Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity

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ABSTRACT

We study the cyclical implications of credit market imperfections in a dynamic, stochastic general equilibrium model wherein firms face persistent shocks to both aggregate and individual productivity. In our model economy, optimal capital reallocation is distorted by two frictions. First, collateralized borrowing constraints limit the investment undertaken by small firms with relatively high productivities. Second, specificity in firm-level capital implies partial investment irreversibilities that lead firms to pursue (S,s) decision rules. This second friction compounds the first in implying that large and relatively unproductive firms carry a disproportionate share of the aggregate capital stock, thereby reducing endogenous aggregate total factor productivity. Moreover, because irreversibilities induce inertia in firm-level capital adjustment, they ensure that the effect of a temporary tightening in financial markets is not quickly reversed.

In the presence of persistent heterogeneity in both capital and total factor productivity, the effects of a financial shock can be amplified and propagated through large and long-lived disruptions to the distribution of capital that, in turn, imply large and persistent reductions in aggregate total factor productivity. This paper seeks to measure the strength of these effects in a calibrated DSGE setting. We find that an unanticipated tightening in borrowing conditions can, on its own, generate a large recession that is far more persistent than the financial shock itself, and the recovery that follows is led by rises in business fixed investment, rather than in household consumption spending.

Keywords: Financial frictions, irreversibilities, (S,s) policies, business cycles

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1 Introduction

Can a large shock to an economy's financial sector produce a large and lasting recession? Can it amplify and propagate the effects of a real shock sufficiently to transform recession into depression? Over the past few years, events in the real and financial sectors of the U.S. and other large, developed economies have been difficult to disentangle. If these conditions have reawakened interest in business cycle research, they have also raised concerns about our existing macroeconomic models' ability to address such topics.

In this paper, we develop a quantitative, dynamic, stochastic general equilibrium model to explore how real and financial shocks interact in determining the size and frequency of aggregate fluctuations. In our model, firms experience persistent shocks to both aggregate and individual productivity, while credit market frictions interact with real frictions to yield persistent disruptions to the efficient allocation of capital across them, and thus persistent reductions in endogenous aggregate productivity. Calibrating our model to aggregate and firm-level data, we use it as a laboratory in which to obtain answers to the questions raised above.

Considering the matter from the perspective of a representative agent model, one might expect that the reductions in aggregate capital implied by a temporary tightening in credit markets could not yield sizeable or long-lived real aggregate effects, since investment is a small fraction of GDP. However, disaggregated data reveals that there is substantial heterogeneity across firms in their individual productivity levels, and that there are real frictions limiting the reallocation of capital across them.[1] Indeed, these elements are essential to understanding microeconomic investment patterns. In light of the first fact, a reduction in credit may sharply reduce aggregate total factor productivity by distorting the allocation of production away from the efficient one, placing too much capital in large, relatively unproductive firms at the expense of smaller firms with higher productivities.[2] To the extent that real frictions slow the reversal of such an allocative disruption, the second fact will compound the first, propagating shocks to the provision of credit.

As mentioned, capital reallocation is distorted by two frictions in our model, one financial

[1] For direct evidence of large and increasing heterogeneity in firm-level productivity, see Comin and Philippon (2005) and the empirical studies cited therein. Elsewhere, Cooper and Haltiwanger (2006) find it is impossible to reproduce microeconomic investment patterns without both large idiosyncratic shocks and adjustment costs limiting capital reallocation.

[2] Restuccia and Rogerson (2008) show that this endogenous TFP effect is an important component in explaining cross-country per-capita GDP differences.

and one real. First, collateralized borrowing constraints limit the investment undertaken by small firms with relatively high productivities. Second, specificity in capital implies partial investment irreversibilities that lead firms to pursue (S,s) rules with respect to their capital adjustments. The second friction further tilts the distribution of production towards larger, less productive firms, further reducing endogenous aggregate total factor productivity. It also exacerbates the direct effects of collateral constraints by reducing the collateral value ascribed to each unit of installed capital. This added element of realism in our setting relative to existing DSGE financial frictions models may be quite important to the transmission and propagation of a financial shock, as we discuss below.

Because specificity in capital induces both downward and upward inertia in firm-level investment activities, and because it tightens the borrowing limits implied by collateralized lending, it ensures that the negative consequences of a temporary tightening in financial markets are not quickly reversed. Given persistent heterogeneity in both capital and total factor productivity, the effects of financial frictions are amplified and propagated through long-lived disruptions to the distribution of capital that, in turn, imply persistent reductions in aggregate productivity. For example, in the presence of only a 5 percent capital irreversibility, we find that steady state output falls by 4 percent when collateralized borrowing limits are introduced. This suggests the potential for large output losses in our model economy following a financial shock, since the long-run GDP reduction in response to a change in borrowing constraints fails to capture sharp transitional reductions associated with reallocation following the shock.

Our primary question in this study is whether a temporary crisis in financial markets can generate a large and persistent drop in aggregate productivity by disrupting the distribution of capital further from that implied by firms' relative productivities, thereby further distorting the distribution of production. We are to our knowledge the first to explore this endogenous TFP channel in a quantitative DSGE setting where real frictions slow the reallocation of capital across firms, and where that reallocation is essential in determining the marginal product of the aggregate stock. In keeping with previous results in the literature, we find that aggregate responses to real shocks are largely unaffected by the presence of financial frictions. However, changes in the distribution of capital can have large and long-lived effects in our model economy. As a result, an unanticipated disruption to the availability of credit can, on its own, generate a large and protracted recession.

We also find that the response to a credit shock is qualitatively different from that following a real shock, both at its impact and in the recovery episode. Unlike the response to a productivity shock, the greatest declines in output, employment and investment do not occur at the onset of a credit crisis, and consumption does not fall immediately. Moreover, once credit conditions return to normal, our model predicts the subsequent recovery will be slow, and it will be led by employment and business fixed investment, rather than household consumption spending. Consumption reverts to its trend far more gradually than does GDP, which further distinguishes the response from that following a real shock.

The remainder of the paper is organized as follows. Section 2 briefly summarizes the literature most closely related to our work. Next, in section 3, we present our model economy. Section 4 provides some analysis useful in developing a numerical algorithm capable of its solution. In section 5, we describe our calibration to moments drawn from postwar U.S. aggregate and firm-level data. Section 6 explores the mechanics of our model in its deterministic steady state and draws some comparisons to the mechanics in a reference model with capital specificity but no financial frictions. Section 7 presents business cycle results, and section 8 concludes.

2 Related literature

To date, there has been little quantitative research examining the channels through which changes in the availability of credit influence macroeconomic series like business investment, employment and production in well-articulated dynamic stochastic general equilibrium settings. There is a large related literature exploring how financial frictions influence the aggregate response to non-financial shocks. Leading this literature, Kiyotaki and Moore (1997) develop a model of credit cycles and show that collateral constraints can have a large role in amplifying and propagating shocks to the value of collateral.[3] Our own work follows in the spirit of Kiyotaki and Moore in that the financial frictions we explore are collateralized borrowing constraints. We adopt this approach in part because collateral appears to have an important role in loan contracts and in part for computational tractability in our heterogeneous firm DSGE setting.

While we assume that firms face collateral constraints, there are well-known alternative approaches. Cooley, Marimon and Quadrini (2004) study constrained-optimal dynamic contracts

[3] Cordoba and Ripoll (2004) and Kocherlakota (2000) argue that these effects are quantitatively minor in calibrated versions of the model.

under limited enforceability. Elsewhere, a large literature examines agency costs as the source of financial frictions.[4] However, these papers do not consider financial shocks as such. Moreover, they abstract from potentially important heterogeneity across firms under which the allocation of capital, and thus credit, becomes relevant.

Over the past few years, several recent studies have begun exploring how financial shocks affect aggregate fluctuations. A leading example is Jermann and Quadrini (2009a), which examines a representative firm model wherein investment is financed using both debt and equity, while costs of adjusting dividends prevent the avoidance of financial frictions. These frictions stem from limited enforceability of intra-temporal debt contracts, which gives rise to endogenous borrowing limits. Specifically, the firm retains its working capital under default, but the lender is able to recover a fraction of the firm's future value. Shocks to the fraction that the lender can confiscate alter the severity of borrowing limits. Measuring these credit shocks, Jermann and Quadrini find that they have been an important source of business cycles.[5] In contrast to the Jermann-Quandrini model, the financial frictions in our setting do not significantly alter the response of the aggregate economy to non-financial shocks. Further, while directly imposing a collateral constraint, we also introduce real frictions in the form of capital specificity.[6] This hinders the reallocation of capital, and makes more gradual the evolution of our nontrivial distribution of capital across firms, thereby protracting the real effects of credit shocks.

Our emphasis on firm-level productivity dispersion is shared by Arellano, Bai and Kehoe (2010), who examine the role of uncertainty shocks in a model with non-contingent debt and equilibrium default. Gomes and Schmid (2009) also develop a model with endogenous default, where firms vary with respect to their leverage, and study the implication for credit spreads.[7] In contrast to these papers, we study ongoing firm-level investment and the aggregate response to credit shocks. We find that credit shocks can generate recessions through changes in aggregate

[4] See Bernanke and Gertler (1989), Carlstrom and Fuerst (1997), and Bernanke, Gertler and Gilchrist (1999).

[5] Jermann and Quadrini (2009b) adapt this model to address the evolving variability of real and financial variables in the past 25 years. In a related setting, Christiano, Motto and Rostagno (2009) study a New Keynesian model with lending subject to agency costs; they too find that financial shocks are an important source of economic fluctuations.

[6] See Veracierto (2002) for a DSGE analysis of how these frictions affect aggregate responses to productivity shocks. Caggese (2007) considers both irreversible capital and collateral constraints; our study is distinguished from his by general equilibrium analysis, partial reversibility in investment, and frictionless within-period borrowing.

[7] Credit spreads are also emphasized by Gertler and Kiyotaki (2010); they study a model where such spreads are driven by agency problems arising with financial intermediaries.

TFP that, in turn, have sharp implications for investment and employment. In emphasizing the endogenous TFP channel, our study is also related to Buera and Shin (2007). They examine the effect of collateral constraints on economic development and show that these frictions can protract the transition to the balanced growth path if capital is initially misallocated.

3 Model

In our model economy, firms face both partial capital fixity and collateralized borrowing limits, which together compound the effects of persistent differences in their total factor productivities to yield substantial heterogeneity in production. We begin our description of the economy with an initial look at the optimization problem facing each firm, then follow with a brief discussion of households and equilibrium. Next, in section 4, we will use a simple implication of equilibrium alongside some immediate observations about firms' optimal allocation of profits across dividends and retained earnings, to characterize the capital adjustment decisions of our firms. This analysis will show how it is possible for us to derive a convenient, computationally tractable algorithm to solve for equilibrium allocations in our model, despite its three-dimensional heterogeneity in production.

3.1 Production, credit and capital adjustment

We assume a large number of firms, each producing a homogenous output using predetermined capital stock k and labor n, via an increasing and concave production function, $y = z\varepsilon F(k, n)$. The variable z represents exogenous stochastic total factor productivity common across firms, while ε is a firm-specific counterpart. For convenience, we assume that ε is a Markov chain, $\varepsilon \in \mathbf{E} \equiv \{\varepsilon_1, \ldots, \varepsilon_{N_\varepsilon}\}$, where $\Pr(\varepsilon' = \varepsilon_j \mid \varepsilon = \varepsilon_i) \equiv \pi_{ij} \geq 0$, and $\sum_{j=1}^{N_\varepsilon} \pi_{ij} = 1$ for each $i = 1, \ldots, N_\varepsilon$. Similarly, $z \in \{z_1, \ldots, z_{N_z}\}$, where $\Pr(z' = z_m \mid z = z_l) \equiv \pi^z_{lm} \geq 0$, and $\sum_{m=1}^{N_\varepsilon} \pi^z_{lm} = 1$ for each $l = 1, \ldots, N_z$.

Because our interest is in understanding how financial constraints interact with the specificity of capital in shaping the investment decisions taken by firms in our economy, we must prevent firms growing so large that none will never again experience a binding borrowing limit. To ensure this does not occur, we impose exit and entry in the model. In particular, we assume that each firm faces a fixed probability, $\pi_d \in (0, 1)$, that it will be forced to exit the economy following production in any given period. Within a period, prior to investment, firms learn whether they

will survive to produce in the next period. Exiting firms are replaced by an equal number of new firms whose initial state will be described below.

At the beginning of each period, a firm is defined by its predetermined stock of capital, $k \in \mathbf{K} \subset \mathbf{R}_+$, by the level of one-period debt it incurred in the previous period, $b \in \mathbf{B} \subset \mathbf{R}$, and by its current idiosyncratic productivity level, $\varepsilon \in \{\varepsilon_1, \ldots, \varepsilon_{N_e}\}$. Immediately thereafter, the firm learns whether it will survive to produce in the next period.[8] Given this individual state, and having observed the current aggregate state, the firm then takes a series of actions to maximize the expected discounted value of the current and future dividends returned to its shareholders, the households in our economy. First, it chooses its current level of employment, undertakes production, and pays its wage bill. Thereafter, it repays its existing debt and, conditional on survival, it chooses its investment, i, current dividends, and the level of debt with which it will enter into the next period, b'. For each unit of debt it incurs for the next period, a firm receives q units of output that it can use toward paying current dividends or investing in its future capital. The relative price q^{-1} reflecting the interest rate at which firms can borrow and lend is a function of the economy's aggregate state, as is the wage rate ω paid to workers. For expositional convenience, we suppress the arguments of these equilibrium price functions until we have described the model further.

In contrast to the typical setting with firm-level capital adjustment frictions, and unlike a typical environment with financial frictions, real and financial frictions are allowed to interact in our model economy. Our firms' borrowing and investment decisions are inter-related, because each firm faces a collateralized borrowing constraint inside of any period. This constraint takes the form: $b' \leq \Theta k$. Two external forces together determine what fraction of its capital stock a firm can borrow against - the degree of specificity in capital and enforceability of financial arrangements. Here, we simply impose both, deferring the question of their foundations for a future study. In particular, we assume that $\Theta = \theta_b \theta_k$, where $\theta_k \in [0, 1]$ is a parameter determining what fraction of a firm's capital stock survives when it is uninstalled and moved to another firm, and $\theta_b \in \mathbf{R}_+$ is the fraction of that collateral firms can borrow against.[9] A financial shock in our model is

[8] We have adopted this timing to ensure there is no equilibrium default in our model, so that all firms borrow at a common real interest rate. Because the only firms borrowing are those that will produce in the next period, and the debt they take on is limited by a collateral constraint, firms are always able to repay their debt in the quantatitative exercises to follow.

[9] Throughout our numerical exercises in section 6, we assume that the degree of capital irreversibility, $1 - \theta_k$, is a fixed technological parameter. In ordinary times when aggregate fluctuations arise from changes in productivity

represented by an unanticipated change in the collateral term, θ_b.

If firm undertakes any nonnegative level of investment, then its capital stock at the start of the next period is determined by a familiar accumulation equation,

$$k' = (1-\delta)\,k + i \quad \text{for } i \geq 0,$$

where $\delta \in (0,1)$ is the rate of capital depreciation, and primes indicate one-period-ahead values. Because there is some degree of specificity in capital, the same equation does not apply when the firm undertakes negative investment. In this case, the effective relative price of investment is θ_k rather than 1, so the accumulation equation is instead:

$$\theta_k k' = \theta_k\,(1-\delta)\,k + i \quad \text{for } i < 0.$$

In the analysis section to follow, we will show how the asymmetry that firms face in the cost of capital adjustment naturally gives rise to two-sided (S,s) investment decision rules. In contrast to a nonconvexity in the capital adjustment technology, this type of adjustment friction implies not only investment inaction among firms within their (S,s) adjustment bands, but also some inertia among firms outside of their (S,s) bands. Because there are no increasing returns in the adjustment technology, but instead a linear penalty for negative adjustments, a firm finding itself with an intolerably high capital stock (given its current productivity), will reduce its stock only to the upper bound of its (S,s) inactivity range. Similarly, a firm with too little capital recognizes that it will incur a linear penalty should it later need to shed capital, so it invests only to the lower bound of its inactivity range.

It should be clear from the discussion above that, alongside its current productivity draw, a firm's capital adjustment may also be influenced by its ability to borrow (now and in the future). This is in turn affected by the capital (collateral) it currently holds. Note also that the firm's current investment decision may influence the level of debt it carries into the next period. These observations imply that we must monitor the distinguishing features of firms along three dimensions: their capital, k, their debt, b, and their idiosyncratic productivity, ε.

We summarize the distribution of firms over (k,b,ε) using the probability measure μ defined on the Borel algebra, $\mathcal{S}$, for the product space $\mathbf{S} = \mathbf{K} \times \mathbf{B} \times \mathbf{E}$. The aggregate state of the economy is then described by (z,μ), and the distribution of firms evolves over time according to a

alone, θ_b is also a fixed parameter. However, we allow for an unanticipated change in θ_b when we consider the aggregate implications of a financial shock that lowers the confidence of lenders.

mapping, Γ, from the current aggregate state; $\mu' = \Gamma(z, \mu)$. The evolution of the firm distribution is determined in part by the actions of continuing firms and in part by entry and exit. Following production in each period, fraction π_d of existing firms exit the economy. These firms invest negatively to shed their remaining capital, returning the proceeds to households, and are replaced by the same number of new firms. Each new firm has zero debt and productivity $\varepsilon_0 \in \mathbf{E}$ drawn from an initial distribution $H(\varepsilon_0)$, and each enters with an initial capital stock $k_0 \in \mathbf{K}$.[10]

We are now in a position to set out the optimization problem solved by each firm in our economy. Let $v_0(k, b, \varepsilon_i; z_l, \mu)$ represent the expected discounted value of a firm that enters the period with (k, b) and firm-specific productivity ε_i, when the aggregate state of the economy is (z_l, μ), just before it learns whether it will survive into the next period. We state the firm's dynamic optimization problem using a functional equation defined by (1) - (4) below.

$$\begin{aligned} v_0(k, b, \varepsilon_i; z_l, \mu) = \pi_d \max_n [z_l \varepsilon_i F(k, n) - \omega(z_l, \mu) n + \theta_k (1 - \delta) k - b] \\ + (1 - \pi_d) v(k, b, \varepsilon_i; z_l, \mu) \end{aligned} \tag{1}$$

After the start of the period, the firm knows which line of (1) will prevail. If it is not continuing beyond the period, the firm simply chooses labor to maximize its current dividend payment to shareholders. Because it will carry no capital or debt into the future, an exiting firm's dividends are its output, less wage payments and debt repayment, together with the remaining capital it can successfully uninstall at the end of the period. The problem conditional on continuation is more involved, because a continuing firm must choose its current labor and dividends alongside its future capital and debt. For expositional convenience, given the partial irreversibility in investment, we begin to describe this problem by defining the firm's value as the result of a binary choice between upward versus downward capital adjustment in (2), then proceed to identify the value associated with each option in (3) and (4).[11]

$$v(k, b, \varepsilon_i; z_l, \mu) = \max\left\{ v^u(k, b, \varepsilon_i; z_l, \mu),\ v^d(k, b, \varepsilon_i; z_l, \mu) \right\} \tag{2}$$

Assume that $d_m(z_l, \mu)$ is the discount factor applied by firms to their next-period expected value if aggregate productivity at that time is z_m and the current aggregate state is (z_l, μ). Taking

[10] We select k_0 below so that each entrant's capital is χ fraction of the typical stock held across all firms in the long-run of our economy.

[11] We could instead describe the firm's problem without the binary max operator by adopting an indicator function determining the relative price of capital as 1 in the event of $k' \geq (1-\delta)k$ and θ_k otherwise. Here, for sake of clarity, we opt for the less concise representation, though we will abandon it at some points below.

as given the evolution of ε and z according to the transition probabilities specified above, and taking as given the the evolution of the firm distribution, $\mu' = \Gamma(z, \mu)$, the firm solves the following two optimization problems to determine its values conditional on (weakly) positive and negative capital adjustment. (Here forward, except where necessary for clarity, we suppress the indices for current aggregate and firm productivity.) In each case, the firm selects its current employment and production, alongside the debt and capital with which it will enter into next period and its current dividends, D, to maximize its expected discounted dividends. As above, dividends are determined by the firm's budget constraint as the residual of its current production and borrowing after its wage bill and debt repayment have been covered, net of its investment expenditures.

Conditional on an upward capital adjustment, the firm solves the following problem constrained by (i) the fact that investment must be non-negative, (ii)-(iii) the requirements that dividends be non-negative and satisfy the firm's budget constraint and (iv) a borrowing limit determined by its collateral.

$$v^u(k, b, \varepsilon_i; z_l, \mu) = \max_{n,k',b',D} \Big[D + \sum_{m=1}^{N_z} \pi^z_{lm} d_m(z_l, \mu) \sum_{j=1}^{N_\varepsilon} \pi_{ij} v_0\left(k', b', \varepsilon_j; z_m, \mu'\right) \Big] \tag{3}$$

$$\text{subject to: } k' \geq (1-\delta)k, \;\; b' \leq \Theta k, \;\; \text{and}$$

$$0 \leq D \leq z_l \varepsilon_i F(k, n) - \omega(z_l, \mu) n + q(z_l, \mu) b' - b - [k' - (1-\delta)k]$$

The downward adjustment problem differs from that above only in that investment must be non-positive and, thus, its relative price is θ_k.

$$v^d(k, b, \varepsilon_i; z_l, \mu) = \max_{n,k',b',D} \Big[D + \sum_{m=1}^{N_z} \pi^z_{lm} d_m(z_l, \mu) \sum_{j=1}^{N_\varepsilon} \pi_{ij} v_0\left(k', b', \varepsilon_j; z_m, \mu'\right) \Big] \tag{4}$$

$$\text{subject to: } k' \leq (1-\delta)k, \;\; b' \leq \Theta k, \text{ and}$$

$$0 \leq D \leq z_l \varepsilon_i F(k, n) - \omega(z_l, \mu) n + q(z_l, \mu) b' - b - \theta_k [k' - (1-\delta)k]$$

Notice that there is no friction associated with the firm's employment choice, since the firm pays its current wage bill after production takes place, and its capital choice for next period also has no implications for current production. Thus, irrespective of their current debt or their continuation into the next period, all firms sharing in common the same (k, ε) combination select the same employment, which we will denote by $N(k, \varepsilon; z, \mu)$, and hence have common production, $y(k, \varepsilon; z, \mu)$. The same cannot be said for the intertemporal decisions of continuing firms, given

the presence of both borrowing limits and irreversibilities. Let $K(k, b, \varepsilon; z, \mu)$ and $B(k, b, \varepsilon; z, \mu)$ represent the choices of next-period capital and debt, respectively, made by firms sharing in common a complete individual type (k, b, ε). We will characterize these decision rules below in section 4.

3.2 Households

The economy is populated by a unit measure of identical households. Household wealth is held as one-period shares in firms, which we denote using the measure λ.[12] Given the prices they receive for their current shares, $\rho_0(k, b, \varepsilon; z, \mu)$, and the real wage they receive for their labor effort, $\omega(z, \mu)$, households determine their current consumption, c, hours worked, n^h, as well as the numbers of new shares, $\lambda'(k', b', \varepsilon')$, to purchase at prices $\rho_1(k', b', \varepsilon'; z, \mu)$. The lifetime expected utility maximization problem of the representative household is listed below.

$$V^h(\lambda; z, \mu) = \max_{c, n^h, \lambda'} \left[U\left(c, 1 - n^h\right) + \beta \sum_{m=1}^{N_z} \pi_{lm}^z V^h\left(\lambda'; z_m, \mu'\right) \right] \tag{5}$$

subject to

$$c + \int_{\mathbf{S}} \rho_1\left(k', b', \varepsilon'; z, \mu\right) \lambda'\left(d\left[k' \times b' \times \varepsilon'\right]\right) \leq \omega(z, \mu) n^h + \int_{\mathbf{S}} \rho_0(k, b, \varepsilon; z, \mu) \lambda\left(d\left[\varepsilon \times k\right]\right).$$

Let $C^h(\lambda; z, \mu)$ describe the household choice of current consumption, and let $N^h(\lambda; z, \mu)$ be the allocation of current available time to working. Finally, let $\Lambda^h(k', b', \varepsilon', \lambda; z, \mu)$ be the quantity of shares purchased in firms that will begin the next period with k' units of capital, b' units of debt, and idiosyncratic productivity ε'.

3.3 Recursive equilibrium

A *recursive competitive equilibrium* is a set of functions,

$$\left(\omega, q, (d_j)_{j=1}^{N_z}, \rho_0, \rho_1, v_0, N, K, B, V^h, C^h, N^h, \Lambda^h\right),$$

that solve firm and household problems and clear the markets for assets, labor and output, as described by the following conditions.

[12] Households also have access to a complete set of state-contingent claims. However, as there is no heterogeneity across households, these assets are in zero net supply in equilibrium. Thus, for sake of brevity, we do not explicitly model them here.

(i) v_0 solves (1) - (4), N is the associated policy function for exiting firms, and (N, K, B) are the associated policy functions for continuing firms

(ii) V^h solves (5), and $\left(C^h, N^h, \Lambda^h\right)$ are the associated policy functions for households

(iii) $\Lambda^h\left(k', b', \varepsilon_j, \mu; z, \mu\right) = \mu'\left(k', b', \varepsilon_j; z, \mu\right)$, for each $\left(k', b', \varepsilon_j\right) \in \mathbf{S}$

(iv) $N^h\left(\mu; z, \mu\right) = \int\limits_{\mathbf{S}} \Big[N\left(k, \varepsilon; z, \mu\right)\Big] \mu(d\left[k \times b \times \varepsilon\right])$

(v) $C^h\left(\mu; z, \mu\right) = \int\limits_{\mathbf{S}} \Big[z\varepsilon F\left(k, N\left(\varepsilon, k; z, \mu\right)\right) - (1-\pi_d)\mathcal{J}\Big(K\left(k, b, \varepsilon; z, \mu\right) - (1-\delta)\, k\Big)\Big(K\left(k, b, \varepsilon; z, \mu\right)$

$$- (1-\delta)\, k\Big) + \pi_d[\theta_k(1-\delta)k - k_0]\Big] \mu(d\left[k \times b \times \varepsilon\right]), \text{ where } \mathcal{J}(x) = \begin{cases} 1 \text{ if } x \geq 0 \\ \theta_k \text{ if } x < 0 \end{cases}$$

(vi) $\mu'\left(D, \varepsilon_j\right) = (1-\pi_d) \int\limits_{\{(k,b,\varepsilon_i) \mid (K(k,b,\varepsilon_i;z,\mu), B(k,b,\varepsilon_i;z,\mu)) \in D\}} \pi_{ij} \mu(d\left[k \times b \times \varepsilon_i\right]) + \pi_d \chi(k_0) H(\varepsilon_j)$, for all $\left(D, \varepsilon_j\right) \in \mathcal{S}$, defines Γ, where $\chi(k_0) = \{1 \text{ if } (k_0, 0) \in D;\ 0 \text{ otherwise}\}$

Using C and N to describe the market-clearing values of household consumption and hours worked satisfying conditions (iv) and (v) above, it is straightforward to show that market-clearing requires that (a) the real wage equal the household marginal rate of substitution between leisure and consumption, $\omega\left(z, \mu\right) = D_2 U\left(C, 1-N\right) / D_1 U\left(C, 1-N\right)$, that (b) the bond price, q^{-1}, equal the expected gross real interest rate, $q\left(z, \mu\right) = \beta \sum\limits_{m=1}^{N_z} \pi_{lm}^z D_1 U\left(C'_m, 1-N'_m\right) / D_1 U\left(C, 1-N\right)$, and that (c) firms' state-contingent discount factors agree with the household discounted marginal utility of consumption across states $d_j\left(z, \mu\right) = \beta D_1 U\left(C'_j, 1-N'_j\right) / D_1 U\left(C, 1-N\right)$. Given these results, we may compute equilibrium by solving a single Bellman equation that combines the firm-level profit maximization problem with these equilibrium implications of household utility maximization, effectively subsuming households' decisions into the problems faced by firms.

Without loss of generality, we assign $p(z, \mu)$ as an output price at which firms value current dividends and payments and correspondingly assume that firms discount their future values by the household subjective discount factor. Given this alternative means of expressing firms'

discounting, the following three conditions ensure all markets clear in our economy.

$$p(z,\mu) = D_1 U(C, 1-N) \tag{6}$$

$$\omega(z,\mu) = D_2 U(C, 1-N)/p(z,\mu) \tag{7}$$

$$q(z,\mu) = \beta \sum_{m=1}^{N_z} \pi_{lm}^z p\left(z_m, \mu'\right)/p(z,\mu) \tag{8}$$

A reformulation of (1) - (4) then yields an equivalent description of a firm's dynamic problem where each firm's value is measured in units of marginal utility, rather than output, with no change in the resulting decision rules. Suppressing the arguments of the price functions, exploiting the fact that the choice of n is independent of the k' and b' choices, and using the indicator function $\mathcal{J}(x) = \{1 \text{ if } x \geq 0 \text{ ; } \theta_k \text{ if } x < 0\}$ to distinguish the relative price of nonnegative versus negative investment, we have:

$$V_0(k,b,\varepsilon_i;z_l,\mu) = \pi_d \max_n p\Big[z_l\varepsilon_i F(k,n) - \omega n + \theta_k(1-\delta)k - b\Big] + (1-\pi_d)V(k,b,\varepsilon_i;z_l,\mu), \tag{9}$$

$$\text{where } V(k,b,\varepsilon_i;z_l,\mu) = \max_{n,k',b',D}\Big[pD + \beta \sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi_{lm}^z \pi_{ij} V_0\left(k',b',\varepsilon_j;z_m,\mu'\right)\Big] \tag{10}$$

$$\text{subject to } 0 \leq D \leq z\varepsilon F(k,n) - \omega n + qb' - b - \mathcal{J}\Big(k' - (1-\delta)k\Big)[k' - (1-\delta)k], \tag{11}$$

$$\text{and subject to } b' \leq \Theta k. \tag{12}$$

4 Analysis

The problem listed in equations (9) - (12) forms the basis for solving equilibrium allocations in our economy, so long as the prices p, ω and q taken as given by our firms satisfy the restrictions in (6) - (8) above.[13] From here, we begin to characterize the decision rules arising from this problem. Each firm chooses its labor $n = N(k,\varepsilon;z,\mu)$ to solve $z\varepsilon D_2 F(k,n;z,\mu) = \omega(z,\mu)$, which immediately returns its current production, $y(k,\varepsilon) = z\varepsilon F(k, N(k,\varepsilon;z,\mu))$, so that any firm of type (k,b,ε) will achieve current profit flows $\pi(k,b,\varepsilon)$ defined below irrespective of its capital adjustment or borrowing decision.

$$\pi(k,b,\varepsilon) \equiv z\varepsilon F(k, N(k,\varepsilon;z,\mu)) - \omega(z,\mu)N(k,\varepsilon;z,\mu) - b \tag{13}$$

[13] Here, and in many instances below, we suppress the z, μ arguments of price functions, decision rules and firm-level state vectors to reduce notation.

The challenging objects to determine are D, k' and b' for continuing firms. Turning to these, we will use a simple observation about the implications of borrowing constraints for the value a firm places on retained earnings versus dividends. As long as the firm places non-zero probability weight on encountering a future state in which its borrowing constraint will bind, the shadow value of retained earnings (which includes the discounted sequence of multipliers on future borrowing constraints) will necessarily exceed the shadow value of current dividends, p.[14] This means that, as long as the firm may face a binding borrowing limit in the future, it will set $D = 0$. In this case, equation 11 establishes that the firm's choice of k' directly implies the level of debt with which it will enter into the next period. We refer to any such firm as a *constrained* firm, and list the resulting univariate problem it solves after deciding it will pay no dividends in the current period.

$$\begin{aligned} V^c\left(k, b, \varepsilon_i; z_l, \mu\right) &= \max_{k' \geq 0} \beta \sum_{m=1}^{N_z} \sum_{j=1}^{N_\varepsilon} \pi_{lm}^z \pi_{ij} V_0\left(k', b', \varepsilon_j; z_m, \mu'\right) \quad \text{subject to:} \\ b' &= \frac{1}{q}\Big[-\pi\left(k, b, \varepsilon\right) + \mathcal{J}\Big(k' - (1-\delta)\, k\Big)\left[k' - (1-\delta)\, k\right]\Big] \\ \text{and } b' &\leq \Theta k \end{aligned} \tag{14}$$

We can make a related observation about the value a firm places on retained earnings versus dividends if it has accumulated sufficient wealth (via $k > 0$ or $b < 0$) such that collateral constraints will never again affect its investment activities. In this case, the sequence of multipliers on all possible future borrowing constraints are zero and the firm is indifferent between allocating profits to savings versus paying dividends. We refer to any such firm as *unconstrained*.

To insure that an unconstrained firm will never again experience a binding borrowing constraint (in any conceivable future state), we will assign any such indifferent firm a savings policy such that no history of z and ε leads to a level of debt exceeding Θk. Given this policy, the firm's savings or debt will not affect its investment. It follows that, for an unconstrained firm, we do not need to know b' to derive k'. We exploit this property and describe unconstrained firms' investment. In doing so, we assume, without loss of generality, that $b' = 0$.[15]

[14] This is easily proved using a sequence approach with explicit multipliers on each constraint; see Caggese (2007).

[15] The actual b' adopted by an unconstrained firm is dictated by the savings policy we specify below in section 4.1.

The firm chooses k' to solve,

$$W\left(k,b,\varepsilon_i;z_l,\mu\right)=\max_{k'\geq 0}\Bigg[p\Big[\pi\left(k,b,\varepsilon\right)-\mathcal{J}\Big(k'-\left(1-\delta\right)k\Big)\left[k'-\left(1-\delta\right)k\right]\Big] \tag{15}$$

$$+\beta\sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon}\pi_{lm}^{z}\pi_{ij}W_0\left(k',0,\varepsilon_j;z_m,\mu'\right)\Bigg],\text{ where}$$

$$W_0\left(k,b,\varepsilon_i;z_l,\mu\right)=\pi_d p\Big[\pi\left(k,b,\varepsilon\right)+\theta_k\left(1-\delta\right)k\Big]+(1-\pi_d)W\left(k,b,\varepsilon_i;z_l,\mu\right).$$

Referring back to equation 13, note that a firm that has just become unconstrained, having entered into the period with some nonzero debt (savings) $b\neq 0$, sees its value linearly reduced (raised) by the associated reduction (rise) in current dividends, which are valued by p. Thus, we can alternatively express the value of any unconstrained firm of type (k,b,ε) as $w\left(k,\varepsilon\right)-pb$, where $w\left(k,\varepsilon\right)\equiv W\left(k,0,\varepsilon\right)$. The firm's beginning-of-period expected value inherits the same property; $W_0\left(k,b,\varepsilon_i;z_l,\mu\right)=w_0\left(k,\varepsilon\right)-pb$, where $w_0\left(k,\varepsilon\right)\equiv W_0\left(k,0,\varepsilon\right)$.

In the next section, we will define the minimum savings policy that is adopted by any unconstrained firm. This policy will define a threshold level of b, as a function of $(k,\varepsilon_i;z_l,\mu)$, such that firms holding debt less than this threshold will be indifferent between paying dividends and retaining earnings.

4.1 Decisions among unconstrained firms

In this section we first characterize the investment policy of an unconstrained firm, then its resultant minimum savings policy. Starting with investment, it is expositionally useful to adopt the following less concise means of representing the problem in (15).

$$W\left(k,b,\varepsilon_i;z_l,\mu\right)=\max\{W^u\left(k,b,\varepsilon_i;z_l,\mu\right),W^d\left(k,b,\varepsilon_i;z_l,\mu\right)\},\text{ where:}$$

$$\begin{aligned}W^u\left(k,b,\varepsilon_i;z_l,\mu\right) &= p\pi(k,b,\varepsilon)+p(1-\delta)k \\ &\quad+\max_{k'\geq(1-\delta)k}\Big[-pk'+\beta\sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon}\pi_{lm}^{z}\pi_{ij}w_0\left(k',\varepsilon_j;z_m,\mu'\right)\Big]\end{aligned} \tag{16}$$

$$\begin{aligned}W^d\left(k,b,\varepsilon_i;z_l,\mu\right) &= p\pi(k,b,\varepsilon)+p\theta_k(1-\delta)k \\ &\quad+\max_{k'\leq(1-\delta)k}\Big[-p\theta_k k'+\beta\sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon}\pi_{lm}^{z}\pi_{ij}w_0\left(k',\varepsilon_j;z_m,\mu'\right)\Big],\end{aligned} \tag{17}$$

where (13) defines $\pi(k, b, \varepsilon)$. In the above, W^u and W^d are both strictly increasing in k. This in turn implies that W and W_0 are strictly increasing functions of the unconstrained firm's capital, as are the w and w_0 functions defined above.

We may characterize the capital decision rule for an unconstrained firm by reference to two target capital stocks, the upward and downward adjustment targets that would solve the problems in (16) and (17), respectively, were there no sign restrictions on investment. Define the upward target, k_u^*, as the capital a firm would choose given a unit relative price of investment, and define the downward target, k_d^*, as the capital a firm would choose given a relative price at θ_k.

$$k_u^*(\varepsilon_i) = \arg\max_{k'} \Big[-pk' + \beta \sum_{m=1}^{N_z} \sum_{j=1}^{N_\varepsilon} \pi_{lm}^z \pi_{ij} w_0 \left(k', \varepsilon_j; z_m, \mu'\right) \Big] \tag{18}$$

$$k_d^*(\varepsilon_i) = \arg\max_{k'} \Big[-p\theta_k k' + \beta \sum_{m=1}^{N_z} \sum_{j=1}^{N_\varepsilon} \pi_{lm}^z \pi_{ij} w_0 \left(k', \varepsilon_j; z_m, \mu'\right) \Big] \tag{19}$$

Notice that each target is independent of current capital and depends only on the aggregate state and the firm's current ε. As such, all unconstrained firms that share in common the same current productivity ε have the same upward and downward target capitals. Note also that, because $\theta_k < 1$ (and because the value function w_0 is strictly increasing in k), the upward adjustment target necessarily lies below the downward target: $k_u^* < k_d^*$.

We are now in a convenient position to retrieve the unconstrained firm's capital decision rule. Given a constant price associated with raising (lowering) its capital stock, and because w_0 is increasing in k, the firm selects a future capital as close to the upward (downward) target as its constraint set allows. Thus, the firm's decision rules conditional on upward adjustment and downward adjustment are as follow.

$$k_u(\varepsilon) = \max\{(1-\delta)k, k_u^*(\varepsilon)\} \quad \text{and} \quad k_d(\varepsilon) = \min\{(1-\delta)k, k_d^*(\varepsilon)\}$$

Given these conditional adjustment rules, we know that an unconstrained firm of type (k, b, ε) selects one of three future capital levels, $k' \in \{k_u^*(\varepsilon), k_d^*(\varepsilon), (1-\delta)k\}$. Which one it selects depends only on where its current capital lies in relation to its two targets.

Recalling that $k_u^*(\varepsilon) < k_d^*(\varepsilon)$, if $k \in \left[\frac{k_u^*(\varepsilon)}{1-\delta}, \frac{k_d^*(\varepsilon)}{1-\delta}\right]$ then $k_u(\varepsilon) = (1-\delta)k = k_d(\varepsilon)$, so the firm makes no adjustment to its capital. If, instead, the firm's capital is sufficiently low that its implied stock for next period under no adjustment lies below the upward target, $k < \frac{k_u^*(\varepsilon)}{1-\delta}$, then $k_u(\varepsilon) = k_u^*(\varepsilon)$, while $k_d(\varepsilon) = (1-\delta)k$. In this case, the firm selects $k_u^*(\varepsilon)$, since $(1-\delta)k$ is in

the constraint set for upward capital adjustment. Finally, if the firm's implied capital for next period under no adjustment lies above the downward target, $k > \frac{k_d^*(\varepsilon)}{1-\delta}$, then $k_d(\varepsilon) = k_d^*(\varepsilon)$, while $k_u(\varepsilon) = (1-\delta)k$. In this case, the firm selects $k_d^*(\varepsilon)$, since $(1-\delta)k$ is in the constraint set for a downward adjustment. Collecting these observations, we have the following (S, s) capital decision rule for an unconstrained firm.

$$K^w(k,\varepsilon) = \begin{cases} k_u^*(\varepsilon; z, \mu) & \text{if } k < \frac{k_u^*(\varepsilon;z,\mu)}{1-\delta} \\ (1-\delta)k & \text{if } k \in \left[\frac{k_u^*(\varepsilon;z,\mu)}{1-\delta}, \frac{k_d^*(\varepsilon;z,\mu)}{1-\delta}\right] \\ k_d^*(\varepsilon; z, \mu) & \text{if } k > \frac{k_d^*(\varepsilon;z,\mu)}{1-\delta} \end{cases} \tag{20}$$

Given the decision rule for capital, we now isolate a minimum level of savings that ensures that an unconstrained firm of type (ε, k) will never be affected by borrowing constraints across all possible future $(\varepsilon'; S')$. Any firm that maintains a level of savings at least equal to the threshold defined by the minimum savings policy will be indifferent to paying additional revenues in the form of dividends, or accumulating further savings. This, in turn, implies that the firm is willing to follow the minimum savings policy.

For an unconstrained firm with a beginning of period level of debt, b, define profits after debt repayment and investment expenditures as

$$D^w(k,b,\varepsilon,S) = \pi(k,b,\varepsilon,S) - \mathcal{J}\Big(K^w(k,\varepsilon_i,S) - (1-\delta)k\Big)[K^w(k,\varepsilon,S) - (1-\delta)k]. \tag{21}$$

Next, let $\widetilde{B}\Big(K^w(k,\varepsilon_i;S), \varepsilon_j; z_m, \mu'(S)\Big)$ define the maximum debt level at which a firm entering next period with capital K^w and (ε_j, z_m) may remain unconstrained. The following pair of equations recursively defines the minimum savings policy, $B^w(k,\varepsilon_i;S)$.

$$B^w(k,\varepsilon_i;S) \equiv \min_{\{\varepsilon_j|\pi_{ij}>0 \ \ and \ \ z_m|\pi^z_{lm}>0\}} \widetilde{B}\Big(K^w(k,\varepsilon_i;S), \varepsilon_j; z_m, \mu'(S)\Big), \tag{22}$$

$$\widetilde{B}(k,\varepsilon;S) \equiv D^w(k,0,\varepsilon;S) + q(S)\min\Big\{B^w(k,\varepsilon_i;S), \theta_b\theta_k k\Big\}. \tag{23}$$

In equation 22, $B^w(k,\varepsilon_i;S)$ is derived as the maximum level of debt with which the firm can exit this period and remain unconstrained next period, given that it adopts the unconstrained capital decision rule. Next, (23) defines the beginning of period maximum debt level under which a firm is unconstrained, using the minimum savings policy function. Notice that $\widetilde{B}$ is increasing in the firm's current profits as these may be used to cover outstanding debt. The minimum operator imposes the borrowing constraint; if the firm does not have sufficient collateral to borrow B^w, it can only be unconstrained this period if it has entered with sufficient savings to finance investment.

Given the capital rule and the minimum savings policy, we can directly retrieve the unconstrained firm's dividends. The firm's value, listed above in (15), may also be expressed as follows.

$$W\left(k,b,\varepsilon_i;z,\mu\right)=p(z,\mu)D^w\left(k,b,\varepsilon;z,\mu\right) \tag{24}$$
$$+\beta\sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon}\pi_{lm}^z\pi_{ij}W_0\left(K^w\left(k,\varepsilon;z,\mu\right),B^w(k,\varepsilon_i;z,\mu),\varepsilon_j;z_m,\mu'(z,\mu)\right),$$
$$\text{where } W_0\left(k,b,\varepsilon_i;z,\mu\right)=\pi_d p(z,\mu)\Big[\pi\left(k,b,\varepsilon;z,\mu\right)+\theta_k\left(1-\delta\right)k\Big]+(1-\pi_d)W\left(k,b,\varepsilon_i;z,\mu\right)$$

4.2 Decisions among constrained firms

We now consider the decisions made by a firm that has not previously attained sufficient wealth to be unconstrained. The first essential step is to establish whether or not the firm has crossed the relevant wealth threshold to become unconstrained. If it has, the decision rules isolated above apply. If it has not, the collateralized borrowing constraint will continue to influence its investment decisions, so that the capital and debt decisions remain intertwined.

To ascertain whether a firm of type (k,b,ε) has become unconstrained, we need only consider whether it is feasible for the firm to adopt the capital rule $K^w\left(k,\varepsilon\right)$ and a level of debt not exceeding that implied by the rule $B^w(k,\varepsilon_i)$, without paying negative dividends in the current period. If the firm of type (k,b,ε) is able to adopt the decision rules in (20) and (22) without violating the non–negativity of dividends, then it achieves the value from (24) above, and it exits the period indistinguishable from any other unconstrained firm that entered the period with (k,ε).

$$V\left(k,b,\varepsilon_i;z_l,\mu\right)=W\left(k,b,\varepsilon_i;z_l,\mu\right)\ \text{ iff } D^w(k,b,\varepsilon_i;z_l,\mu)+q(z_l,\mu)\min\left\{B^w(k,\varepsilon_i;z_l,\mu),\theta_b\theta_k k\right\}\geq 0$$

Any constrained firm that can adopt the decision rules of an unconstrained firm will always choose to do so, since $V\leq W$. However, when the inequality above cannot be satisfied, the firm remains constrained. For any such firm surviving beyond the current period, $V\left(k,b,\varepsilon_i;z_l,\mu\right)=V^c\left(k,b,\varepsilon_i;z_l,\mu\right)$. To isolate the decisions made by a continuing constrained firm facing the problem in (14), we again find it useful to adopt a less concise representation.

$$V^c\left(k,b,\varepsilon_i;z_l,\mu\right)=\max\{V^u\left(k,b,\varepsilon_i;z_l,\mu\right),V^d\left(k,b,\varepsilon_i;z_l,\mu\right)\},\quad\text{where:} \tag{25}$$

$$
\begin{aligned}
V^u\left(k,b,\varepsilon_i;z_l,\mu\right) &= \max_{k'\geq(1-\delta)k} \beta \sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi^z_{lm}\pi_{ij} V_0\left(k',b'_u(k'),\varepsilon_j;z_m,\mu'\right), \text{ with} \qquad (26)\\
b'_u(k') &\equiv \frac{1}{q(z_l,\mu)}\Big(-\pi\left(k,b,\varepsilon_i\right)+\left[k'-(1-\delta)\,k\right]\Big)\\
\text{subject to:} \quad & b'_u(k')\leq \Theta k
\end{aligned}
$$

$$
\begin{aligned}
V^d\left(k,b,\varepsilon_i;z_l,\mu\right) &= \max_{k'\leq(1-\delta)k} \beta \sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi^z_{lm}\pi_{ij} V_0\left(k',b'_d(k'),\varepsilon_j;z_m,\mu'\right), \text{ with} \qquad (27)\\
b'_d(k') &\equiv \frac{1}{q(z_l,\mu)}\Big(-\pi\left(k,b,\varepsilon_i\right)+\theta_k\left[k'-(1-\delta)\,k\right]\Big)\\
\text{subject to:} \quad & b'_d(k')\leq \Theta k
\end{aligned}
$$

We approach the constrained firm's problem as follows. First, given its (k,ε), we isolate a cutoff debt level under which (26) is a feasible option. The lowest choice of k' permitted by the non-negativity constraint on investment is $(1-\delta)\,k$. If this choice is not affordable given the firm's borrowing constraint, it cannot undertake even a trivial upward capital adjustment. Recalling the definition of $\pi\left(k,b,\varepsilon\right)$, this is the case if $\frac{1}{q}[b+\omega N\left(k,\varepsilon\right)-z\varepsilon F\left(k,N\left(k,\varepsilon\right)\right)] > \Theta k$. Thus, among any group of firms sharing a common (k,ε), only those with debt not exceeding $b^T(k,\varepsilon)$ can consider an upward adjustment, where the threshold debt level is:

$$
b^T(k,\varepsilon)\equiv q\theta_b\theta_k k+z\varepsilon F\left(k,N\left(k,\varepsilon\right)\right)-\omega N\left(k,\varepsilon\right).
$$

Firms with $b>b^T(k,\varepsilon)$ do not solve (26); for them, $V^c\left(k,b,\varepsilon;z,\mu\right)=V^d\left(k,b,\varepsilon;z,\mu\right)$.

To solve the problems (26) - (27), we identify the maximum capitals permitted by the borrowing constraint under upward versus downward capital adjustment, and then impose the relevant sign restrictions on investment to arrive at the constraint sets associated with each option.

$$
\begin{aligned}
\overline{k}_u(k,b,\varepsilon) &\equiv (1-\delta)\,k+\Big[q\theta_b\theta_k k+\pi\left(k,b,\varepsilon\right)\Big]\\
\overline{k}_d(k,b,\varepsilon) &\equiv (1-\delta)\,k+\frac{1}{\theta_k}\Big[q\theta_b\theta_k k+\pi\left(k,b,\varepsilon\right)\Big]\\
\Lambda^u(k,b,\varepsilon) &= \left[(1-\delta)\,k,\overline{k}_u(k,b,\varepsilon)\right]\\
\Lambda^d(k,b,\varepsilon) &= \max\left\{0,\min\{(1-\delta)\,k,\overline{k}_d(k,b,\varepsilon)\right\}
\end{aligned}
$$

Substituting in the debt implied by each capital choice and making use of our findings above, we

may express the constrained firm's value as follows.

$$V^c(k,b,\varepsilon_i;\cdot) = \max\{V^u(k,b,\varepsilon_i;z_l,\mu), V^d(k,b,\varepsilon_i;z_l,\mu)\}, \quad \text{where:} \tag{28}$$

$$V^u(k,b,\varepsilon_i;\cdot) = \max_{k'\in\Lambda^u(k,b,\varepsilon)} \beta \sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi^z_{lm}\pi_{ij} V_0\left(k', \frac{k'-(1-\delta)k-\pi(k,b,\varepsilon_i)}{q(z,\mu)}), \varepsilon_j; z_m, \mu'(z,\mu)\right),$$

$$V^d(k,b,\varepsilon_i;\cdot) = \max_{k'\in\Lambda^d(k,b,\varepsilon)} \beta \sum_{m=1}^{N_z}\sum_{j=1}^{N_\varepsilon} \pi^z_{lm}\pi_{ij} V_0\left(k', \frac{\theta_k[k'-(1-\delta)k]-\pi(k,b,\varepsilon_i)}{q(z,\mu)}, \varepsilon_j; z_m, \mu'(z,\mu)\right),$$

and where:

$$\begin{aligned} V_0(k,b,\varepsilon_i;S) &= \pi_d p(S)[\pi(k,b,\varepsilon;S)+\theta_k(1-\delta)k] + (1-\pi_d)V(k,b,\varepsilon_i;S), \\ V(k,b,\varepsilon_i;S) &= \begin{cases} W(k,b,\varepsilon;S) \text{ if } D^w(k,b,\varepsilon,S)+q(S)\min\{B^w(k,\varepsilon_i;S),\theta_b\theta_k k\}\geq 0 \\ V^c(k,b,\varepsilon_i;S) \text{ otherwise} \end{cases} \end{aligned}$$

Denoting the capitals that solve the conditional adjustment problems above by $\widehat{k}^u(k,b,\varepsilon_i;\cdot)$ and $\widehat{k}^d(k,b,\varepsilon_i;\cdot)$, and recalling $D^c(\cdot)=0$, we obtain the following decision rules for capital and debt.

$$K^c(k,b,\varepsilon_i;S) = \begin{cases} \widehat{k}^u(k,b,\varepsilon_i;S) & \text{if } V^c(k,b,\varepsilon_i;S) = V^u(k,b,\varepsilon_i;S) \\ \widehat{k}^d(k,b,\varepsilon_i;S) & \text{if } V^c(k,b,\varepsilon_i;S) = V^d(k,b,\varepsilon_i;S) \end{cases} \tag{29}$$

$$\begin{aligned} B^c(k,b,\varepsilon_i;S) &= \frac{1}{q(S)}[\mathcal{J}(K^c(k,b,\varepsilon_i;S)-(1-\delta)k)[K^c(k,b,\varepsilon_i;S)-(1-\delta)k] \\ &\quad -\pi(k,b,\varepsilon_i;S)] \end{aligned} \tag{30}$$

The numerical algorithm we use to solve our model is an extension of that described in Khan and Thomas (2003, 2008) using the analysis above. More specifically, our solution involves repeated application of the contraction mapping implied by (28) to solve the constrained firm value function V^c, given the price functions $p(z,\mu)$, $\omega(z,\mu)$ and $q(z,\mu)$ and the laws of motion implied by $\Gamma(z,\mu)$, (π_{ij}) and (π^z_{lm}). In each instance, the starting point is solving (24) to isolate the unconstrained firm value function W, which serves as an input for V^c.

5 Calibration

In the sections to follow, we will consider how the mechanics of our model with real and financial frictions compare to those in two relevant reference models - one where there are no borrowing limits ($\theta_b \to \infty$) and one where there are neither financial nor real frictions ($\theta_b \to \infty$,

$\theta_k = 1$). These two reference models will help us to isolate how much the *interaction* between credit constraints and micro-level capital rigidities influences our economy's aggregate dynamics. Aside from the values of θ_b and θ_k, all three models share a common parameter set that is selected in our full model to best match moments drawn from postwar U.S. aggregate and firm-level data. To be clear, we do not re-calibrate the reference models; thus, the average capital/output ratio, hours worked, and other important aspects of these economies are allowed to vary as each friction is eliminated.

5.1 Functional forms

Across our model economies, we assume that the representative household's period utility is the result of indivisible labor (Rogerson (1988)): $u(c, L) = \log c + \varphi L$. The firm-level production function is Cobb-Douglas: $z\varepsilon F(k, n) = z\varepsilon k^{\alpha} n^{\nu}$. The initial capital stock of each entering firm is a fixed χ fraction of the typical stock held across all firms in the long-run of our full economy; that is, $k_0 = \chi \int k\widetilde{\mu}(d\,[k \times b \times \varepsilon])$, where $\widetilde{\mu}$ represents the steady-state distribution therein.

In specifying our exogenous stochastic process for aggregate productivity, we begin by assuming a continuous shock following a mean zero AR(1) process in logs: $\log z' = \rho_z \log z + \eta'_z$ with $\eta'_z \sim N\left(0, \sigma^2_{\eta_z}\right)$. Next, we estimate the values of ρ_z and σ_{η_z} from Solow residuals measured using NIPA data on US real GDP and private capital, together with the total employment hours series constructed by Prescott, Ueberfeldt, and Cociuba (2005) from CPS household survey data, over the years 1959-2002, and we discretize the resulting productivity process using a grid with 3 shock realizations ($N_z = 3$) to obtain (z_l) and (π^z_{lm}). We determine the firm-specific productivity shocks (ε_i) and the Markov Chain governing their evolution (π_{ij}) similarly by discretizing a log-normal process, $\log \varepsilon' = \rho_\varepsilon \log \varepsilon + \eta'$ using 7 values ($N_\varepsilon = 7$). The selection of these shocks' persistence and volatility is described in section 5.3.

5.2 Aggregate targets

We set the length of a period to correspond to one year, and we determine the values of β, ν, δ, α, φ and θ_b using moments from the aggregate data as follows. First, we set the household discount factor, β, to imply an average real interest rate of 4 percent, consistent with recent findings by Gomme, Ravikumar and Rupert (2008). Next, the production parameter ν is set to yield an average labor share of income at 0.60 (Cooley and Prescott (1995)). The depreciation rate,

δ, is taken to imply an average investment-to-capital ratio of roughly 0.069, which corresponds to the average value for the private capital stock between 1954 and 2002 in the U.S. Fixed Asset Tables, controlling for growth. Given this value, we determine capital's share, α, so that our model matches the average private capital-to-output ratio over the same period, at 2.3, and we set the parameter governing the preference for leisure, φ, to imply an average of one-third of available time is spent in market work. Finally, we select the parameter governing the extent of financial frictions in our model, θ_b, to imply an average debt-to-assets ratio at 0.366, which matches that of nonfarm nonfinancial businesses over 1952-05 in the Flow of Funds.

5.3 Firm-level targets

The parameters we determine using moments drawn from firm-level data are the exit rate, π_d, the fraction of the steady-state aggregate capital stock held by each entering firm, χ, the extent of reversibilty in capital, θ_k, and the persistence and variability of the firm-specific productivity shocks, ρ_ε and σ_η. We set π_d at 0.10, +-so that 10 percent of firms enter and exit the economy each year. Next, we set $\chi = 0.10$ so that entering firms are, on average, one-tenth the size of the typical firm in our economy (Davis and Haltiwanger (1992)).

We choose θ_k, ρ_ε and σ_η jointly to reproduce three aspects of establishment-level investment data documented by Cooper and Haltiwanger (2006) based on a 17-year sample drawn from the Longitudinal Research Database. These targets are (i) the average mean investment rate (i/k) across establishments: 0.122, (ii) the average standard deviation of investment rates: 0.337, and (iii) the average serial correlation of investment rates: 0.058.[16] While our models has life-cycle aspects affecting firms' investments, the Cooper and Haltiwanger (2006) dataset includes only large manufacturing establishments that remain in operation throughout their sample period, Thus, in undertaking this part of our calibration, we must select an appropriate model-generated sample for comparability with their sample. This we do by simulating a large number of firms for 30 years, retaining only those firms that survive throughout, and then restricting the dates over which investment rates are measured to eliminate life-cycle effects.

[16] While not a target in the calibration, our model also closely matches a fourth moment drawn from the Cooper and Haltiwanger study, the fraction of establishment-year observations wherein a positive investment spike ($i/k > 0.20$) occurs: 0.186.

5.4 Resulting parameters

The table below lists the parameter set obtained from our calibration.

β	ν	δ	α	φ	ρ_z	σ_{η_z}	θ_b	π_d	χ	θ_k	ρ_ε	σ_η
0.96	0.60	0.065	0.27	2.15	0.852	0.014	1.35	0.10	0.10	0.95	0.653	0.135

Note that these parameters imply minimal real frictions in our model economy, with only a 5 percent loss incurred in uninstalling capital, and a moderate degree of financial frictions, with firms able to take on debt up to 135 percent of the value of their capital. Also note that firm-level shocks are far more volatile and less persistent than aggregate shocks. Given these aspects of the calibration, our model gives rise to a stationary distribution of firms over (k, b, ε) wherein roughly 86 percent of firms are constrained when one applies the definition from section 4 above. By contrast, the fraction of firms facing a currently binding borrowing limit is 27 percent.

6 Steady state

We begin by considering the implications of borrowing limits and irreversibilities for the typical decisions made in our economy. Figure 1 overviews the stationary distribution of firms in the baseline case of our full model, presenting three slices of the full distribution. In the top panel, we see the distribution of firms over capital and debt-to-capital levels at the lowest firm-level productivity, while the middle and bottom present the counterparts at the median and highest levels of productivity.

Note that each panel of figure 1 appears to have two essentially separate distributions. The first distribution (in the foreground) has a distinctly curved shape and reflects an inverse relation between firms' capital stocks and their savings rates. This distribution corresponds to older, wealthier firms that are unconstrained and following the minimum savings policy described in section 4. Elsewhere, the 10 percent of firms newly entering the economy each period are scattered across each ε level according to the ergodic productivity distribution. These firms enter with zero debt and very low initial capital (roughly 0.14), and are found in a large spike near the left edge of each panel.

After its first date in production, each new firm begins to take on debt in effort to build up its capital. In the absence of the collateralized borrowing limits, young firms would immediately take on a large, temporary debt that would allow them to jump to the capital stock selected by

unconstrained firms with the same current productivity level. Here, however, firms with little collateral have a relatively limited ability to borrow, so their capital accumulation is necessarily gradual. As a result, ripples of these entering firms slowly move into higher ranges of k and b/k as they age. In the figure, these young firms are found along the back edge of each panel; as they mature, they steadily raise their capital while maintaining a roughly constant borrowing rate typically below the maximum permitted.

Those firms that survive long enough eventually reach a level of capital such that they can adopt the unconstrained capital choices consistent with their current productivity while beginning to reduce their debt. Those surviving longer still will, at some point, attain a level of capital and savings such that their investment decisions become impervious to borrowing limits. At this point, they join the distribution of unconstrained firms. As would be expected, the mean capital among constrained firms rises with firm-level productivity; the same is true for unconstrained firms, though this is somewhat harder to see given the perspective of the figure.

The life-cycle aspects of our model described above may be seen from figure 2. There we display the average capital and debt choices within a cohort of (initially) $25,000$ firms as they age. Notice that the typical firm raises its capital and debt over its first six periods of life. Thereafter, starting in period 7, it begins to reduce its debt and begins financing the remaining rise in its capital stock fully out of earnings. By age 16, the typical firm has become a net saver, and thereafter joins the distribution of permanently unconstrained firms.

Figure 3 is the no-financial frictions counterpart to figure 2, depicting the average capital among the same cohort of firms in a version of our model where the collateral constraint is removed, so debt becomes irrelevant to investment. As in the previous figure, the cohort enters the economy with low initial capital. However, in this case, young firms can immediately reach their unconstrained capital targets for the start of the next period. Thus, we see a much larger initial rise in capital between dates 1 and 2 relative to figure 2. Notice, however, that the elimination of financial frictions does not entirely eliminate life-cycle aspects from our model. Firms still face a small real friction that causes bands of inaction in investment. Thus, as a firm transits from one ε to a lower one nearby, it will at times choose not to lower its capital stock, given the forfeit of 5 percent of any capital uninstalled. Likewise, when a firm's relative productivity rises, it is slow to respond fully to that rise given the partial irreversibility in investment. As a result, we see the average capital stock of the cohort gradually continuing to rise from age

2 to age 7. Nonetheless, this rise is quite modest relative to that between age 1 and 2; after taking into account the implications of irreversibity, all but the newest firms here operate at a scale appropriate to their productivity. The quantitative impact of the more efficient allocation of production this implies is that steady state output rises by 4 percent relative to our full economy, with measured TFP rising roughly 1 percent.

Returning to our full economy with both frictions in place, figure 4 illustrates the pure effects of the irreversibility in cases where it does not interact with the financial friction in our economy. Here, we summarize the capital choices made by unconstrained firms entering the period with various levels of capital (measured on the x-axis) and debt (measured on the y-axis), conditional on a current productivity draw. The top panel depicts firms entering with the lowest productivity value, the middle panel shows those with the median value, and the bottom panel shows those the highest productivity. The z-axis in each panel reports an indicator variable that takes on a value of 1 for unconstrained firms that invest positively to the upward target capital consistent with their current productivity, a value of 2 for those investing negatively to the relevant downward target, and a value of 5 for those that remain inactive with respect to their capital, setting investment to zero. (Areas along the floor of each panel are combinations of (k, b) where firms are not unconstrained.)

The region of (k, b) where firms invest to their upward target expands into higher current capital levels as one looks from the top panel downward, since rises in current productivity predict higher marginal product of capital schedules next period. To the left of these regions are the areas with zero investment induced by the irreversibility in capital. While the loss associated with uninstalling capital in our economy is only 5 percent, it nonetheless makes some firms quite reluctant to shed capital. Those with higher current productivities are more so, given the persistence in ε alongside depreciation. As such, the inactivity region expands to higher capital levels as productivity rises, while the region associated with downward investment shrinks, finally disappearing from view by the bottom panel.

Note that figure 4 is largely an expositional device. It depicts the capital choice adopted by unconstrained firms at each potential firm-level state rather than at states actually populated in the economy's stationary distribution. Restricting consideration to those states, the actual fraction of all firms that are (permanently) unconstrained and adjust to the upward target consistent with their productivity is 5 percent, the fraction that are unconstrained and remain inactive is 7

percent, and the fraction undertaking negative investment is 2 percent.

Figure 5 is analogous to figure 4. Again conditional on currently productivity, it illustrates the capital decisions taken by firms, this time considering those that are affected by both the real friction in our economy and the financial one. Such firms are located in regions of the (k, b) space to the right and back where capital is low and/or debt is high. (Areas along the floor of each panel are combinations of (k, b) where firms are unconstrained.)

Constrained firms investing positively to the maximum capital permitted by their ability to borrow (below their upward target capital) are reflected by a value of 3 on the z-axis. These are firms with higher current productivity, comparatively low capital, and comparatively high debt. They make up 27 percent of the population in our model's steady state and are the only firms facing a currently binding borrowing limit. Looking just left and in front of that region, we see firms with slightly higher capital (or slightly lower debt) that adjust to their upward capital targets. This region, reflected by a value of 1 on the z-axis, expands into higher values of capital as ε rises, since the target itself rises. In the stationary distribution, roughly 25 percent of firms are of this type. Finally, looking further left in each panel, we have firms selecting inaction with respect to investment due to the irreversibility (with a z-axis value of 5), and thereafter those whose capital is sufficiently high relative to their productivity that they disinvest (with a z-value of 2). These categories represent 27 and 6 percent of firms in our model's stationary distribution.

7 Results

We begin to examine business cycle results by first considering the effect each friction in our economy has on its typical business cycle. Table 1 presents some commonly reported business cycle statistics derived from an HP-filtered 5041 period simulation of our model economy under the assumption that aggregate productivity shocks are the only source of aggregate fluctuations, table 2 presents the corresponding moments when we eliminate financial frictions, and table 3 is the same economy with neither collateral constraints nor capital specificity. As expected, each friction acts to reduce the average levels of output, capital, and consumption over our simulation. Most notably, average output rises by roughly 4.1 percent when financial frictions are stripped away, then another 2.3 percent when the irreversibility is also eliminated.

Moving to consider second moments, there are some small differences across the three tables. Output volatility rises between our full economy and the counterpart model without limits to

borrowing, and it rises again between that model and the one with no frictions. Despite this, as each friction is lifted, the representative household grows more effective in smoothing its consumption. As the contemporaneous correlation between consumption and production is slightly weakened from one table to the next, consumption's standard deviation (raw and relative) falls. Elsewhere, the volatility of hours worked rises steadily, and the hours series is marginally more correlated with output as each friction is eliminated. The same monotone pattern does not follow for investment expenditures, however. There, the relative standard deviation falls from 3.83 percent to 3.77 percent as the financial friction is stripped away, allowing the inertia associated with irreversibility more prominence, while it rises to 4.04 percent when the irreversibility is eliminated.

While we have mentioned some minor differences in the business cycle moments across tables 1 through 3, two points are surely more important. The first is that the business cycle moments drawn from our full model in table 1 are similar to those of a typical real business cycle model without its complications (table 3). Output volatility is roughly 2 percent, consumption is about half as volatile as output, and investment roughly four times as volatile as output. We also see the customary strong positive contemporaneous correlations with output in consumption, investment, hours and wages. While the usual difficulties of excessive investment volatility and weak hours volatility are a bit more pronounced here relative to some representative firm real business cycle models, these distinctions come from our differing returns to scale in production rather than either friction we mean to study; the same features are present in table 3 with both removed.

This brings us to our second point. Despite the differences noted above, the second moments across all three tables are quite similar on the whole. Comparing table 1 to table 2, in particular, it appears that the typical business cycle in our economy is relatively impervious to some ordinary, ongoing degree of financial frictions. This observation is reinforced by figure 6, which presents our full model economy's impulse responses following a persistent negative shock to the exogenous component of total factor productivity. As may be seen from the close match between the exogenous and measured TFP series in the top panel, a persistent real shock has only very minimal implications for the endogenous component of aggregate productivity. Thus, when we examine output, consumption, employment and investment, we see impulse responses closely matching those that would represent the counterpart economy without real or financial frictions summarized in table 3. In particular, just as in the frictionless model, there are immediate declines in all four series and, aside from the hump-shaped consumption series, we see the largest

responses at the impact of the shock, with each series thereafter monotonically reverting to its long-run level.

A productivity shock on its own in our model economy, as in the frictionless economy, does not capture the macroeconomic changes observed in the U.S. economy over the most recent recession, which the NBER dates as having begun in the fourth quarter of 2007. Figure 7 illustrates the changes in GDP, consumption, investment and measured TFP, plotting each series' percent deviations relative to their 2007Q4 levels. There, we see that the initial response in GDP was negligible, while real personal consumption expenditure actually rose by roughly 1 percent and stayed high until 2008Q4. Moreover, the immediate declines in investment were modest relative to what came later. While total private investment fell immediately, this was initially entirely driven by housing. Non-residential investment did not begin to fall until 2008Q3, at which point it began to drop off sharply relative to the more gradual declines in GDP and consumption.

We have seen that our model economy behaves quite similarly to a frictionless model, and fails to generate the patterns observed in the most recent U.S. recession, so long as its aggregate fluctuations arise solely from changes in exogenous productivity. However, our main interest is to understand what happens when the extent of financial frictions suddenly and unexpectedly grows more severe than is normal. We explore this question via a series of impulse response figures to which we turn now.

Figure 8 depicts our economy's response to a financial crisis, absent any technology shock. More specifically, it is the response to a 55 percentage point drop in the value of firms' collateral, as generated by a reduction in θ_b, which implies a 25 percent reduction in new debt issuance. In this exercise, we assume that firms predict a return to normal financial conditions will ultimately occur. Each period, they place 40 percent probability weight on a full financial recovery in the subsequent period. Thus, when the shock occurs in period 1, they expect it will persist for 2.5 years.

Although the distribution of capital is predetermined when the financial shock hits in year 1, the top left panel of figure 8 reveals that aggregate production immediately falls by about 1.5 percent (relative to its simulated mean in normal financial times). This is, of course, a direct consequence of the 2.4 percent fall in the labor input (top right panel), which is, in turn, a reaction to the reduced expected return to investment (bottom right panel). With the sudden reduction in credit, there is a drop in the fraction of firms that are permanently financially unconstrained

and a sharp rise in the fraction of firms facing currently binding borrowing limits.[17] Underlying these changes, young firms are now far more hindered in their investment activities relative to the pre-shock economy, and thus will take considerably longer to outgrow financial frictions and begin producing at a scale consistent with their productivities. Moreover, some mature firms that, in the pre-shock economy, had been adopting the unconstrained capital decisions now find their collateral insufficient to prevent financial frictions once again influencing their investment plans. These larger constrained firms initially exhibit life-cycle investment similar to that in their youth, accumulating capital in effort to outgrow the new financial friction irrespective of their productivities.

Notice that, unlike the response that would follow a negative productivity shock, consumption does not immediately fall when the financial shock hits our economy. Anticipating a more distorted distribution of production over coming years, and thus unusually low endogenous total factor productivity (in the lower right panel), the representative household in our economy expects a lowered return to saving. This leads to a 0.5 percent rise in consumption at the impact of the shock, and also a rise in leisure. This effect of reduced future TFP is compounded by the fact that the initial aggregate capital stock is roughly 9 percent above that consistent with the tighter borrowing conditions, which further encourages consumption and leisure. The fall in investment (at lower left) does not support consumption for long, however; consumption falls to its pre-shock level by year 3, then steadily declines for roughly 8 more years before it levels off. Elsewhere, labor falls at the impact of the shock as described above. Thereafter, given the severe misallocation of capital at the start of date 2, alongside reductions in the total capital stock, the marginal product of labor drops. This leads to further large reductions in employment. By year 3, employment is 3.9 percent below its pre-shock level, and it does not rise back to the level consistent with the new financial setting until around period 15. This long adjustment period is a reflection of the time that it takes for the capital distribution to settle, as may be inferred from the measured TFP response in the lower right panel.

On balance, we take the following observation from figure 8. A tightening of collateral constraints alone, a purely financial shock, is capable of large and persistent real effects in our model

[17] If these financial conditions remained permanently, the resulting stationary distribution would have 51 percent of firms constrained in their current upward capital adjustments and 1 percent of firms forced to undertake some negative investment to repay outstanding debt. In ordinary financial times, by contrast, these percentages are 27 and 0, respectively.

economy. In the example we have shown here, the misallocation of capital arising from tight financial conditions is compounded by the reductions in aggregate capital, productivity, and labor that it causes. As a result, there are protracted adjustments in aggregate quantities lasting a decade or more, and GDP is ultimately reduced by 3.6 percent, while aggregate consumption is reduced by 1.3 percent.

We next consider what implications the prolonged financial crisis from above can have if its onset is followed by a 1 standard deviation negative technology shock. As seen in the lower right panel of Figure, the exogenous component of TFP falls one year after the financial shock hits, and thereafter gradually reverts to its mean. Were credit markets functioning as normal when this TFP shock appeared, output would fall 3.8 percent, labor would fall 2 percent, and the half-life of the output response would be roughly 5 years. In this case, however, with tight credit markets disrupting the economy in the background, the effects of this otherwise ordinary negative productivity shock look more dramatic. With employment and production already contracting due to the increased inefficiency in capital allocation, labor drops to 5.5 percent below its average at the impact of the productivity shock, while GDP drops to 6.7 percent below average. Thereafter, although exogenous TFP is smoothly rising back to trend, the financial crisis continues to hold real quantities down. Until borrowing conditions return to normal, total production will remain nearly 4 percent below trend.

To this point, we have considered the implications of a *persistent* financial crisis, in that borrowing conditions do not recover throughout the exercises depicted in figures 8 and 9. As such, a natural question we have not yet addressed is this: "What should we expect to see in the recovery following a financial crisis?" We explore this question in figure 10. There, the same shock to the value of collateral hits the economy in date 1, and agents have the same expectations regarding financial recovery, as described above. The financial shock remains in place for 4 periods; thereafter, beginning in date 5, we allow a complete recovery of financial conditions, returning the value of collateral to normal.[18]

Three aspects of the responses in figure 10 are worthy of note. First, so long as GDP or consumption is adopted as our measure, the effects of a financial crisis are not rapidly reversed. Although credit markets are operating perfectly normally in year 5, GDP is still 3.1 percent below trend in that date. Moreover, it does not fully regain its pre-shock average level until year 12,

[18] We omit the negative TFP shock from this exercise for expositional simplicity, as we have seen above that its implications do not add unexpected or noteworthy features to the impulse responses.

while consumption takes far longer to return to its average. The slow recovery of output and consumption after real and financial frictions have been restored to their ordinary levels arises in part from the fact that the distribution of capital does not immediately settle back to its pre-shock state. As a result, aggregate productivity remains below normal until year 8, as seen in the third panel of the figure. This compounds the fact that the aggregate capital stock is more that 5 percent below its usual level by the start of the recovery.

Second, consumption does not begin to recover in date 5. Given a high demand for investment goods, and output's failure to rebound rapidly, households actually allow their consumption to fall for an additional period and thereafter raise it only very slowly. Third, during this episode, it is the labor input that drives the recovery. Anticipating the subsequent rise in endogenous productivity, and thus a raised return to savings, households abruptly raise their hours worked from 3.6 percent to only 1.1 percent below normal within date 5. In the next date, the allocation of capital across firms has begun to move back toward the long-run distribution, and the resulting improvement in productivity directly encourages a further large rise in the labor input. At this point, it overshoots its average level by just over 1 percent. Thereafter, it remains high for many periods while the capital stock is being rebuilt.

We may draw several conclusions about the implications of financial shocks from this third model-based exercise. First, absent any real shock to the economy, a temporary financial crisis on its own can generate a recession that is not only large, but persistent. Because tight borrowing conditions deliver a long-lived disruption to the distribution of capital, and thus to endogenous aggregate productivity, their aftermath is a long and anemic recovery in output and consumption of the sort one would never expect to see following a TFP shock. Moreover, when conditions in the financial sector do revert to normal, it is not household consumption expenditure, but instead business fixed investment, that leads the recovery, with this in turn derived from sharp increases in employment.

8 Concluding remarks

We have developed a dynamic stochastic general equilibrium model with collateralized borrowing constraints to explore how real and financial shocks interact in shaping aggregate fluctuations. In our model there is nontrivial heterogeneity in production; firms face persistent idiosyncratic shocks to their total factor productivity and irreversibilities in investment dampen

capital reallocation across firms. The extent of these real frictions is chosen to be consistent with microeconomic evidence on establishment level investment dynamics. Financial frictions impede capital reallocation from larger firms that are relatively unproductive, but less hindered by borrowing constraints, to smaller firms. In the steady state, the resultant change in the distribution of production reduces aggregate total factor productivity, and thus output, relative to an economy without collateralized borrowing constraints.

We find that the typical business cycle may be relatively unaffected by financial frictions. Nonetheless, a sharp reduction in lending brought about by an exogenous tightening of collateral requirements leads to a large, protracted recession in our model economy. This recession is qualitatively different from that which follows a technology shock, and it more closely resembles the recession recently observed in the US in several respects. The drop in GDP, employment and business investment is not greatest at the start of the recession; these series continue declining over subsequent dates, so that the overall responses are non-monotone. Furthermore, consumption actually rises slightly at the start of the recession, and it does not drop below average for several periods. Finally, in contrast to the response following a technology shock, once borrowing conditions return to normal, the recovery that follows is gradual and led by employment and business fixed investment. Household consumption recovers slowly.

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FIGURE 1. Steady state distribution in the full model

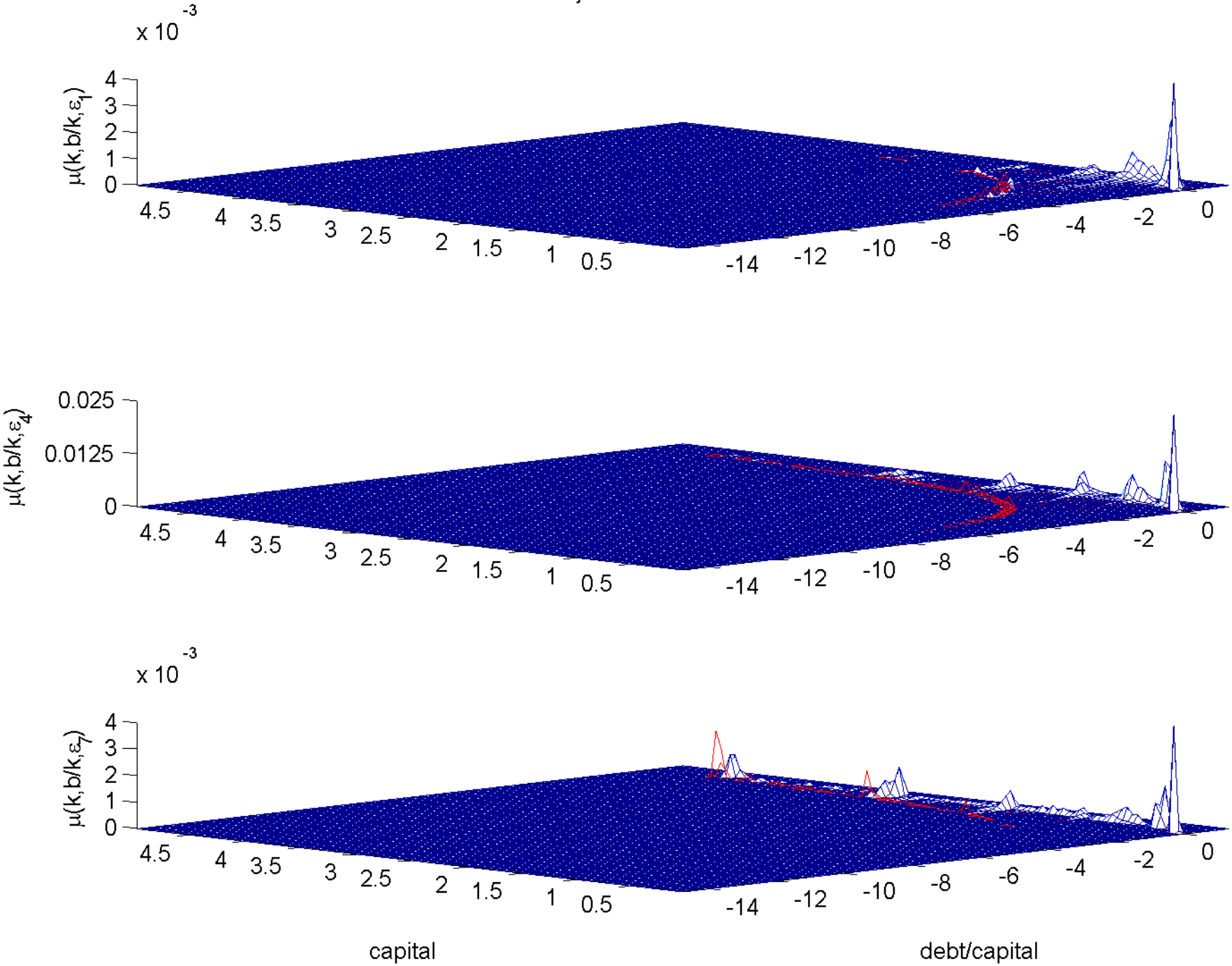

FIGURE 2. Cohort in steady state

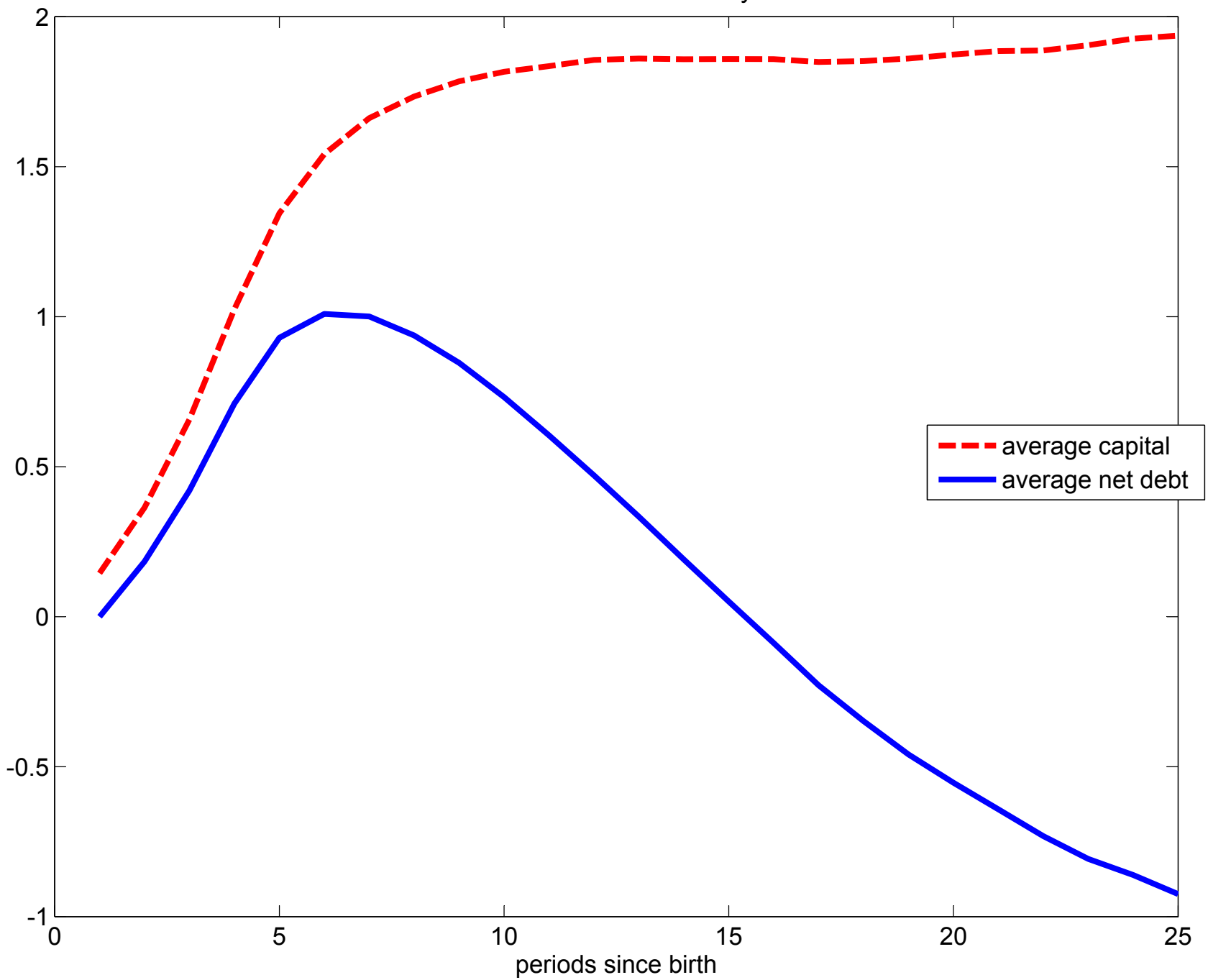

FIGURE 3. Cohort in no-financial-frictions steady state

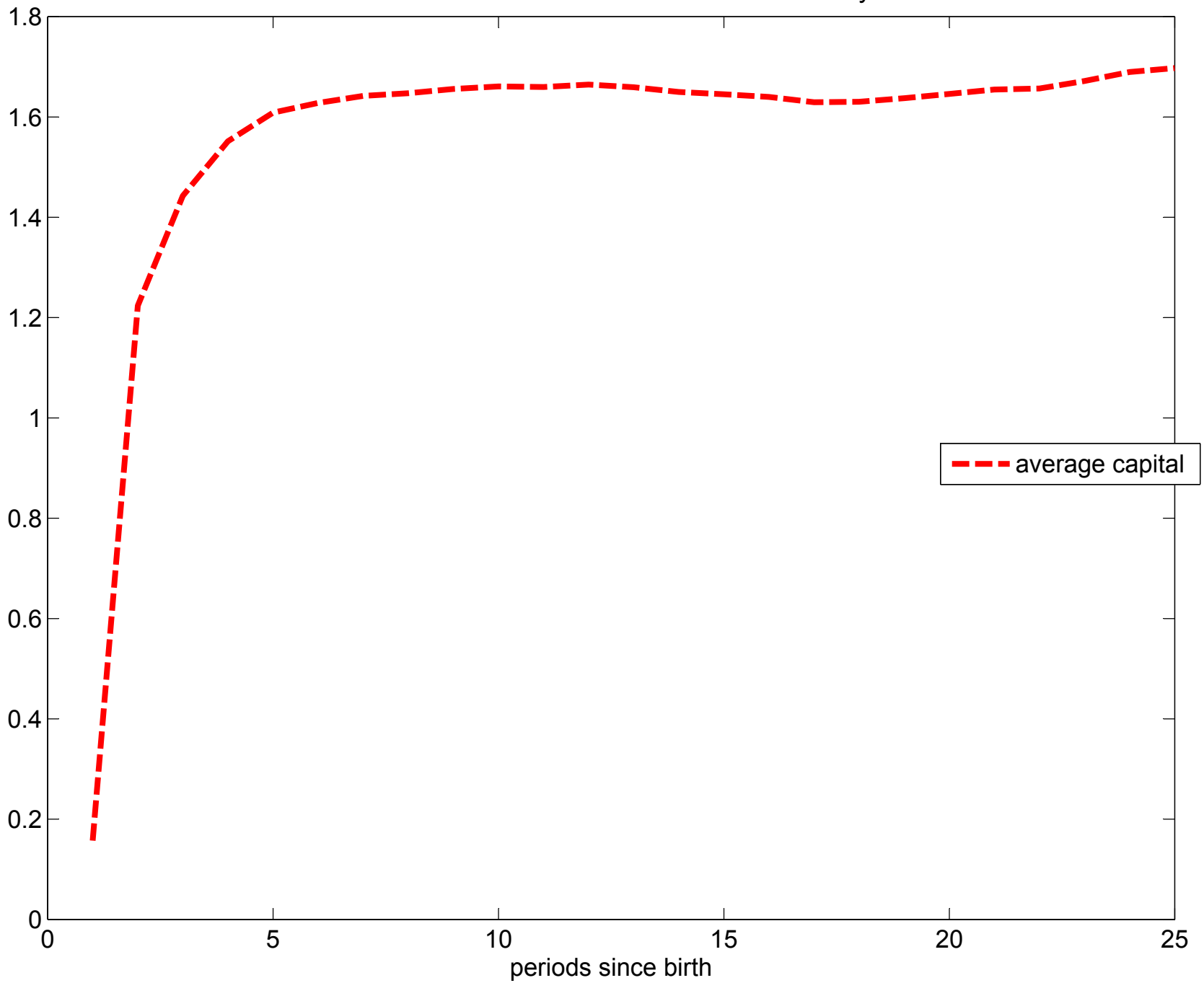

FIGURE 4. Capital choices among unconstrained firms

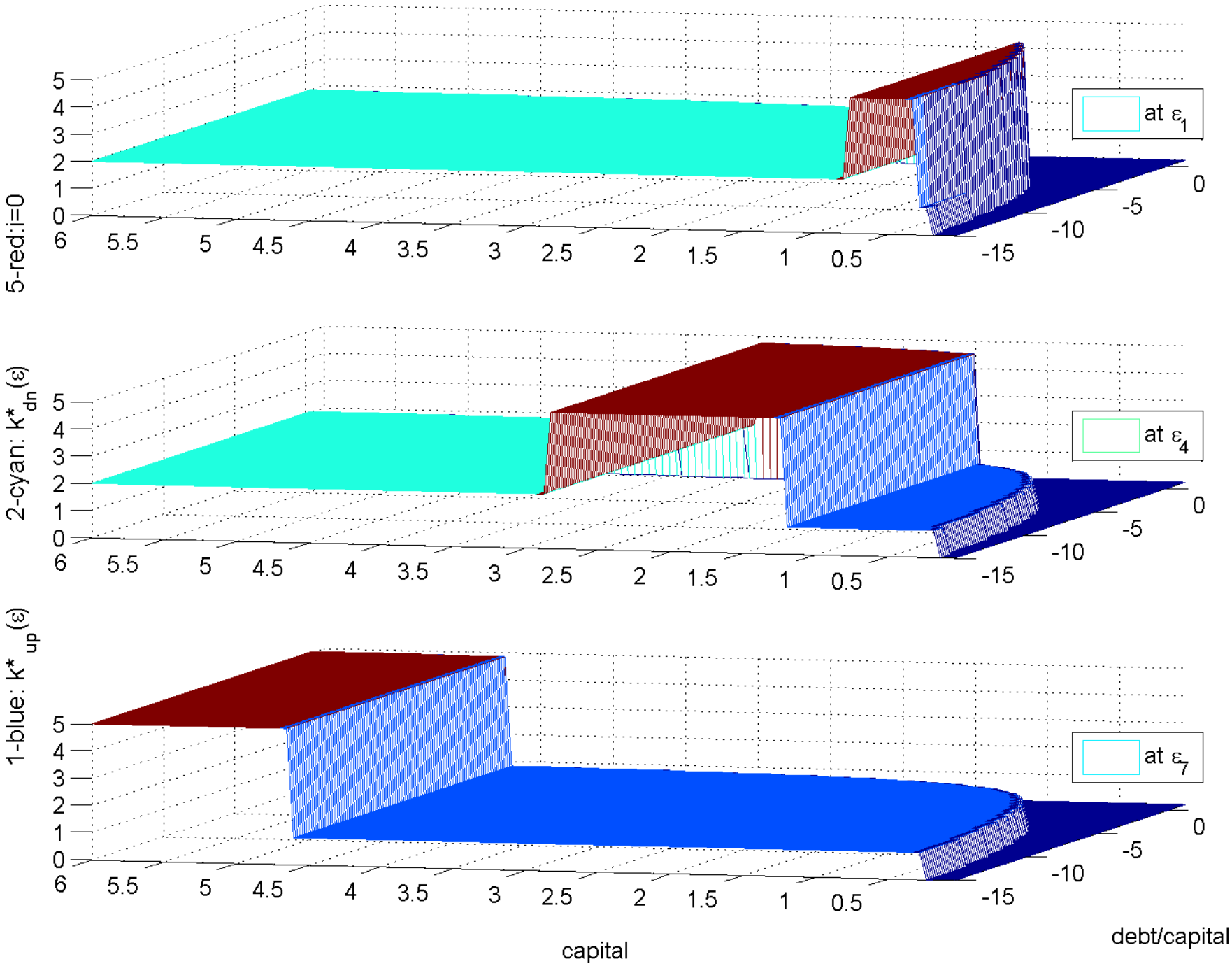

FIGURE 5. Capital choices among constrained firms

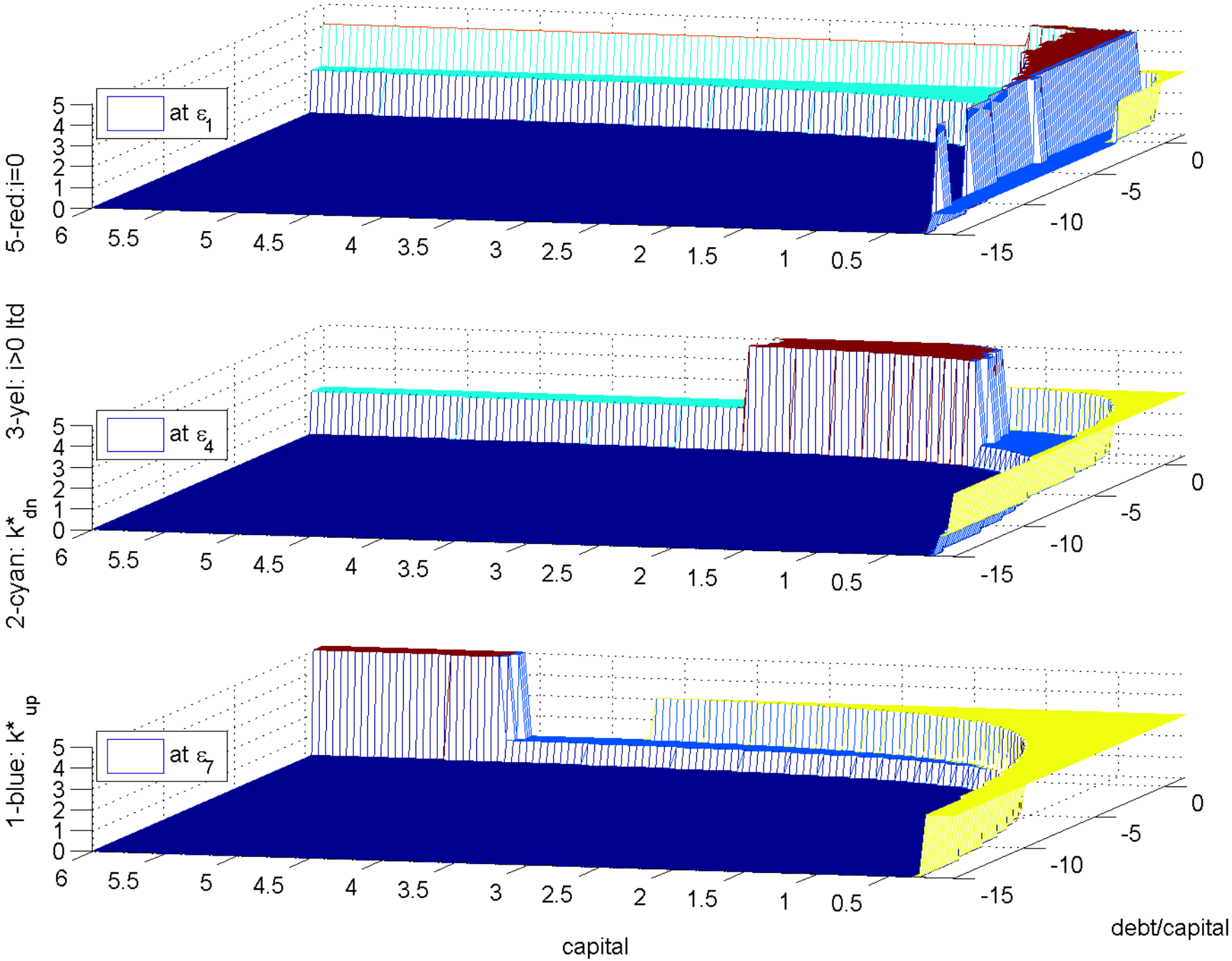

FIGURE 6. Pure technology shock

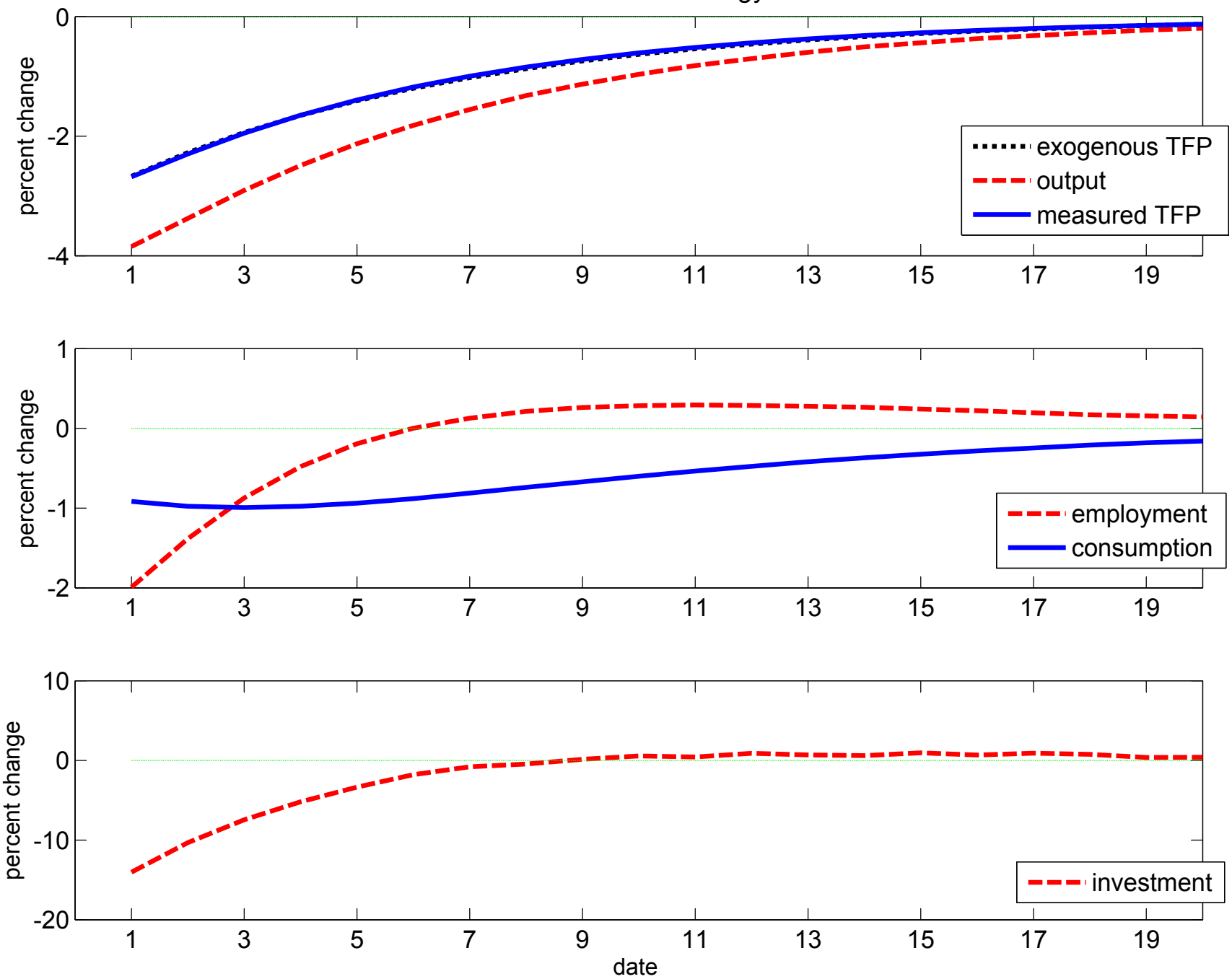

FIGURE 7. GDP, Consumption, Investment and Measured TFP over the Recent Recession

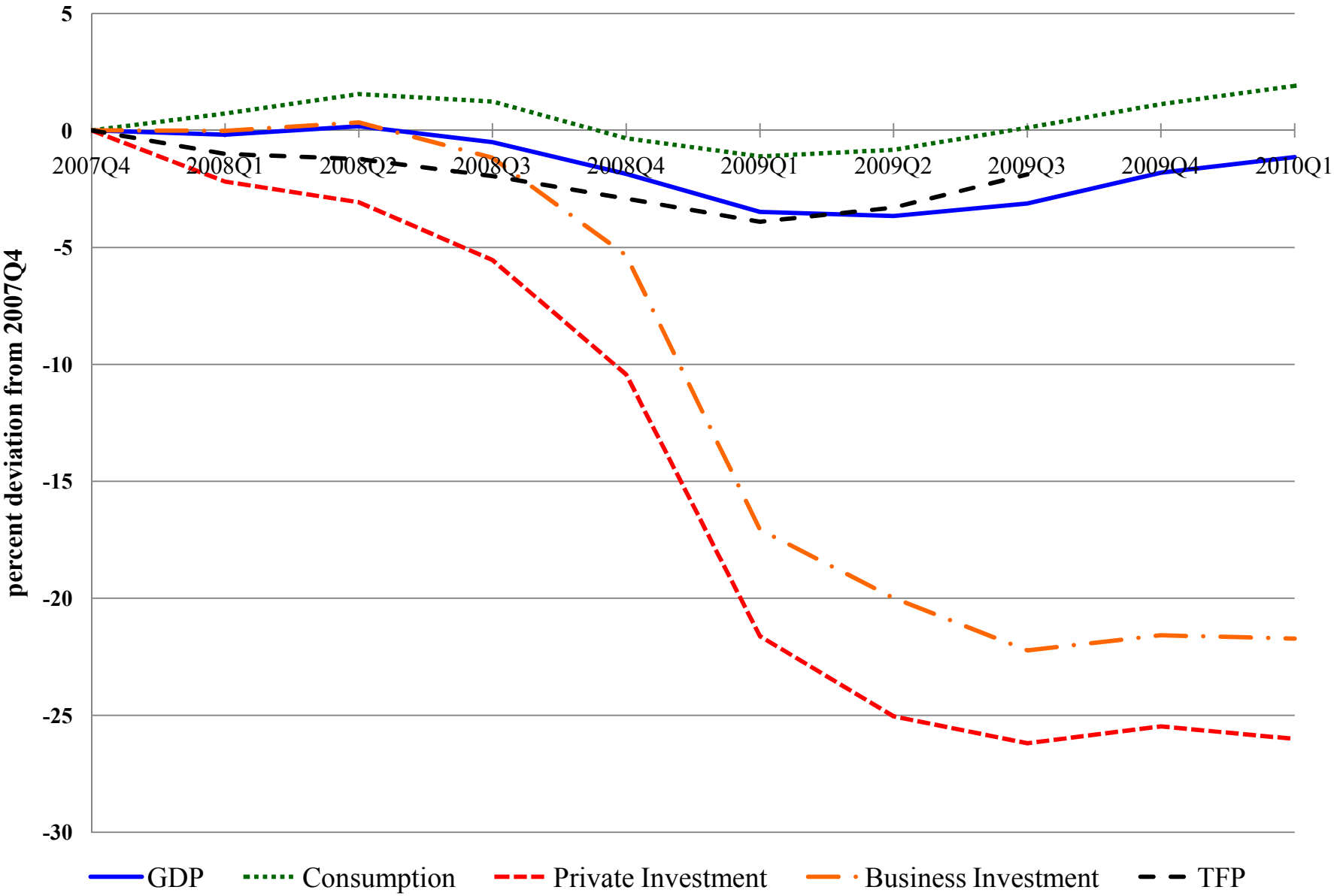

FIGURE 8. Persistent financial crisis

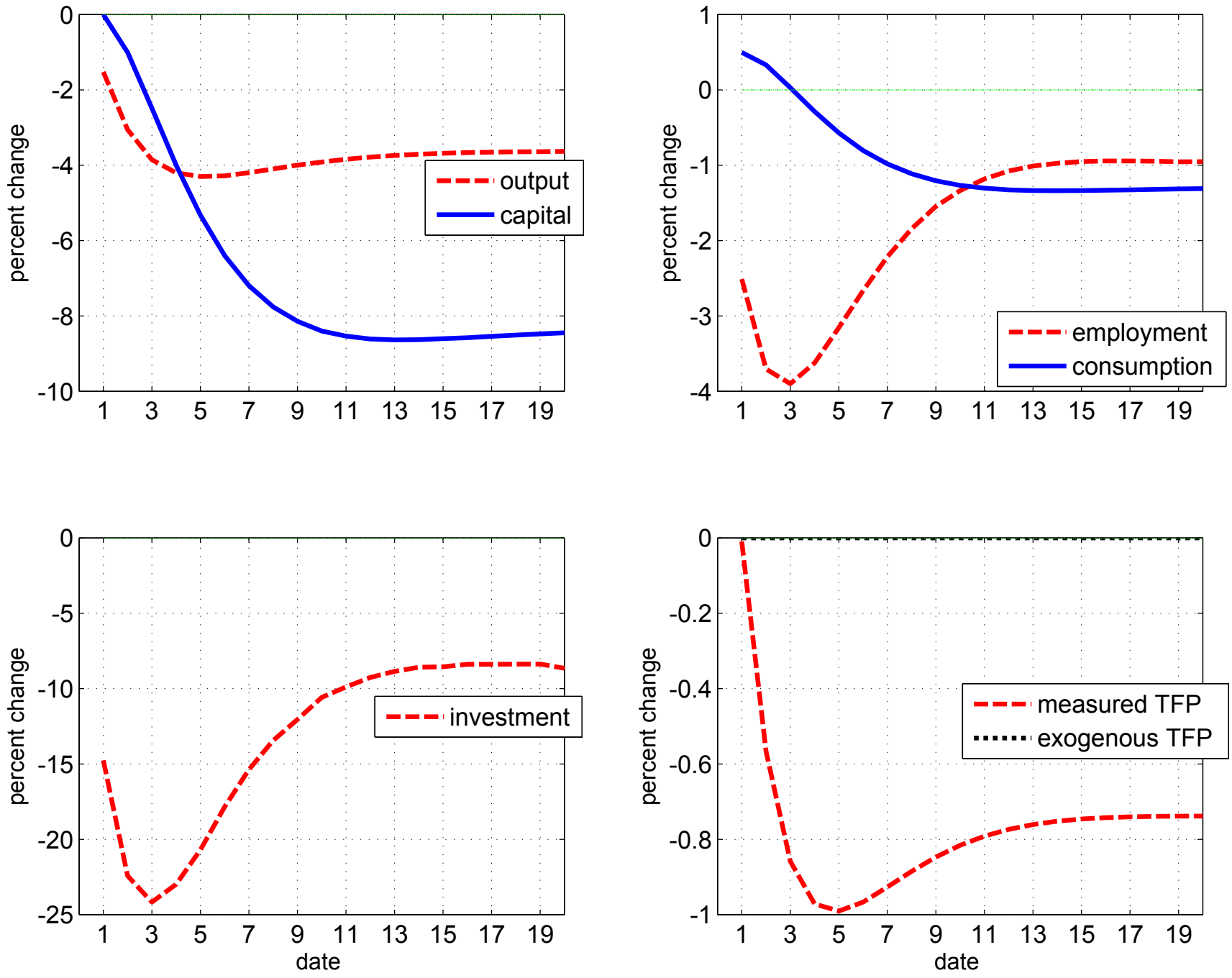

FIGURE 9. Financial crisis with a technology shock

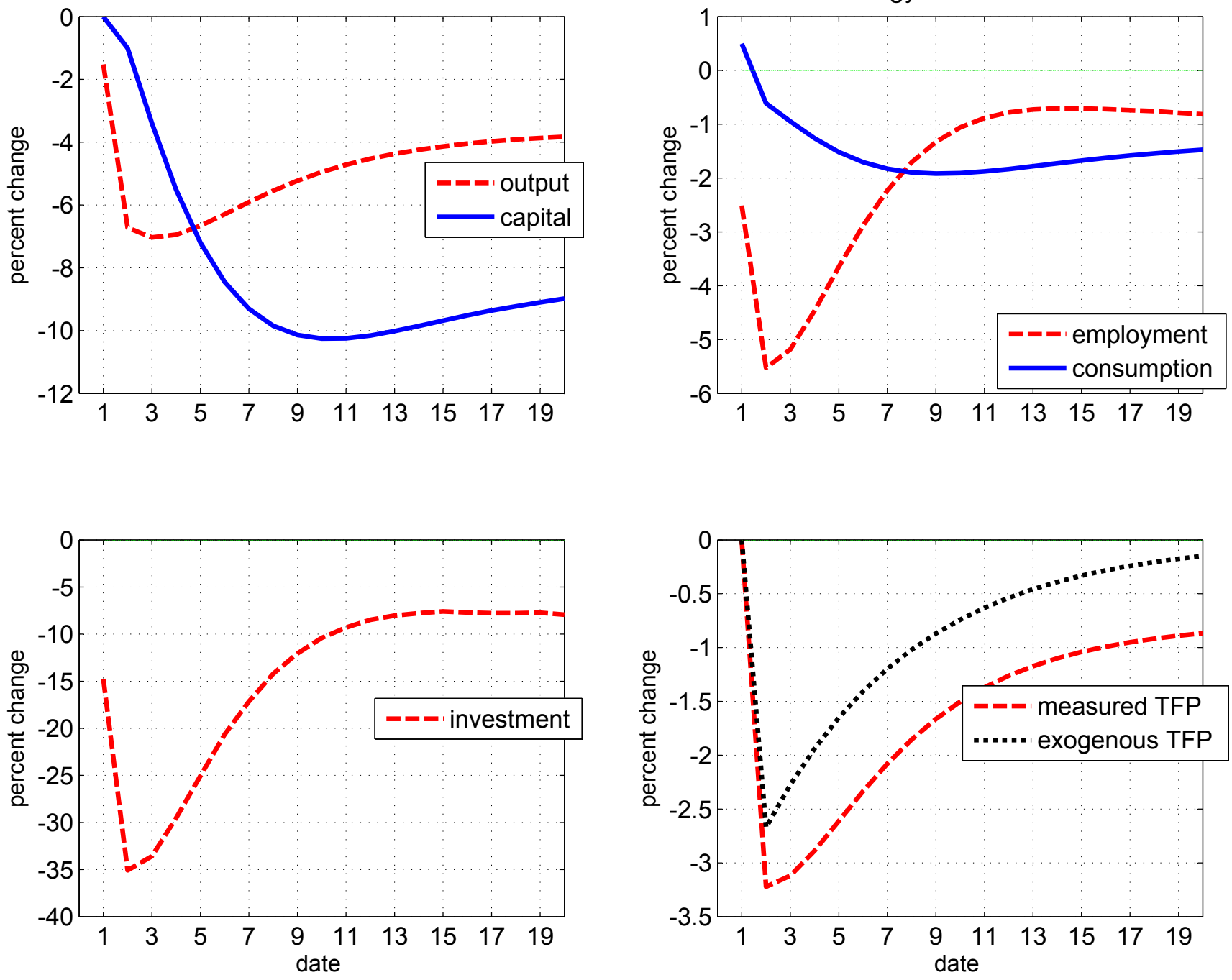

FIGURE 10. Financial crisis and recovery

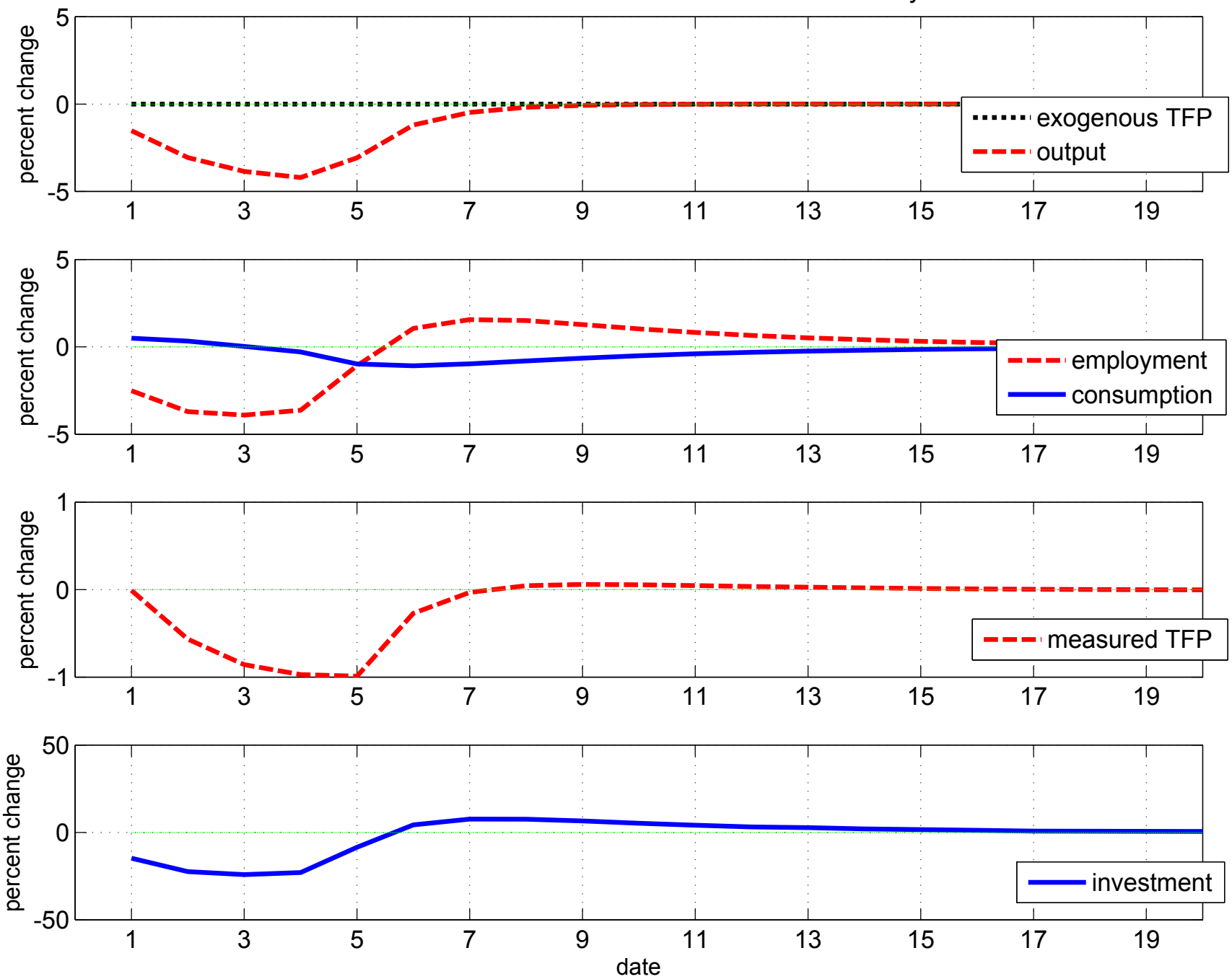

TABLE 1. Business Cycles in the Full Economy

$x =$	Y	C	I	N	K	r
mean(x)	0.581	0.487	0.094	0.333	1.321	0.042
σ_x/σ_Y	(1.919)	0.514	3.834	0.547	0.477	0.476
$corr(x, Y)$	1.000	0.939	0.968	0.946	0.066	0.665

TABLE 2. Business Cycles Without Financial Frictions

$x =$	Y	C	I	N	K	r
mean(x)	0.605	0.502	0.103	0.336	1.438	0.042
σ_x/σ_Y	(1.955)	0.497	3.768	0.568	0.471	0.454
$corr(x, Y)$	1.000	0.930	0.969	0.948	0.062	0.675

TABLE 3. Business Cycles Without Financial or Real Frictions

$x =$	Y	C	I	N	K	r
mean(x)	0.619	0.518	0.101	0.333	1.555	0.042
σ_x/σ_Y	(1.972)	0.479	4.037	0.588	0.451	0.440
$corr(x, Y)$	1.000	0.918	0.968	0.950	0.047	0.682