

A Theory of Credit Scoring and Competitive Pricing of Default Risk¹

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Abstract

We propose a theory of unsecured consumer credit where: (i) borrowers have the legal option to default; (ii) defaulters are not exogenously excluded from future borrowing; (iii) there is free entry of lenders; and (iv) lenders cannot collude to punish defaulters. In our framework, limited credit or credit at higher interest rates following default arises from the lender's optimal response to limited information about the agent's type and earnings realizations. The lender learns from an individual's borrowing and repayment behavior about his type and encapsulates his reputation for not defaulting in a credit score. We take the theory to data choosing the parameters of the model to match key data moments such as the overall and subprime delinquency rates. We test the theory by showing that our underlying framework is broadly consistent with the way credit scores affect unsecured consumer credit market behavior. The framework can be used to shed light on household consumption smoothing with respect to transitory income shocks and to examine the welfare consequences of legal restrictions on the length of time adverse events can remain on one's credit record.

1 Introduction

It is well known that lenders use credit scores to regulate the extension of consumer credit. People with high scores are offered credit on more favorable terms. People who default on their loans experience a decline in their scores and, therefore, lose access to credit on favorable terms. People who run up debt also experience a decline in their credit scores and have to pay higher interest rates on new loans. While credit scores play an important role in the allocation of consumer credit, credit scoring has not been adequately integrated into the theoretical literature on consumption smoothing and asset pricing. This paper attempts to remedy this gap.¹

We propose a theory of unsecured consumer credit where: (i) borrowers have the legal option to default; (ii) defaulters are not exogenously excluded from future borrowing; (iii) there is free entry of lenders; and (iv) lenders cannot collude to punish defaulters. We use the framework to try to understand why households typically face limited credit or credit at higher interest rates following default and why this changes over time. We show such outcomes arise from the lender's optimal response to limited information about the agent's type and earnings realizations. The lender learns from an individual's borrowing and repayment behavior about his type and encapsulates his reputation for not defaulting in a credit score.

The legal environment surrounding the U.S. unsecured consumer credit market is characterized by the following features. Individual debtors have can file for bankruptcy under Chapter 7 which permanently discharges net debt (liabilities minus assets above statewide exemption levels). A Chapter 7 filer is ineligible for a subsequent Chapter 7 discharge for 6 years. During that period, the individual is forced into Chapter 13 which is typically a 3-5 year repayment schedule followed by discharge. Over two-thirds of household bankruptcies in the U.S. are Chapter 7. The Fair Credit Reporting Act requires credit bureaus to exclude the filing from credit reports after 10 years (and all other adverse items after 7 years).

¹One important attempt to remedy this deficiency in the consumption smoothing literature is Gross and Souleles [15]. That paper empirically tests whether consumption is excessively sensitive to variations in credit limits taking into account a household's risk characteristics embodied by credit scores.

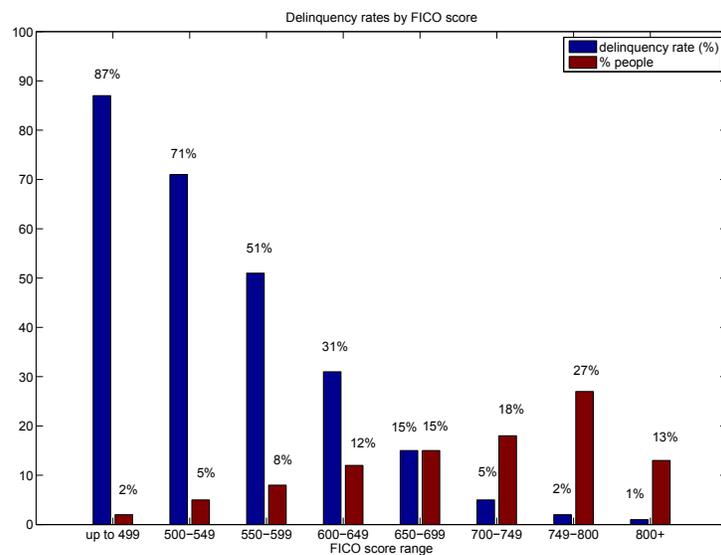
Beginning with the work of Athreya [2], there has been a growing number of papers that have tried to understand bankruptcy data using quantitative, heterogeneous agent models (for example Chatterjee, et. al. [8], Livshits, et. al. [20]). For simplicity, these models have assumed that an individual is exogenously excluded from borrowing while a bankruptcy remains on his credit record. This exclusion restriction is often modelled as a Markov process and calibrated so that on average the household is excluded for 10 years, after which the Fair Credit Reporting Act requires that it be stricken from the household's record. This assumption is roughly consistent with the findings by Musto [21] who documents the following important facts: (1) households with low credit ratings face very limited credit lines (averaging around \$215) prior to and \$600 following the removal of a bankruptcy flag; (2) for households with medium and high credit ratings, their average credit lines were a little over \$800 and \$2000 respectively prior to the year their bankruptcy flag was removed from their record; and (3) for households with high and medium credit ratings, their average credit lines jumped nearly doubled to \$2,810 and \$4,578 in the year that the bankruptcy flag was removed from their record.²

While this exogenous exclusion restriction is broadly consistent with the empirical facts, a fundamental question remains. Since a Chapter 7 filer is ineligible for a subsequent Chapter 7 discharge for 6 years (and at worst forced into a subsequent Chapter 13 repayment schedule), why don't we see more lending to those who declare bankruptcy? If lenders believe that the Chapter 7 bankruptcy signals something relatively permanent about the household's unobservable characteristics, then it may be optimal for lenders to limit future credit. But if the circumstances surrounding bankruptcy are temporary (like a transitory, adverse income shock), those individuals who have just shed their previous obligations may be a good future credit risk. Competitive lenders use current repayment and bankruptcy status to try to infer an individual's future likelihood of default in order to correctly price loans. There is virtually no existing work embedding this inference problem into a quantitative, dynamic model.

Given commitment frictions, it's important for a lender to assess the probability that a borrower

²These numbers are actually drawn from Table III, panel A of Musto's Wharton working paper #99-22.

Figure 1: Delinquency Rates in the Population



will fail to pay back – that is, assess the risk of default. In the U.S., lenders use *credit scores* as an index of the risk of default. The credit scores most commonly used are produced by a single company, the Fair Isaac and Company, and are known as FICO scores.³ These scores range between 300 and 850, where a higher score signals a lower probability of default. Scores under 620, which account for roughly one quarter of the population with scores, are called “subprime”.⁴ There is ample empirical evidence that households with subprime credit scores are more likely to default. Figure 1 provides one such example. As discipline on our theory, we require our framework to match key credit market facts like that in Figure 1.

A FICO score aggregates information from an individual’s credit record like his payment history (most particularly the presence of adverse public records such as bankruptcy and delinquency) and current amounts owed.⁵ It’s also worth noting the kinds of information that are not used in

³Over 75% of mortgage lenders and 80% of the largest financial institutions use FICO scores in their evaluation and approvals process for credit applications.

⁴<http://www.privacyrights.org/fs/fs6c-CreditScores.htm>.

⁵The score also takes into account the length of a person’s credit history, the kinds of credit accounts (retail credit, installment credit etc.) and the borrowing capacity (or line of credit) on each account.

credit scores. By law, credit scores cannot use information on race, color, national origin, sex, and marital status. Further, FICO scores do not use age, assets, salary, occupation, and employment history.

These scores appear to affect the extension of consumer credit in four primary ways.

1. Credit terms (e.g. interest rates) improve with a person's credit score.
2. The presence of adverse public records (e.g. a bankruptcy) lowers an individual's score and removal can substantially raise it.
3. Taking on more debt (paying off debt) tends to lower (raise) credit scores.
4. Credit scores are mean reverting.

The Fico website (<http://www.myfico.com/myfico/Credit Central/LoanRates.asp>) documents the negative relationship between FICO scores and average interest rates on loans. Item 2 is consistent with evidence provided in Musto [21], as well as Fisher, et. al. [13]. Using data from the PSID and SCF, Fisher, et. al. document that a higher percentage of post-bankruptcy households were denied access to credit. Musto found (p.735) “there is a strong tenth year effect for the best initial credits...these consumers move ahead of 19% of the nonfiler population in apparent credit-worthiness when their flags are removed.” Furthermore, he states (p.740) “...the boost translates to significant new credit access for these filers over the ensuing year”. Items 1 and 2 taken together imply that an individual who fails to pay back an unsecured loan will experience an adverse change in the terms of (unsecured) credit. Thus, a failure to pay back a loan adversely impacts the terms of credit and may result in outright denial of credit. Item 3 is consistent with the advice given by FICO for improving one's credit score.⁶ Item 3 in conjunction with item 1 indicates that even absent default, the terms of credit on unsecured credit worsen as an individual gets further into debt – people face a rising marginal cost of funds. Item 4 is documented by Musto [21].

⁶To improve a score, FICO advises to “Keep balances low on credit card and ‘other revolving credit’” and “[p]ay off debt rather than moving it around”. Source:www.myfico.com/CreditEducation/ImproveYour Score

These facts suggest the following characterization of the workings of the unsecured consumer credit market. Given the inability of borrowers to commit to pay back, lenders condition the terms of credit (including whether they lend at all) on an individual's credit history encapsulated by a credit score. Individuals with higher scores are viewed by lenders as less likely to default and receive credit on more attractive terms. A default may signal something about the borrower's future ability to repay and leads to a drop in the individual's credit score. Consequently, post-default access to credit is available on worse terms and may not be available at all. Even absent default, greater indebtedness may signal something about the borrower's future ability to repay which subsequently leads to a lower credit score and worse terms of credit.

There is now a fairly substantial literature (beginning with Kehoe and Levine [19]) on how and to what extent borrowing can occur when agents cannot commit to pay back. This literature typically assumes that a default triggers permanent exclusion from credit markets. A challenge for this literature is to specify a structure with free entry of lenders and where lenders cannot collude to punish defaulters that can make quantitative sense of the characterization of a competitive unsecured consumer credit market with on-the-equilibrium-path default offered in the previous paragraphs. This paper takes steps toward meeting this challenge.⁷ We consider an environment with a continuum of infinitely-lived agents who at any point in time may be one of two types that affect their earnings realizations and preferences. An agent's type is drawn independently from others and follows a persistent two-state Markov process. Importantly, a person's type and earnings realizations are unobservable to the lender.⁸

These people interact with competitive financial intermediaries that can borrow in the international credit market at some fixed risk-free rate and make one-period loans to individuals at an interest rate that reflects that person's risk of default.⁹ Because differences in earnings distributions and preferences bear on the willingness of each type of agent to default, intermediaries

⁷In Chatterjee, et.al. [9] we show that credit can be supported even in a finite horizon model where trigger strategies cannot support credit.

⁸Ausubel [4] documents adverse selection in the credit market both with respect to observable and unobservable household characteristics.

⁹Our earlier paper Chatterjee, et. al. [8] shows that there is not a big gain to relaxing the fixed risk-free rate assumption.

must form some assessment of a person's type which is an input into his credit score. We model this assessment as a Bayesian inference problem: intermediaries use the recorded history of a person's actions in the credit market to update their prior probability of his or her type and then charge an interest rate that is appropriate for that posterior. The fundamental inference problem for the lender is to assess whether a borrower or a defaulter is a chronically "risky" type or just experiencing a temporary shortfall in earnings. A rational expectations equilibrium requires that a lender's perceived probability of an agent's default must equal the objective probability implied by the agent's decision rule. Incorporating this equilibrium Bayesian credit scoring function into a dynamic incomplete markets model is the main technical challenge of our paper.

We model the pricing of unsecured consumer loans in the same fashion as in our predecessor paper Chatterjee, *et.al.* [8]. As in that paper, all one-period loans are viewed as discount bonds and the price of these bonds depend on the size of the bond. This is necessary because the probability of default (for any type) will depend on the size of the bond (i.e., on the person's liability). If the bond price is independent of the size of the loan and other characteristics, as it is in Athreya [2], then large loans which are more likely to be defaulted upon must be subsidized by small loans which are less likely to be defaulted upon. But with competitive credit markets, such cross subsidization of pooling contracts will fail to be an equilibrium. This reasoning is corroborated by recent empirical work by Edelberg [12] who finds that there has been a sharp increase in the cross-sectional variance of interest rates charged to consumers.

In Chatterjee, *et.al.* [8], we also assumed that the price of a one-period bond depended on certain observable household type characteristics like whether households were blue or white collar workers. Here we assume those characteristics are not observable but instead assume that the bond depends on the agent's probability of repayment, in other words, his credit score. The probability of repayment depends on the posterior probability of a person being of a given type *conditional* on selling that particular sized bond. This is necessary because the two types will not have the same probability of default for any given sized bond and a person's asset choice is potentially informative about the person's type. With this asset market structure, competition implies that the

expected rate of return on each type of bond is equal to the (exogenous) risk-free rate.

This is possibly the simplest environment one could imagine that could make sense of the observed connection between credit history and the terms of credit. Suppose it turns out that, in equilibrium, one type of person, say type g , has a lower probability of default. Then, under competition, the price of a discount bond (of any size) could be expected to be positively related to the probability of a person being of type g . Further, default will lower the *posterior* probability of being of type g because type g people default less frequently. This provides the basis for a theory why people with high scores are offered credit on more favorable terms. This would explain the fact that people with high scores are offered credit on more favorable terms.

There are two strands of existing literature to which our paper is closely related. The first strand relates to Diamond's [11] well-known paper on acquisition of reputation in debt markets. Besides differences in the environment (e.g. preferences in his case are risk neutral), the main difference is that here the decision to default is endogenous while in Diamond it happens exogenously. The second strand relates to the paper of Cole, Dow and English [10] on sovereign debt.¹⁰ In their setting a sovereign who defaults is shut out of international credit markets until such time as the sovereign makes a payment on the defaulted debt. Chapter 7 bankruptcy law, which we consider here, results in discharge of uncollateralized debt.¹¹ Further, the law does not permit individuals to simultaneously accumulate assets during the discharge of debt granted by the bankruptcy court.¹²

Our framework has the ability to address an interesting question that arises from Musto's empirical work. What are the effects on consumption smoothing and welfare of imposing legal restrictions (like the Fair Credit Reporting Act), which requires adverse credit information (like a bankruptcy) to be stricken from one's record after a certain number of years (10 in the U.S.)?

¹⁰Athreya et. al. [3] also consider a signalling model but assume anonymity so that past asset market choices encapsulated in a type score cannot be used as a prior when calculating posteriors associated with current asset market choices.

¹¹Given the choice between Chapter 7 and 13, individuals would choose to file Chapter 13 only if they wished to keep assets they would lose under a Chapter 7 filing. Since borrowers in our model have negative net worth (there is only one asset), Chapter 7 is always the preferred means to file for bankruptcy.

¹²This fact rules out the purchase of consumption insurance from savings in the period of discharge studied by Bulow and Rogoff [7].

Specifically, Musto p. 726 states that his empirical “results bear on the informational efficiency of the consumer credit market, the efficacy of regulating this market with reporting limits, and the quality of postbankruptcy credit access, which in turn bears on the incentive to file in the first place.” He finds p. 747 “the removal of the flag leads to excessive credit, increasing the eventual probability of default. This is concrete evidence that the flag regulation has real economic effects. This is market efficiency in reverse.” We use our model to assess this efficiency concern. In a world of incomplete markets and private information, flag removal may provide insurance to impatient agents in our framework that competitive intermediaries may not be able to provide. Hence extending the length of time that bankruptcy flags remain on credit records may not necessarily raise ex-ante welfare. This issue echoes Hart’s [16] examples where the opening of a market in a world of incomplete markets may make agents worse off and Hirschleifer’s [18] finding regarding the potential inefficiency of revealing information.

The paper is organized as follows. Section 2 describes a model economy where there are no restrictions on information about asset market behavior, defines an equilibrium, and discusses existence. Section 3 describes a model economy where there are restrictions on what information on asset market behavior can be kept in an agents credit history. In particular, we assume that information can be kept only for a finite amount of time and that there are partitions on what asset transactions are recorded. These restrictions on information are intended to capture the requirement that adverse events be stricken from an individual’s credit history and the fact that credit scores are based on debt transactions rather than assets in the current system. Section 4 estimates parameters of the model of Section 3 to match certain key moments in the data. Section 5 studies the properties of the model. Section 6 assesses the welfare consequences of restrictions on asset market information used by credit scoring agencies like that in the model of Section 3 compares to the unrestricted case of Section 2. This exercise sheds some light on the impact of the Fair Credit Reporting Act.

2 Model Economy 1

2.1 People, Preferences and Endowments

Time is discrete and indexed by $t = 0, 1, 2, \dots$. There is a unit measure of infinitely-lived people alive at each date. At each date, a person can be one of two types, denoted $i_t \in \{g, b\}$. An individual of type g (or b) at time t can become an individual of type b (or g) at the beginning of time $t + 1$ with probability $\Gamma_{\{i_{t+1}=b, i_t=g\}} \in (0, 1)$ (or $\Gamma_{bg} \in (0, 1)$), respectively.¹³ Let γ denote the unconditional probability that an individual is of type g . An individual of type i_t draws her endowment e_t independently (across time and agents) from a probability space $(E, \mathcal{B}(E), \Phi_i)$, where $E = [\underline{e}, \bar{e}] \subset \mathbb{R}_{++}$ is a strictly positive closed interval and $\mathcal{B}(E)$ is the Borel sigma algebra generated by E . Further, we assume Φ_i is absolutely continuous with respect to the Lebesgue measure on E and the density $\phi_i(\bar{e}) > 0$ for some i .

Denote the life-time utility from a non-negative stream of current and future consumption $\{c_t, c_{t+1}, c_{t+2}, \dots\}$ of an individual who is of type i_t by $U_i(c_t, c_{t+1}, c_{t+2}, \dots, \theta_t)$ where $\theta_t \in \Theta$ is an independent (across time and agents) time preference shock drawn at time t from a finite set with probability mass function Λ . For each i , $U_i(c_t, c_{t+1}, c_{t+2}, \dots, \theta_t)$ is defined by the recursion

$$U_i(c_t, c_{t+1}, c_{t+2}, \dots, \theta_t) = u_i(c_t) + \beta_i \theta_t \sum_{j, \theta_{t+1}} \Gamma_{ji} U_j(c_{t+1}, c_{t+2}, c_{t+3}, \dots, \theta_{t+1}) \Lambda(\theta_{t+1}) \quad (1)$$

where, for all i , $u_i(c_t) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a bounded, continuous, twice differentiable and strictly concave function with bounded derivatives and $\beta_i \in [0, 1)$.

Importantly, we assume that a person's type i_t , endowment e_t , and time preference shock θ_t are unobservable to others.

¹³This is a similar assumption to Phelan [22], who studies reputation acquisition by a government.

2.2 Default Option and Market Arrangement

There is a competitive credit industry that accepts deposits and makes loans to individuals. We assume that there is a finite set $L \subset \mathbb{R}$ of possible loans or deposits (L contains negative and positive elements as well as 0). If an individual takes out a loan $\ell_{t+1} < 0$ at time t there is some probability p_t that the individual will repay ℓ_{t+1} units of goods at time $t + 1$. If $\ell_{t+1} > 0$ then the individual makes a deposit which we assume that the intermediary promises to pay back with probability $p_t = 1$ for simplicity.

A probability of repayment $p_t < 1$ reflects the possibility of default on the part of the individual. We model the default option to resemble, in procedure, a Chapter 7 bankruptcy filing. If an individual defaults, the individual's beginning of period liabilities are set to zero (i.e., the individual's debt is discharged) and the individual is not permitted to enter into new contracts in the period of default.

There is a competitive market in financial contracts. The unit price of a financial contract (ℓ_{t+1}, p_t) is $q(\ell_{t+1}, p_t)$. For $\ell_{t+1} < 0$, $q(\ell_{t+1}, p_t) \cdot (-\ell_{t+1})$ is the amount received by an individual at time t who promises to pay ℓ_{t+1} next period with probability p_t . For $\ell_{t+1} > 0$, $q(\ell_{t+1}, 1) \cdot \ell_{t+1}$ is the amount handed over by the individual at time t in return for the certain promise to receive ℓ_{t+1} next period.

As noted earlier, there are two types of people in this economy. Let $s_t \in [0, 1]$ be the prior probability at time t that a person is of type g . Beliefs about an individual's type are important to lenders because the probability of repayment on a consumer loan may (and will) vary across types. An important part of the market arrangement is the existence of an agency that collects information on financial transactions of every individual and, using this information, estimates the probability s_t that a given individual is of type g at time t . We call an individual's estimated repayment probability the individual's *credit score*. We call the agency that computes this score the *credit scoring agency*. And, we call the type probability s on which the credit score (or the

repayment probability) is based on an individual's *type score*.¹⁴

Thus the existence of the credit scoring agency implies the presence of two functions that are part of the market arrangement. First, there is a *credit scoring function* $p(\ell_{t+1}, \psi)$ which gives the estimated probability of repayment on a loan $\ell_{t+1} < 0$ taken out by an individual with type score ψ . And, second, there is a *type score updating function* $\psi^{(d_t, \ell_{t+1})}(\ell_t, s_t)$ which gives an individual's type score at the start of next period conditional on having begun the current period with asset ℓ_t and type score s_t and choosing (d_t, ℓ_{t+1}) – a choice of default corresponds to the 2-tuple $(1, 0)$ and choice of loan/deposit ℓ_{t+1} corresponds to the 2-tuple $(0, \ell_{t+1})$ (the precise definitions of these functions will be given in the next section).

2.3 Decision Problems

2.3.1 People

Let a current variable, say a_t , be denoted a and let next period's variable a_{t+1} be denoted a' . In the special case of assets/liabilities we will let ℓ_{t+1} be denoted y and ℓ_t be denoted x . Let $\mathbb{Y} = \{(d, y) : (d, y) \in (0 \times L) \text{ or } (d, y) = (1, 0)\}$ be the set of possible (d, y) choices (recall that a person can borrow or save only if she does not default and if she defaults then she cannot borrow or save).

Each individual takes as given

- the price function $q(y, p) : \{L_{--} \times [0, 1]\} \cup \{L_+ \times \{1\}\} \rightarrow R$,
- the credit scoring function $p(y, s') : L_{--} \times [0, 1] \rightarrow [0, 1]$, and
- the type scoring function $\psi^{(d, y)}(x, s) : \mathbb{Y} \times L \times [0, 1] \rightarrow [0, 1]$.

¹⁴Nothing depends on the assumption that there are only two types. With $I > 2$ types, we could let s_t be a $I - 1$ length vector (and correspondingly ψ be a vector valued function). Even in this case, the credit score p_t is just the probability of repayment on a loan.

We can now develop the recursive formulation of an individual's decision problem. The state variables for an individual are (i, e, θ, x, s) . We begin with the definition of the set of feasible actions.

Definition 2.1 *Given (e, x, s) , the set of feasible actions is a finite set $B(e, x, s; q, p, \psi) \subset \mathbb{Y}$ that contains: (i) all $(0, y)$ where $y < 0$ such that $c = e + x - q(y, p(y, s')) \cdot y \geq 0$, where $s' = \psi^{(d,y)}(x, s)$; (ii) all $(0, y)$ where $y \geq 0$ such that $c = e + x - q(y, 1) \cdot y \geq 0$; and (iii) if $x < 0$ it also contains $(1, 0)$.*

Observe that the feasible action set does not depend on i nor θ since these are not directly known either to financial intermediaries or to the credit scoring agency. The credit scoring agency assigns probability ψ to the individual being of type g and the set of feasible actions does depend on these probabilities. The dependence of the feasible action set on the functions p , q and ψ is noted.

We permit randomization so individuals choose probabilities over elements in the set of feasible actions. We will use $m^{(d,y)} \in [0, 1]$ to denote the probability mass on the element $(d, y) \in \mathbb{Y}$ and m as the choice probability vector. Let $B_+(e, x, s; q, p, \psi) \subseteq B(e, x, s; q, p, \psi)$ denote the set of (d, y) choices that yield strictly positive consumption.

Definition 2.2 *Given (e, θ, x, s) the feasible choice set $M(e, \theta, x, s; q, p, \psi)$ is the set of all $m \geq 0$ such that: (i) $m^{(d,y)} = 0$ for all $(d, y) \notin B(e, x, s; q, p, \psi)$; (ii) $m^{(d,y)} \geq \epsilon$ for all $(d, y) \in B_+(e, x, s; q, p, \psi)$; and (iii) $\sum_{(d,y) \in \mathbb{Y}} m^{(d,y)} = 1$.*

In order to keep the type score updating function well defined across all actions (thereby avoiding having to supply an exogenous set of off-the-equilibrium-path beliefs), we assume each feasible probability vector m assigns at least some small probability $\epsilon > 0$ on every action that yields strictly positive consumption. In addition, we will assume that $\bar{e} + \ell_{min} - \ell_{max} > 0$. These assumptions guarantee that every (d, y) choice yields strictly positive consumption for some agent and,

therefore, will be chosen with positive probability in any equilibrium. We interpret these outcomes as people making “tiny mistakes”, similar to the “trembling hand” assumption made in Selten [23].

Given (i, e, θ, x, s) and the functions p, q and ψ , the current-period return of a type i individual from choosing a feasible action $(0, y)$ is

$$R_i^{(0,y)}(e, x, s; q, p, \psi) = \begin{cases} u_i(e + x - q(y, p(y, \psi^{(0,y)}(x, s))) \cdot y & \text{if } y < 0 \\ u_i(e + x - q(y, 1) \cdot y & \text{if } y \geq 0 \end{cases}$$

and the current-period return from choosing $(1, 0)$ (if this choice is feasible) is

$$R_i^{(1,0)}(e, x, s; q, p, \psi) = u_i(e).$$

Denote by $V_i(e, \theta, x, s; q, p, \psi) : E \times \Theta \times L \times [0, 1] \rightarrow R$ the value function of a type i individual. Then, a currently type- i individual’s recursive decision problem is given by

$$V_i(e, \theta, x, s; q, p, \psi) = \max_{m \in M(e, \theta, x, s; q, p, \psi)} \sum_{(d,y)} \left[\begin{array}{c} R_i^{(d,y)}(e, x, s; q, p, \psi) \\ + \beta_i \theta \sum_{j \in \{g,b\}, \theta' \in \Theta} \Gamma_{ji} \left\{ \int_E V_j(e', \theta', y, \psi^{(d,y)}(x, s); q, p, \psi) \Phi_j(de') \right\} \Lambda(\theta') \end{array} \right] \cdot m^{(d,y)}. \quad (2)$$

Denote the optimal decision correspondence by $M_i^*(e, \theta, x, s; q, p, \psi)$ and a given selection from this correspondence by $m_i^*(e, \theta, x, s; q, p, \psi)$.

2.3.2 Financial Intermediary

The (representative) financial intermediary has access to an international credit market where it can borrow or lend at the risk-free interest rate $r \geq 0$. The intermediary operates in a competitive market and takes the price function $q(y, p)$ as given. The profit $\pi(y, p)$ on financial contract of type

(y, p) is:

$$\pi(y, p) = \begin{cases} (1+r)^{-1}p \cdot (-y) - q(y, p) \cdot (-y) & \text{if } y < 0 \\ q(y, 1) \cdot y - (1+r)^{-1} \cdot y & \text{if } y \geq 0 \end{cases} \quad (3)$$

Let $\mathcal{B}(L \times [0, 1])$ be the Borel sets of $L \times [0, 1]$. Let \mathcal{A} be the set of all measures defined on the measurable space $(L \times [0, 1], \mathcal{B}(L \times [0, 1]))$. For $\alpha \in \mathcal{A}$, $\alpha(y, P)$ is the measure of financial contracts of type $(y, P) \in \mathcal{B}(L \times [0, 1])$ sold by the financial intermediary. The decision problem of the financial intermediary is:

$$\max_{\alpha \in \mathcal{A}} \int \pi(y, p) d\alpha(y, p).$$

2.3.3 Credit Scoring Agency

We do not explicitly model the process by which the credit scoring agency computes type scores and credit scores. Instead, we impose restrictions on the outcome of this process. Specifically we assume that (i) $p(y, s')$ is the fraction of people with loan y and type score s' who repay and (ii) $\psi^{(d,y)}(x, s)$ is the fraction of type g among people who start with assets x , type score s , and choose (d, y) .

Denoting the fraction of type i agents choosing action (d, y) by $P_i^{(d,y)}$, we have

$$P_i^{(d,y)}(\theta, x, s; q, p, \psi) = \int m_i^{(d,y)}(e, \theta, x, s; q, p, \psi) \Phi_i(de). \quad (4)$$

Then, condition (i) implies

$$\begin{aligned} p(y, s') &= s' \cdot \left[1 - \sum_{\theta'} \Lambda(\theta') P_g^{(1,0)}(\theta', y, s'; q, p, \psi) \right] \\ &\quad + (1 - s') \cdot \left[1 - \sum_{\theta'} \Lambda(\theta') P_b^{(1,0)}(\theta', y, s'; q, p, \psi) \right]. \end{aligned} \quad (5)$$

Further, condition (ii) implies

$$\begin{aligned} \psi^{(d,y)}(x, s; q, p, \psi) = & \\ (1 - \Gamma_{bg}) & \left[\frac{\sum_{\theta} \Lambda(\theta) P_g^{(d,y)}(\theta, x, s; q, p, \psi) s}{\sum_{\theta} \Lambda(\theta) P_g^{(d,y)}(\theta, x, s; q, p, \psi) s + \sum_{\theta} \Lambda(\theta) P_b^{(d,y)}(\theta, x, s, q, p, \psi) (1 - s)} \right] \\ + \Gamma_{gb} & \left[\frac{\sum_{\theta} \Lambda(\theta) P_b^{(d,y)}(\theta, x, s; q, p, \psi) (1 - s)}{\sum_{\theta} \Lambda(\theta) P_g^{(d,y)}(\theta, x, s; p, q, \psi) s + \sum_{\theta} \Lambda(\theta) P_b^{(d,y)}(\theta, x, s; q, p, \psi) (1 - s)} \right]. \end{aligned} \quad (6)$$

2.4 Equilibrium

We can now give the definition of a stationary recursive competitive equilibrium.

Definition 2.3 *A stationary recursive competitive equilibrium is: (i) a pricing function $q^*(y, p)$; (ii) a credit scoring function $p^*(y, s')$; (iii) a type scoring function $\psi^{*(d,y)}(x, s)$; and (iv) decision rules $m_i^*(e, \theta, x, s; q^*, p^*, \psi^*)$ such that*

- D1. $m_i^*(e, \theta, x, s; q^*, p^*, \psi^*)$ is a selection from $M_i^*(e, \theta, x, s; q^*, p^*, \psi^*)$,
- D2. $q^*(y, p)$ is such that $\pi(y, p; q^*(y, p)) = 0$ in (3) $\forall (y, p)$,
- D3. $p^*(y, s')$ satisfies condition (5) for $m_i^*(e, \theta, x, s; q^*, p^*, \psi^*)$, $i \in \{g, b\}$,
- D4. For all (d, y) , $\psi^{*(d,y)}(x, s)$ satisfies (6) for $m_i^*(e, \theta, x, s; q^*, p^*, \psi^*)$, $i \in \{g, b\}$.

2.5 Existence

To simplify the analysis and focus on variables of primary interest, the following preliminary lemma shows that the price function q , being a linear function of p , shares the continuity properties of the scoring functions p and ψ .

Lemma 2.1 *If the credit scoring function $p = p(y, s')$ is continuous in the type score s' for each y , and the type scoring function $s' = \psi^{(d,y)}(x, s)$ is continuous in the current type score s , then the price function $q = q(y, p(y, \psi^{(d,y)}(x, s)))$ is continuous in s for each (d, y) and x .*

Proof. Follows from the assumed continuity of p and s' and the zero profit condition in (3). ■

Given Lemma 2.1, the equilibrium problem reduces to finding a pair of functions $p^*(y, s')$ and $s' = \psi^{*(d,y)}(x, s)$ such that *D1-D4* hold. To prove existence we take the following steps.

S1. The function ψ is defined on $\Omega = \{0, 1\} \times L \times L \times [0, 1]$. Since we want both ψ and p to share the same domain, we extend p in the following way: For $d = 0$ and $y < 0$, $p^{(d,y)}(x, s') = p(y, s')$ for all x ; for $d = 0$ and $y \geq 0$, $p^{(d,y)}(x, s') = 1$ for all x and s' ; for $d = 1$ and $y = 0$, $p^{(d,y)}(x, s') = 0$ for all x and s' . Observe that the extension preserves the continuity of p with respect to s' , given d, y, x .

S2. Stack the functions to create the vector valued function:

$$f : \Omega \rightarrow [0, 1]^2 \equiv \begin{bmatrix} f^1(\omega) \\ f^2(\omega) \end{bmatrix} \equiv \begin{bmatrix} p(\omega) \\ \psi(\omega) \end{bmatrix},$$

where $\omega \in \Omega$. Let F be the set of all such functions and let K be the set of all such functions which are continuous in s (since the other components of ω are discrete, the functions are trivially continuous in those arguments). Let $\|f\| = \max\{\sup_{\omega} f^1, \sup_{\omega} f^2\}$. Observe that K is a closed (in the max-sup norm), bounded, and convex subset of F .

S3. Define an operator $T(f) : K \rightarrow F$ in the following way. Given $f \in K$, solve the individual's problem to get $m_i^*(e, \theta, x, s; f)$. Then use (5) and (6) to get $T^1(f)$ and $T^2(f)$, respectively (to get $T^2(f)$ we need to extend the the “output” function over to Ω as in step *S1* above).

S4. Prove the following properties regarding T and K : i) $T(K) \subseteq K$, ii) T is a continuous operator, and iii) $T(K)$ is an equicontinuous family.

S5. Use Schauder's Fixed point theorem to prove the existence of a fixed point f such that $T(f) = f$.

Note that Ω is a bounded subset of R^4 . Moreover, by definition $K \subset C(\Omega)$, where $C(\Omega)$ is the space of bounded continuous functions on Ω , with the supnorm. The existence proof using Schauder's theorem (see for example Stokey and Lucas [24], pg. 520) requires verifying the conditions on the operator T and the family $T(K)$ given in step S4. In the following lemmas, we verify these conditions. Inspection of equations (4)-(5) which define the operator T suggests that the continuity property of the operator is closely related to that of the decision rule $m_i^*(e, \theta, x, s; f)$ through (4).

In Appendix 7.1 we prove the main existence result.

Theorem 2.1 *A recursive competitive equilibrium specified in Definition 2.3 exists.*

A sketch of the proof is as follows. Lemma 7.1 applies a generalized Theorem of the Maximum by Ausubel and Deneckere [5] to show that the decision correspondence $M_i^*(e, \theta, x, s; f)$ is a non-empty, compact valued, upper hemi-continuous correspondence in e and s for a given $f \in K$. The generalized version requires only upper hemi-continuity of the feasible choice set. Lemma 7.2 uses results in Araujo and Mas-Colell [1] to show that $M_i^*(e, \theta, x, s; f)$ is single valued and continuous almost everywhere in E for a given $f \in K$. To establish equicontinuity of $T(K)$, we will use a Lipschitz argument. As an input into this argument, Lemma 7.3 proves a local Lipschitz property of decision rules $m_i^*(e, \theta, x, s; f)$ in s . This result follows from the fact that action set is finite, which implies that for a small enough change in s there is no change in actions except at a countable number of earnings levels. Lemma 7.4 establishes that $P_i^{(d,y)}(\theta, x, s; f)$ is well defined and continuous in s . Intuitively, integrating over e in equation (4) "smooths out" any discontinuities in the selection $m_i^*(\cdot; f)$. Lemma 7.5 establishes that $p^*(y, s')$ and $\psi^{*(d,y)}(x, s; f)$ are continuous in s , which follows from equations (5), (6) and Lemma 7.4. Lemma 7.6 establishes that $P_i^{(d,y)}(\theta, x, s; f)$ has Lipschitz constant 1 in s for any f . The Lemma extends the local Lipschitz property of decision

rules in Lemma 7.3 globally to $P_i^{(d,y)}(\cdot; f)$. Intuitively, if $P_i^{(d,y)}(\cdot; f)$ fails to be globally Lipschitz, there must be some interval on s where the local Lipschitz property is contradicted. Since this holds for any f , the family of functions $\{P_i^{(d,y)}(\cdot; f)\}_{f \in K}$ is uniformly Lipschitz continuous. After establishing some algebraic properties of Lipschitz functions in Lemma 7.7, Lemmas 7.8 and 7.9 prove that $\{p(\cdot; f)\}_{f \in K}$ and $\{\psi^{(d,y)}(\cdot; f)\}_{f \in K}$ are also uniformly Lipschitz. Having proven that f is Lipschitz then allows us to prove equicontinuity in Lemma 7.10. Finally, Theorem 2.1 establishes that the conditions for equilibrium in Definition 2.3 are satisfied.

3 Model Economy 2

Now we describe a model economy where there are restrictions on what information on asset market behavior can be kept in an agents credit history. In particular, we assume that information can be kept only for a finite amount of time and that there are partitions on what asset transactions are recorded. These restrictions on information are intended to capture the requirement that adverse events be stricken from an individual's credit history and the fact that credit scores are based upon data on liabilities rather assets. We also assume that there are regulatory and technological reasons that restrict what credit scoring agencies and intermediaries can observe about an individual's priors.¹⁵

An individual's history of asset market actions (asset choices and default decisions) at the beginning of period t is given by (ℓ_t, h_t^T) where $h_t^T = (d_{t-1}, \ell_{t-1}, d_{t-2}, \dots, \ell_{t+1-T}, d_{t-T}) \in \{0, 1\} \times L \times \{0, 1\} \times \dots \times L \times \{0, 1\} \equiv \mathcal{H}^T$, the set of possible histories of finite length $T \geq 1$. This definition directly incorporates the restriction that information can only be kept for a finite number, denoted T , periods. We formalize the restrictions on observability of asset transactions via partitions on $L \times \mathcal{H}^T$. Because all feasible actions are taken with at least probability ϵ , all feasible (ℓ_t, h_t^T) are possible along the equilibrium path. Let the particular subsets (or blocks) of the partition of $L \times \mathcal{H}^T$ be denoted $\Xi^T = \{H_1, \dots, H_k\}$ which by the assumptions that L and T are finite is itself

¹⁵For instance, prices which incorporate priors are considered proprietary and are excluded from standard credit histories.

a finite set. The restriction that data on assets (which we take as $\ell_t \in L_{++}$) are not included in the credit scoring agency's information set is modelled by a measurability assumption that (ℓ_t, h_t^T) is constant on each block of Ξ^T . As an example, suppose that $T = 1$ and $L = \{\ell_-, 0, \ell_+^1, \ell_+^2\}$ with $\ell_- < 0 < \ell_+^1 < \ell_+^2$. Then $L \times \mathcal{H}^{T=1} = \{(0, 1), (\ell_-, 0), (0, 0), (\ell_+^1, 0), (\ell_+^2, 0)\}$ and the measurability assumption requires $H_1 = \{(0, 1)\}$, $H_2 = \{(\ell_-, 0)\}$, $H_3 = \{(0, 0)\}$ and $H_4 = \{(\ell_+^1, 0), (\ell_+^2, 0)\}$. To conserve on notation, let $H(\ell_t, h_t^T)$ denote one of the partition blocks H_1, \dots, H_k . We use a similar notation, that is $A(\ell_{t+1}, d_t)$ is a partition block, to denote what an intermediary can observe regarding an individual's current actions (ℓ_{t+1}, d_t) . For the case where $L = \{\ell_-, 0, \ell_+^1, \ell_+^2\}$, the partition block is given (coincidentally) by $A_1 = \{(0, 1)\}$, $A_2 = \{(\ell_-, 0)\}$, $A_3 = \{(0, 0)\}$ and $A_4 = \{(\ell_+^1, 0), (\ell_+^2, 0)\}$.

How does this change in the environment affect decision problems? Since these informational restrictions are only on the credit scoring agency (as well as the financial intermediary since it uses credit scores as an input into its pricing calculations), the individual's problem is basically identical to what we had in section 2.3.1. In particular, we simply substitute h^T for s in the individual state (i, e, θ, ℓ, s) . Note that since h^T is a finite object, the state space is now finite except for exogenous earnings. We can also define the endogenous measure of individuals across the state space by $\mu_i(e, \theta, \ell, h^T)$.

The informational restrictions on the credit scoring agency affect both the credit scoring and type scoring functions in (5)-(4). In particular now the type scoring function is given by

$$\begin{aligned} \psi(\ell', d, \ell, h^T) &= (1 - \Gamma_{bg}) \left[\frac{\hat{P}_g(\ell', d, \ell, h^T) \cdot s^T}{\hat{P}_g(\ell', d, \ell, h^T) \cdot s^T + \hat{P}_b(\ell', d, \ell, h^T) \cdot (1 - s^T)} \right] \\ &+ \Gamma_{gb} \left[\frac{\hat{P}_b(\ell', d, \ell, h^T) \cdot (1 - s^T)}{\hat{P}_g(\ell', d, \ell, h^T) \cdot s^T + \hat{P}_b(\ell', d, \ell, h^T) \cdot (1 - s^T)} \right] \end{aligned} \quad (7)$$

where the prior of an agent's type is calculated from the population distribution

$$s^T(\ell, h^T) = \sum_{(\tilde{\ell}, \tilde{h}^T) \in H(\ell, h^T)} \left[\int_{E_g} \sum_{\theta} \mu_g(e, \theta, \tilde{\ell}, \tilde{h}^T) \Phi_g(de) \Lambda(\theta) \right] \quad (8)$$

and

$$\hat{P}_i(\ell', d, \ell, h^T) = \sum_{(\tilde{\ell}', \tilde{d}) \in A(\ell', d), (\tilde{\ell}, \tilde{h}^T) \in H(\ell, h^T)} P_i(\tilde{\ell}', \tilde{d}, \tilde{\ell}, \tilde{h}^T) \quad (9)$$

Then the credit scoring function is just as before

$$\begin{aligned} p(\ell', \psi) &= \psi(\ell', d, \ell, h^T) \cdot \left[\sum_{\theta'} \int [1 - m_g(1, 0; e', \theta', \ell', h^{T'}, q, p, \psi)] \Phi_g(de') \Lambda(\theta') \right] \\ &+ (1 - \psi(\ell', d, \ell, h^T)) \cdot \left[\sum_{\theta'} \int [1 - m_b(1, 0; e', \theta', \ell', h^{T'}, q, p, \psi)] \Phi_b(de') \Lambda(\theta') \right]. \end{aligned} \quad (10)$$

As can be easily seen, the key difference from (6)-(4) simply arises from the measurability restrictions in (8)-(9) and we use information on the distribution of agents in the economy μ to construct the “prior” likelihood that an agent with (ℓ, h^T) is of type g .

4 Moment Matching

According to the Fair Credit Reporting Act, a bankruptcy filing stays on an individual’s credit record for 10 years. To keep the state space workable, we assume $T = 2$ so that a model period corresponds to 5 years. The discount rate β for both types is set to be 0.99. The risk-free interest rate r is set to satisfy $\beta(1 + r) = 1$. We assume the time preference shock can take two values $\theta \in \{0, 1\}$ so that agents who receive the low shock are myopic for one period. This implies we need only pin down one probability, namely $\Lambda(0)$. The utility function takes the form $u(c) = c^{1-\varphi}/(1-\varphi)$. Thus, in this calibration we will abstract from preference differences between types. We assume that the “tremble” parameter is $\varepsilon = 0.0001$. This is the probability that agents will play a suboptimal but feasible action by mistake.

We assume an earnings process for type i that is Beta-distributed, $e_i \sim Be(\nu_i, \eta_i)$. Each agent takes a random draw from an endowment distribution conditional on her type. We use simulated

method of moments to estimate the parameters of the endowment process for each type to match the earnings gini index, mean-to-median earnings ratio, autocorrelation of earnings, and the percentage of earnings for the first to third quintiles. We use data from the PSID 1996-2001 to construct those statistics. Average annual earnings in the two survey years (1996 and 2001) are calculated and we multiply these numbers by five to get the average five-year earnings estimates for 1996-2001 and 2001-2005.

The parameters (ν_i, η_i) for the earnings process are estimated to be $(2.6570, 4.0642)$ for type g and $(1.0153, 24.4051)$ for type b . These estimated coefficients imply that type g earn more on average (0.40) than type b (0.04). The probability of type g switching to type b is estimated to be 0.0104, while the probability of type b switching to type g is 0.0149. This yields an invariant distribution where 0.59 of agents are type g . Table 1 summarizes the estimated parameter values and the targeted and predicted earnings statistics. The standard errors are based on a monte carlo from a simulation with 7500 agents, roughly the same as in the PSID.

Table 1: Earnings Statistics (PSID 1996-2001) and Parameter Values

Statistics	Target	Model	Parameter	Estimate (s.e.)
Gini index	0.54	0.50	ν_b	1.0153 (0.0616)
Mean/median	1.40	1.21	η_b	24.4051 (2.1358)
Autocorrelation	0.67	0.60	ν_g	2.6570 (0.1440)
1st quintile share	0.17	0.99	η_g	4.0642 (0.2208)
2nd quintile share	6.77	4.52	Γ_{gb}	0.0149 (0.0009)
3rd quintile share	14.73	16.30	Γ_{bg}	0.0104 (0.0007)

Taking the earnings parameters as given, we then estimate the remaining parameters by matching data moments on delinquency and wealth statistics. The distribution of delinquency rates from TransUnion in Figure 1 allows us to construct the moments for overall and subprime delinquency rates. Other statistics including the debt-to-earnings ratio, asset-to-earnings ratio, and percentage in debt are obtained from the 2004 SCF.¹⁶

¹⁶The credit scoring function in equation (10) is defined after agents make their asset decisions. Therefore, only agents in debt have a credit score other than 1. To account for the distribution of credit scores in Figure 1, which

The set of asset choices L includes one borrowing level (\underline{x}), zero, and two saving levels (\bar{x}_1 and \bar{x}_2). We estimate the borrowing level to be -0.0033 while the two saving levels are 0.1078 and 0.5683. Therefore, the five elements in \mathbb{Y} are $\{(1, 0), (0, -0.0033), (0, 0), (0, 0.1078), (0, 0.5683)\}$. The probability of the time preference shock is estimated to be 5% for both types. The CRRA coefficient is 6.4618. Table 2 summarizes the model statistics and parameter values.

Table 2: Model Statistics (TransUnion and SCF) and Parameter Values

Statistics	Target	Model	Parameter	Estimates
Overall delinquency rate	29.23%	31.28%	\underline{x}	-0.0033
Subprime (bottom 27%) del. rate	75.74%	54.56%	\bar{x}_1	0.1078
Debt to earnings ratio	0.002	0.001	\bar{x}_2	0.5683
Asset to earnings ratio	1.36	1.35	$\Lambda(0)$	0.0500
Percentage in debt	6.7	5.4	φ	6.4618

5 Model Properties

Since credit scores are based on observed asset market decisions, we start by listing the equilibrium decision rules of agents. With $T = 2$, there are 13 possible (x, h^2) partitions.

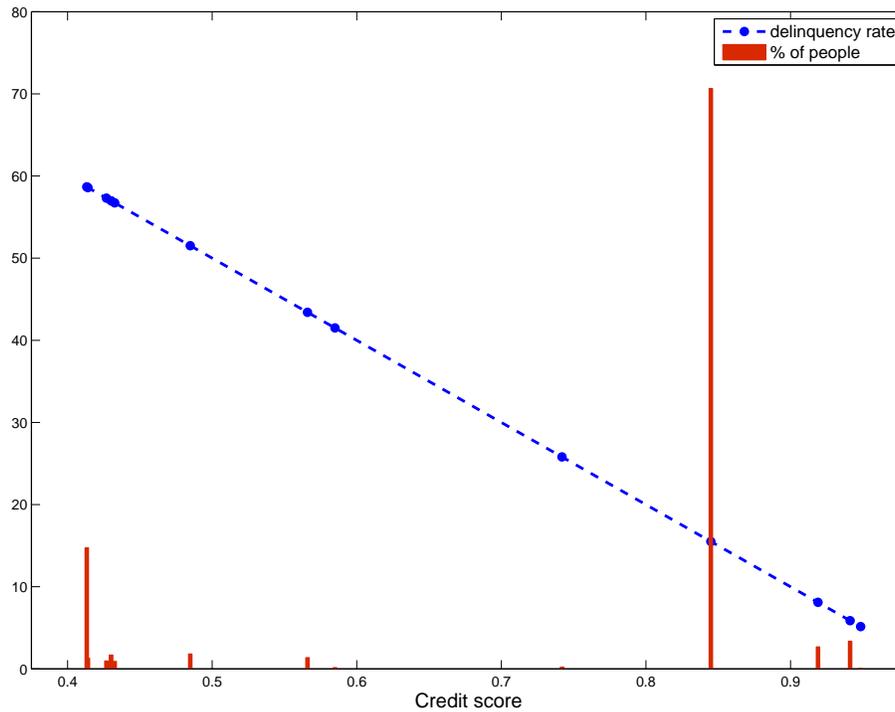
If agents experience a time preference shock (i.e. $\theta = 0$), they become perfectly myopic. In this case, they will default if they are in debt and will borrow if they are not in debt regardless of their earnings and any other characteristics. If agents do not receive the time preference shock (i.e., $\theta = 1$), their decision rules depend on their state/history tuple. When in debt, type g agents default for low earnings or save to \bar{x}_1 or \bar{x}_2 with higher earnings, while type b agents default for a larger set of low earnings or save to \bar{x}_1 . With zero assets, type g agents continue with zero assets or save to \bar{x}_1 or \bar{x}_2 , while type b agents borrow when earnings are very low, continue with zero assets for intermediate earnings, or save to \bar{x}_1 at high earnings. With savings, both types continue to save. In

includes everyone (even those who have positive net assets), we re-define the credit scoring function as the probability of repayment if the agent borrows. See the appendix for the formula.

equilibrium, every action is taken by some agents (i.e. there is no state that is infeasible for both types of agents).

The decision rules imply the distribution of credit scores graphed in Figure 2. As in the data, the distribution puts more weight on high scores which have lower likelihood of default.

Figure 2: Distribution of credit scores



Given the earnings distributions, the decision rules also imply certain properties for the type scoring function: an observation of default is more likely to come from a bad type individual; an observation of borrowing is more likely to come from a bad type individual too. Since the credit scoring function depends on the type scoring function via (10), this behavior translates into implications for credit scores. Figure 3 graphs the mapping between type scores and credit scores. The red dotted line plots the linear regression between the two and illustrates that there is not a perfect fit. This can be seen from equations (7)-(10). If the equilibrium borrowing actions are

independent of the state/history tuple, then there is a direct mapping between s^T and ψ in (7). Hence, the higher is s^T , the higher is ψ . Because type g agents default less often, this translates via (10) into higher p . However, since the equilibrium borrowing action depends upon the state/history tuple, the type scores do not map perfectly into credit scores. We can nonetheless still clearly see from Figure 3 that type scores and credit scores are highly positively correlated. The correlation coefficient weighted by the distribution measure is 0.9948.

One way to test the model is to see if it can predict the four key properties of credit score facts stated in section 1. To do so, the dynamics of credit scores need to be constructed.

1. Interest rates fall as a person's credit score rises.

Since higher credit scores mean a higher probability of repayment and intermediaries earn zero profits, this implies a negative relation between credit scores and interest rates as in the data (see Figure 4).

2. Default lowers a person's score, removal raises it.

Default lowers an individual's type score because type b are more likely to default than type g . Figure 5 graphs the percentage change in credit scores after default. We can see that the fact holds for all possible state and history tuples, because all the percentage changes are negative. On average, credit scores drop by 48% from 0.82 to 0.43 after default. Furthermore, the model prediction is also consistent with the fact documented by Fair Issac Corp,

*Someone that had spotless credit and a very high FICO score could expect a huge drop in their score. On the other hand, someone with many negative items already listed on their credit report might only see a modest drop in their score.*¹⁷

For instance, for an agent who has a low credit score at 0.48 before default, her credit score will drop by 11% after default. However, for an agent who has a high credit score at 0.94 before default, her credit score will have a dramatic drop by 54% after default.

¹⁷<http://www.myfico.com/crediteducation/questions/bankruptcy-fico-score.aspx>

Figure 3: Mapping between type scores and credit scores

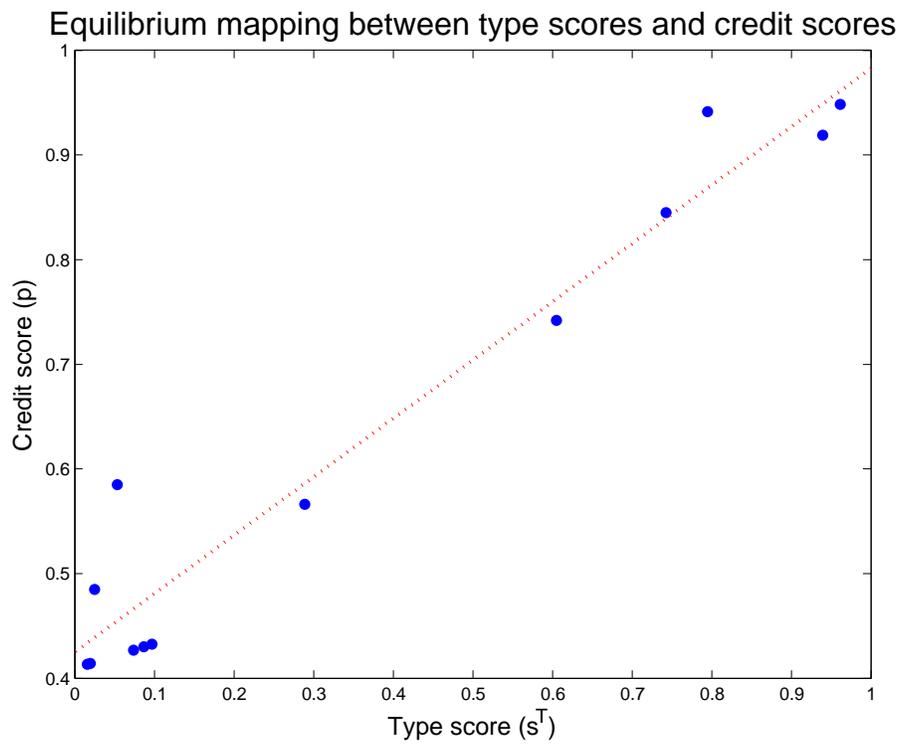


Figure 4: Credit scores and interest rates

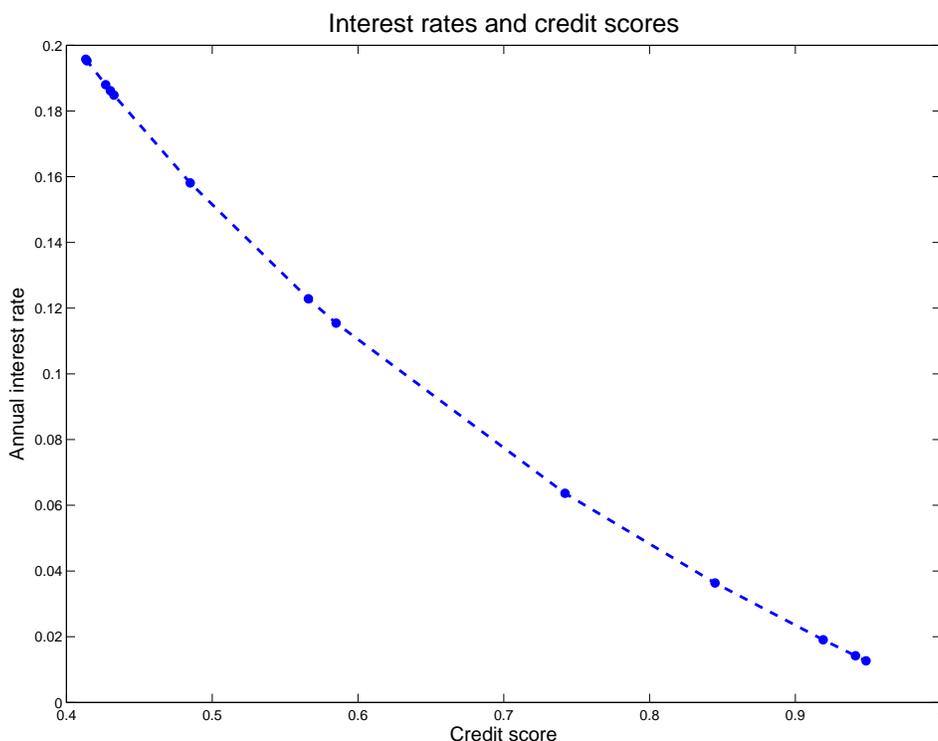
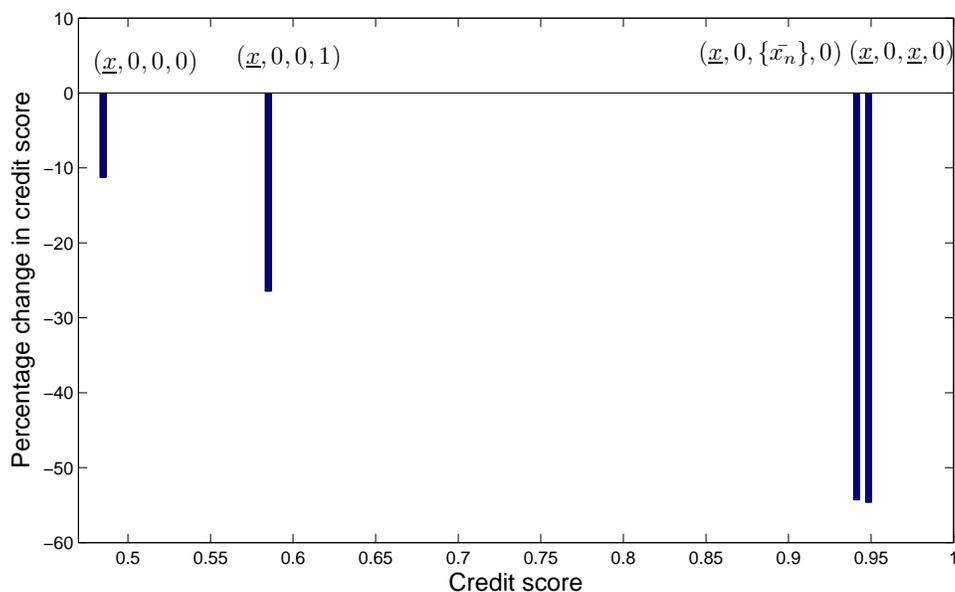


Figure 6 graphs the percentage change in credit scores when the default flag is removed from an agent’s credit history. The blue bars correspond to changes when the agent chooses to borrow, while the green and red bars correspond to changes when the agent chooses to have zero assets or save. As we can see from the graph, agents have higher credit scores once their default history is erased in most cases except for the state and history tuples $(\underline{x}, 0, 0, 1)$ or $\{(\bar{x}_1, 0, 0, 1), (\bar{x}, 0, 0, 1)\}$ when they choose to zero assets. These (green) cases, however, only happen as a consequence of a tremble.

As in Musto [21], we can compute the changes in percentile of the distribution of credit scores following a removal of the bankruptcy flag from one’s record. Musto [21] categorized bankrupt households according to their initial post-default percentage in the distribution of credit scores and kept track of them for ten years (the length of time the bankruptcy record stays in their credit history by the FCRA). Since $T = 2$, there is not a lot of variation in

Figure 5: Percentage change in credit score after default

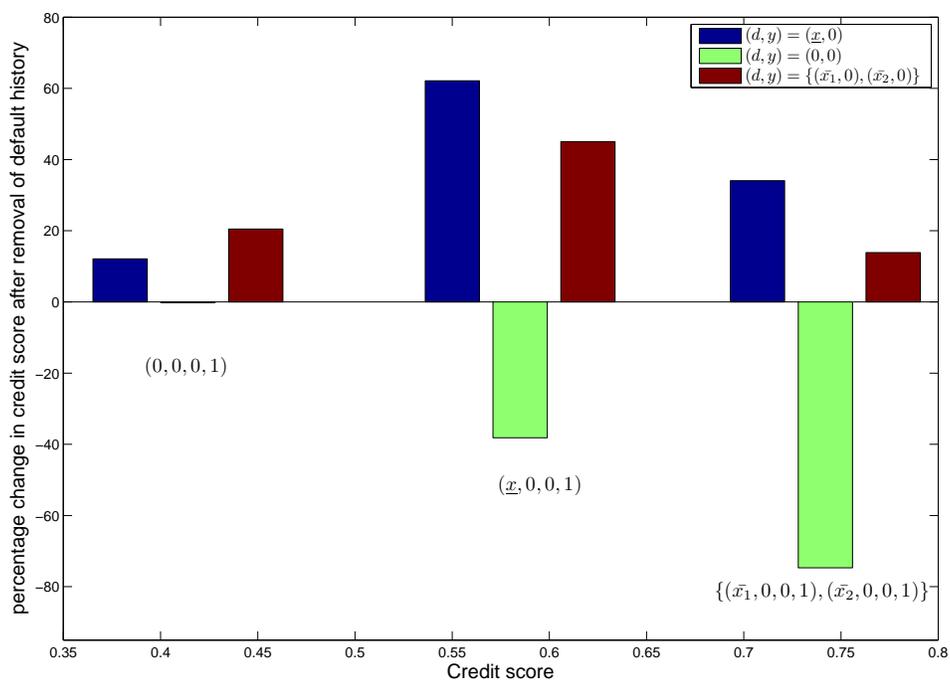


state/history tuples after default; here it is simply $(0, (1, \underline{x}, 0))$ and this falls within the first quintile of the distribution. After two model periods when their default record is erased, an individual's new credit score on average increases 6% in one 5-year period (1.2% annually). Musto found that for individuals in the first quintile of credit scores, they jumped ahead of 5% of households post default annually. However, these households are not the group in which people are mostly affected by the information restriction. If we raise $T > 2$ we should find more heterogeneity in post default scores which would map to Musto's dataset better.

3. Taking on more debt (paying off debt) tends to lower (raise) credit scores.

Figure 7 graphs the percentage change in credit score after an agent takes on more debt. In the model, since borrowing only arises when hit with $\theta = 0$ (except in one unlikely event) and θ shocks are iid, assessment following borrowing rises since the population proportion of good types is 0.59. This makes it hard to match the prediction that increasing indebtedness lowers scores. On average, credit scores rise by 3% from 0.76 to 0.78 after households go into debt. Obviously, the sparse parameterization of the model does not match this fact well.

Figure 6: Percentage change in credit score after default flag removal

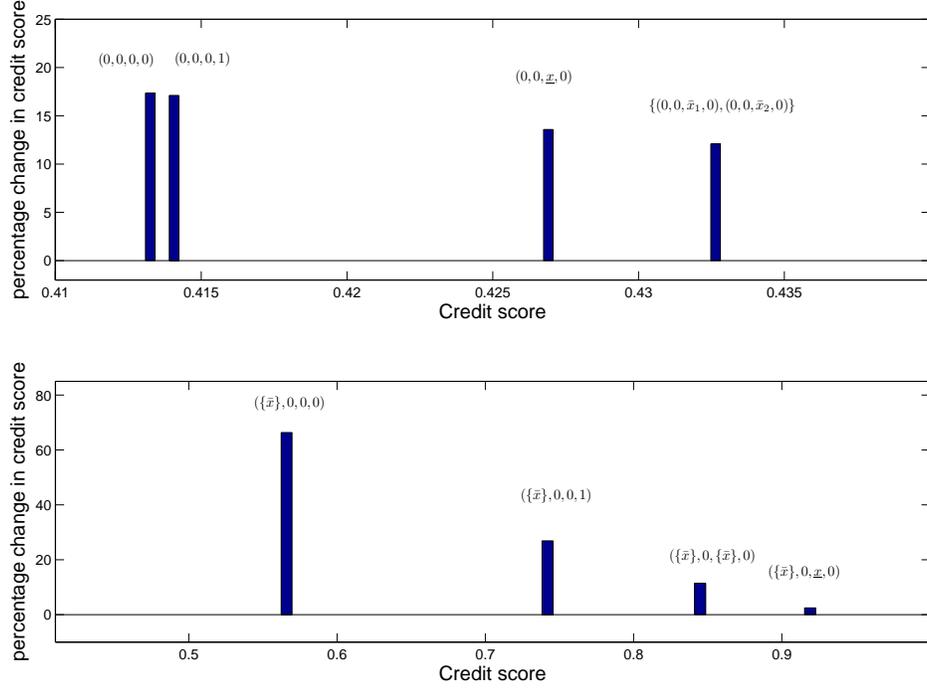


On the other hand, the percentage change in credit score after an agent pays off her debt is graphed in Figure 8. The model predicts the fact well when when an agent chooses zero assets after paying off her debt as illustrated by the red bars. The fact does not hold when an agent choose to save after paying off her debt as illustrated by the blue bars (this case, however, is a suboptimal but feasible action). On average, credit scores rise by 59% from 0.49 to 0.78 after agents pay off debt.

4. Scores are mean reverting.

Figure 9 graphs the average credit score given current credit scores using the equilibrium de-

Figure 7: Percentage change in credit score after borrowing



cision rules.¹⁸ It can be seen that agents with lower (higher) credit scores tend to have higher (lower) credit scores next period. Therefore, the linear regression line has a flatter slope at 0.8 than the 45 degree line.

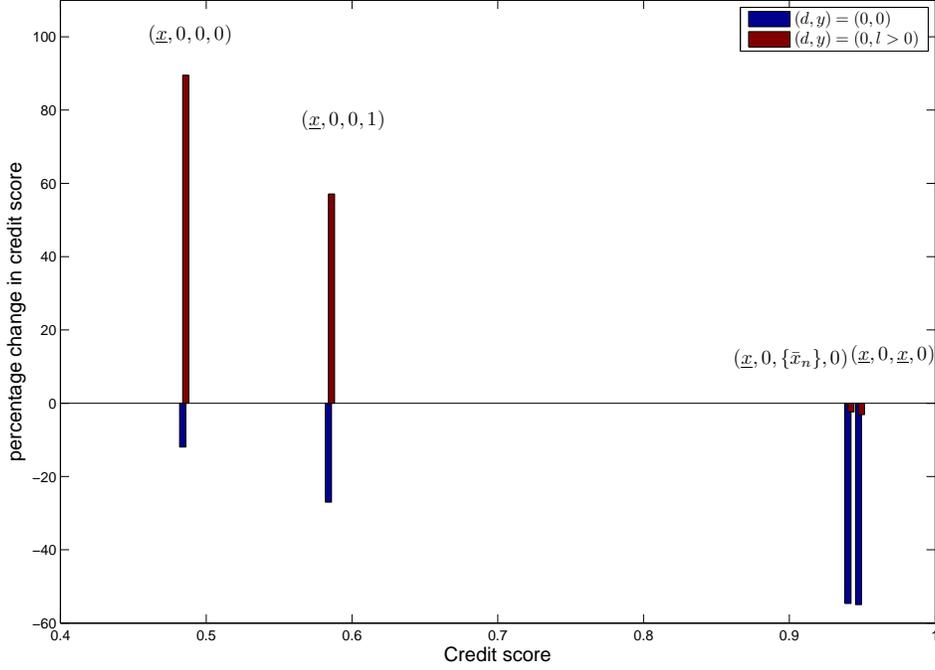
6 Policy Experiment

Here we use the model to address a question about the welfare consequences of imposing legal restrictions (like the Fair Credit Reporting Act), which requires adverse credit information (like a

¹⁸The average next-period credit score given (x, h^T) is calculated as

$$\frac{\sum_i \left[\int_{E_i} \sum_{\theta, (d, y)} p(\underline{x}, \psi(\underline{x}, 0, d, y, l, d_{-1})) m_i(d, y; e, \theta, x, h^T) \mu_i(e, \theta, l, h^T) \Phi_i(de) \Lambda(\theta) \right]}{\sum_i \left[\int_{E_i} \sum_{\theta} \mu_i(e, \theta, l, h^T) \Phi_i(de) \Lambda(\theta) \right]}$$

Figure 8: Percentage change in credit score after paying off debt

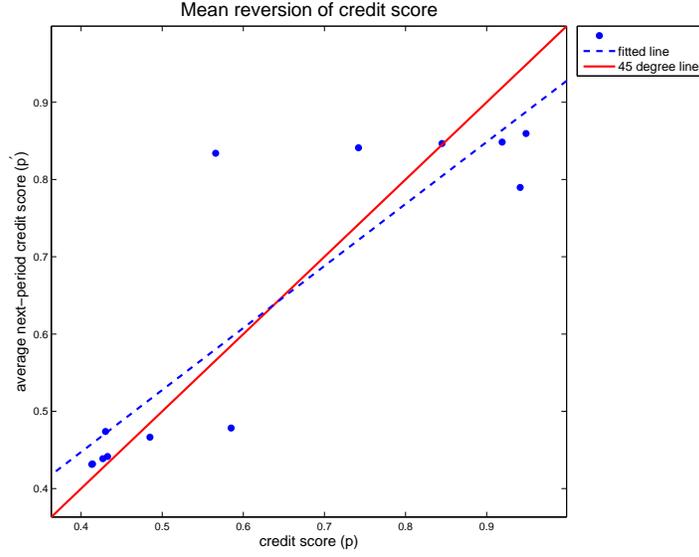


bankruptcy) to be stricken from one’s record after a certain number of years (10 in the U.S.). As discussed in the introduction, in a world of incomplete markets and private information, flag removal may provide insurance to impatient agents in our framework that competitive intermediaries may not be able to provide. Hence extending the length of time that a bankruptcy flag remains on one’s credit record may not necessarily raise ex-ante welfare. This issue is similar to Hart’s [16] examples where the opening of a market in a world of incomplete markets may make agents worse off and Hirschleifer’s [18] finding regarding the potential inefficiency of revealing information.

To assess this question, we compute consumption equivalents using the following formulas. Say the EPDV of utility starting in state (i, e, θ, x, h^T) for a given T is given by

$$V_i(e, \theta, x, h^{T=2}; T = 2) = E_i \left[\sum_{t=0}^{\infty} (\beta\theta)^t \frac{c_t(i, e, \theta, h^{T=2}; T = 2)^{1-\varphi}}{1 - \varphi} \right].$$

Figure 9: Mean reversion of credit scores



To assess how much a type i agent with earnings e and time preference shock θ in history $(x, h^{T=2})$ would be willing to pay forever to be in a regime where $T = \infty$ and there are no partitions, for each $(i, e, \theta, x, h^{T=2})$ we compute $\lambda_i(e, \theta, x, h^{T=2})$ such that

$$\begin{aligned} V_i(e, \theta, x, h^\infty; \infty) &= E_i \left[\sum_{t=0}^{\infty} (\beta\theta)^t \frac{[(1 + \lambda_i(e, \theta, x, h^{T=2}))c_t(i, e, \theta, x, h^{T=2}; T = 2)]^{1-\varphi}}{1 - \varphi} \right] \\ &= (1 + \lambda_i(e, \theta, x, h^T))^{1-\varphi} V_i(e, \theta, x, h^{T=2}; T = 2) \end{aligned}$$

or

$$\lambda_i(e, \theta, x, h^{T=2}) = \left[\frac{V_i(e, \theta, x, h^\infty; \infty)}{V_i(e, \theta, x, h^{T=2}; T = 2)} \right]^{1/(1-\varphi)} - 1.$$

Then the total welfare gain/loss is given by

$$\sum_{i, e, \theta, x, h^{T=2}} \lambda_i(e, \theta, x, h^{T=2}) \mu_i(e, \theta, x, h^{T=2}).$$

We use the same parameterization in the calibrated model for the $T = \infty$ world with no partitions. As a whole, the economy is worse off without the legal restriction (specifically, the welfare loss is 0.0001). Table 3 reports the average consumption equivalents by types and time preference shock. Type g on average would prefer to lift the legal restriction on information, while type b on average are worse off and must be compensated if the legal restriction is removed.

Table 3: CE by types and shocks

$\theta \backslash i$	g	b
1	0.0420e-3	-0.5266e-3
0	0.0650e-3	-0.1072e-3

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7 Appendix

7.1 Existence Proof

We start by showing in Lemma 7.1 that the decision correspondence $M_i^*(e, \theta, x, s; f)$, which is a member of the simplex over $\{0, 1\} \times L$ with full support on the feasible set $B(e, \theta, x, s; f)$, is well-defined and has standard properties. We apply a generalized Theorem of the Maximum by Ausubel and Deneckere [5] because the standard continuity assumption of the feasible choice correspondence is not satisfied in our environment due to the presence of strictly positive trembles. In particular, the feasible choice set is not necessarily lower hemi-continuous in e, s, f . To see why, consider a sequence p_n converging to p , such that there is an action \hat{y} that delivers strictly positive consumption for all p_n but delivers zero consumption for p . Then, every $m_i(\cdot; p_n) \in M_i(\cdot; p_n)$ assigns at least ϵ probability weight to \hat{y} (i.e., $m_i^{(0, \hat{y})}(\cdot; p_n) \geq \epsilon$) but there exists at least one feasible $\tilde{m}_i(\cdot; p) \in M_i(\cdot; p)$ which assigns zero probability to \hat{y} (i.e., $\tilde{m}_i^{(0, \hat{y})}(\cdot; p) = 0$). Therefore, there does not exist any feasible sequence of $m_i(\cdot; p_n)$ that converges to $\tilde{m}_i(\cdot; p)$.

Lemma 7.1 *The decision correspondence $M_i^*(e, \theta, x, s; f)$ is a non-empty, compact valued, upper hemi-continuous correspondence in e, s , and f , for each (θ, x) , and $i \in \{g, b\}$.*

Proof. We verify the hypotheses of the Generalized Theorem of the Maximum by Ausubel and Deneckere [5].

Claim 1: The feasible choice set $M(e, \theta, x, s; f)$ is non-empty for each $(e, \theta, x, s; f)$. This is because the default option is always feasible for individuals with debt (i.e. those who start with $x < 0$) and that an individual with positive assets can always afford positive amounts of consumption.

Claim 2: $M(e, \theta, x, s; f)$ is a compact valued correspondence since the feasible choice set is a closed simplex.

Claim 3: To show that $M(e, \theta, x, s; f)$ is uhc, pick arbitrary sequences $(e_n, s_n, f_n) \rightarrow (e, s, f)$ and $m_n \in M(e_n, \theta, x, s_n; f_n)$, and find a subsequence $(e_{n_k}, s_{n_k}, f_{n_k})$ such that $m_{n_k} \rightarrow m$ and $m \in M(e, \theta, x, s; f)$, where the convergence of f_n to f is in the sup-norm metric. To prove the claim, suppose m does not belong in $M(e, \theta, x, s; f)$. Then there exists (d, y) such that $m^{(d,y)} > 0$ for some (d, y) not in $B(e, x, s; f)$. Since every element of the sequence is a probability vector, the limit must be a probability vector also. Hence, the only way this vector can be infeasible is if it assigns positive weight to some point with negative consumption, i.e., $c = e + x - q(y, p(y, \psi^{(d,y)}(x, s; f))) < 0$. But this implies that for sufficiently large n , $c_n = e_n + x - q(y, p(y, \psi^{(d,y)}(x, s_n; f_n))) < 0$. Hence $m_n^{(d,y)} = 0$ for all sufficiently large n , which contradicts $m^{(d,y)} > 0$.

Claim 4: The objective function in the decision problem is continuous in e, s , and f for each (θ, x) . To see this, define the following operator \widehat{T} corresponding to the optimization problem:

$$\begin{aligned} (\widehat{TV})_i(e, \theta, x, s; f) = & \max_{m \in M(e, \theta, x, s; f)} \sum_{(d,y)} [R_i^{(d,y)}(e, \theta, x, s, f) \\ & + \beta_i \theta \sum_{j \in \{g, b\}, \theta'} \Gamma_{ji} \int_E V_j(e', \theta', y, \psi^{(d,y)}(x, s); f) \Phi_j(de') \Lambda(\theta')] \cdot m^{(d,y)}. \end{aligned}$$

Let $Z = E \times \Theta \times L \times [0, 1]$ denote the product space. Observe that Z is a compact set, since it is a product of finite number of compact sets.

Let $B(Z)$ denote the space of bounded functions defined over Z , and similarly $C(Z)$ denote the space of continuous functions. Moreover, $C(Z) \subseteq B(Z)$ since a continuous function defined over a compact set is bounded. The space of continuous and bounded functions $C(Z)$ endowed with the sup norm defines a complete metric space. Since Z is compact, R is a bounded function for each (θ, x) . Starting with a bounded function $V \in B(Z)$, the operator \widehat{T} updates to another bounded function \widehat{TV} . This also holds true for a $V \in C(Z)$, since a continuous function on a compact domain is bounded, so that $\widehat{T} : C(Z) \rightarrow B(Z)$. Next we show that the operator maps $C(Z)$ into itself using the generalized Maximum Theorem, which will yield existence and uniqueness of the value function and that the choice correspondence has the properties given in the statement of the

current lemma.

Given a $V \in C(Z)$ consider the terms in the objective function. Since q is continuous in s , by Lemma 2.1, R is continuous in s . Moreover, R is also continuous in e since current utility is continuous in current consumption and the latter is linear in e . Moreover, since $V \in C(Z)$ is continuous in s' and $s' := \psi \in K$ is a continuous function of s , then $V = \widehat{T}V$ is continuous in s . Furthermore, $V \in C(Z)$ does not depend on the current endowment e because the transition probabilities Γ_{ji} do not depend on e and next period's draw e' is independent. Therefore, being a sum of continuous functions, the objective function is continuous.

Together Claims 1 to 4 constitute the hypotheses of the Theorem of Maximum in Ausubel and Deneckere [5]. One consequence of this theorem is that $\widehat{T}V$ is continuous in e, s for each (θ, x) . Therefore, the operator \widehat{T} maps $C(Z)$ into itself. Moreover, this operator is monotone and is a contraction of modulus less than one, which are Blackwell's sufficient conditions to establish that \widehat{T} is a contraction. Therefore, existence and uniqueness of a fixed point V of \widehat{T} in $C(Z)$ follows from the contraction mapping theorem. The other important consequence of the theorem is that the choice correspondence $M_i^*(\cdot, \theta, x, \cdot; f)$ is a non-empty, compact valued and upper hemi-continuous (uhc) correspondence in e, s for each (θ, x) . ■

We next use some results by Araujo and Mas-Colell [1] to show that M_i^* is single-valued and continuous except at a set of points of Lebesgue measure zero in E_i . We establish:

Lemma 7.2 $M_i^*(e, \theta, x, s; f)$ is single valued and continuous almost everywhere (a.e.) in E_i for each (θ, x, s, f) .

Proof. Fixing (θ, x, s, f) we verify that Assumptions 1 to 4 and the Sondermann Condition on pages 115–116 of Araujo and Mas-Colell [1] are satisfied for the objective function corresponding to the individual optimization problem

$$F(m, e) := \sum_{(d,y)} [R_i^{(d,y)}(e, \theta, x, s, f) + \beta_i \theta W_i(y, \psi^{(d,y)}(x, s))] m^{(d,y)}.$$

where

$$W_i(y, \psi^{(d,y)}(x, s)) = \sum_{j \in \{g, b\}, \theta'} \Gamma_{ji} \int_E V_j(e', \theta', y, \psi^{(d,y)}(x, s)) \Phi_j(de') \Lambda(\theta') \quad \forall i \in \{g, b\}$$

with m and e in the roles of $x \in X$ and $a \in \mathcal{E}$, respectively, as well as Φ_i in the role of the probability measure ν , in the statement of Theorem 1 of Araujo and Mas-Colell. If the conditions hold for our model economy, then we can conclude that the $M_i^*(e, \theta, x, s; f)$ is single-valued almost everywhere in e .

1. Assumption 1: that $(X \times X) \setminus \Delta$ is a Lindelöf space, where $\Delta = \{(x, y) \in X \times X : x = y\}$, which is necessary for a countable open cover, holds. In our case the set of all feasible choices is $X := \bigcup_{i \in \{g, b\}} \overline{M}_i$ where $\overline{M}_i := M_i(\bar{e}_i, \theta, x, s, f)$. To see this note that $e^1 < e^2$ implies that $M_i(e^1, \theta, x, s, f) \subseteq M_i(e^2, \theta, x, s, f)$ for $i \in \{g, b\}$ and $e \in E_i$. Note also that X is compact and so is the product space $X \times X$. Moreover, $X \times X \setminus \Delta$ is compact, being a closed subset of a compact set. Consequently, $X \times X \setminus \Delta$ is a Lindelöf space, since the latter is a weakening of compactness (See section 7.2 of Gemignani [14]).
2. Assumption 2: that $F : X \times \mathcal{E} \rightarrow R$ is a continuous function, holds. As shown in Claim 4 of Lemma 7.1 the objective function is continuous.
3. Assumption 3: that for every i , $x \in X$ and $a \in \mathcal{E}$, $\partial_a F(x, a)$ exists and depends continuously on x and a - holds. To see this, dropping the i index for notational ease, note that a small change in e has an effect only in the current period through its direct effect on the set of consumption choices. Continuation values are unaffected by a small change in e . This is because the transition probabilities Γ_{ji} do not depend on e and the next period's draw e' is independent. Therefore, $\partial F_e(m, e) = \partial_e R_i = \sum_{(d,y)} u'(c^{(d,y)}) \cdot m^{(d,y)}$, which by the assumption that $u(\cdot)$ is continuously differentiable in c , by the fact that consumption is linear in e , and linearity in m implies that the expression varies continuously in e and m .
4. Assumption 4: that ν is a product probability measure, each factor being absolutely continuous with respect to Lebesgue measure, holds. By assumption our probability measure Φ_i is

absolutely continuous with respect to the Lebesgue measure.

5. **Sondermann Condition (SC):** that if $F(x, a) = F(y, a)$, $x \neq y$, then $\partial_{a_i}(F(x, a) - F(y, a)) \neq 0$ for some i , holds. Suppose, to the contrary that $F(m, e) = F(\hat{m}, e)$ and $m \neq \hat{m}$ implies that $\partial F_e(m, e) := \sum_{(d,y)} u'(c^{(d,y)})m^{(d,y)} = \sum_{(d,y)} u'(c^{(d,y)})\hat{m}^{(d,y)} =: \partial F_e(\hat{m}, e)$ for all e . This means that $\sum_{(d,y)} u'(c^{(d,y)}) (m^{(d,y)} - \hat{m}^{(d,y)}) = 0$. Since $\sum_{(d,y)} (m^{(d,y)} - \hat{m}^{(d,y)}) = 0$ because both m and \hat{m} are probability vectors that sum to 1, the sub-vector $u'(c^{(d,y)})$ composed of all (d, y) for which $(m^{(d,y)} - \hat{m}^{(d,y)}) \neq 0$ must be proportional to the unit sub-vector. This is because both the unit sub-vector and the $u'(c^{(d,y)})$ sub-vector are both orthogonal to $(m^{(d,y)} - \hat{m}^{(d,y)})$. But, provided there is at least one pair of actions in this sub-vector, say (d, y) and (\tilde{d}, \tilde{y}) for which $c^{(d,y)} \neq c^{(\tilde{d}, \tilde{y})}$, this proportionality will contradict the strict concavity of u . Hence, $\partial_e(F(m, e) - F(\hat{m}, e)) \neq 0$.

Together, items 1 to 5 verify that for each (θ, x, s) the hypotheses for Theorem 1 of Araujo and Mas-Colell [1] are satisfied. Consequently, $M_i^*(e, \theta, x, s; f)$ is single-valued a.e in E_i .

Next we show that the uhc correspondence M_i^* that is a.e. single-valued in E_i is continuous a.e. in E_i . To see this, we pick an arbitrary convergent sequence in the domain $e_n \rightarrow e$ and show that $M^*(e_n, \cdot) \rightarrow M_i^*(e, \cdot)$ in e a.e. Let M^* be single-valued at e and $M^*(e)$ be the value. By upper hemi-continuity there exists a subsequence such that $e_{n_k} \rightarrow e$ and $M^*(e_{n_k}, \cdot) \rightarrow M^*(e)$. Since the limit of any such subsequence is unique, the original sequence converges to the same limit, that is, $M^*(e_n, \cdot) \rightarrow M^*(e)$. This shows that $M^*(e, \cdot)$ is continuous at the set of points where it is single-valued. But the latter set has probability one. Therefore, $M^*(e, \cdot)$ is continuous in e a.e.

Hence, the claim of the lemma follows. ■

Since we will establish equicontinuity using a Lipschitz condition, the next lemma proves that small changes in s satisfy a Lipschitz condition on decision rules with Lipschitz constant 1 almost everywhere. Given upper hemicontinuity in Lemma 7.1 and single-valuedness a.e. in e in Lemma 7.2, the result follows from the finite action set.

Lemma 7.3 For a given (θ, x, f) and any s , there exists a $\delta_s(f) > 0$ such that for any $s' \neq s$ and $|s - s'| \leq \delta_s(f)$, $|m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f)| \leq |s - s'|$ a.e. in E_i .

Proof. For a given (θ, x, f) , suppose to the contrary that there exists s and $s' \neq s$ such that for any $\delta_s(f) > 0$ with $|s - s'| < \delta_s(f)$ and an associated positive measure set $E_i(s, s'; f)$,

$$|m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f)| > |s - s'| \text{ for each } e \in E_i(s, s'; f), \quad (11)$$

where $E(s, s'; f)$ is a set of e for which both $m_i^{*(d,y)}(e, \theta, x, s; f)$ and $m_i^{*(d,y)}(e, \theta, x, s'; f)$ are single-valued. The latter is possible since $m_i^{*(d,y)}(\cdot, s; f)$ is single-valued a.e. in e both at s and s' by Lemma 7.2.

The following steps lead to the desired contradiction.

Step1. Since $s \neq s'$, (11) implies that $m_i^{*(d,y)}(e, \theta, x, s; f) \neq m_i^{*(d,y)}(e, \theta, x, s'; f)$ for each $e \in E_i(s, s'; f)$. This further implies that

$$|m_i^{*(d,y)}(e, s; f) - m_i^{*(d,y)}(e, s'; f)| \geq \epsilon \text{ for each } e \in E_i(s, s'; f), \quad (12)$$

where ϵ is the tremble parameter. This follows since $m_i^{*(d,y)}(e, \theta, x, s; f)$ and $m_i^{*(d,y)}(e, \theta, x, s'; f)$ are single valued for each $e \in E_i(s, s'; f)$, the action set has a finite number of elements, and the smallest possible difference in probability mass assigned to actions is ϵ .

Step2. Since $m_i^{*(d,y)}(e, \theta, x, s; f)$ is single-valued in e by Lemma 7.2, uhc of $m_i^{*(d,y)}(e, \theta, x, s; f)$ at s by Lemma 7.1 implies that for an open ball of radius $\epsilon/2$ around $m_i^{*(d,y)}(e, \theta, x, s; f)$ there exists an open ball of radius $\hat{\delta}_s(f) > 0$ around s such that

$$|m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f)| \leq \epsilon/2 \quad (13)$$

for every s' such that $|s - s'| < \hat{\delta}_s(f)$.

Step3. Since (11) must hold for any $\delta_s(f) > 0$, if we pick s' in (*Step1*) satisfying $|s - s'| < \hat{\delta}_s(f)$ then (13) in (*Step2*) contradicts (12).

■

Next we prove that $P_i^{(d,y)}(\theta, x, s; f)$ in equation (4) is well defined. Given the continuity result of Lemma 7.2, the next lemma also establishes the continuity of $P_i(\cdot)$ in s . Intuitively, the integral “smooths out” the discontinuities in $m_i^*(\cdot, \theta, x, s; f)$.

Lemma 7.4 *Given the measure Φ_i , observable characteristics (θ, x, s) and price and scoring functions f , the measure $P_i^{(d,y)}(\theta, x, s; f)$ of individuals choosing (d, y) given in equation (4) is well defined for all i . Further, $P_i^{(d,y)}(\theta, x, s; f)$ is continuous in s for each (d, y) , x , $f := (\psi^{old}, p^{old}) \in K$ and $i \in \{g, b\}$.*

Proof. For the first part of the lemma, we know by Lemma 7.1 that $M_i^*(e, \theta, x, s; f)$ is a compact valued and uhc correspondence. From the Measurable Selection Theorem (Stokey and Lucas, Theorem 7.6), there exists a function $m_i^*(e, \theta, x, s; f)$, measurable with respect to $\mathcal{B}(E_i)$, such that $m_i^*(e, \theta, x, s; f) \in M_i^*(e, \theta, x, s; f)$. Furthermore, $m_i^*(e, \cdot) \leq 1$ and Φ_i is a probability measure. Therefore $m_i^{*(d,y)}$ is Φ_i integrable and $\int m_i^{*(d,y)}(e, \theta, x, s; f)\Phi_i(de)$ exists.

For the second part of the proof, fix θ, x and f . Pick \hat{e} and \hat{s} . Assume that $M_i^*(\hat{e}, \theta, x, \hat{s}; f)$ is single-valued. Therefore $m_i^*(\hat{e}, \theta, x, \hat{s}; f) = M_i^*(\hat{e}, \theta, x, \hat{s}; f)$. Let $s_n \rightarrow \hat{s}$. We claim that $m_i^*(\hat{e}, \theta, x, s_n; f) \rightarrow m_i^*(\hat{e}, \theta, x, \hat{s}; f)$. Suppose not, then for any $\epsilon > 0$ there exists a subsequence $m_i^*(\hat{e}, \theta, x, s_{n_k})$ such that $|m_i^*(\hat{e}, \theta, x, s_{n_k}) - m_i^*(\hat{e}, \theta, x, \hat{s})| > \epsilon$ for all n_k . But $m_i^*(\hat{e}, \theta, x, s_{n_k})$ is a selection from $M_i^*(\hat{e}, \theta, x, s_{n_k})$. So, by the uhc of M_i^* , the subsequence must contain a subsequence converging to a point in $M_i^*(\hat{e}, \theta, x, \hat{s}; f)$. But the latter contains only $m_i^*(\hat{e}, \theta, x, \hat{s}; f)$. Thus there must be some N such that $|m_i^*(\hat{e}, \theta, x, s_N) - m_i^*(\hat{e}, \theta, x, \hat{s})| < \epsilon$, a contradiction. Therefore, $m_i^*(\hat{e}, \theta, x, s_n; f) \rightarrow m_i^*(\hat{e}, \theta, x, \hat{s}; f)$.

Now consider $P_i^{(d,y)}(\theta, x, s_n; f) = \int m_i^{*(d,y)}(e, \theta, x, s_n; f)\Phi_i(de)$. Then (i) $m_i^{*(d,y)}(e, \theta, x, s_n; f) \rightarrow m_i^{*(d,y)}(e, \theta, x, s; f)$ for all e for which $m_i^{*(d,y)}(e, \theta, x, s; f)$ is single-valued, and therefore, for e

a.e., and (ii) $m_i^*(e, \theta, x, s_n; f) \leq 1$. Therefore, by the Lebesgue Dominated Convergence Theorem, $\lim_n P_i^{(d,y)}(\theta, x, s_n; f) = \lim_n \int m_i^{*(d,y)}(e, \theta, x, s_n; f) \Phi_i(de) = \int \lim m_i^{*(d,y)}(e, \theta, x, s_n; f) \Phi_i(de) = \int m_i^{*(d,y)}(e, \theta, x, s; f) \Phi_i(de) = P_i^{(d,y)}(\theta, x, s; f)$. ■

Given the continuity of $P_i(\cdot)$ in s by Lemma 7.4, the new scoring functions $\psi^{new} = T^1(f)$ and $p^{new} = T^2(f)$, which are obtained from the old scoring functions $f := (\psi^{old}, p^{old})$ by applying the operator T as defined in equations (5) and (6), are continuous in s . This is because from (5) and (6) the new scoring functions are continuous functions of $P_i(\cdot)$. This result is summarized in the following Lemma 7.5 and will be used below in Lemma 7.6 to show the continuity properties of T .

Lemma 7.5 $\psi^{new} = T^1(f)$ and $p^{new} = T^2(f)$ is continuous in s , for each (d, y, θ, x) , $f := (\psi^{old}, p^{old}) \in K$ and $i \in \{g, b\}$.

The next lemma establishes that $P_i^{(d,y)}(\cdot, s)$ has Lipschitz constant 1. The proof uses the fact from Lemma 7.3 that small changes in s yield small changes in decision rules a.e. for any s . We show in the first part of the proof that this implies that small changes in s yield small changes in P at any s and then extend this to all changes in s via an argument similar to a nondifferentiable version of the Mean Value Theorem. In particular, the standard Mean Value Theorem is often used to prove theorems that make global conclusions about a function on an interval starting from local hypotheses about derivatives at points of the interval. Here we extend that idea to make global conclusions about the Lipschitz constant without assuming differentiability of P . Further, the lemma establishes that the family of functions $\{P(\cdot; f)\}_{f \in K}$ is uniformly Lipschitz continuous, which uses the following:

Definition 7.1 (Uniform Lipschitz Continuity) We say that the family of functions $\{P(\cdot; f)\}_{f \in K}$ is uniformly Lipschitz continuous if each function in the family is Lipschitz continuous and has the same Lipschitz constant for any $f \in K$.

Lemma 7.6 For any given (θ, x, s) and any $f \in K$, $|P_i^{(d,y)}(\theta, x, s; f) - P_i^{(d,y)}(\theta, x, s'; f)| \leq |s - s'|$ whenever $s \neq s'$.

Proof. First we establish that for any given s and f , there exists a $\delta_s(f)$ such that $|P_i^{(d,y)}(\theta, x, s; f) - P_i^{(d,y)}(\theta, x, s'; f)| \leq |s - s'|$ whenever $|s - s'| \leq \delta_s(f)$ and $s \neq s'$. To see this, fix an f . For any s, s' with $s \neq s'$,

$$\begin{aligned} & |P_i^{(d,y)}(\theta, x, s; f) - P_i^{(d,y)}(\theta, x, s'; f)| \\ &= \left| \int_E \left(m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f) \right) \Phi(de) \right| \\ &\leq \int_E \left| m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f) \right| \Phi(de), \end{aligned} \quad (14)$$

where the equality follows from the definition of $P(\cdot)$ and the inequality follows from the Jensen's inequality since $|\cdot|$ is a convex function. By Lemma 7.3, there exists a $\delta_s(f) > 0$ such that $|m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f)| \leq |s - s'|$ a.e. whenever $|s - s'| \leq \delta_s(f)$. For such s and s' , $\int_E \left| m_i^{*(d,y)}(e, \theta, x, s; f) - m_i^{*(d,y)}(e, \theta, x, s'; f) \right| \Phi(de) \leq |s - s'|$, which from (14) implies that $|P_i^{(d,y)}(\theta, x, s; f) - P_i^{(d,y)}(\theta, x, s'; f)| \leq |s - s'|$ whenever $|s - s'| \leq \delta_s(f)$.

Next we extend the argument to all $s \neq s'$ and not just those where $|s - s'| \leq \delta_s(f)$. In particular, for any given f , fix $s, s' \in [0, 1]$ with $s \neq s'$ and assume without loss of generality that $s' > s$. For an arbitrary $z \in \mathfrak{R}$ define a function $g_z : [0, 1] \rightarrow \mathfrak{R}$ as a product in the following way:

$$g_z(\tilde{s}) := z \cdot \left(P_i^{(d,y)}(\theta, x, \tilde{s}; f) - P_i^{(d,y)}(\theta, x, s; f) - (\tilde{s} - s) \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right).$$

Note that by construction, $g_z(s) = g_z(s') = 0$. Since by Lemma 7.4 we have $P_i^{(d,y)}(\theta, x, s; f)$ is a continuous function of s , g_z is continuous in \tilde{s} . Moreover, restricted to a compact subset $[s, s']$ of $[0, 1]$, g_z is also continuous in \tilde{s} on that subset. Therefore, by the Weierstrass Theorem (Aliprantis and Border [6], page 40), there exists an interior point $\xi \in (s, s')$ at which g_z attains a maximum or a minimum. Therefore, there are two cases to consider depending on whether ξ is a minimum or a maximum.

Case 1. g_z attains a minimum at ξ . If ξ is a minimum,

$$\liminf_{\substack{\tilde{s}_n \rightarrow \xi \\ \tilde{s}_n > \xi}} \frac{g_z(\tilde{s}_n) - g_z(\xi)}{\tilde{s}_n - \xi} \geq 0. \quad (15)$$

This holds because the \liminf is well-defined and for each \tilde{s}_n , the numerator is non-negative since ξ is a minimum and the denominator is non-negative since the sequence of $\{\tilde{s}_n\}$ was chosen such that $\tilde{s}_n > \xi$. Using the definition of the function g_z the latter implies that

$$\begin{aligned} g_z(\tilde{s}) - g_z(\xi) &= z \cdot \left(P_i^{(d,y)}(\theta, x, \tilde{s}; f) - P_i^{(d,y)}(\theta, x, \xi; f) \right) \\ &\quad - z \cdot \left((\tilde{s} - \xi) \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right) \geq 0 \end{aligned}$$

or, for any s'

$$z \left(P_i^{(d,y)}(\theta, x, \tilde{s}; f) - P_i^{(d,y)}(\theta, x, \xi; f) \right) \geq z \left((\tilde{s} - \xi) \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right).$$

In particular, this condition for any $\tilde{s} > \xi$ and linearity imply that

$$z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, \tilde{s}; f) - P_i^{(d,y)}(\theta, x, \xi; f)}{\tilde{s} - \xi} \right) \geq z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right).$$

Since this condition is true for any $\tilde{s} > \xi$, then for any sequence of \tilde{s}_n converging to ξ from above we know

$$\liminf_{\substack{\tilde{s}_n \rightarrow \xi \\ s'_n > \xi}} z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, \tilde{s}_n; f) - P_i^{(d,y)}(\theta, x, \xi; f)}{\tilde{s}_n - \xi} \right) \geq z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, s'_n; f) - P_i^{(d,y)}(\theta, x, s; f)}{s'_n - s} \right). \quad (16)$$

Moreover, by the definition of the absolute value function $|\cdot|$,

$$z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, \tilde{s}_n; f) - P_i^{(d,y)}(\theta, x, \xi; f)}{\tilde{s}_n - \xi} \right) \leq |z| \frac{|P_i^{(d,y)}(\theta, x, \tilde{s}_n; f) - P_i^{(d,y)}(\theta, x, \xi; f)|}{|\tilde{s}_n - \xi|}. \quad (17)$$

Since $\tilde{s}_n \rightarrow \xi$, for sufficiently large n 's we have $|\tilde{s}_n - \xi| < \delta_\xi^f$ and hence by the first part of this proof we know $\frac{|P^{(d,y)}(\tilde{s}_n) - P^{(d,y)}(\xi)|}{|\tilde{s}_n - \xi|} \leq 1$. From (17), the latter implies that for all sufficiently large n 's, $z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, \tilde{s}_n; f) - P_i^{(d,y)}(\theta, x, \xi; f)}{\tilde{s}_n - \xi} \right) \leq |z|$, which combined with (16) yields the desired inequality

$$|z| \geq z \cdot \left(\frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right). \quad (18)$$

Case2. If, on the other hand, ξ is a maximum, then by an analogous argument we can show (15) for a sequence \tilde{s}_n converging to ξ from below,

$$\liminf_{\substack{\tilde{s}_n \rightarrow \xi \\ \tilde{s}_n < \xi}} \frac{g_z(\tilde{s}_n) - g_z(\xi)}{\tilde{s}_n - \xi} \geq 0.$$

Using this condition and following an analogous argument made in *Case1* establishes that (18) also holds in *Case2*.

We have established that for an arbitrary z the condition in (18) holds. Therefore, the condition holds for any z . In particular, it holds for $z = \left(\frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right)$; that is,

$$\left| \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right| \geq \left| \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right|^2$$

or, equivalently, $\left| \frac{P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f)}{s' - s} \right| \leq 1$. Rearranging now shows that for any given $f \in K$, $\left| P_i^{(d,y)}(\theta, x, s'; f) - P_i^{(d,y)}(\theta, x, s; f) \right| \leq |s' - s|$. The uniform Lipschitz property follows from the independence of this condition from a particular f . ■

Having established that P is Lipschitz for any f , we now need to establish that f , or in particular ψ and p which are functions of P , are also Lipschitz. The next lemma establishes certain properties of functions of Lipschitz functions.

Lemma 7.7 *If $g : [0, 1] \rightarrow [0, 1]$ and $\hat{g} : [0, 1] \rightarrow [0, 1]$ are Lipschitz continuous functions with the same Lipschitz constant Z , then: (i) their product $h := g\hat{g}$ is also Lipschitz continuous with Lipschitz constant is $2Z$; and (ii) their sum $h := g + \hat{g}$ is also Lipschitz continuous and its Lipschitz constant is $2Z$.*

Proof. Part (i). We must show that there exists a $\hat{Z} > 0$ such that for any s, s' with $s \neq s'$ $|h(s) - h(s')| \leq \hat{Z}|s - s'|$ and $\hat{Z} = 2Z$. Note that

$$\begin{aligned}
|h(s) - h(s')| &= |g(s)\hat{g}(s) - g(s')\hat{g}(s')| \\
&= |g(s)\hat{g}(s) - g(s)\hat{g}(s') + g(s)\hat{g}(s') - g(s')\hat{g}(s')| \\
&\leq |g(s)\hat{g}(s) - g(s)\hat{g}(s')| + |g(s)\hat{g}(s') - g(s')\hat{g}(s')| \\
&= g(s)|\hat{g}(s) - \hat{g}(s')| + \hat{g}(s')|g(s) - g(s')| \\
&\leq (g(s) + \hat{g}(s'))Z|s - s'| \\
&\leq \hat{Z}|s - s'|, \text{ where } \hat{Z} = 2Z.
\end{aligned}$$

The first equality follows from the definition of h . The second by adding and subtracting a term. The third equality follows since $g(s)$ and $\hat{g}(s)$ are non-negative. The first inequality uses the triangle inequality, the second uses the fact that g and \hat{g} are Lipschitz continuous with Lipschitz constant Z . The last inequality uses the fact that g and \hat{g} take values in $[0, 1]$. Part (ii). A similar (even simpler) argument to above. ■

Lemma 7.8 $\{\psi^{(d,y)}(x, s; f)\}_{f \in K}$ is uniformly Lipschitz continuous.

Proof. By Lemma 7.6, $P_i^{(d,y)}(\theta, x, s; f)$ is uniformly Lipschitz continuous. By Lemma 7.7, $h_i(s; f) := P_i^{(d,y)}(\theta, x, s; f) \cdot s$ is Lipschitz continuous since both $P_i^{(d,y)}(\theta, x, s; f)$ and the identity map $s \rightarrow s$ are both Lipschitz continuous. Note that from the definition of $\psi^{(d,y)}(x, s; f)$ in (6), it is of the form $\frac{h_i(s; f)}{\hat{h}_i(s; f)}$ where $h_i(s; f)$ and $\hat{h}_i(s; f)$ are Lipschitz continuous (by Lemma 7.7 being finite sums of functions of the form $P_i^{(d,y)}(\theta, x, s; f) \cdot s$). Moreover, $h_i(s; f)$ belongs to a family that is uniformly Lipschitz continuous. This is because $h_i \in \{P_i^{(d,y)}(\theta, x, \cdot; f) \cdot s\}_{\{f \in K\}}$ where $\{P_i^{(d,y)}(\cdot; f)\}_{\{f \in K\}}$ is a uniformly Lipschitz continuous family and the mapping $s \rightarrow P_i^{(d,y)}(\cdot, s; f)s$ is Lipschitz continuous. Therefore, $h_i(\cdot)$ and $\hat{h}_i(\cdot)$ with Lipschitz constants κ and $\hat{\kappa}$ belong to uniformly Lipschitz continuous families with a constant, say, $\bar{\kappa} \geq \max\{\kappa, \hat{\kappa}\}$. Consider

$$\begin{aligned}
|\psi(s) - \psi(s')| &= \left| \frac{h(s)}{\hat{h}(s)} - \frac{h(s')}{\hat{h}(s')} \right| \\
&= \left| \frac{h(s)\hat{h}(s') - \hat{h}(s)h(s')}{\hat{h}(s)\hat{h}(s')} \right| \\
&\leq \frac{|h(s)\hat{h}(s') - \hat{h}(s)h(s')|}{D} \\
&= \frac{|h(s)\hat{h}(s') - h(s')\hat{h}(s') + h(s')\hat{h}(s') - \hat{h}(s)h(s')|}{D} \\
&\leq \frac{\hat{h}(s') |h(s) - h(s')| + h(s') |\hat{h}(s) - \hat{h}(s')|}{D} \\
&\leq \frac{(\hat{h}(s') + h(s'))\bar{\kappa} |s - s'|}{D} \\
&\leq \tilde{\kappa} |s - s'|, \text{ where } \tilde{\kappa} = \frac{2\bar{\kappa}}{D}
\end{aligned}$$

where $D := \inf_{\{(d,y) \in \{0,1\} \times L\}} \{P_g^{(d,y)}(\theta, x, s)\}$. The first and second equalities are effectively by definition and the third follows by adding and subtracting a term. The first inequality uses the fact that $P_g^{(d,y)}(\theta, x, s) > 0$ since all actions are feasible for a type g agent given the assumption $\bar{e}_g + \ell_{min} - \ell_{max} > 0$. The second inequality is a consequence of applying the triangle inequality and

recognizing that h_i and \hat{h} are non-negative. The third inequality results from Lipschitz continuity of h_i and \hat{h} with the same constant. Finally, the last inequality follows from the fact that h_i and \hat{h} take values in $[0, 1]$. This shows that $\{\psi^{(d,y)}(\cdot; f)\}_{\{f \in K\}}$ is a uniformly Lipschitz continuous family. ■

Lemma 7.9 $\{p(y, s'; f)\}_{\{f \in K\}}$ is uniformly Lipschitz continuous.

Proof. From its definition in (5), $p(y, s'; f)$ is a finite sum of terms involving $P_i^{(0,y)}(x, s; f)$ or the identity map $s \rightarrow s$ or the product of the two. Since the identity map is uniformly Lipschitz continuous and so is P_i by Lemma 7.6, their product is also uniformly Lipschitz continuous by Lemma 7.7. Lemma 7.7 applied to the sum of these functions establishes that $\{p(\cdot; f)\}_{\{f \in K\}}$ is a uniformly Lipschitz continuous family. ■

We now establish the properties of the operator T in step S4. These properties are required by Schauder's fixed point theorem which is the key ingredient of the main existence result in Theorem 2.1 below.

Lemma 7.10 For the operator $T : K \rightarrow F$ defined in step 3: (i) $T(K) \subseteq K$ (ii) $T(K)$ is continuous in the sup-norm; and (iii) $T(K)$ is an equicontinuous family.

Proof. To see part (i), starting with a pair of continuous functions $f = (\psi^{old}, p^{old}) \in K$, the application of the operator T through (6) and (4) updates to a new type scoring function $\psi^{new} = T^1(f)$ which is continuous by lemma 7.5 and has the properties of the scoring function ψ given in step S2. Moreover, the operator T yields the new credit scoring function $p^{new} = T^2(f)$ from (5) and (4), which is continuous by lemma 7.5 and has the properties of the credit scoring function p given in steps S1 and S2. Therefore, $T(f) = (\psi^{new}, p^{new}) \in K$. Since $f \in K$ is arbitrary, we have that $T(K) \subseteq K$.

To see part (ii), pick an arbitrary sequence of functions that converges in K , say, $f_n \rightarrow f$ in the sup-norm. We need to show that $f_n^{new} := T f_n$ converges to $f^{new} := T f$ in the sup-norm, that is

$\sup_{s \in [0,1]} |f_n^{new}(s) - f^{new}(s)| \rightarrow 0$ as $n \rightarrow \infty$. By the definition of convergence in the sup-norm, for an arbitrary $s \in [0, 1]$, $f_n(s) \rightarrow f(s)$.

Observe that for an arbitrary s a variation in $f(s)$ as f changes in K and a variation in s for a given f has the same effects on the feasible choice set and on the objective function for the recursive decision problem given by (2). More formally, from Definition 2.1 (i.e. the condition that defines the feasible action set B) and Lemma 2.1, we know that the budget set varies continuously with $f = (\psi, p)$ for a given s in a similar way as it varies continuously with s for a given f by the continuity of ψ and p in s and that of q in p . Given that, it therefore follows from Definition 2.2 that these variations have the same continuous effect on the feasible choice set. By analogous arguments, they have the same effect on objective function. Therefore, the arguments made in Lemmas 7.1, 7.2, 7.4, and 7.5 for s for an arbitrary f work analogously for $f \in K$ for an arbitrary s . This shows, in particular, that $f_n^{new}(s) \rightarrow f^{new}(s)$ for each s . Moreover, since the domain of the functions is compact, the convergence is uniform and hence $Tf_n \rightarrow Tf$ in the sup-norm, showing the continuity of T .

To see part (iii), since $Tf(s) = (\psi(s; f), p(s; f))$ and $T(K) = \{\psi(\cdot; f), p(\cdot; f)\}_{\{f \in K\}}$, it follows that $T(K)$ is a uniformly Lipschitz continuous family by lemmas 7.8 and 7.9. But establishing uniform Lipschitz continuity is sufficient for establishing equicontinuity. In particular, by definition, a family of functions K is equicontinuous if given an $\varepsilon > 0$, there exists a (single) $\delta > 0$ such that $|f(s) - f(s')| < \varepsilon$ whenever $|s - s'| < \delta$ for all $f \in K$ (see, for example, Kolmogorov and Fomin, page 102). But this condition is implied by uniform Lipschitz property. To see this, let K be a uniformly Lipschitz continuous family, with a Lipschitz constant, say, κ . Therefore, $|f(s) - f(s')| < \kappa|s - s'|$ for all $f \in K$. For a given ε , choosing $\delta = \frac{\varepsilon}{\kappa}$ shows that the equicontinuity property is satisfied. ■

Having established the key properties of the operator T , we end with the main existence result.

Theorem 2.1 *A recursive competitive equilibrium specified in Definition 2.3 exists.*

Proof. The set of functions K as specified in step $S1$ and step $S2$ is a convex and closed subset of a continuous function defined on Ω . These properties of K together with the properties of the operator T as defined in step $S3$ that are established in Lemma 7.10 constitute sufficient conditions for Schauder's fixed point theorem. Consequently there exists a pair of credit scoring p^* and type scoring ψ^* functions that is a fixed point of the operator T , i.e., $T(p^*, \psi^*) = (p^*, \psi^*)$. The existence of a fixed point to this operator establishes the existence of a competitive equilibrium as specified in Definition 2.3. The claim then follows by verifying conditions $D1$ to $D4$ in Definition 2.3. Given a pair of scoring functions (p^*, ψ^*) that is a fixed point of the operator T , a pricing function q^* is found by solving the zero profit condition for each (y, p) : $\pi(y, p; q^*(y, p)) = 0$, verifying condition $D2$ in Definition 2.3. Moreover, given these price and score functions (q^*, p^*, ψ^*) and individual characteristics (e, θ, x, s) , from Definition 2.1 and 2.2, $M_i(e, \theta, x, s; q^*, p^*, \psi^*)$ defines the feasible choice set. From Lemma 7.1, the selection $m_i^*(e, \theta, x, s, q^*, p^*, \psi^*)$ is feasible and solves the decision problem in (2), which verifies condition $D1$. Finally, by the definition of the operator T and the existence of a fixed point of that operator, for $m_i^*(e, \theta, x, s, q^*, p^*, \psi^*)$, p^* and ψ^* solves (5) and (6), respectively, verifying conditions $D3$ and $D4$. ■

7.2 Algorithm to compute $T=\infty$ equilibrium with no partitions

1. Set grid points for endowments and scores.
 - (a) There are 220 endowment grid points equally spaced between the bounds of the endowment distribution for each type.
 - (b) There are twenty score grid points equally spaced between Γ_{LH} and $1 - \Gamma_{HL}$.
2. Start iteration $j = 1$ with a set of initial guesses for the price function $q^j(y, p)$, the credit scoring function $p^j(y, s')$, and the type scoring function $\psi^j(d, y, x, s)$.
3. Given the individual state (e, x, s) , solve for the feasible actions set $B^j(e, x, s; q^j, p^j, \psi^j)$.

4. Solve for $V_i^j(e, \theta, x, s; q^j, p^j, \psi^j, W_i)$ by value function iteration. If s' is not on grids, linear interpolation is used for $W_i(y, s')$. The solution gives the set of optimal decision rule $m_i^j(d, y; e, \theta, x, s, q^j, p^j, \psi^j) \in M_i^j(e, \theta, x, s; q^j, p^j, \psi^j)$.
5. Given $m_i^j(e, \theta, x, s; q^j, p^j, \psi^j)$, calculate $\psi^{j+1}(d, y; x, s, q^j, p^j, \psi^j)$.
6. Given $\psi^{j+1}(d, y; x, s, q^j, p^j, \psi^j)$, calculate $p^{j+1}(y, s')$ and $q^{j+1}(y, p^{j+1})$.
7. Start iteration $j + 1$ by using $q^{j+1}(y, p)$, $p^{j+1}(y, s')$, and $\psi^{j+1}(d, y, x, s)$ as the new set of initial guesses. Repeat until they converge.
8. (optional) Solve for the stationary distribution $\mu_i(e, \theta, x, s)$ according to $m_i(d, y; e, \theta, x, s, q, p, \psi)$ and $\psi(d, y, x, s)$. These distribution are defined recursively by

$$\mu_{i'}(e', \theta''(d, y, x, s)) = \sum_{i, \theta} \left(\Gamma_{i'i} \cdot f(e'|i') \cdot \Lambda(\theta') \int_e m_i(d, y; e, \theta, x, s, q, p, \psi) \mu_i(\Phi(de), \theta, x, s) \right). \quad (19)$$

7.3 Algorithm to compute T=2 equilibrium with partitions

1. Set grids for endowments. There are 220 endowment grid points equally spaced between the bounds of the endowment distribution for each type.
2. Create history tuples $h^{T=2} = (d_{-1}, x_{-1}, d_{-2})$. The set of history tuples is denoted as $\mathcal{H} = \{(0, 0, 1), (0, \underline{x}, 0), (1, \underline{x}, 0), (0, 0, 0), (0, \bar{x}_1, 0), (0, \bar{x}_2, 0)\}$.
3. List all possible action/history pairs consistent with the partition that financial intermediaries can only observe default and borrowing. This will be useful in the later calculations of updating functions.

We have now four tables and 43 applicable cells. A cell is marked NA if it is not an applicable action/history pair (for instance, a household can not default with non-negative assets which is why there are NAs in two rightmost columns in Table 1). Because there are partitions, one

cell may include more than one possible action/history pair (for instance, in Table 1, the cell in the fifth row and first column includes the asset/history tuples $\{(\underline{x}, 0, \bar{x}_1, 0), (\underline{x}, 0, \bar{x}_2, 0)\}$). In that case, every action/history tuple in that same cell must have the same score due to the measurability restriction.

- Table 1: For $(d, y) = (1, 0)$, the possible histories are

$h^{T=2} \setminus x$	\underline{x}	0	$\{\bar{x}_1, \bar{x}_2\}$
$(0, 0, 1)$	OK	NA	NA
$(0, \underline{x}, 0)$	OK	NA	NA
$(1, \underline{x}, 0)$	NA	NA	NA
$(0, 0, 0)$	OK	NA	NA
$\{(0, \bar{x}_1, 0), (0, \bar{x}_2, 0)\}$	OK	NA	NA

- Table 2: For $(d, y) = (0, \underline{x})$, the possible histories are

$h^{T=2} \setminus x$	\underline{x}	0	$\{\bar{x}_1, \bar{x}_2\}$
$(0, 0, 1)$	OK	OK	OK
$(0, \underline{x}, 0)$	OK	OK	OK
$(1, \underline{x}, 0)$	NA	OK	NA
$(0, 0, 0)$	OK	OK	OK
$\{(0, \bar{x}_1, 0), (0, \bar{x}_2, 0)\}$	OK	OK	OK

- Table 3: For $(d, y) = (0, 0)$, the possible histories are

$h^{T=2} \setminus x$	\underline{x}	0	$\{\bar{x}_1, \bar{x}_2\}$
$(0, 0, 1)$	OK	OK	OK
$(0, \underline{x}, 0)$	OK	OK	OK
$(1, \underline{x}, 0)$	NA	OK	NA
$(0, 0, 0)$	OK	OK	OK
$\{(0, \bar{x}_1, 0), (0, \bar{x}_2, 0)\}$	OK	OK	OK

- Table 4: For $(d, y) = \{(0, \bar{x}_1), (0, \bar{x}_2)\}$, the possible histories are

$h^{T=2} \setminus x$	\underline{x}	0	$\{\bar{x}_1, \bar{x}_2\}$
(0, 0, 1)	OK	OK	OK
(0, \underline{x} , 0)	OK	OK	OK
(1, \underline{x} , 0)	NA	OK	NA
(0, 0, 0)	OK	OK	OK
$\{(0, \bar{x}_1, 0), (0, \bar{x}_2, 0)\}$	OK	OK	OK

4. Start iteration $j = 1$ with a set of initial guesses for the price function $q^j(y, p)$, the credit scoring function $p^j(y, \psi^j(d, y, x, h^{T=2}))$, and the type scoring function $\psi^j(d, y, x, h^{T=2})$.
5. Given the individual state (e, x, s) , solve for the feasible actions set $B^j(e, x, h^{T=2}; q^j, p^j, \psi^j)$.
6. Solve for $V_i^j(e, \theta, x, h^{T=2}; q^j, p^j, \psi^j, W_i)$ by value function iteration. The solution gives the set of optimal decision rule $m_i^j(e, \theta, x, h^{T=2}; q^j, p^j, \psi^j) \in M_i^j(e, \theta, x, h^{T=2}; q^j, p^j, \psi^j)$.
7. Given $M_i^j(e, \theta, x, h^{T=2}; q^j, p^j, \psi^j)$, solve for stationary distribution $\mu_i^j(e, \theta, x, h^{T=2})$.

$$\mu_{i'}^j(e', y, d, x, d_{-1}) = \sum_{i, \theta, (x_{-1}, d_{-2})} \left(\Gamma_{i'i} \cdot f(e'|i') \cdot \Lambda(\theta') \int_e m_i(d, y; e, \theta, x, d_{-1}, x_{-1}, d_{-2}, q, p, \psi) \mu_i(\Phi(de), \theta, x, d_{-1}, x_{-1}, d_{-2}) \right). \quad (20)$$

8. Given $m_i^j(e, \theta, x, h^{T=2}; q^j, p^j, \psi^j)$ and $\mu_i^j(e, \theta, x, h^{T=2})$, calculate $\psi^{j+1}(d, y; x, h^{T=2}, q^j, p^j, \psi^j)$ with respect to the partition blocks listed in step 3.
9. Given $\psi^{j+1}(d, y; x, h^{T=2}, q^j, p^j, \psi^j)$, calculate $p^{j+1}(y, \psi^{j+1}(d, y; x, h^{T=2}, q^j, p^j, \psi^j))$ and $q^{j+1}(y, p^{j+1})$.
10. Start iteration $j + 1$ by using $q^{j+1}(y, p)$, $p^{j+1}(y, \psi^{j+1}(d, y, x, h^{T=2}))$, and $\psi^{j+1}(d, y, x, h^{T=2})$ as the new set of initial guesses. Repeat until they converge.
11. With the distribution, the type score of an agent $s^T(x, h^T)$ can be calculated according to equation (8).

7.4 Credit scoring function

We re-define the credit scoring function as \tilde{p} in order to account for the distribution of credit scores in Figure 1 as follows:

$$\tilde{p}(x, h^T) = \frac{p(\underline{x}, \psi(\underline{x}, 0, x, h^T)) \left[\hat{P}_g(\underline{x}, 0, x, h^T) s^T + \hat{P}_b(\underline{x}, 0, x, h^T) (1 - s^T) \right]}{\hat{P}_g(\underline{x}, 0, x, h^T) s^T + \hat{P}_b(\underline{x}, 0, x, h^T) (1 - s^T)}. \quad (21)$$

The denominator is the fraction of agents in tuple (x, h^T) who borrow, and the numerator is the fraction of agents in the same tuple who borrow and pay back their debt.