell,1,s,0}(e; \alpha, q, w) \subseteq B_{\ell,0,s,0}(e; \alpha, q, w)$ and because v satisfies (29) it follows that $(Tv)(\ell, 1, s, e; \alpha, q, w) < (Tv)(\ell, 0, s, e; \alpha, q, w)$.

To establish that $\mathcal{T}$ preserves the "default at zero consumption" property (30), by Assumption A1 and the fact that $\mathcal{T}$ satisfies the boundedness property it follows that

$$u\left((1-\gamma)e_{\min} \cdot w_{\min}, s\right) - u(0, s) > \beta\,\rho\,\left[(Tv)(\ell_{\max}, 0, s, e; \alpha, q, w) - (Tv)(0, 1, s, e; \alpha, q, w)\right].$$

Re-arranging gives:

$$u\left((1-\gamma)e_{\min} \cdot w_{\min}, s\right) + \beta\,\rho\,(Tv)(0, 1, s, e; \alpha, q, w) > u(0, s) + \beta\,\rho\,(Tv)(\ell_{\max}, 0, s, e; \alpha, q, w).$$

Having thus established $\mathcal{T}(\mathcal{V}) \subset \mathcal{V}$, we now show that $\mathcal{T}$ is a contraction with modulus $\beta\rho$. The first step is to establish the analogue of the Blackwell monotonicity and discounting properties. Monotonicity: Let v, $v' \in \mathcal{V}$ and $v(e; \alpha, q, w) \leq v'(e; \alpha, q, w)$ for all e, α, q, w. From the definition of the $\mathcal{T}$ operator it's clear that $(\mathcal{T}\ v) \leq (\mathcal{T}v')$. Discounting: It's also clear that for any $\kappa \in R_+^{N_\mathcal{L}}$, $[\mathcal{T}(v + \kappa)]\ (e; \alpha, q, w) = (\mathcal{T}v)(e; \alpha, q, w) + \beta\,\rho\,\kappa$. To prove that $\mathcal{T}$ is a contraction mapping one must simply adapt the final part of the proof of Theorem 3.3 in Stokey-Lucas-Prescott to a vector-valued function. Specifically, from the definition of $\|\cdot\|$, it follows that for any $v, v' \in \mathcal{V}$, $v(e; \alpha, q, w) \leq v'(e; \alpha, q, w) + \overline{\|v - v'\|}$, where $\overline{\|v - v'\|}$ is a $N_\mathcal{L}$-element vector with $\|v - v'\|$ as each component element. Hence, $(\mathcal{T}\ v) \leq \left[\mathcal{T}(v' + \overline{\|v - v'\|})\right] = (\mathcal{T}v') + \beta\rho\overline{\|v - v'\|}$. Reversing the roles of v and v' gives $(\mathcal{T}v\ ') \leq (\mathcal{T}v) + \beta\rho\overline{\|v - v'\|}$. Combining these two inequalities shows that $(\mathcal{T}\ v) - (\mathcal{T}v') \leq \beta\rho\overline{\|v - v'\|}$ for all e, q, w and $(\mathcal{T}v') - (\mathcal{T}v) \leq \beta\rho\overline{\|v - v'\|}$ for all e, q, w . Hence, $\sup_{e,\alpha,q,w} \mid (\mathcal{T}v)(e; \alpha, q, w) - (\mathcal{T}v')(e; \alpha, q, w) \mid \leq \beta\,\rho\,\overline{\|v - v'\|}$. Therefore,

$$\max_{\ell,h,s}\left\{\sup_{e,\alpha,q,w} \mid (\mathcal{T}v)(e; \alpha, q, w) - (\mathcal{T}v')(e; \alpha, q, w) \mid\right\} \leq \beta\,\rho\overline{\|v - v'\|}.$$

Hence, $\|(\mathcal{T}v) - (\mathcal{T}v')\| \leq \beta\,\rho\,\|v - v'\|$. This establishes that $\mathcal{T}$ is a contraction mapping with modulus $\beta\rho$.

■

Theorem 1 (Existence of a Recursive Solution to the Household Problem). There exists a unique $v^* \in \mathcal{V}$ such that $v^* = \mathcal{T}(v^*)$. Furthermore: (i) v^* is bounded and increasing in ℓ and e; (ii) a bad credit record reduces v^*; (iii), the optimal policy correspondence implied by $\mathcal{T}(v^*)$ is compact-valued and upper hemi-continuous; and (iv) provided $u(0, s)$ is sufficiently low, default is strictly preferable to zero consumption and consumption is strictly positive.

Proof. Existence and uniqueness of v^*, as well as properties (i), (ii), and (iv) follow directly from Lemmas A1, A3, A4, and the Corollary to Lemma A3.

Define the optimal policy correspondence to be

$$\chi_{\ell,h,s}(e;\alpha,q,w) = \{(c,\ell',d) \in B_{\ell,h,s,d}(e;\alpha,q,w) : (c,\ell',d) \text{ attains } v^*_{\ell,h,s}(e;\alpha,q,w)\}.$$

To establish the first part of (iii), note that the correspondence $\chi_{\ell,h,s}(e;\alpha,q,w)$ is bounded because c is bounded between 0 and $e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}$, and $(\ell',d) \in L \times \{0,1\}$. To prove that $\chi_{\ell,h,s}(e;\alpha,q,w)$ is closed, let $\{c_n,\ell'_n,d_n\}$ be a sequence in $\chi_{\ell,h,s}(e;\alpha,q,w)$ converging to $(\overline{c},\overline{\ell}',\overline{d})$. Since (ℓ',d) are elements of finite sets, $\exists\eta$ such that $\forall n > \eta$, $(\ell'_n,d_n) = (\overline{\ell}',\overline{d})$. Given that $(c_n,\overline{\ell}',\overline{d})$ attains $v^*_{\ell,h,s}(e;\alpha,q,w)$, $\forall n > \eta$ we have $c_n = c^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e;\alpha,q,w)$. Therefore, $\overline{c} = c^{\overline{\ell}',\overline{d}}_{\ell,h,s}(e;\alpha,q,w)$ and $(\overline{c},\overline{\ell}',\overline{d}) \in \chi_{\ell,h,s}(e;\alpha,q,w)$.

To establish the second part of property (iii), let $\{\ell_n,h_n,s_n,e_n,\alpha_n,q_n,w_n\} \to (\overline{\ell},\overline{h},\overline{s},\overline{e},\overline{\alpha},\overline{q},\overline{w})$. Since $\mathcal{L}$ is finite we can fix $(\ell_n,h_n,s_n) = (\overline{\ell},\overline{h},\overline{s})$ and simply consider $e_n,\alpha_n,q_n,w_n \to \overline{e},\overline{\alpha},\overline{q},\overline{w}$. Let $\{c_n,\ell'_n,d_n\} \in \chi_{\overline{\ell},\overline{h},\overline{s}}(e_n;\alpha_n,q_n,w_n)$. Since the correspondence is compact valued there must exist a subsequence $\{c_{n_k},\ell'_{n_k},d_{n_k}\}$ converging to $(\overline{c},\overline{\ell}',\overline{d})$. Furthermore, since ℓ' and d take on only a finite number of values, $\exists\,\eta$ such that $\forall n_k > \eta$,

$$(c_{n_k},\ell'_{n_k},d_{n_k}) = \left(c^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k};\alpha_{n_k},q_{n_k},w_{n_k}),\overline{\ell}',\overline{d}\right).$$

By optimality,

$$\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k},\alpha_{n_k},q_{n_k},w_{n_k};\omega^*(\alpha_{n_k},q_{n_k},w_{n_k})) = v^*_{\overline{\ell},\overline{h},\overline{s}}(e_{n_k};\alpha_{n_k},q_{n_k},w_{n_k}).$$

Then, by continuity of $\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}$, $v^*_{\overline{\ell},\overline{h},\overline{s}}$ and ω^* with respect to e,α,q and w we have

$$\phi^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(\overline{e},\overline{\alpha},\overline{q},\overline{w};\omega^*(\overline{\alpha},\overline{q},\overline{w})) = v^*_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w}).$$

Therefore,

$$\left(\overline{c} = c^{\overline{\ell}',\overline{d}}_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w}),\overline{\ell}',\overline{d}\right) \in \chi_{\overline{\ell},\overline{h},\overline{s}}(\overline{e};\overline{\alpha},\overline{q},\overline{w})$$

and the correspondence is u.h.c.

■

Theorem 2 (Existence of a Unique Invariant Distribution). For $(\alpha,q,w) \in A \times Q \times W$ and any measurable selection from the optimal policy correspondence, there exists a unique $\mu_{(\alpha,q,w)} \in \mathcal{M}(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$ such that $\mu_{(\alpha,q,w)} = \Upsilon_{(\alpha,q,w)}\mu_{(\alpha,q,w)}$.

Proof. By the Measurable Selection Theorem, there exists an optimal policy rule that is measurable with respect to any measure in $\mathcal{M}(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$. Therefore, $G^*_{(\alpha,q,w)}$ is well-defined. To establish this lemma we then simply need to verify that $G^*_{(\alpha,q,w)}$ satisfies the conditions stipulated in Theorem 11.10 of Stokey and Lucas. The first condition is that $G^*_{(\alpha,q,w)}$ satisfies Doeblin's condition (which states that there is a finite measure φ on $(\mathcal{L} \times E, 2^{\mathcal{L}} \times \mathcal{B}(E))$, an integer $I \geq 1$, and a number $\varepsilon > 0$, such that if $\varphi(Z) \leq \varepsilon$, then $G^{*I}_{(\alpha,q,w)}((\ell,h,s,e),Z) \leq 1-\varepsilon$, for all (ℓ,h,s,e)). It is sufficient to show that GN satisfies the Doeblin condition (see Exercise 11.4.g of Stokey and Lucas). Observe that since GN is independent of (ℓ,h,s,e), we can pick $\varphi(Z) = GN((\ell,h,s,e),Z)$. Then GN satisfies the Doeblin condition for $I = 1$ and $\varepsilon < \frac{1}{2}$.

Second, we need to show that if Z is any set of positive φ-measure, then for each (ℓ,h,s,e), there exists $n \geq 1$ such that $G^{*n}_{(\alpha,q,w)}((\ell,h,s,e),Z) > 0$. To see this, observe that if $\varphi(Z) > 0$, then $GN((\ell,h,s,e),Z) > 0$ for any (ℓ,h,s,e). Therefore, $G^{*1}_{(\alpha,q,w)}((\ell,h,s,e),Z) > 0$.

■

A.2 Results for Theorems 3 and 4

We turn now to the proof of Theorem 3. We give a formal definition of the maximal default set and then establish two key lemmas. The maximal default $\overline{D}^*_{\ell,h,s}(\alpha,q,w) = \{e : v^*_{\ell,h,s}(e;\alpha,q,w) = \phi^{0,1}_{\ell,h,s}(e,\alpha,q;\omega^*)\}$, where ω^* is $\omega(v^*)$.

Lemma A5. Let $e \in E\backslash\overline{D}^*_{\ell,0,s}(0,q,w)$, $e > \widehat{e}$, and $\ell - \zeta(s) < 0$. If $e \in \overline{D}^*_{\ell,0,s}(0,q,w)$, then $c^*_{\ell,0,s}(\widehat{e};0,q,w) > \widehat{e} \cdot w$.

Proof. Since $\widehat{e} \in E\backslash\overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) + \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) > u(\widehat{e} \cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{31}$$

Let $\Delta = (e - \widehat{e}) \cdot w > 0$. The pair $\{\underline{c} = c^*_{\ell,0,s}(\widehat{e};0,q,w) + \Delta, \underline{\ell'} = \ell'^*_{\ell,0,s}(\widehat{e};0,q,w)\}$ clearly belongs in $B_{\ell,0,s,0}(e;0,q,w)$. Then by optimality, utility obtained by not defaulting when labor efficiency is e must satisfy the inequality

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega^*_{\tilde{\ell'}_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \geq u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w), \tag{32}$$

where $\tilde{c}_{\ell,0,s,0}(e;0,q,w)$ and $\tilde{\ell'}_{\ell,0,s,0}(e;0,q,w)$ are the optimal choices of c and ℓ' conditional on not defaulting. Since $e \in \overline{D}^{\,*}_{\ell,h,s}(0,q,w)$,

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right) + \beta\rho\omega_{\tilde{\ell'}_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w) \leq u(e \cdot w,s) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{33}$$

By (32) and the fact that $\widehat{e} \cdot w + \Delta = e \cdot w$, (33) can be re-written

$$u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) \leq u(\widehat{e} \cdot w + \Delta,\mu) + \beta\rho\omega^*_{0,1,s}(0,q,w). \tag{34}$$

Then (34) minus (31) implies

$$\begin{aligned} u(\underline{c},s) + \beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w) - u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right) - \beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w) \\ < u(\widehat{e} \cdot w + \Delta,s) + \beta\rho\omega^*_{0,1,s}(0,q,w) - u(\widehat{e} \cdot w,s) - \beta\rho\omega^*_{0,1,s}(0,q,w). \end{aligned} \tag{35}$$

Or, by definition of $(\underline{c}, \underline{\ell'})$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w)+\Delta,s\right)-u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right)<u(\widehat{e}\cdot w+\Delta,s)-u(\widehat{e}\cdot w,s).$$

Since $u(\cdot,s)$ is strictly concave, the last inequality implies $c^*_{\ell,0,s}(\widehat{e};0,q,w)>\widehat{e}\cdot w$. ■

Lemma A6. Let $\widehat{e}\in E\backslash\overline{D}^*_{\ell,0,s}(0,q,w)$, $e<\widehat{e}$, and $\ell-\zeta(s)<0$. If $e\in\overline{D}^*_{\ell,0,s}(0,q,w)$, then $c^*_{\ell,0,s}(\widehat{e};0,q,w)<\widehat{e}\cdot w$.

Proof. Since $\widehat{e}\in E\backslash\overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right)+\beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w)>u(\widehat{e}\cdot w,s)+\beta\rho\omega^*_{0,1,s}(0,q,w). \tag{36}$$

Let $\Delta=(\widehat{e}-e)\cdot w>0$. Consider the quantity $c^*_{\ell,0,s}(\widehat{e};0,q,w)-\Delta$. If $c^*_{\ell,0,s}(\widehat{e};0q,w)-\Delta\leq 0$ then it must be the case that $c^*_{\ell,0,s}(\widehat{e};0,q,w)<\widehat{e}\cdot w$ because $\widehat{e}\cdot w-\Delta=e\cdot w>0$. So, we only need to consider the case where $c^*_{\ell,0,s}(\widehat{e};0,q,w)-\Delta>0$. The pair $\{\underline{c}=c^*_{\ell,0,s}(\widehat{e};0,q,w)-\Delta$, $\underline{\ell'}=\ell'^*_{\ell,0,s}(\widehat{e};0,q,w)\}$ clearly belongs in $B_{\ell,0,s,0}(0,q,w)$. Then by optimality, utility obtained by not defaulting when labor efficiency is e must satisfy the inequality

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right)+\beta\rho\omega^*_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w)\geq u(\underline{c},s)+\beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w), \tag{37}$$

where, once again, $\tilde{c}_{\ell,0,s,0}(e;0,q,w)$ and $\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w)$ are the optimal choices of c and ℓ' conditional on not defaulting. Since $e\in\overline{D}^*_{\ell,0,s}(0,q,w)$,

$$u\left(\tilde{c}_{\ell,0,s,0}(e;0,q,w),s\right)+\beta\rho\omega^*_{\tilde{\ell}'_{\ell,0,s,0}(e;0,q,w),0,s}(0,q,w)\leq u(e\cdot w,s)+\beta\rho\omega^*_{0,1,s}(0,q,w). \tag{38}$$

By (37) and the fact that $\widehat{e}\cdot w-\Delta=e\cdot w$, (38) can be rewritten

$$u(\underline{c},s)+\beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w)\leq u(\widehat{e}\cdot w-\Delta,s)+\beta\rho\omega^*_{0,1,s}(0,q,w). \tag{39}$$

Then (39) minus (36) implies

$$\begin{aligned}u(\underline{c},s)+\beta\rho\omega^*_{\underline{\ell'},0,s}(0,q,w)-u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right)-\beta\rho\omega^*_{\ell'^*_{\ell,0,s}(\widehat{e};0,q,w),0,s}(0,q,w)\\<u(\widehat{e}\cdot w-\Delta,s)+\beta\rho\omega^*_{0,1,s}(0,q,w)-u(\widehat{e}\cdot w,s)-\beta\rho\omega^*_{0,1,s}(0,q,w).\end{aligned}$$

Or, by definition of $(\underline{c}, \underline{\ell'})$,

$$u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w),s\right)-u\left(c^*_{\ell,0,s}(\widehat{e};0,q,w)-\Delta,s\right)>u(\widehat{e}\cdot w,s)-u(\widehat{e}\cdot x-\Delta,s). \tag{40}$$

Since $u(\cdot,s)$ is strictly concave, the last inequality implies $c^*_{\ell,0,s}(\widehat{e};0,q,w)-\Delta<\widehat{e}\cdot w-\Delta$, or, $c^*_{\ell,0,s}(\widehat{e};0,q,w)<\widehat{e}\cdot w$. ■

Theorem 3 (The Maximal Default Set is a Closed Interval). If $\overline{D}^*_{\ell,0,s}(0,q,w)$ is non-empty, it is a closed interval.

Proof. First, consider the case $h = 0$. If $\ell - \zeta(s) \geq 0$, then $\overline{D}^*_{\ell,0,s}(0, q, w) = \varnothing$. If $\ell - \zeta(s) < 0$, l et $e_L = \inf \overline{D}^*_{\ell,0,s}(0, q, w)$ and $e_U = \sup \overline{D}^*_{\ell,0,s}(0, q, w)$. Since $\overline{D}^*_{\ell,0,s}(0, q, w) \subset E$, which is bounded, both e_L and e_U exist by the Completeness Property of R . If $e_L = e_U$, the default set contains only one element $e = e_L = e_U$ and the result is trivially true. Suppose, then, that $e_L < e_U$. Let $\widehat{e} \in (e_L, e_U)$ and assume that $\widehat{e} \notin \overline{D}^*_{\ell,0,s}(0, q, w)$. Then there is an $e \in \overline{D}^*_{\ell,0,s}(0, q, w)$ such that $e > \widehat{e}$ (if not, then $e_U = \widehat{e}$ which contradicts the assertion that $\widehat{e} \in (e_L, e_U)$). Then, by Lemma A5, $c^*_{\ell,0,s}(\widehat{e}; 0, q, w) > \widehat{e} \cdot w$. Similarly, there is an $e \in \overline{D}^*_{\ell,0,s}(0, q, w)$ such that $e < \widehat{e}$. Then, by Lemma A6, $c^*_{\ell,0,s}(\widehat{e}; 0, q, w) < \widehat{e} \cdot w$. But $c^*_{\ell,0,s}(\widehat{e}; 0, q, w)$ cannot be both greater and less than $\widehat{e} \cdot w$. Hence, the assertion $\widehat{e} \notin \overline{D}^*_{\ell,0,s}(0, q, w)$ must be false and $(e_L, e_U) \subset \overline{D}^*_{\ell,0,s}(0, q, w)$. To show that $e_U \in \overline{D}^*_{\ell,0,s}(0, q, w)$, pick a sequence $\{e^n\} \subset (e_L, e_U)$ converging to e_U. Then, $v^*_{\ell,0,s}(e_n; 0, q, w) - u(e_n \cdot w, s) = \beta\rho\omega^*_{0,1,s}(0, q, w)$ for all n. Since e_U is clearly in E, by the continuity of $v^*_{\ell,0,s}(e; 0, q, w)$ and u, it follows that $\lim_{n\to\infty}\{v^*_{\ell,0,s}(e_n; 0, q, w) - u(e_n \cdot w, s)\} = v^*_{\ell,0,s}(e_U; 0, q, w) - u(e_U \cdot w, s)$. Since every element of the sequence $\{v^*_{\ell,0,s}(e_n; 0, q, w) - u(e_n \cdot w, s)\}$ is equal to $\beta\rho\omega^*_{0,1,s}(0, q, w)$, it must be the case that $v^*_{\ell,0,s}(e_U; 0, q, w) - u(e_U \cdot w, s) = \beta\rho\omega^*_{0,1,s}(0, q, w)$. Hence, $e_U \in \overline{D}^*_{\ell,0,s}(0, q, w)$. By analogous reasoning, $e_L \in \overline{D}^*_{\ell,0,s}(0, q, w)$. Hence, $[e_L, e_U] \subseteq \overline{D}^*_{\ell,0,s}(0, q, w)$. But by the definition of e_L and e_U, $\overline{D}^*_{\ell,0,s}(0, q, w) \subset [e_L, e_U]$. Hence $[e_L, e_U] = \overline{D}^*_{\ell,0,s}(0, q, w)$. Next consider the case $h = 1$. If $\ell - \zeta(s) \geq 0$, then $\overline{D}^*_{\ell,0,s}(0, q, w) = \varnothing$. If $\ell - \zeta(s) < 0$, then $\overline{D}^*_{\ell,0,s}(0, q, w) = E$. ■

Theorem 4 (Maximal Default Set Expands with Liabilities). If $\ell^0 > \ell^1$, then $\overline{D}^*_{\ell,0,s}(0, q, w) \subseteq \overline{D}^*_{\ell^1,h,s}(\alpha, q, w)$.

Proof. Suppose $e \in \overline{D}^*_{\ell,0,s}(0, q, w)$. Since $v^*_{\ell,0,s}(e; \alpha, q, w)$ is increasing in ℓ, $v^*_{\ell^0,0,s}(e; \alpha, q, w) \geq v^*_{\ell^1,0,s}(e; \alpha, q, w)$. But $v^*_{\ell^0,0,s}(e; \alpha, q, w) = u(e \cdot w, s) + \beta\rho\omega^*_{0,1,s}(\alpha, q, w)$. Since default is also an option at ℓ^1, it must be the case that $v^*_{\ell^1,0,s}(e; \alpha, q, w) = u(e \cdot w, s) + \beta\rho\omega^*_{0,1,s}(\alpha, q, w)$. Hence any e in $\overline{D}^*_{\ell,0,s}(0, q, w)$ is also in $\overline{D}^*_{\ell,0,s}(0, q, w)$.■

A.3 Results for Theorems 5 and 6

We now turn to the proof of existence of equilibrium. For the environment with $\alpha > 0$, all conditions in Definition 2 remain the same except for the goods market clearing condition (ix) which we now call (ixA):

$$\int c^*_{\ell,h,s}(e; \alpha, q^*, w^*) d\mu^* + K^* + \int \frac{\zeta(s)}{m^*} d\mu^* =$$
$$F(N^*, K^*) + (1 - \delta) K^* - \gamma w^* \int e\mu^*(d\ell, 1, ds, de)$$
$$-\alpha w^* \int (e - e_{\min}) \cdot d^*_{\ell,0,s}(e; \alpha, q^*, w^*)\mu^*(d\ell, 0, ds, de).$$

Lemma A7. The goods market clearing condition (ixA) is implied by the other conditions for an equilibrium in Definition 2.

Proof. First note that the household budget sets (2)-(5) imply

$$c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \cdot \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*) \cdot \left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]$$
$$= \left[e\,(1-\gamma h) - \alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + (\ell - \zeta(s))\cdot\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right].$$

Then aggregating over all households yields

$$\int\left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]d\mu^*\right\}$$
$$+\int\left\{\zeta(s)\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^*$$
$$=\int\left\{\left[e\,(1-\gamma h)-\alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* \tag{41}$$

Condition (v) along with (41) imply

$$\int\left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]d\mu^*\right\}$$
$$+\int\left\{\zeta(s)\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^*$$
$$=\int\left\{\left[e\,(1-\gamma h)-\alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^*$$
$$+\int\left\{\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\zeta(s) + \ d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\} - \zeta(s)/m^*\right\}d\mu^*$$

or

$$\int\left\{c^*_{\ell,h,s}(e;\alpha,q^*,w^*) + q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)\left[1 - d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^* + \int\frac{\zeta(s)}{m^*}d\mu^* \tag{42}$$
$$=\int\left\{\left[e\,(1-\gamma h)-\alpha(e-e_{\min})(1-h)\cdot d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\cdot w^* + \ell\cdot\left[1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\right]\right\}d\mu^*$$
$$+\int\left\{d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\max\{\ell,0\}\right\}d\mu^*.$$

Since $d^*_{\ell,h,s}(e;\alpha,q^*,w^*)=1$ implies $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=0$, it follows that $\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d^*_{\ell,h,s}(e;\alpha,q^*,w^*) = 0$ for all ℓ,h,s,e. Hence, the left hand side of (42) can be written

$$\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s}\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int\frac{\zeta(s)}{m^*}d\mu^*.$$

Next the first term on the right hand side can be written

$$w^*\left[\int e d\mu^* - \gamma\int e\mu^*(d\ell,1,ds,de) - \alpha\int(e-e_{\min})\cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de)\right]$$

Finally, the remaining term on the right hand side of (42) can be written

$$\sum_{\ell,s}\ell\int(1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de)+\sum_{\ell\geq 0,s}\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\ell\mu^*(\ell,dh,s,de)$$
$$=\sum_{\ell,s}\ell\int\mu^*(\ell,dh,s,de)-\sum_{\ell<0,s}\ell\int d^*_{\ell,h,s}(e;\alpha,q^*,w^*)\mu^*(\ell,dh,s,de)$$
$$=\sum_{\ell>0,s}\ell\int\mu^*(\ell,dh,s,de)+\sum_{\ell<0,s}\ell\int(1-d^*_{\ell,h,s}(e;\alpha,q^*,w^*))\mu^*(\ell,dh,s,de) \tag{43}$$

Next, observe that for $x\neq 0$, we have from (x), (6), and (vii)

$$\int\mu^*(x,dh',\sigma,de';q^*,w^*)$$
$$=\rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\sum_{h'}H^*(\ell,h,s,e;h')\int_E\Phi(e'|\sigma)de'\Gamma(s;\sigma)\right]d\mu^*$$
$$=\rho\int\left[\mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=x\}}\Gamma(s;\sigma)\right]\mu^*(d\ell,dh,ds,de)$$
$$=\rho\sum_s a^*_{x,s}\Gamma(s;\sigma).$$

Hence, the first term in (43):

$$\sum_{x>0,\sigma}x\int\mu^*(x,dh,\sigma,de)=\sum_{x>0,\sigma}x\rho\sum_s a^*_{x,s}\Gamma(s;\sigma)$$
$$=\rho\sum_{x>0,s}xa^*_{x,s}\sum_\sigma\Gamma(s;\sigma)$$
$$=\rho\sum_{x>0,s}xa^*_{x,s}$$

Now consider the second term in (43):

$$\int(1-d^*_{x,h,\sigma}(e;\alpha,q^*,w^*))\mu^*(x,dh,\sigma,de)$$
$$=\int\mu^*(x,dh,\sigma,de)-\int d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\mu^*(x,dh,\sigma,de)$$

We can re-write the latter part of this expression as

$$\int d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\mu^*(x,dh,\sigma,de;\alpha,q^*,w^*)=$$
$$\rho\int\left[\mathbf{1}_{\{(\ell,\eta,s,\varepsilon):(\ell'^*_{\ell,\eta,s}(\varepsilon;q^*,w^*)=x\}}\sum_h H(\ell,\eta,s,\varepsilon;h)\int_E d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\Phi(e|\sigma)de\Gamma(s;\sigma)\right]\mu^*(d\ell,d\eta,ds,d\varepsilon)$$

Since $x < 0$, it follows that $\eta = 0$ and $h = 0$ so that $H(\ell, 0, s, \varepsilon; 0) = 1$ and $H(\ell, 0, s, \varepsilon; 1) = 0, \forall \ell, s, \varepsilon$. Therefore

$$\begin{aligned}
&\int d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\mu^*(x,dh,\sigma,de;\alpha,q^*,w^*) \\
&= \rho \int \left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}} \int_E \sum_h H(\ell,0,s,\varepsilon;h) d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\Phi(e|\sigma)de\Gamma(s;\sigma) \right] \mu^*(d\ell,0,ds,d\varepsilon) \\
&= \rho \int \left[\mathbf{1}_{\{(\ell,0,s,\varepsilon):(\ell'^*_{\ell,0,s}(\varepsilon;q^*,w^*)=x\}} \int_E d^*_{x,0,\sigma}(e;\alpha,q^*,w^*)\Phi(e|\sigma)de\Gamma(s;\sigma) \right] \mu^*(d\ell,0,ds,d\varepsilon).
\end{aligned}$$

Let $p^{*\sigma}_x = \int_E d^*_{x,0,\sigma}(e;\alpha,q^*,w^*)\Phi(e|\sigma)de$ be the probability of default on a loan of size x by households with characteristic σ. Then

$$\begin{aligned}
&\int d^*_{x,h,\sigma}(e;\alpha,q^*,w^*)\mu^*(x,dh,\sigma,de;\alpha,q^*,w^*) \\
&= \sum_s \rho \int \left[\mathbf{1}_{\{(\ell,0,s,e):(\ell'^*_{\ell,0,s}(e;\alpha,q^*,w^*)=x\}} p^{*\sigma}_x \Gamma(s;\sigma) \right] \mu^*(d\ell,0,s,de;\alpha,q^*,w^*) \\
&= \rho \sum_s p^{*\sigma}_x \Gamma(s;\sigma) a^*_{x,s}.
\end{aligned}$$

The second equality follows from (vii) recognizing that $\mu^*(Z) = 0$ for all $Z \in L_{--} \times \{1\} \times S \times \mathcal{B}(E)$. Thus the second part of (43) can be written

$$\begin{aligned}
&\sum_{x>0,\sigma} x \int \mu^*(x,dh,\sigma,de) + \sum_{x<0,\sigma} x \int (1 - d^*_{x,h,\sigma}(e;\alpha,q^*,w^*))\mu^*(x,dh,\sigma,de) \\
&= \rho \sum_{x>0,s} x a^*_{x,s} + \rho \sum_{x<0,s} x a^*_{x,s} - \sum_{x<0,s} x \rho \sum_\sigma p^{*\sigma}_x \Gamma(s;\sigma) a^*_{x,s} \\
&= \rho \left[\sum_{x>0,s} x a^*_{x,s} + \sum_{x<0,s} x a^*_{x,s}(1 - p^*_{x,s}) \right].
\end{aligned}$$

Thus, re-writing (42) we have

$$\begin{aligned}
&\int c^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* + \int \frac{\zeta(s)}{m^*} d\mu^* \\
&= w^* \int e d\mu^* - \gamma w^* \int e \mu^*(d\ell,1,ds,de) - \alpha w^* \int (e - e_{\min}) \cdot d^*_{\ell,0,s}(e;\alpha,q^*,w^*)\mu^*(d\ell,0,ds,de) \\
&+ \rho \sum_{\ell,s} \ell a^*_{\ell,s}(1 - p^*_{\ell,s}).
\end{aligned}$$

But

$$\begin{aligned}
\int q^*_{\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*),s} \ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)d\mu^* &= \sum_{\ell'} \int \mathbf{1}_{\{(\ell,h,s,e):(\ell'^*_{\ell,h,s}(e;\alpha,q^*,w^*)=\ell'\}} q_{\ell',s} \ell' \mu^*(d\ell,dh,ds,de) \\
&= \sum_{\ell',s} q^*_{\ell',s} a^*_{\ell',s} \ell' \\
&= K^*
\end{aligned}$$

where the last inequality follows from (20). Another implication of (20) is

$$(1 + r^* - \delta) K^* = \rho \sum_{(\ell',s)\in L\times S} (1 - p^*_{\ell',s}) a^*_{\ell',s} \ell'.$$

Thus, we have

$$\begin{aligned}
&\int c^*_{\ell,h,s}(e; \alpha, q^*, w^*) d\mu^* + K^* + \int \frac{\zeta(s)}{m^*} d\mu^* \\
&= w^* N^* - \gamma w^* \int e\mu^*(d\ell, 1, ds, de) + (1 + r^* - \delta) K^* \\
&= F(N^*, K^*) + (1 - \delta) K^* - \gamma w^* \int e\mu^*(d\ell, 1, ds, de) - \alpha w^* \int (e - e_{\min}) \cdot d^*_{\ell,0,s}(e; \alpha, q^*, w^*) \mu^*(d\ell, 0, ds, de).
\end{aligned}$$

So that the goods market clears.

■

Next, we establish that for $\alpha > 0$, the set of (ℓ, h, s, e) for which a household is indifferent between two or more courses of action is finite. Given (ℓ, h, s), define the set of e for which the household is indifferent between any two distinct, feasible actions (ℓ', d) and $(\overline{\ell}', \overline{d})$ as

$$\begin{aligned}
&I^{(\ell',d),(\overline{\ell}',\overline{d})}_{\ell,h,s}(\alpha, q, w, v^*) \\
&\equiv \{e \in E : \phi^{(\ell',d)}_{\ell,h,s}(e; \alpha, q, w, v^*) = \phi^{(\overline{\ell}',\overline{d})}_{\ell,h,s}(e; \alpha, q, w, v^*), c^{(\ell',d)}_{\ell,h,s}(e; \alpha, q, w) \geq 0, c^{(\overline{\ell}',\overline{d})}_{\ell,h,s}(e; \alpha, q, w) \geq 0\}.
\end{aligned}$$

Lemma A8. If $\alpha > 0$, $I^{(\ell',d),(\overline{\ell}',\overline{d})}_{\ell,h,s}(\alpha, q, w, v^*)$ contains at most two elements.

Proof. (i) Let $e \in I^{(\ell',0),(\overline{\ell}',0)}_{\ell,h,s}(\alpha, q, w)$. Since $\omega^*_{\ell',0,s}(\alpha, q, w) \neq \omega^*_{\overline{\ell}',0,s}(\alpha, q, w)$, it follows that $\Delta \equiv u(w \cdot e(1 - \gamma h) + \ell - \zeta(s) - q_{\ell',s}\ell', s) - u(w \cdot e(1 - \gamma h) + \ell - \zeta(s) - q_{\overline{\ell}',s}\overline{\ell}', s) \neq 0$. Therefore, consumption under each of the two actions must be different. Since $u(\cdot)$ is strictly concave, an equal change in consumption from these two different levels must lead to unequal changes in utility. Therefore, for $y \neq 0$ we must have that $u(w \cdot [\widehat{e} + y](1 - \gamma h) + \ell - \zeta(s) - q_{\ell',s}\ell', s) - u(w \cdot [\widehat{e} + y](1 - \gamma h) + \ell - \zeta(s) - q_{\overline{\ell}',s}\overline{\ell}', s) \neq \Delta$. Hence there can be at most one e for which $\phi^{\ell',0}_{\ell,0,s}(e, q, w; \omega^*) = \phi^{\overline{\ell}',0}_{\ell,0,s}(e, q, w; \omega^*)$.

(ii) Let $e \in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha, q, w)$ and let $y > 0$.

(a): Suppose that $u(w \cdot e + \ell - \zeta(s) - q_{\ell',s}\ell', s) - u(w \cdot [e_{\min} + (1 - \alpha)(e - e_{\min})], s) = \Delta \geq 0$. Given $\alpha > 0$, it follows that $u'(w \cdot [e + y] + \ell - \zeta(s) - q_{\ell',s}\ell', s) < u'(w \cdot [e_{\min} + (1 - \alpha)(e + y - e_{\min})], s)$. Now observe that $u(w \cdot [e + y] + \ell - \zeta(s) - q_{\ell',s}\ell', s)$ is $u(w \cdot e + \ell - \zeta(s) - q_{\ell',s}\ell', s) + \int_0^y u'(w \cdot [e + x] + \ell - \zeta(s) - q_{\ell',s}\ell', s) dx$ and $u(w \cdot [e_{\min} + (1 - \alpha)(e + y - e_{\min})], s)$ is $u(w \cdot [e_{\min} + (1 - \alpha)(e - \underline{e})], s) +$

$\int_0^y u'(w\cdot[e_{\min}+(1-\alpha)(e+x-e_{\min})],s)dx$. Therefore, $u(w\cdot[e+y]+\ell-\zeta(s)-q_{\ell',s}\ell',s)-u(w\cdot[e_{\min}+(1-\alpha)(e+y-e_{\min})],s)<\Delta$. Hence $e+y\notin I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. On the other hand, it's possible that there is a $z>0$ such that $e-z\in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. If so, $\int_0^z u'(w\cdot[e-x]\,\ell-\zeta(s)-q_{\ell',s}\ell',s)dx=\int_0^z u'(w\cdot[e_{\min}+(1-\alpha)(e-x-e_{\min})],s)dx$. Since $\Delta\geq 0$, we have $u'(w\cdot e+\ell-\zeta(s)-q_{\ell',s}\ell',s)<u'(w\cdot[e_{\min}+(1-\alpha)(e-e_{\min})],s)$. Therefore, $w\cdot[e-z]+\ell\zeta(s)-q_{\ell',s}\ell'<w\cdot e_{\min}+(1-\alpha)(e-z-e_{\min})$. Then, given $\alpha>0$ $u(w\cdot[e-z-y]+\ell-\zeta(s)-q_{\ell',s}\ell',s)-u(w\cdot[e_{\min}+(1-\alpha)(e-z-y-e_{\min})],s)\neq\Delta$ because would be taking *more* consumption away from the l.h.s. than from the r.h.s. when the l.h.s. already has less. Therefore, $I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$ can have at most two elements.
(b): Suppose that $u(w\cdot e+\ell-\zeta(s)-q_{\ell',s}\ell',s)-u(w\cdot[e_{\min}+(1-\alpha)(e-e_{\min})],s)=\Delta<0$. Then, given $\alpha>0$, $u'(w\cdot[e-y]+\ell-\zeta(s)-q_{\ell',s}\ell',s)<u'(w\cdot[e_{\min}+(1-\alpha)(e-y-e_{\min})],s)$. By an argument analogous to the first part of (a) we can establish that $e-y\notin I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. On the other hand, it is possible that there is a $z>0$ such that $e+z\in I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$. If so, then $u'(w\cdot[e+x]+\ell-\zeta(s)-q_{\ell',s}\ell',s)dx=\int_0^z u'(w\cdot[e_{\min}+(1-\alpha)(e+x-e_{\min})],s)dx$. By an argument analogous to the second part of (a) we can establish that $w\cdot[e+z]\,\ell-\zeta(s)-q_{\ell',s}\ell'>w\cdot[e_{\min}+(1-\alpha)(e+z-e_{\min})]$. Therefore, given $\alpha>0$, $u(w\cdot[e+z+y]+\ell-\zeta(s)-q_{\ell',s}\ell',s)-u(w\cdot[e_{\min}+(1-\alpha)(e+z+y-e_{\min})],s)\neq\Delta$ because we would be giving more consumption to the l.h.s than to the r.h.s. when the l.h.s. already has more. Therefore, $I^{(\ell',0),(0,1)}_{\ell,0,s}(\alpha,q,w)$ can have at most two elements. ■

Define

$$E^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)\equiv\{e\in E:\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell',d^*_{\ell,h,s}(e;\alpha,q,w)=d\}$$

as the subset of E that returns (ℓ',d) as the optimal decision and define

$$ES^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)$$
$$\equiv\left\{e\in E^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*):\Delta(e)\equiv\left[\phi^{\ell',d}_{\ell,h,s}(e,\alpha,q,w,v^*)-\max_{(\widetilde{\ell}',\widetilde{d})\neq(\ell',d)}\phi^{\widetilde{\ell}',\widetilde{d}}_{\ell,h,s}(e,\alpha,q,w,v^*)\right]>0\right\}$$

the subset of $E^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)$ for which (ℓ',d) is strictly better than any other action.

Lemma A9. For $\alpha>0$, $E^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)\backslash ES^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)$ is a finite set.

Proof. Observe that

$$\left\{E^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)\backslash ES^{\ell',d}_{\ell,h,s}(\alpha,q,w,v^*)\right\}\subseteq\bigcup_{(\widetilde{\ell}',\widetilde{d})\neq(\ell',d)}I^{(\ell',d),(\widetilde{\ell}',\widetilde{d})}_{\ell,h,s}(\alpha,q,w,v^*).$$

Since $I^{(\ell',d),(\widetilde{\ell}',\widetilde{d})}_{\ell,h,s}(\alpha,q,w,v^*)$ are finite sets by Lemma A8, the result follows.

■

Lemma A10. Let $Z \in 2^{\mathcal{L}} \times \mathcal{B}(E)$ and $(\ell_n, h_n, s_n, e_n, \alpha_n, q_n, w_n) \to (\ell, h, s, e, \alpha, q, w)$. If $\alpha > 0$, for all but a set (ℓ, h, s, e) of $\mu_{(\alpha,q,w)}$-measure zero: (i) $\ell'^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) \to \ell'^*_{\ell,h,s}(e; \alpha, q, w)$ and $d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) \to d^*_{\ell,h,s}(e; \alpha, q, w)$; and (ii) $\lim_{n\to\infty} G^*_{(\alpha_n,q_n,w_n)}(\ell_n, h_n, s_n, e_n, Z) = G^*_{(\alpha,q,w)}(\ell, h, s, e, Z)$.

Proof. Since $\mathcal{L}$ is finite, it follows that there is some η such that for all $n \geq \eta$, $(\ell_n, h_n, s_n) = (\ell, h, s)$. Without loss of generality, we simply consider the sequence $(e_n, \alpha_n, q_n, w_n) \to (e, \alpha, q, w)$.

It is convenient to begin with (ii). By (8),

$$G^*_{(\alpha,q,w)}(\ell, h, s, e, Z) = \rho GS^*_{(\alpha,q,w)}(\ell, h, s, e, Z) + (1-\rho) GN(\ell, h, s, e, Z).$$

From (7), notice that $\lim_{n\to\infty} GN((\ell, h, s, e_n), Z) = GN((\ell, h, s, e), Z)$ for all (ℓ, h, s, e) since $GN((\ell, h, s, e), Z)$ is actually independent of $(\ell, h, s, e, \alpha, q, w)$.

Next consider $GS^*_{(\alpha_n,q_n,w_n)}(\ell, h, s, e_n, Z)$. For $h = 1$ and $\ell - \zeta(s) \geq 0$,

$$\begin{aligned}
&GS^*_{(\alpha_n,q_n,w_n)}(\ell, 1, s, e_n, Z) \\
&= \int_{Z_s \times Z_s \times Z_e} \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} H^*_{(\alpha_n,q_n,w_n)}(\ell, 1, s, e_n, dh') \Phi(e'|s') de' \Gamma(s, ds') \\
&= \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} \cdot \int_{Z_h} \left[\lambda h' + (1-\lambda)(1-h')\right] dh' \cdot \int_{Z_s \times Z_e} \Phi(e'|s') de' \Gamma(s, ds').
\end{aligned}$$

For all other cases,

$$\begin{aligned}
&GS^*_{(\alpha_n,q_n,w_n)}(\ell, h, s, e_n, Z) \\
&= \int_{Z_s \times Z_s \times Z_e} \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} H^*_{(\alpha_n,q_n,w_n)}(\ell, h, s, e_n, dh') \Phi(e'|s') de' \Gamma(s, ds') \\
&= \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} \times \\
&\int_{Z_h} \left[h' d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) + (1-h')(1 - d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n)\right] dh' \cdot \int_{Z_s \times Z_e} \Phi(e'|s') de' \Gamma(s, ds').
\end{aligned}$$

Hence it is sufficient to establish that $\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} \to \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w) \in Z_\ell\}}$ and $d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) \to d^*_{\ell,h,s}(e; \alpha, q, w)$ except on a set of $\mu_{(\alpha,q,w)}$-measure zero.

Consider the set of efficiency levels $ES_{\ell,h,s}(\alpha, q, w, v^*)$ for which, given $(\ell, h, s, \alpha, q, w)$, the household strictly prefers some action (ℓ', d). That is,

$$ES_{\ell,h,s}(\alpha, q, w, v^*) = \left\{ e \in E : e \in \bigcup_{(\ell',d)} ES^{\ell',d}_{\ell,h,s}(\alpha, q, w, v^*) \right\}.$$

For any $e \in ES_{\ell,h,s}(\alpha, q, w, v^*)$, consider the sequence $\left\{\ell'^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n), d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n)\right\}$ where $(e_n, \alpha_n, q_n, w_n) \to (e, \alpha, q, w)$. Since $\left\{\ell'^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n), d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n)\right\}$ lies in a compact subset of R^2, we can extract a subsequence $\left\{\ell'^*_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}), d^*_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k})\right\}$

converging to $(\overline{\ell}', \overline{d})$. Furthermore, since ℓ' and d belong to finite sets, there must be some η such that for $n_k \geq \eta$, $(\ell'^*_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}), d^*_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k})) = (\overline{\ell}', \overline{d})$. Therefore, for $n_k \geq \eta$

$$\phi^{\overline{\ell}', \overline{d}}_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}, v^*) = v^*_{\ell,h,s}(e_{n_k}; \alpha_{n_k}, q_{n_k}, w_{n_k}).$$

Hence taking limits of both sides and using continuity of ϕ and v^* established in Lemma A2 and Theorem 1 respectively,

$$\phi^{\overline{\ell}', \overline{d}}_{\ell,h,s}(e; \alpha, q, w, v^*) = v^*_{\ell,h,s}(e; \alpha, q, w).$$

But since $e \in ES_{\ell,h,s}(\alpha, q, w, v^*)$, it follows that $(\overline{\ell}', \overline{d}) = (\ell'^*_{\ell,h,s}(e; \alpha, q, w), d^*_{\ell,h,s}(e; \alpha, q, w))$. Since the set of efficiency levels for which there is indifference can be expressed as

$$\bigcup_{(\ell',d)} \left\{ E^{\ell',d}_{\ell,h,s}(\alpha, q, w, v^*) \backslash ES^{\ell',d}_{\ell,h,s}(\alpha, q, w, v^*) \right\},$$

by Lemma A9 it follows that this set is finite and therefore of $\mu_{(\alpha,q,w)}$-measure zero. Hence (i) follows which in turn establishes that $\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e_n;\alpha_n,q_n,w_n) \in Z_\ell\}} \to \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w) \in Z_\ell\}}$ and $d^*_{\ell,h,s}(e_n; \alpha_n, q_n, w_n) \to d^*_{\ell,h,s}(e; \alpha, q, w)$ a.s.

■

The next step is to establish the weak convergence of the invariant distribution $\mu_{(\alpha,q,w)}$ w. r.t α, q and w. Theorem 12.13 of Stokey and Lucas provide sufficient conditions under which this holds. However, because the probability measure $G^*_{(\alpha,q,w)}((\ell, h, s, e_n), \cdot)$ need not converge weakly to $G^*_{(\alpha,q,w)}((\ell, h, s, e), \cdot)$ for $e_n \to e$ if the household is indifferent between two courses of action at (ℓ, h, s, e), condition (b) of the Theorem is not satisfied. To get around this problem, we use Theorem 12.13 to establish the weak convergence of an invariant distribution $\pi_{(\alpha,q,w)}(\ell, h, s)$ with the property that $\mu_{(\alpha,q,w)}(\ell, h, s, e) = \pi_{(\alpha,q,w)}(\ell, h, s)\Phi(e|s)$. Since $\Phi(e|s)$ is independent of (α, q, w), the weak convergence of $\mu_{(\alpha,q,w)}$ w.r.t (α, q, w) follows.

We begin by defining a finite state Markov chain $P^*_{(\alpha,q,w)}$ over the space (ℓ, h, s). Let

$$P^*_{(\alpha,q,w)} \left[(\ell, h, s), (\ell', h', s') \right] \equiv \int_E G^*_{(\alpha,q,w)}((\ell, h, s, e), \left(\ell', h', s', E\right))\Phi(e|s)de. \tag{44}$$

Then

$$P^*_{(\alpha,q,w)} \left[(\ell, h, s), (\ell', h', s') \right] = \int_E \left[\begin{array}{c} \rho GS^*_{(\alpha,q,w)}((\ell, h, s, e), (\ell', h', s', E)) \\ +(1-\rho) GN((\ell, h, s, e), (\ell', h', s', E)) \end{array} \right] \Phi(e|s)de$$

$$= \int_E \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w) = \ell'\}} H^*_{(\alpha,q,w)}(\ell, h, s, e, h')\Gamma(s, s')\Phi(e'|s')de' \\ +(1-\rho) \int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}} \psi(s', de') \end{array} \right] \Phi(e|s)de$$

$$= \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w) = \ell'\}} H^*_{(\alpha,q,w)}(\ell, h, s, e, h')\Gamma(s, s')\Phi(e|s)de \\ +(1-\rho) \int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}} \psi(s', de') \end{array} \right] \tag{45}$$

where the first two equalities follow from the definitions of (8), (6) and (7), and the third equality follows using $\int_E \Phi(e'|s')de' = 1 = \int_E \Phi(e|s)de$. To see that $P^*_{(\alpha,q,w)}$ is a Markov chain, note that by definition $G^*_{(\alpha,q,w)} \geq 0$ and

$$\begin{aligned}
&\int_{\mathcal{L}} P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right] d\ell' dh' ds' \\
&= \begin{array}{l} \rho \int_{\mathcal{L}} \left[\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Gamma(s,s')\Phi(e|s)de\right] d\ell' dh' ds' \\ \qquad\qquad +(1-\rho)\int_{\mathcal{L}}\left[\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',de')\right] d\ell' dh' ds' \end{array} \\
&= \begin{array}{l} \rho \int_{E} \left[\int_{\mathcal{L}} \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Gamma(s,s')d\ell' dh' ds'\right] \Phi(e|s)de \\ \qquad\qquad +(1-\rho)\int_{\mathcal{L}}\left[\mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',E)\right] d\ell' dh' ds' \end{array} \\
&= \rho \int_E \Phi(e|s)de + (1-\rho)\int_S \psi(s',E)ds' = 1.
\end{aligned}$$

Lemma A11. $P^*_{(\alpha,q,w)}$ induces a unique invariant distribution $\pi_{(\alpha,q,w)}$ on $(\mathcal{L}, 2^{\mathcal{L}})$.

Proof. The proof follows by applying Theorem 11.4 in Stokey and Lucas. Let $\widehat{s} \in S$ be such that $\psi(\widehat{s}, E) > 0$. Since newborns must be of *some* type, such an $\widehat{s}$ exists. Then $P^*_{(\alpha,q,w)}\left[(\ell,h,s),(0,0,\widehat{s})\right] \geq (1-\rho)\psi(\widehat{s},E) > 0$, $\forall \ell,h,s$. Therefore

$$\varepsilon = \sum_{(\ell',h',s')} \left\{ \min_{(\ell,h,s)} P\left[(\ell,h,s),(\ell',h',s')\right] \right\} \geq (1-\rho)\psi(\widehat{s},E) > 0$$

which satisfies the requirement of Theorem 11.4 (for $N = 1$).

■

Lemma A12. If $(\alpha_n, q_n, w_n) \in A \times Q \times W$ is a sequence converging to $(\alpha, q, w) \in A \times Q \times W$ where $\alpha_n, \alpha > 0$, then the sequence $\pi_{(\alpha_n,q_n,w_n)}$ converges weakly to $\pi_{(\alpha,q,w)}$.

Proof. The proof follows by applying Theorem 12.13 in Stokey and Lucas. Part a of the requirements follows since $\mathcal{L}$ is compact. Part b requires that $P^*_{(\alpha_n,q_n,w_n)}\left[(\ell_n,h_n,s_n),\cdot\right]$ converges weakly to $P^*_{(\alpha,q,w)}\left[(\ell,h,s),\cdot\right]$ as $(\ell_n,h_n,s_n,\alpha_n,q_n,w_n) \to (\ell,h,s,\alpha,q,w)$. By Theorem 12.3d of Stokey and Lucas it is sufficient to show that for any (ℓ',h',s'), $\lim_{n\to\infty} P^*_{(\alpha_n,q_n,w_n)}\left[(\ell_n,h_n,s_n),(\ell',h',s')\right] = P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right]$. Since $\mathcal{L}$ is finite, without loss of generality consider the sequence $(\alpha_n,q_n,w_n) \to (\alpha,q,w)$. But from (44)

$$\begin{aligned}
&\lim_{n\to\infty} P^*_{(\alpha_n,q_n,w_n)}\left[(\ell,h,s),(\ell',h',s')\right] \\
&= \lim_{n\to\infty} \int_E G^*_{(\alpha_n,q_n,w_n)}((\ell,h,s,e),(\ell',h',s',E))\Phi(e|s)de.
\end{aligned}$$

By Lemma A10, the sequence of integrable functions $G^*_{(\alpha_n,q_n,w_n)}((\ell,h,s,e),(\ell',h',s',E))$ of e converges almost everywhere to the measurable function $G^*_{(\alpha,q,w)}((\ell,h,s,e),(\ell',h',s',E))$ of e. Since $G^*_{(\alpha_n,q_n,w_n)}((\ell,h,s,e),(\ell',h',s',E)) \leq 1$, by the Lebesgue Dominated Convergence Theorem (Theorem 7.10 in Stokey and Lucas),

$$\begin{aligned}
&\lim_{n\to\infty}\int_E G^*_{(\alpha_n,q_n,w_n)}((\ell,h,s,e),\big(\ell',h',s',E\big))\Phi(e|s)de \\
&= \int_E G^*_{(\alpha,q,w)}((\ell,h,s,e),\big(\ell',h',s',E\big))\Phi(e|s)de \\
&= P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right].
\end{aligned}$$

Part c requires that for each (α,q,w), $P^*_{(\alpha,q,w)}$ induce a unique invariant measure; this follows from Lemma A11.

■

Lemma A13. $\mu_{(\alpha,q,w)}(\ell,h,s,e) = \pi_{(\alpha,q,w)}(\ell,h,s)\Phi(e|s)$.

Proof. Let $m_{(\alpha,q,w)}(\ell,h,s)$ be the function implicitly defined by

$$m_{(\alpha,q,w)}(\ell,h,s)\Phi(e|s) = \mu_{(\alpha,q,w)}(\ell,h,s,e) \tag{46}$$

and let $Z' = \ell' \times h' \times s' \times J'$. Then

$$\begin{aligned}
&\mu_{(\alpha,q,w)}(Z') \qquad (47)\\
&= (\Upsilon_{(\alpha,q,w)}\mu_{(\alpha,q,w)})(Z') = \int_{L\times H\times S\times E} G^*_{(\alpha,q,w)}(\ell,h,s,e,Z')d\mu_{(\alpha,q,w)} \\
&= \int_{L\times H\times S\times E}\Big[\rho GS^*_{(\alpha,q,w)}((\ell,h,s,e),Z') + (1-\rho)GN((\ell,h,s,e),Z')\Big]d\mu_{(\alpha,q,w)} \\
&= \int_{L\times H\times S\times E}\left[\begin{array}{c}\rho\int_{J'}\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Gamma(s,s')\Phi(e'|s')de' \\ +(1-\rho)\int_{J'}\mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',de')\end{array}\right]\mu_{(\alpha,q,w)}(d\ell,dh,ds,de) \\
&= \int_{L\times H\times S}\left[\begin{array}{c}\rho\int_E\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Gamma(s,s')\int_{J'}\Phi(e'|s')de' \\ +(1-\rho)\int_E\mathbf{1}_{\{(\ell',h')=(0,0)\}}\int_{J'}\psi(s',de')\end{array}\right]\Phi(de|s)m_{(\alpha,q,w)}(d\ell,dh,ds) \\
&= \int_{L\times H\times S}\left[\begin{array}{c}\rho\int_E\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Phi(de|s)\Gamma(s,s')\int_{J'}\Phi(e'|s')de' \\ +(1-\rho)\mathbf{1}_{\{(\ell',h')=(0,0)\}}\int_{J'}\psi(s',de')\end{array}\right]m_{(\alpha,q,w)}(d\ell,dh,ds)
\end{aligned} \tag{48}$$

where the first equality follows as a consequence of μ^* being fixed point, the second equality follows from the definition in (9), the fifth follows by using (46), and the sixth follows recognizing $\int_E \Phi(de|s) = 1$.

Then letting $J' = E$ in (48)

$$\begin{aligned}
&\mu_{(\alpha,q,w)}(Z') \\
&= \int_{L\times H\times S} \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Phi(de|s)\Gamma(s,s')m_{(\alpha,q,w)}(d\ell,dh,ds) \int_E \Phi(e'|s')de' \\ +(1-\rho)\mathbf{1}_{\{(\ell',h')=(0,0)\}} \int_E \psi(s',de')m_{(\alpha,q,w)}(d\ell,dh,ds) \end{array} \right] \\
&= \int_{L\times H\times S} \left[\begin{array}{c} \left\{\rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}} H^*_{(\alpha,q,w)}(\ell,h,s,e,h')\Phi(de|s)\Gamma(s,s')\right\} \\ + \left\{(1-\rho)\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s',de')\right\} \end{array} \right] m_{(\alpha,q,w)}(d\ell,dh,ds) \\
&= \int_{L\times H\times S} P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right] m_{(\alpha,q,w)}(d\ell,dh,ds)
\end{aligned}$$

where where the second equality follows since $\int_E \Phi(e'|s')de' = 1$ and the third follows by definition (45).

By (46)

$$\begin{aligned}
\mu_{(\alpha,q,w)}(\ell',h',s',E) &\equiv m_{(\alpha,q,w)}(\ell',h',s')\int_E \Phi(e'|s')de' \\
&= m_{(\alpha,q,w)}(\ell',h',s'),
\end{aligned}$$

so that

$$\mu_{(\alpha,q,w)}(Z') = m_{(\alpha,q,w)}(\ell',h',s') = \int_{L\times H\times S} P^*_{(\alpha,q,w)}\left[(\ell,h,s),(\ell',h',s')\right] m_{(\alpha,q,w)}(d\ell,dh,ds).$$

The second equality implies that $m_{(\alpha,q,w)}(\ell',h',s')$ is a fixed point of the Markov chain whose transition function is $P^*_{(\alpha,q,w)}$. Therefore, $m_{(\alpha,q,w)}(\ell,h,s) = \pi_{(\alpha,q,w)}(\ell,h,s)$ from which the result follows.

■

Lemma A14. If $(\alpha_n,q_n,w_n) \in A\times Q\times W$ is a sequence converging to $(\alpha,q,w) \in A\times Q\times W$ where $\alpha_n,\alpha > 0$, then the sequence $\mu_{(\alpha_n,q_n,w_n)}$ converges weakly to $\mu_{(\alpha,q,w)}$.

Proof. Since $\Phi(e|s)$ is independent of (α,q,w), the result follows from Lemmas A12 and A13.

■

Lemma A15. Let $\alpha > 0$, $K_{(\alpha,q,w)} \equiv \sum_{(\ell',s)\in L\times S} \ell' q_{\ell',s} \int \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}\mu_{(\alpha,q,w)}(d\ell,dh,s,de)$, $N_{(\alpha,q,w)} \equiv \int e d\mu_{(\alpha,q,w)}$, and $p_{(\alpha,q,w)}(\ell',s) \equiv \int d^*_{\ell',0,s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s;ds')de'$. Then $K_{(\alpha,q,w)}$, $N_{(\alpha,q,w)}$, and $p_{(\alpha,q,w)}(\ell',s)$ are continuous with respect to (α,q,w).

Proof. To prove $K_{(\alpha,q,w)}$ is continuous, we know by Lemma A13,

$$\int_{L\times H\times E} \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\mu_{(\alpha_n,q_n,w_n)}(d\ell,dh,s,de)$$
$$=\sum_{\ell,h}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\Phi(de|s)\pi_{(\alpha_n,q_n,w_n)}(\ell,h,s).$$

By Lemma A12, $\lim_{n\to\infty}\pi_{(\alpha_n,q_n,w_n)}(\ell,h,s)=\pi_{(\alpha,q,w)}(\ell,h,s)$. By Lemma A10, $\mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\to \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}$ except possibly for a finite number of points in E. By the Lebesgue Dominated Convergence Theorem (Stokey and Lucas Theorem 7.10), $\lim_{n\to\infty}\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)=\ell'\}}\Phi(de|s) = \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell'\}}\Phi(de|s)$. Then, since $K_{(\alpha_n,q_n,w_n)}$ is the sum of a finite number of products each of which converge, the sum converges as well.

To prove $N_{(\alpha,q,w)}$ is continuous, simply apply Lemma A14.

To prove $p_{(\alpha,q,w)}(\ell',s)$ is continuous, note that by Lemma A10 $d^*_{\ell,h,s}(e;\alpha_n,q_n,w_n)\to d^*_{\ell,h,s}(e;\alpha,q,w)$ except possibly for a finite number of points in E. By the Lebesgue Dominated Convergence Theorem, $\lim_{n\to\infty}\int_{E\times S} d^*_{\ell',0,s'}(e';\alpha_n,q_n,w_n)\Phi(e'|s')\Gamma(s;ds')de' = \int_{E\times S} d^*_{\ell',0,s'}(e';\alpha,q,w)\Phi(e'|s')\Gamma(s;ds')de'$.

■

Finally, we define the vector-valued function whose fixed point gives us a candidate equilibrium price vector. At this point, we need to be explicit about the upper and lower bounds of the sets W and the upper bound of the set Q.

Assumption A2. Assume that $q_{\max}=\rho\left(1+F_K(\ell_{\max},e_{\min})-\delta\right)^{-1}$, $w_{\min}=b$ and $w_{\max}=F_N(\ell_{\max},e_{\min})$.

Note that our earlier assumption that $\ell_{\max}$ is such that $F_K(\ell_{\max},e_{\min})>\delta$ guarantees that $q_{\max}$ is strictly positive.

Let $\Omega^\alpha: Q\times W\to R^{N_L\cdot N_S+1}$ be given by[35]

$$\Omega^\alpha(q,w)\equiv\begin{bmatrix}\Omega^\alpha_{\ell'\geq 0,s}(q,w)\\ \Omega^\alpha_{\ell'<0,s}(q,w)\\ \Omega^\alpha_{w}(q,w)\end{bmatrix} \tag{49}$$

where

$$\Omega^\alpha_{\ell'\geq 0,s}(q,w)=\begin{cases}\rho\left(1+F_K\left(K_{(\alpha,q,w)},N_{(\alpha,q,w)}\right)-\delta\right)^{-1} & \text{for } K_{(\alpha,q,w)}>0\\ 0 & \text{for } K_{(\alpha,q,w)}\leq 0\end{cases},$$

$$\Omega^\alpha_{\ell'<0,s}(q,w)=\begin{cases}\rho\left(1-p_{(\alpha,q,w)}(\ell',s)\right)\left(1+F_K\left(K_{(\alpha,q,w)},N_{(\alpha,q,w)}\right)-\delta\right)^{-1} & \text{for } K_{(\alpha,q,w)}>0\\ 0 & \text{for } K_{(\alpha,q,w)}\leq 0\end{cases},$$

[35] $\overline{w}$ can always be made to exceed $\underline{w}$ by placing assumptions on the production technology.

and

$$\Omega^\alpha_w(q,w) = \begin{cases} F_N\left(K_{(\alpha,q,w)}, N_{(\alpha,q,w)}\right) & \text{for } K_{(\alpha,q,w)} > 0 \\ F_N\left(0, N_{(\alpha,q,w)}\right) & \text{for } K_{(\alpha,q,w)} \leq 0 \end{cases} .$$

A fixed point of this function is an equilibrium price vector provided an $m^* \geq 1$ can be found for which condition (v) in Definition 2 is satisfied.

Lemma A16. For $\alpha > 0$, there exists $(q^*, w^*) \in Q \times W$ such that $(q^*, w^*) = \Omega^\alpha(q^*, w^*)$.

Proof. The set $Q \times W$ is compact. By Assumption A2, $\Omega^\alpha(q,w) \subset Q \times W$. To see this, observe that by Assumption 1(iii) $F_K(\ell_{\max}, e_{\min})$ is the lowest marginal product of capital possible in this economy and therefore, $q_{\max}$ is highest price on deposits possible. The lower bound on wages is the lower bound on the marginal product of labor in Assumption 1(v) and the upper bound on wages is, by Assumption 1(iii) again, the highest marginal product of labor possible in this economy.

Next we need to establish that $\Omega^\alpha(q,w)$ is continuous in q and w. Note that $N_{(\alpha,q,w)}$ is always strictly positive since it is bounded below by $e_{\min}$. First, consider (α, q, w) such that $K_{(\alpha,q,w)} > 0$ and let $(q_n, w_n) \to (q, w)$. By Lemma A15 and continuity of F_K and F_N, it follows that $\Omega^\alpha(q_n, w_n) \to \Omega^\alpha(q,w)$. Second, consider (α, q, w) such that $K_{(\alpha,q,w)} < 0$. Then for any $\varepsilon > 0$ there exists η such that for all $n \geq \eta$,

$$\rho\left(1 + F_K\left(K_{(\alpha,q_n,w_n)}, N_{(\alpha,q_n,w_n)}\right) - \delta\right)^{-1} \leq \rho\left(1 + F_K\left(K_{(\alpha,q_n,w_n)}, e_{\min}\right) - \delta\right)^{-1} < \varepsilon$$

Therefore, since ε can be made arbitrarily small, $\Omega^\alpha_{\ell'<0,s}(q_n, w_n) \to 0 = \Omega^\alpha_{\ell'<0,s}(q,w)$ and $\Omega^\alpha_{\ell'\geq 0,s}(q_n, w_n) \to 0 = \Omega^\alpha_{\ell'\geq 0,s}(q,w)$. Furthermore, there exists η such that for all $n \geq \eta$, $\Omega^\alpha_w(q_n, w_n) = F_N(0, N_{(\alpha,q_n,w_n)})$. Therefore, by Lemma A15 and continuity of F_N, it follows that $\Omega^\alpha_w(q_n, w_n) \to \Omega^\alpha_w(q,w)$. Third, consider (α, q, w) such that $K_{(\alpha,q,w)} = 0$. Then $\Omega^\alpha_{\ell'<0,s}(q_n, w_n) \to 0 = \Omega^\alpha_{\ell'<0,s}(q,w)$ and $\Omega^\alpha_{\ell'\geq 0,s}(q_n, w_n) \to 0 = \Omega^\alpha_{\ell'\geq 0,s}(q,w)$ by an argument similar to the above case where $K_{(\alpha,q,w)} < 0$. Furthermore, for any $\varepsilon > 0$ there exists η such that for all $n \geq \eta$, $K_{(\alpha,q_n,w_n)} < \varepsilon$ and hence

$$F_N\left(\varepsilon, N_{(\alpha,q_n,w_n)}\right) \geq \Omega^\alpha_w(q_n, w_n) \geq F_N\left(0, N_{(\alpha,q_n,w_n)}\right).$$

Therefore, by Lemma A15 and continuity of F_N, it follows that

$$F_N\left(\varepsilon, N_{(\alpha,q,w)}\right) \geq \lim_{n\to\infty} \Omega^\alpha_w(q_n, w_n) \geq F_N\left(0, N_{(\alpha,q,w)}\right).$$

Since ε can be arbitrarily small, it follows that $\Omega^\alpha_w(q_n, w_n) \to F_N\left(0, N_{(\alpha,q,w)}\right) = \Omega^\alpha_w(q,w)$.

The result follows by Brouwer's fixed point theorem.

■

Lemma A17. $\ell_{\max} \geq K_{(\alpha,q^*,w^*)} > 0$.

Proof. If $K_{(\alpha,q^*,w^*)} = 0$, then $q^*_{\ell',s} = 0$ for all ℓ' by (49). Hence, the optimal decision for households with $\ell \geq 0$ is to choose $\ell' = \ell_{\max}$ and the optimal decision for households with $\ell < 0$ is either to choose default today and choose $\ell_{\max}$ tomorrow or pay back and choose $\ell_{\max}$ today. Therefore, within at most one period the invariant distribution will have all its mass on points with $(\ell_{\max}, h, s, e)$. Hence $K_{(\alpha,q^*,w^*)} = \ell_{\max}$. But this implies that $\Omega^{\alpha}_{\ell' \geq 0,s}(0, w^*) = \rho\left(1 + F_K\left(\ell_{\max}, N_{(0,w^*)}\right) - \delta\right)^{-1} > 0$, which yields a contradiction. Hence $K_{(\alpha,q^*,w^*)} > 0$. Since the asset holding of each household is bounded above by $\ell_{\max}$, it follows that $\ell_{\max} \geq K_{(\alpha,q^*,w^*)}$.

■

Lemma A18. There exists a steady state competitive equilibrium with $\alpha > 0$.

Proof. For $\alpha > 0$, we know there exists $(q^*, w^*) = \Omega^{\alpha}(q^*, w^*)$ by Lemma A16. Then provided (v) is satisfied, all the conditions for a competitive steady state equilibrium in Definition 2 are satisfied by construction of Ω^{α}. Observe that if the hospital sector has strictly positive revenue in the steady state, that is

$$\int \left[(1 - d^*_{\ell,h,s}(e; \alpha, q^*, w^*))\zeta(s) + d^*_{\ell,h,s}(e; \alpha, q^*, w^*) \max\{\ell, 0\}\right] d\mu^* > 0, \tag{50}$$

then we can always choose $m^* \geq 1$ to satisfy condition (v). Since we have assumed that every surviving household has a strictly positive probability of experiencing a medical expense and $K_{(\alpha,q^*,w^*)} > 0$ by Lemma A17, (50) is satisfied.

■

For a given pair of optimal decision rules $(\ell'^*_{\ell,h,s}(e; \alpha, q, w), d^*_{\ell,h,s}(e; \alpha, q, w))$ define the (optimal) probability of choosing (ℓ', d) given (ℓ, h, s) and (α, q, w) as

$$x^{(\ell',d)}_{(\ell,h,s)}(\alpha, q, w) \equiv \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha,q,w)=\ell', d^*_{\ell,h,s}(e;\alpha,q,w)=d\}} \Phi(de|s)$$

and define the $2 \cdot N_L \cdot N_{\mathcal{L}}$-element vector of choice probabilities $x(\alpha, q, w)$ by

$$x(\alpha, q, w) \equiv \{x^{(\ell',d)}_{(\ell,h,s)}(\alpha, q, w)\ \forall\ (\ell, h, s) \in \mathcal{L} \text{ and } (\ell', d) \in L \times \{0, 1\}\}.$$

Let a sequence of costs $\alpha_n \to 0$ with $\alpha_n > 0$. For each α_n, let $(q^*_n, w^*_n) \in Q \times W$ be the equilibrium price vector. Since $Q \times W$ is compact, we can extract a subsequence $(q^*_{n_k}, w^*_{n_k})$ converging to $(\overline{q}, \overline{w}) \in Q \times W$. Let the corresponding sequence of measurable optimal decision rules and the sequence of optimal choice probability vectors be

$$\{c^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}),\ \ell'^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}),\ d^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\} \text{ and } \left\{x(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\right\}.$$

Since each term in the sequence $\{x(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\}$ is in $[0, 1]^{2 \cdot N_L \cdot N_{\mathcal{L}}}$ we can extract a subsequence converging to some $\overline{x} \in [0, 1]^{2 \cdot N_L \cdot N_{\mathcal{L}}}$. In the following key Lemma, we construct measurable decision rules that are optimal given $\alpha = 0$ and $(\overline{q}, \overline{w})$ and consistent with the limiting choice probabilities.

Lemma A19. For all $(\ell, h, s) \in \mathcal{L}$, there exist measurable functions $\Big(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\Big)$ such that for all $(\ell', d) \in L \times \{0,1\}$,

$$\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}} \Phi(de|s) = \overline{x}^{(\ell',d)}_{(\ell,h,s)},$$

and

$$\big(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\big) \in \chi_{\ell,h,s}(e; 0; \overline{q}, \overline{w}).$$

Proof. Order the elements of $L \times \{0,1\}$ in some fashion.

Stage 1: Let (ℓ', d) be the first element in $L \times \{0,1\}$ and let $E_1 = E \in \mathcal{B}(E)$. Consider the sequence $\Big\{x^{(\ell',d)}_{(\ell,h,s)\,n_k}\Big\}$ converging to $\overline{x}^{(\ell',d)}_{(\ell,h,s)}$. (i) If $\Big\{x^{(\ell',d)}_{(\ell,h,s)\,n_k}\Big\} = \overline{x}^{(\ell',d)}_{(\ell,h,s)}$ for all n_k, let

$$\overline{E} = \{e : \ell'^{*}_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) = \ell', d^*_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) = d\}.$$

Set

$$\ell'_{\ell,h,s}(e) = \ell',\ d_{\ell,h,s}(e) = d \text{ and } c_{\ell,h,s}(e) = c^{(\ell',d)}_{\ell,h,s}(e; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) \text{ iff } E_1 \cap \overline{E}.$$

(ii) If (i) does not apply and there is some k_1 such that $\left\{x^{(\ell',d)}_{(\ell,h,s)\,n_{k_1}}\right\} > \overline{x}^{(\ell',d)}_{(\ell,h,s)}$, pick another subsequence $\{n_{k_j}\}$ converging to $\overline{x}^{(\ell',d)}_{(\ell,h,s)}$ with the property that $x^{(\ell',d)}_{(\ell,h,s)\,n_{k_j}} \geq x^{(\ell',d)}_{(\ell,h,s)\,n_{k_{j+1}}}$ for all $j \geq 1$. Then,

$$x^{(\ell',d)}_{(\ell,h,s)\,n_{k_j}} = \int \mathbf{1}_{\{E^{(\ell',d)}_{(\ell,h,s)}(\alpha_{n_{k_j}}, q^*_{n_{k_j}}, w^*_{n_{k_j}})\}} \Phi(de|s)$$

and by construction of the subsequence

$$\left\{E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_{k_j}}, q^*_{n_{k_j}}, w^*_{n_{k_j}}\right)\right\} \supseteq \left\{E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_{k_{j+1}}}, q^*_{n_{k_{j+1}}}, w^*_{n_{k_{j+1}}}\right)\right\}.$$

Because optimal decision rules are measurable selections from the optimal policy correspondence, each element of the subsequence is Borel measurable and, therefore,

$$\overline{E} = \left\{\bigcap_{n_{k_j}} \left(E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_{k_j}}, q^*_{n_{k_j}}, w^*_{n_{k_j}}\right)\right)\right\}$$

is Borel measurable as well. Set

$$\ell'_{\ell,h,s}(e) = \ell',\ d_{\ell,h,s}(e) = d \text{ and } c_{\ell,h,s}(e) = \lim_{j\to\infty} c^{(\ell',d)}_{\ell,h,s}(e; \alpha_{n_{k_j}}, q^*_{n_{k_j}}, w^*_{n_{k_j}}) \text{ iff } e \in \overline{E}.$$

By Theorem 7.1 of Stokey and Lucas,

$$\int \mathbf{1}_{\{\overline{E}\}}\Phi(de|s) = \lim_{j\to\infty} x^{(\ell',d)}_{(\ell,h,s)\,n_{k_{j+1}}} = \overline{x}^{(\ell',d)}_{(\ell,h,s)}.$$

Finally, if neither (i) or (ii) applies, it follows that $\left\{x^{(\ell',d)}_{(\ell,h,s)\,n_k}\right\} \leq \left\{x^{(\ell',d)}_{(\ell,h,s)\,n_{k+1}}\right\}$. Then

$$\left\{E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}\right)\right\} \subseteq \left\{E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_{k+1}}, q^*_{n_{k+1}}, w^*_{n_{k+1}}\right)\right\}$$

is an increasing sequence of Borel sets and therefore

$$\overline{E} = \left\{\bigcup_{n_k}\left(E_1 \cap E^{(\ell',d)}_{(\ell,h,s)}\left(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}\right)\right)\right\}$$

is also Borel. Now, set

$$\ell'_{\ell,h,s}(e) = \ell',\ d_{\ell,h,s}(e) = d \text{ and } c_{\ell,h,s}(e) = \lim_{k\to\infty} c^{(\ell',d)}_{\ell,h,s}(e;\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) \text{ iff } e \in \overline{E}.$$

By Theorem 7.1 of Stokey and Lucas

$$\int \mathbf{1}_{\{\overline{E}\}}\Phi(de|s) = \lim_{k\to\infty} x^{(\ell',d)}_{(\ell,h,s)\,n_k} = \overline{x}^{(\ell',d)}_{(\ell,h,s)}$$

Stage 2: Set $E_2 = \left(E_1 \backslash \overline{E}\right) \in \mathcal{B}(E)$. If E_2 is empty, stop. Otherwise, pick the second element in $L \times \{0,1\}$ and repeat the steps in stage 1 with E_2 in place of E_1.

Stage 3: Set $E_3 = E_2 \backslash \overline{E}$. If E_3 is empty, stop. Otherwise pick the third element in $L \times \{0,1\}$ and repeat the steps in stage 1 with E_3 in place of E_1.

⋮ ⋮ ⋮

Stage $2 \cdot N_L$: Set $E_{2\cdot N_L} = \left(E_{2\cdot N_L - 1} \backslash \overline{E}\right) \in \mathcal{B}(E)$. If $E_{2\cdot N_L}$ is empty, stop. Otherwise, pick the last element in $L \times \{0,1\}$ and repeat the steps in stage 1 with $E_{2\cdot N_L}$ in place of E_1.

Stage $2 \cdot N_L + 1$: Set $E_{2\cdot N_L + 1} = \left(E_{2\cdot N_L} \backslash \overline{E}\right) \in \mathcal{B}(E)$. If $E_{2\cdot N_L+1}$ is empty, stop. Otherwise, assign to elements of $E_{2\cdot N_L+1}$ any actions that are optimal given ℓ, h, s and $\alpha = 0, q = \overline{q}$ and $w = \overline{w}$.

Since an e that is assigned an action in one of the stages cannot reappear in a later stage, every $e \in E$ is assigned just one element in $L \times \{0,1\}$. Thus the construction clearly yields measurable *functions* $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right)$ defined on E.

If the construction terminates in one of the intermediate stages or if it terminates because $E_{2\cdot N_L+1}$ is empty then evidently

$$\forall(\ell', d) \in L \times \{0,1\}, \int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}}\Phi(de|s) = \overline{x}^{(\ell',d)}_{(\ell,h,s)}.$$

The same is true if the contruction terminates after elements of $E_{2\cdot N_L+1}$ are assigned optimal actions. To see this note that for all $\{n_k\}$, $\Sigma_{(\ell',d)\in L\times\{0,1\}} x_{\ell,h,s}(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) = 1$ and, therefore, the sum of limiting probabilities $\Sigma_{(\ell',d)\in L\times\{0,1\}}\overline{x}_{\ell,h,s}$ must also equal 1. But this implies that $\int_E \mathbf{1}_{\{\widetilde{E}\}}\Phi(de|s) = 1$, where $\widetilde{E} = \bigcup_{i=1}^{2\cdot N_L} E_i$. Since $\widetilde{E}\cap E_{2\cdot N_L+1} = \emptyset$, it follows that $\int_E \mathbf{1}_{\{E_{2\cdot N_L+1}\}}\Phi(de|s) = 0$. Therefore,

$$\begin{aligned}
&\int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}}\Phi(de|s)\\
&= \int \mathbf{1}_{\left\{\widetilde{E}\cap\{e:\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}\right\}}\Phi(de|s) + \int \mathbf{1}_{\left\{E_{2\cdot N_L+1}\cap\{e:\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}\right\}}\Phi(de|s)\\
&= \int \mathbf{1}_{\left\{\widetilde{E}\cap\{e:\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}\right\}}\Phi(de|s)\\
&= \overline{x}^{(\ell',d)}_{(\ell,h,s)}.
\end{aligned}$$

Next, we prove that these measurable functions are feasible for $\alpha = 0, q = \overline{q}$ and $w = \overline{w}$. Observe that if the (discrete) action (ℓ', d) is assigned to a point e, say $\widehat{e}$, then by construction there is a non-negative subsequence $\left\{c^{(\ell',d)}_{\ell,h,s}(\widehat{e};\alpha_m, q^*_m, w^*_m)\right\}$ such that

$$c^{(\ell',d)}_{\ell,h,s}(\widehat{e}) = \lim_{m\to\infty} c^{(\ell',d)}_{\ell,h,s}(\widehat{e};\alpha_m, q^*_m, w^*_m).$$

Therefore

$$c^{(\ell',d)}_{\ell,h,s}(e) \geq 0.$$

Furthermore, because the function $c^{(\ell',d)}_{\ell,h,s}(e;\alpha,q,w)$ is continuous in all arguments we also have that

$$c^{(\ell',d)}_{\ell,h,s}(e) = c^{(\ell',d)}_{\ell,h,s}(e;0,\overline{q},\overline{w}).$$

Hence the functions $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right)$ are feasible for $\alpha = 0, q = \overline{q}$ and $w = \overline{w}$.

Finally, we prove that $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right)$ are optimal for $\alpha = 0, q = \overline{q}$ and $w = \overline{w}$. Observe that if the (discrete) action (ℓ', d) is assigned to a point e, say $\widehat{e}$, then by construction there is a subsequence such that

$$\phi^{\ell',d}_{\ell,h,s}(\widehat{e},\alpha_m, q^*_m, w^*_m;\omega^*(\alpha_m, q^*_m, w^*_m)) - v^*_{\ell,h,s}(\widehat{e},\alpha_m, q^*_m, w^*_m) = 0.$$

Since $\phi^{\ell',d}_{\ell,h,s}$, ω^* and $v^*_{\ell,h,s}$ are continuous in all arguments, it follows that

$$\phi^{\ell',d}_{\ell,h,s}(\widehat{e},0,\overline{q},\overline{w};\omega^*(0,\overline{q},\overline{w})) = v^*_{\ell,h,s}(\widehat{e};0,\overline{q},\overline{w})$$

Therefore $\left(c_{\ell,h,s}(e), \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e)\right) \in \chi_{\ell,h,s}(e;0;\overline{q},\overline{w})$.

■

We now establish the analogs of Lemma A12, A14 and A15 for the sequence $\left\{\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}\right\}$ converging to $(0, \overline{q}, \overline{w})$.

Lemma A20. Let $\overline{\pi}_{(0,\overline{q},\overline{w})}$ be the invariant distribution of the Markov chain $\overline{P}$ defined by the decision rules $(\ell'_{\ell,h,s}(e), d_{\ell,h,s}(e))$. Then the sequence $\pi_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}$ converges weakly to $\overline{\pi}_{(0,\overline{q},\overline{w})}$.

Proof. We apply Theorem 12.13 in Stokey and Lucas. Part a of the requirements follows since $\mathcal{L}$ is compact. Part b requires that $P^*_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}\left[(\ell_n, h_n, s_n), \cdot\right]$ converge weakly to $\overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell, h, s), \cdot\right]$ as $(\ell_n, h_n, s_n, \alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) \to (\ell, h, s, 0, \overline{q}, \overline{w})$. By Theorem 12.3d of Stokey and Lucas it is sufficient to show that for any (ℓ', h', s'),

$$\lim_{k\to\infty} P^*_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}\left[(\ell_n, h_n, s_n), (\ell', h', s')\right] = \overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell, h, s), (\ell', h', s')\right].$$

By definition

$$\begin{aligned} & P^*_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}\left[(\ell, h, s), (\ell', h', s')\right] \\ & = \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})=\ell'\}} H^*_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \\ +(1-\rho)\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s', de') \end{array}\right] \end{aligned}$$

and

$$H^*_{(q,w)}(\ell, h, s, e, h'=1) = \left\{\begin{array}{ll} 1 & \text{if } d^*_{\ell,h,s}(e;q,w) = 1 \\ \lambda & \text{if } d^*_{\ell,h,s}(e;q,w) = 0 \text{ and } h = 1 \\ 0 & \text{if } d^*_{\ell,h,s}(e;q,w) = 0 \text{ and } h = 0 \end{array}\right.,$$

$$H^*_{(q,w)}(\ell, h, s, e, h'=0) = \left\{\begin{array}{ll} 0 & \text{if } d^*_{\ell,h,s}(e;q,w) = 1 \\ 1-\lambda & \text{if } d^*_{\ell,h,s}(e;q,w) = 0 \text{ and } h = 1 \\ 1 & \text{if } d^*_{\ell,h,s}(e;q,w) = 0 \text{ and } h = 0 \end{array}\right..$$

By construction, the Markov chain $\overline{P}$ is

$$\begin{aligned} & \overline{P}\left[(\ell, h, s), (\ell', h', s')\right] \\ & = \left[\begin{array}{c} \rho \int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell'\}} H^*_{(0,\overline{q},\overline{w})}(\ell, h, s, e, h')\Phi(de|s)\Gamma(s, s') \\ +(1-\rho)\int_E \mathbf{1}_{\{(\ell',h')=(0,0)\}}\psi(s', de') \end{array}\right] \end{aligned}$$

where $H^*_{(0,\overline{q},\overline{w})}(\ell, h, s, e, h')$ is determined by $d_{\ell,h,s}(e)$.

Since $\mathcal{L}$ is finite, without loss of generality consider the sequence $(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) \to (0, \overline{q}, \overline{w})$. Since the second term on the r.h.s. is independent of (α, q, w), it is sufficient to consider the limiting behavior of the integral

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})=\ell'\}} H^*_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}(\ell, h, s, e, h')\Phi(de|s).$$

For $h=0$ and $h'=0$, this integral in P^* is

$$\int_E \mathbf{1}_{\{\ell'^*_{\ell,0,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} H^*_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}(\ell,0,s,e,0)\Phi(de|s)$$
$$= \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} \mathbf{1}_{\{d^*_{\ell,0,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=0\}} \Phi(de|s)$$
$$= \int_E \mathbf{1}_{\{\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=0\}} \Phi(de|s)$$
$$= x^{(\ell',0)}_{(\ell,0,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k})$$

and in $\overline{P}$ it is

$$\int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell'\}} H^*_{(0,\overline{q},\overline{w})}(\ell,0,s,e,0)\Phi(de|s)$$
$$= \int_E \mathbf{1}_{\{\ell'_{\ell,0,s}(e)=\ell', d_{\ell,0,s}(e)=0\}} \Phi(de|s).$$

By Lemma A19 we have

$$\lim_{k\to\infty} x^{(\ell',0)}_{(\ell,0,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k}) = \overline{x}^{(\ell',0)}_{(\ell,0,s)} = \int_E \mathbf{1}_{\{\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=0\}} \Phi(de|s)$$

Hence

$$\lim_{k\to\infty} P^*_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\left[(\ell,0,s),(\ell',0,s')\right] = \overline{P}_{(0;\overline{q},\overline{w})}\left[(\ell,0,s),(\ell',0,s')\right].$$

The remaining cases can be dealt with in exactly the same way. Here we simply note which choice probabilities are involved and omit the details. For $h=0$ and $h'=1$, the integral in $P^*$is

$$= \int_E \mathbf{1}_{\{\ell'^*_{\ell,0,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,0,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=1\}} \Phi(de|s)$$
$$= x^{(\ell',1)}_{(\ell,0,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k})$$

For $h=1$ and $h'=0$, the integral

$$= (1-\lambda)\int_E \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=0\}} \Phi(de|s)$$
$$= (1-\lambda)x^{(\ell',0)}_{(\ell,1,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k})$$

For $h=1$ and $h'=1$, the integral

$$= \int_E \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=1\}} \Phi(de|s)$$
$$+ \lambda\int_E \mathbf{1}_{\{\ell'^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,1,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=0\}} \Phi(de|s)$$
$$= x^{(\ell',1)}_{(\ell,1,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k}) + \lambda x^{(\ell',0)}_{(\ell,1,s)}(\alpha_{n_k},q^*_{n_k},w^*_{n_k}).$$

■

Lemma A21. Let $\overline{\mu}_{(0,\overline{q},\overline{w})}$ be the invariant distribution corresponding to the decision rules $\ell'_{\ell,h,s}(e)$ and $d_{\ell,h,s}(e)$. Then, the sequence $\mu_{(\alpha_{n_k},q^*_{nk},w^*_{nk})}$ converges weakly to $\overline{\mu}_{(0,\overline{q},\overline{w})}$.

Proof. Since $\Phi(e|s)$ is independent of (α, q, w), the result follows from Lemmas A13 and A20.

■

Lemma A22. Let $K_{(0,\overline{q},\overline{w})} \equiv \sum_{(\ell',s)\in L\times S} \ell' \overline{q}_{\ell',s} \int \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}} \overline{\mu}_{(0,\overline{q},\overline{w})}(d\ell, dh, s, de)$, $N_{(0,\overline{q},\overline{w})} \equiv \int e d\overline{\mu}_{(0,\overline{q},\overline{w})}$, and $p_{(0,\overline{q},\overline{w})}(\ell', s) \equiv \int d_{\ell',0,s'}(e')\Phi(e'|s')\Gamma(s; ds')de'$. Then

$$\lim_{k\to\infty} K(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) = K_{(0,\overline{q},\overline{w})},$$
$$\lim_{k\to\infty} N(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}) = N_{(0,\overline{q},\overline{w})},$$
$$\lim_{k\to\infty} p_{\alpha_{n_k},q^*_{n_k},w^*_{n_k}}(\ell', s) = p_{(0,\overline{q},\overline{w})}(\ell', s).$$

Proof. To prove that $K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}$ converges to $K_{(0,\overline{q},\overline{w})}$ note that we know by Lemma A13,

$$\int_{L\times H\times E} \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} \mu_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}(d\ell, dh, s, de)$$
$$= \sum_{\ell,h} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} \Phi(de|s)\pi_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}(\ell, h, s).$$

By Lemma A19

$$\lim_{n_k\to\infty} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=d\}} \Phi(de|s)$$
$$= \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}} \Phi(de|s).$$

Since

$$\int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} \Phi(de|s)$$
$$= \sum_{d\in\{0,1\}} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=d\}} \Phi(de|s),$$

then

$$\lim_{n_k\to\infty} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell'\}} \Phi(de|s)$$
$$= \lim_{n_k\to\infty} \sum_{d\in\{0,1\}} \int_E \mathbf{1}_{\{(\ell'^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=\ell', d^*_{\ell,h,s}(e;\alpha_{n_k},q^*_{n_k},w^*_{n_k})=d\}} \Phi(de|s)$$
$$= \sum_{d\in\{0,1\}} \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell', d_{\ell,h,s}(e)=d\}} \Phi(de|s).$$
$$= \int_E \mathbf{1}_{\{(\ell'_{\ell,h,s}(e)=\ell'\}} \Phi(de|s).$$

Next, by Lemma A20,

$$\lim_{n\to\infty} \pi_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})}(\ell, h, s) = \pi_{(0,\overline{q},\overline{w})}(\ell, h, s).$$

Therefore $\lim_{n_k\to\infty} K_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})} = K_{(0,\overline{q},\overline{w})}$.

To prove $\lim_{n_k\to\infty} N_{(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})} = N_{(0,\overline{q},\overline{w})}$, simply apply Lemma A21.

To prove $\lim_{n_k\to\infty} p_{\alpha_{n_k}, q^*_{n_k}, w^*_{n_k}}(\ell', s) = p_{(0,\overline{q},\overline{w})}(\ell', s)$, note that by Lemma A19

$$\lim_{n_k\to\infty} \int_E d^*_{\ell',0,s'}(e'; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\Phi(de'|s') = \int_E d_{\ell',0,s'}(e')\Phi(de'|s').$$

Thus,

$$\lim_{n_k\to\infty} \int_{E\times S} d^*_{\ell',0,s'}(e'; \alpha_{n_k}, q^*_{n_k}, w^*_{n_k})\Phi(de'|s')\Gamma(s; ds')$$
$$= \int_{E\times S} d_{\ell',0,s'}(e')\Phi(de'|s')\Gamma(s; ds').$$

■

Theorem 5 (Existence). A steady state competitive equilibrium exists.

Proof. For the sequence $\{q^*_{n_k}, w^*_{n_k}\}$ converging $(\overline{q}, \overline{w})$, let $\left(\ell'_{\ell,h,s}(e), d_{\ell,h,s}(e), c_{\ell,h,s}(e)\right)$ be the decision rules whose existence is guaranteed in Lemma A19. Using $\overline{q}, \overline{w}, \ell'_{\ell,h,s}(e), d_{\ell,h,s}(e), c_{\ell,h,s}(e)$ we will construct a collection

$$\{\overline{q}, \overline{w}, \overline{\ell}'_{\ell,h,s}(e; \overline{q}, \overline{w}), \overline{d}_{\ell,h,s}(e; \overline{q}, \overline{w}), \overline{c}_{\ell,h,s}(e; \overline{q}, \overline{w}), \overline{r}, \overline{i}, \overline{p}, \overline{m}, \overline{N}, \overline{K}, \overline{a}, \overline{B}, \overline{\mu}\}$$

that satisfies all the conditions of steady state equilibrium in Definition 2.

Given $\overline{q}, \overline{w}$, the conditions we satisfy by construction are:

(i) $\overline{c}_{\ell,h,s}(e; \overline{q}, \overline{w}) = c_{\ell,h,s}(e)$, $\overline{\ell}'_{\ell,h,s}(e; \overline{q}, \overline{w}) = \ell'_{\ell,h,s}(e)$, and $\overline{d}_{\ell,h,s}(e; \overline{q}, \overline{w}) = d_{\ell,h,s}(e)$. By Lemma A19 these decision rules solve the household's optimization problem for $\alpha = 0, q = \overline{q}$,and $w = \overline{w}$.

(x) $\overline{\mu} = \mu_{(\overline{q},\overline{w})} = \Upsilon_{(\overline{q},\overline{w})}\mu_{(\overline{q},\overline{w})}$ (where Υ is based on $(\overline{\ell}'_{\ell,h,s}(e; \overline{q}, \overline{w}), \overline{d}_{\ell,h,s}(e; \overline{q}, \overline{w}))$;

(vi) $\overline{N} = \int e d\overline{\mu}$;

(vii) $\overline{a}_{\ell',s} = \int \mathbf{1}_{\{(\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w})=\ell'\}}\mu_{(\overline{q},\overline{w})}(d\ell, dh, s, de)$;

(viii) $\overline{K} = \sum_{(\ell',s)\in L\times S} \overline{q}_{\ell',s}\ell' \int \mathbf{1}_{\{(\overline{\ell}'_{\ell,h,s}(e;\overline{q},\overline{w})=\ell'\}}\mu_{(\overline{q},\overline{w})}(d\ell, dh, s, de)$;

(v) $\overline{m} = \left[\int \left[(1 - \overline{d}_{\ell,h,s}(e; \overline{q}, \overline{w}))\zeta(s) + \overline{d}_{\ell,h,s}(e; \overline{q}, \overline{w}) \max\{\ell, 0\}\right] d\mu_{(\overline{q},\overline{w})}\right]^{-1} \cdot \int \zeta(s)d\mu_{(\overline{q},\overline{w})}$;

(iib) $\overline{r} = \frac{\partial F(\overline{K},\overline{N})}{\partial \overline{K}}$.

(iv) $\overline{p}_{\ell',s} = \int \overline{d}_{\ell',0,s'}(e';\overline{q},\overline{w})\Phi(e'|s')\Gamma(s;ds')de'$ for $\ell' < 0$ and $\overline{p}_{\ell',s} = 0$ for $\ell' \geq 0$.

The conditions we must verify are:

(iia)

$$\overline{w} = \frac{\partial F(\overline{K},\overline{N})}{\partial \overline{N}}.$$

Since $(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})$ are equilibrium prices and $K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})} > 0$ by Lemma A17, then for all n_k :

$$f(w^*_{n_k}, K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}) \equiv w^*_{n_k} - F_N\left(K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right) = 0.$$

Observe that f is continuous in all arguments because F_N is continuous. Therefore

$$\lim_{n_k\to\infty} f(w^*_{n_k}, K_{(\alpha,q^*_{n_k},w^*_{n_k})}, N_{(\alpha,q^*_{n_k},w^*_{n_k})}) = \overline{w} - F_N\left(\overline{K},\overline{N}\right) = 0$$

since by Lemma A22 we know $\lim_{n_k\to\infty}\left(K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right) = \left(K_{(0,\overline{q},\overline{w})}, N_{(0,\overline{q},\overline{w})}\right) = \left(\overline{K},\overline{N}\right)$ by construction.

(iii)

$$\overline{q}_{\ell',s} = \frac{\rho(1-\overline{p}_{\ell',s})}{1+\overline{r}-\delta}.$$

Since $(\alpha_{n_k}, q^*_{n_k}, w^*_{n_k})$ are equilibrium prices and $K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})} > 0$ by Lemma A17, then for all n_k and $\ell' \geq 0$:

$$\begin{aligned}
&f\left(\left(q^*_{\ell'\geq 0,s}\right)_{n_k}, K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)\\
&\equiv \left(q^*_{\ell'\geq 0,s}\right)_{n_k} - \rho\left(1+F_K\left(K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)-\delta\right)^{-1} = 0.
\end{aligned}$$

Observe that f is continuous in all arguments because F_K is continuous. Therefore, by Lemma A22 again

$$\begin{aligned}
&\lim_{n_k\to\infty} f\left(\left(q^*_{\ell'\geq 0,s}\right)_{n_k}, K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)\\
&= \overline{q}_{\ell'\geq 0,s} - \rho\left(1+F_K\left(\overline{K},\overline{N}\right)-\delta\right)^{-1} = 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&f\left(\left(q^*_{\ell'<0,s}\right)_{n_k}, K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)\\
&\equiv \left(q^*_{\ell'<0,s}\right)_{n_k} - \frac{\rho\left(1-\int d^*_{\ell',0,s'}(e',\alpha_{n_k},q^*_{n_k},w^*_{n_k})\Phi(de'|s')\Gamma(s;ds')\right)}{1+F_K\left(K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}, N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)-\delta} = 0.
\end{aligned}$$

By the choice of $\overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})$ and Lemma A19,

$$\lim_{n_k\to\infty}\int d^*_{\ell',0,s'}(e',\alpha_{n_k},q^*_{n_k},w^*_{n_k})\Phi(de'|s')=\int \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})\Phi(de'|s').$$

Therefore by Lemma A22

$$\lim_{n_k\to\infty} f\left(\left(q^*_{\ell'<0,s}\right)_{n_k},K_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})},N_{(\alpha_{n_k},q^*_{n_k},w^*_{n_k})}\right)$$
$$=\overline{q}_{\ell'<0,s}-\frac{\rho\left(1-\int \overline{d}_{\ell,h,s}(e;\overline{q},\overline{w})\Phi(de'|s')\Gamma(s;ds')\right)}{1+F_K\left(\overline{K},\overline{N}\right)-\delta}=0.$$

Finally, since the collection satisfies all conditions for an equilibrium except condition (ix), it follows from Lemma A7 that (ix) is satisfied as well:

$$\int \overline{c}_{\ell,h,s}(e;\overline{q},\overline{w})d\overline{\mu}+\int\frac{\zeta(s)}{\overline{m}}d\overline{\mu}+\delta\overline{K}=F(\overline{K},\overline{N})-\gamma\overline{w}\int e\overline{\mu}(d\ell,1,ds,de).$$

■

Theorem 6 (Characterization of Equilibrium Prices) In any steady state competitive equilibrium: (i) $q^*_{\ell',s}=\rho(1+r^*-\delta)^{-1}$ for $\ell'\geq 0$; (ii) if the grid for L is sufficiently fine, there exists $\ell^0<0$ such that $q^*_{\ell^0,s}=\rho(1+r^*-\delta)^{-1}$; (iii) if the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, $0>\ell^1>\ell^2$ implies $q^*_{\ell^1,s}\geq q^*_{\ell^2,s}$; (iv) when $\ell_{\min}\leq-\left[e_{\max}\cdot w_{\max}\right]\left[(1+r^*-\delta)/(1-\rho+r^*-\delta)\right]$, $q^*_{\ell_{\min},s}=0$.

Proof. (i) Follows from condition (iii) in the definition of competitive equilibrium; (ii) Let the grid be fine enough so that there is at least one $\ell^0<0$ for which $w_{\min}\cdot e_{\min}+\ell^0>0$. For a household, the utility from defaulting on a loan of size ℓ^0 can be expressed as:

$$u(e\cdot w,s)\ +\ \beta\rho\int u\left(c^*_{0,1,s'}(e';q^*,w^*),s'\right)\Phi(de'|s')\Gamma(s,ds')$$
$$+\ (\beta\rho)^2\int\left[\lambda\omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),1,s'}(q^*,w^*)+(1-\lambda)\omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),0,s'}(q^*,w^*)\right]\Phi(de'|s')\Gamma(s,ds')$$

Since $w_{\min}\cdot e_{\min}+\ell^0>0$,an alternative to not defaulting is to pay off the loan, consume the remaining endowment, and in the following period set consumption equal to

$$c^*_{0,1,s'}(e';q^*,w^*)+\gamma e'.$$

The utility from this course of action is:

$$u(e\cdot w+\ell_0,s)\ +\ \beta\rho\int u\left(c^*_{0,1,s'}(e';q^*,w^*)+\gamma e',s'\right)\Phi(de'|s')\Gamma(s,ds')$$
$$+\ (\beta\rho)^2\int\omega^*_{\ell'^*_{0,1,s'}(e';q^*,w^*),0,s'}(q^*,w^*)\Phi(de'|s')\Gamma(s,ds').$$

In view of (29), the utility-gain from not defaulting must be at least as large as

$$u(e \cdot w + \ell_0, s) - u(e \cdot w, s)$$
$$+ \beta\rho \int \left[u\left(c^*_{0,1,s'}(e'; q^*, w^*) + \gamma e', s'\right) - u\left(c^*_{0,1,s'}(e'; q^*, w^*), s'\right)\right] \Phi(de'|s')\Gamma(s, ds'). \quad (51)$$

Since consumption is bounded above by $e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}$ and the $u(\cdot, s)$ is strictly concave for each s , the integral in the above expression is bounded *below* by

$$\int [u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min} + \gamma e', s') - u(e_{\max} \cdot w_{\max} + \ell_{\max} - \ell_{\min}, s')]\Phi(de'|s')\Gamma(s, ds').$$

Notice that the above integral is strictly positive and independent of the fineness of the grid for L. Therefore, since $u(\cdot, s)$ is continuous, the expression in 51 will be strictly positive if $\ell^0 < 0$ is sufficiently close to zero. Hence, for a sufficiently fine grid there exists an $\ell^0 < 0$ for which defaulting is not optimal and $q^*_{\ell^0,s} = \rho(1 + r^* - \delta)^{-1}$; (iii) If the set of efficiency levels for which a household is indifferent between defaulting and not defaulting is of measure zero, by Theorem 4 (the default set expands with liabilities) it follows that

$$d^*_{\ell^2,0,s}(e, q^*, w^*) \geq d^*_{\ell^1,0,s}(e, q^*, w^*)$$

for all e except, possibly, for those in a set of $\Phi(e|s)$-measure zero. Therefore

$$\int d^*_{\ell^2,0,s}(e, q^*, w^*)\Phi(de|s) = p_{\ell^2,s} \geq p_{\ell^1,s} = \int d^*_{\ell^1,0,s}(e, q^*, w^*)\Phi(de|s)$$

and the result follows; (iv) Set $\ell_{\min} \leq -[e_{\max} \cdot w_{\max}]\,[(1 + r^* - \delta)/(1 - \rho + r^* - \delta)]$. If a household has characteristics s, loan $\ell_{\min}$ and endowment $e \cdot w$ then its consumption, conditional on not defaulting, is bounded above by $e \cdot w + \ell_{\min} - \zeta(s) + \max_{\ell' \in L}\{-q^*_{\ell',s} \cdot \ell'\}$. Since $e \cdot w \leq e_{\max} \cdot w_{\max}$, $-\zeta(s) \leq 0$, and $\max_{\ell' \in L}\{-q_{\ell',s} \cdot \ell'\} \leq -\rho/(1 + r^* - \delta) \cdot \ell_{\min}$, consumption conditional on not defaulting is bounded above by $e_{\max} \cdot w_{\max} + \ell_{\min} - \rho/(1 + r^* - \delta) \cdot \ell_{\min} \leq 0$. This means either that the set $B_{\ell_{\min},0,\eta,0}(e, q)$ is either empty or that the only feasible consumption is zero consumption. In the first case default is the only option and in the second case it's the best option by (30). Therefore in any competitive equilibrium $q^*_{\ell_{\min},s}$ must be zero.

■

B Reduction in the Punishment Period from 10 Years to 5 Years

In this appendix we report the welfare consequences of a policy experiment in which the average number of years a person is excluded from borrowing after filing for bankruptcy is reduced from 10 years to 5.

Table 16 reports the results for the baseline model. For comparison purposes, the results for the "means-testing" policy reform is also reported. Not surprisingly, the policy change increases the welfare of households that are currently shut out of the credit market. The gain is significant – a little under 0.8% in flow consumption and *all* such households support this policy. In contrast, households who have a good credit record lose from this policy, on average. The loss is small. However, opposition to this policy is not uniform – some indebted households support it.

Table 16: Welfare Comparisons for the Baseline Model for a Shorter Exclusion Period

	5-year exclusion	Means-testing
Average % gain in flow consumption		
With bad credit record	0.78%	0.76%
With good credit record and debt	-0.37%	7.79%
With good credit record and no debt	-0.06%	1.38%
Total	-0.05%	1.67%
% of households in favor of reform		
With bad credit record	100.0%	100.0%
With good credit record and debt	8.4%	100.0%
With good credit record and no debt	0.0%	100.0%
Total	2.7%	100.0%

The policy has the effect of reducing the costs of bankruptcy and therefore encourages it. Consequently, the interest rate on loans is higher in equilibrium. Higher interest rates is the reason why all households that are not currently in debt lose from this policy – they all anticipate borrowing sometime in the future at higher interest rates. For households that are already in debt the policy raises the debt burden. But some of these indebted households (those with a large debt) anticipate declaring bankruptcy in the future with high probability and therefore benefit from the lower exclusion period. For most borrowers, however, the higher cost of borrowing imposes a loss and they do not support the policy.

Table 17 summarizes the welfare effects for the other economies. Overall there is a small loss in terms of the flow consumption across all economies. The average welfare loss is 0.05% in the EP economy, 0.01% in the EL economy, and 0.02% in the EPL economy. The difference across different

Table 17: Welfare Comparisons for Other Economies of a Shorter Exclusion Period

Shock	Preference Shock		Liability Shock		Total
	Yes	No	Yes	No	
EP Economy					
Proportion of households	0.160	0.840			1.000
Average % gain in flow consumption					
With bad credit record	0.67%	0.79%			0.77%
With good credit record and debt	-0.67%	-0.30%			-0.34%
With good credit record and no debt	-0.09%	-0.05%			-0.06%
Total	-0.08%	-0.04%			-0.05%
% of agents in favor of reform					
With bad credit record	100.0%	100.0%			100.0%
With good credit record and debt	0.0%	14.0%			12.6%
With good credit record and no debt	0.0%	0.0%			0.0%
Total	3.0%	3.7%			3.5%
EL economy					
Proportion of households			0.012	0.988	1.000
Average % gain in flow consumption					
With bad credit record			1.21%	1.37%	1.36%
With good credit record and debt			1.30%	-0.52%	-0.50%
With good credit record and no debt			0.28%	-0.04%	-0.03%
Total			0.37%	-0.01%	-0.01%
% of agents in favor of reform					
With bad credit record			100.0%	100.0%	100.0%
Non-delinquent but in debt			96.7%	3.8%	4.9%
Non-delinquent and not in debt			43.2%	14.1%	14.4%
Total			48.1%	16.7%	17.1%
EPL economy					
Proportion of households	0.160	0.840	0.012	0.988	1.000
Average % gain in flow consumption					
With bad credit record	0.91%	1.15%	0.99%	1.11%	1.11%
With good credit record and debt	-0.84%	-0.31%	1.14%	-0.39%	-0.37%
With good credit record and no debt	-0.09%	-0.04%	0.15%	-0.05%	-0.05%
Total	-0.07%	-0.01%	0.24%	-0.02%	-0.02%
% of agents in favor of reform					
With bad credit record	100.0%	100.0%	100.0%	100.0%	100.0%
With good credit record and debt	1.2%	11.5%	96.9%	9.4%	10.4%
With good credit record and no debt	7.7%	11.8%	35.2%	10.8%	11.1%
Total	11.5%	15.6%	41.1%	14.6%	14.9%

groups is also similar. The households that are currently shut out of the credit market are the only households who gain uniformly – and gain substantially. All the other households, on average, lose from the reform. Among them, indebted households, on average, lose more than others.

Opposition to the reform is not uniform. As in the baseline model some households gain from a reduction in the exclusion period. In the EP economy 3.5% of households support the reform, similar to the proportion for the baseline economy (2.7%). In contrast, 17% of households support the reform in the EL economy and 15% support it in the EPL economy. In the EL and EPL economies more than 10% of households without debt support the reform, while none of these households support the reform in the baseline and the EP economies. This difference comes about because in the presence of liability shocks, households might be forced to file for bankruptcy in the next period even if they have no debt in the current period. Households without debt who support the reform are blue-collar households with relatively small asset holdings.