FEDERAL RESERVE BANK
PHILADELPHIA

Working Papers
RESEARCH DEPARTMENT

Price-Level Determination Under the Gold Standard

Jesús Fernández-Villaverde
University of Pennsylvania, NBER, and CEPR

Daniel Sanches
Federal Reserve Bank of Philadelphia Research Department

WP 24-06

PUBLISHED
February 2024

ISSN: 1962-5361

DOI: https://doi.org/10.21799/frbp.wp.2024.06

Price-Level Determination Under the Gold Standard

Jesús Fernández-Villaverde*
University of Pennsylvania, NBER, and CEPR

Daniel Sanches†
Federal Reserve Bank of Philadelphia

January 9, 2024

Abstract

We present a micro-founded monetary model of a small open economy to examine the behavior of money, prices, and output under the gold standard. In particular, we formally analyze Hume's celebrated price-specie flow mechanism. Our framework incorporates the influence of international trade on the money supply in the Home country through gold flows. In the short run, a positive correlation exists between the quantity of money and the price level. Additionally, we demonstrate that money is non-neutral during the transition to the steady state, which has implications for welfare. While the gold standard exposes the Home country to short-term fluctuations in money, prices, and output caused by external shocks, it ensures long-term price stability as the quantity of money and prices only temporarily deviate from their steady-state levels. We discuss the importance of policy coordination for achieving efficiency under the gold standard and consider the role of fiat money in this environment. We also develop a version of the model with two large economies.

Keywords: Gold standard; specie flows; non-neutrality of money; long-run price stability; inelastic money supply.

JEL classifications: E42, E58, G21.

*Email address: jesusfv@upenn.edu.

†Email address: daniel.sanches@phil.frb.org. The views expressed in this paper are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System. Philadelphia Fed working papers are free to download at https://philadelphiafed.org/research-and-data/publications/working-papers.

1 Introduction

The price-specie flow mechanism, developed by Hume ([1752] 1977), is one of the oldest and most celebrated models in economics. Hume identified a mechanism that links the quantity of money in an economy to its prices and the impact of price changes on balance-of-trade surpluses and deficits. Classical economists further built upon Hume's insights, considering the money supply to consist of precious metals like gold and silver coins. The adoption of a multilateral gold standard by most advanced countries in the early 1870s renewed economists' interest in monetary affairs, both domestically and internationally.[1] The gold standard remained the focal point of monetary economics until the early 1930s.

Although the classical gold standard was abandoned following the Great Depression, discussions of viable monetary systems based on precious metals have persisted, and scholars have dedicated significant attention to studying their properties. For example, Friedman and Schwartz (1963) documented the U.S. experience under the gold standard after the country resumed specie payments in 1879, and Gallarotti (1995) analyzed the institutions associated with the gold standard during its classical period (1880 to 1914).

While the gold standard continued to be viewed by many scholars as an important benchmark for analyzing the international monetary system, there have been few attempts to model a commodity money system in a modern dynamic general equilibrium framework. In particular, we are unaware of any theoretical analysis of specie flows and their relation to the money supply, prices, and output in modern models where money is essential in the sense of Wallace (2001).

To address this gap in the literature, this paper presents a random matching model of a small open economy to examine the dynamics of prices and output under the gold standard. The model incorporates decentralized trade in domestic and international markets and the absence of a monetary authority controlling the money supply within the Home country. A durable commodity (gold) serves as a medium of exchange for bilateral transactions among agents. Consequently, the Home country's total money supply hinges on the international trade pattern: imports lead to gold outflows, and exports result in gold inflows. The price level in the Home country is determined by the reciprocal of the value of goods in terms of gold.

A model with random matching and bargaining is particularly suitable for characterizing the price-specie flow mechanism under a commodity money system. In addition to giving money an essential role in exchange without ad hoc constraints, a random matching model formalizes the intensive and extensive margins of trade that are important to account for trade flows across countries. This is especially important when we consider international economic integration.

Our primary objective is to explore the features of the gold standard as a monetary frame-

[1] Friedman and Schwartz (1963), Bordo (1981), Redish (1990), Bordo and Kydland (1995), Gallarotti (1995), and Eichengreen (2019) provide excellent analyses of the operation of the gold standard.

work for domestic and international transactions.[2] Motivated by the theoretical and empirical literature, we are interested in the following questions: Can the gold standard generate short- and long-run price stability? Can the gold standard provide an efficient monetary arrangement? How do economies under the gold standard respond to external trade shocks?

Our analysis uncovers three crucial properties of the gold standard. First, the gold standard ensures long-run price stability: the price level in the Home country consistently converges to its long-run equilibrium value. Inflation and deflation are merely temporary phenomena. Formally, we establish that the dynamic system under the gold standard behaves like a saddle. The price level in the Home country acts as a jump variable, while the quantity of money in a country remains predetermined because it depends on the inflows and outflows of gold resulting from international trade. If the initial quantity of money is below the steady-state level, the economy experiences temporary inflation as the quantity of money increases during the transition period. This price dynamic leads to a short-term surplus in exports, causing the Home country to accumulate gold temporarily. Over time, the inflation rate decreases as the price level converges to its long-run value from below. Conversely, if the initial quantity of money in the Home country exceeds the steady-state level, the domestic economy endures a temporary deflation. In this scenario, imports surpass exports in the short run, temporarily causing the Home country to lose gold. The deflation rate is decreasing along the transition path, given that the price level converges to the steady-state level from above.

Second, money is non-neutral under the gold standard. When the quantity of money in the Home country starts below the steady-state level, the Home country temporarily accumulates gold along the transition path. Hence, the money supply rises over time. The output in the bilateral matches involving domestic traders monotonically increases in the transition path to the steady state as the price level in the Home country rises. Money is non-neutral because new money can only enter the Home country if exporters bring in more gold than importers take out (we abstract from the possibility of domestic gold production). Conversely, when the quantity of money starts above the steady-state level, the Home country loses gold temporarily to the rest of the world, and the money supply declines. The output in the bilateral matches involving domestic traders monotonically decreases in the transition path. In this case, a declining money supply is associated with decreasing output and a falling price level in the Home country.

Third, the Home country is vulnerable to external shocks due to the inelasticity of the money supply under the gold standard. A trade shock (e.g., an unanticipated and permanent change in the degree of economic integration between the Home country and the rest of the world) triggers

[2] The standard textbook treatment of the gold standard interprets it as a multilateral system of fixed exchange rates. Indeed, the implied fixed exchange rates among countries under the gold standard are a feature of the system. But the gold standard is a stronger proposition. Its key properties are the definition of national currencies as specific amounts of gold and the ensuing commitment to follow the rules of this system regarding the evolution of the money supply, both of which go beyond a multilateral system of fixed exchange rates.

specie movements that result in either temporary inflation or deflation as the quantity of money adjusts to the new steady state. Because money is non-neutral, these price movements have real effects. At least some Home-country agents will be strictly worse off in the transition path.

Our results align with the empirical studies on the macroeconomic properties of the gold standard. Comparing the classical gold standard with the post-war period, Bordo (1981) and Schwartz (1986) find that it is not clear that the classical gold standard delivered greater short-run price stability, but they unequivocally conclude that it delivered remarkable long-run price stability (i.e., prices consistently reverted to their mean) compared with other regimes. Using data from early modern Europe, Palma (2022) documents that monetary expansions under a commodity standard had a substantial and persistent effect on economic activity.

The vulnerability of the gold standard to external shocks is reported in Ford (1960), Gallarotti (1995), and Bordo et al. (2001), among others. These authors find that external shocks, such as changes in trade policy or the onset of a financial crisis in a major gold-standard country, set in motion specie movements in the world economy that affect the quantity of money, prices, and output in the domestic economy of a country under the gold standard. Our framework permits us to perform some of these experiments, and our findings align with these empirical studies.

Our welfare analysis shows that the laissez-faire equilibrium under the gold standard is inefficient. In view of this inefficiency, we consider policy interventions in the Home and Foreign countries. The policy options available are limited because of the impossibility of changing the money supply under the gold standard. Thus, we study an intervention where governments pay a flow return on money holdings, as in Lagos and Wright (2003). This policy intervention can be viewed as a subsidy to gold holders, a policy compatible with adherence to the gold standard.[3]

We find that synchronizing domestic and foreign policy is crucial for efficiency. In particular, it is not possible to implement the efficient allocation under the gold standard if domestic and foreign policies are not perfectly synchronized. We then provide the conditions for efficiency under synchronized policies and show how the policy-induced flow returns on gold holdings can be constructed to implement the efficient allocation in the long run.

If international cooperation is unattainable (or if interest payments on gold holdings are incentive-infeasible), we consider other policy options, such as the introduction of fiat money. Specifically, we characterize the equilibrium when the Home country government issues fiat money and follows a policy rule. This hybrid monetary system allows the concurrent circulation of fiat money and gold as media of exchange. Our main new finding is that the demonetization of gold can be endogenously attained by choosing a specific class of monetary policy rules. Moreover, the demonetization of gold can result in the efficient allocation, even though the implementation of optimal monetary policy in this environment is not unique.

[3]A historical example of this policy is the different gold devices implemented by central banks during the classical gold standard to induce local gold holdings. For instance, gold arbitragers were given interest-free loans and subsidized transportation services.

To complete our analysis, we develop a version of the model with two large economies. This version of the model has a symmetric steady state in which the world gold stock is evenly distributed across the two economies, and the price level in each country is constant over time. We characterize the dynamic system describing the world equilibrium of the model and argue that any adjustment mechanism necessarily involves gold outflows from the country that starts with an above-average quantity of gold to the other country or region, as predicted by the price-specie flow mechanism. However, we cannot guarantee the monotonicity of the price level and the quantity of money during the transition path to the symmetric steady state (as we have determined for the small open economy) without further imposing restrictions on preferences and technologies. We leave this characterization for future work.

This paper is organized as follows. Section 2 links our analysis to the previous literature. Section 3 describes and characterizes a small open economy model under the gold standard. Section 4 discusses policy options in the domestic and foreign countries under the gold standard and the conditions for efficiency. Section 5 introduces fiat money in the environment. Section 6 presents an extension of the model where gold has non-monetary uses. Section 7 develops a model of the world economy with two large countries or regions. Section 8 concludes.

2 Related Literature

Sargent and Wallace (1983) provide an early analysis of commodity money in an overlapping generations model. In their framework, commodity money is a capital good that can be converted into a consumption good or accumulated over time. Among other conclusions, they find that a commodity money system is generally inefficient. Velde and Weber (2000) study a bimetallic system in which gold and silver coexist as media of exchange. Their study focuses on the sustainability of bimetallism in a single country and the long-run welfare benefits of unilaterally moving from a bimetallic to a monometallic system. In contrast, we analyze specie flows in a small open economy and study their implications for money, prices, and output in the short and long run.

Another significant contribution to the literature comes from Kiyotaki et al. (1993), who build upon the first-generation monetary search model of Kiyotaki and Wright (1989) to study the existence of equilibria in which fiat currencies can serve as international media of exchange. Trejos and Wright (2001) extend their analysis by including endogenous prices through bargaining theory and provide insights into the features of the environment that make it more likely that a given currency circulates internationally. Devereux and Shi (2013) present a related analysis of the emergence of a single currency as an international medium of exchange. Our analysis builds on their contributions. However, in our framework, money and goods are both divisible, and we focus on the international circulation of commodity money. Building on the third-generation monetary search model developed by Lagos and Wright (2003), Lagos and Wright (2005), and

Rocheteau and Wright (2005), we derive price and output dynamics that rationalize the price-specie flow mechanism under a commodity standard.

Velde et al. (1999) study commodity money with heterogeneous assets under imperfect information in the tradition of the second-generation monetary search model developed by Shi (1995) and Trejos and Wright (1995). Velde et al. (1999) are interested in rationalizing Gresham's law as an equilibrium phenomenon and providing a potential explanation for the so-called debasement puzzle. However, their model is not designed to consider international commerce and monetary relations, given that it assumes a closed system with a fixed amount of durable assets.

Zhang (2014) develops an information-based theory of international fiat currencies to study monetary policy coordination and other issues in international economics. Gomis-Porqueras et al. (2017) show that the threat of counterfeiting can result in the determinacy of the nominal exchange rate under informational frictions. Zhu (2022) studies an environment with aggregate shocks to show that the exchange rate regime can be non-neutral in a two-country world economy. These models provide micro-foundations for monetary exchange in the context of international fiat currencies. In comparison, our analysis focuses on commodity money and the effects of international specie movements on domestic prices and output.

Geromichalos and Simonovska (2014) construct a two-country model in which internationally traded assets represent claims on future consumption and facilitate transactions in markets with imperfect credit. In their model, the liquidity properties of assets matter for asset prices, output, and consumption, which allows the authors to account for the so-called home asset bias in international finance. In contrast to our paper, their analysis focuses on steady states.

Farhi and Maggiori (2018) develop a model of the international monetary system under different scenarios, including the gold exchange standard that prevailed in the interwar period. They find that gold scarcity can inefficiently decrease output via sticky prices and that any attempt to introduce monetary assets will result in a self-fulfilling collapse of the international system. Their analysis rationalizes Keynes's argument against Britain's return to the gold standard at the prewar parity. Importantly, our work does not assume nominal rigidities and focuses on the dynamic adjustment of the domestic economy to specie flows under the gold standard.

Sargent (2019) presents a simple general equilibrium framework that explores the coexistence of commodity and fiat monies, offering insights into several episodes in monetary history. Our paper concentrates instead on a pure commodity money system and emphasizes the importance of dynamic analysis in understanding cross-border monetary movements and welfare considerations.

Later in the analysis, we study the policy options available under the gold standard to attain efficiency. We consider a subsidy to gold holders as a policy-induced flow return on gold holdings, a policy proposed by Lagos and Wright (2003). Hence, our analysis is related to studies of economies in which real assets are valued for their liquidity properties, such as Geromichalos et al. (2007), Lagos (2010), Geromichalos and Herrenbrueck (2016), Geromichalos et al. (2016), and Altermatt et al. (2021). However, our main policy result –that efficiency under the gold standard necessarily requires international policy coordination– is new.

Although we do not consider the implications of domestic mining of precious metals for the dynamics of prices and output in our model, the analysis in Choi and Rocheteau (2022) could be helpful in formalizing mining in our framework (see also Choi and Rocheteau, 2021). We believe this could be an interesting extension of our research.

3 A Gold Standard Economy

We now present a monetary model for a small open economy under the gold standard. The Home country is negligible relative to the Foreign country, representing the rest of the world. In particular, asset movements to and from the Home country will not affect prices in the Foreign country. Our focus is on the construction of equilibria in the Home country. Although we do not provide all the equilibrium conditions for the Foreign country, we need to provide some structure for that country because we will develop a search model characterized by bilateral exchange, where Home country buyers trade with Foreign country sellers.

Model. Our framework builds on the New Monetarist model developed in Lagos and Wright (2005) and Rocheteau and Wright (2005) and subsequently reviewed in Williamson and Wright (2010) and Lagos et al. (2017).

Time is discrete and continues indefinitely. Each period is divided into two subperiods: the first with a decentralized market (DM) and the second with a centralized market (CM). Within each subperiod, a perishable commodity is produced and consumed. The world economy is divided into two countries or regions: Home and Foreign. The Home country is a small open economy (SOE), while the Foreign country represents the rest of the world. We refer to the decentralized market in the Home (Foreign) country as the Home (Foreign) DM and to the centralized market in the Home (Foreign) country as the Home (Foreign) CM. The Home country consists of a measure $[0, 2-\lambda]$ of buyers and a measure $[0, 1]$ of sellers, where $\lambda \in (.5, 1)$.

At the beginning of each period, a random measure $1-\lambda$ of Home country buyers travel abroad and are absent from the Home country for one period. We refer to those buyers as *international traders*. These traders participate in the Foreign DM and CM and return to the Home country with probability 1 in the following period. When an international trader in period t returns to the Home country in period $t+1$, we refer to him as a *returning international trader*. We call the Home country buyers who stay in the Home country *domestic traders*. These traders participate in the Home DM and CM. Returning international traders also participate in the Home DM and CM. Each buyer in the Home country finds out whether he will be an international or domestic trader at date $t+1$ in the period-t CM. In summary, in any given period, the Home country buyers are divided into three groups or types: domestic traders (with measure λ), international traders (with measure $1-\lambda$), and returning international traders (with measure $1-\lambda$). See Figure 1 for a summary of timing.

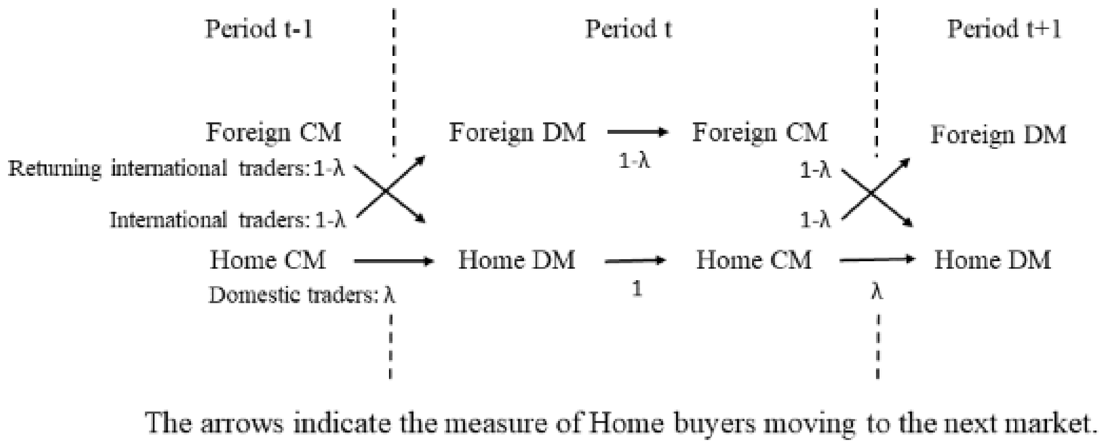

Figure 1: Home country buyers

The DM (in the first subperiod) is characterized by bilateral matches and bargaining. A domestic trader will find a Home country seller with probability $\alpha \in (0,1)$. An international trader will find a Foreign country seller with probability δ. A returning international trader will find a Home country seller with probability ω. Assume that $0 \leq \delta, \omega \leq \alpha$. These assumptions imply that international trade matches are harder to find than domestic ones and that returning traders might have lost some of their ability to match with Home sellers due to local changes during their absence.

As we will see, the international trader will take assets out of the Home country to make purchases in the period-t Foreign DM. The returning international trader (who was an international trader in period $t-1$) will bring assets to the Home country to make purchases in the period-t Home DM. More concretely, the international trader is an importer in the period-t Foreign DM and an exporter in the period-t Foreign CM. The international trader acquires assets in the period-$(t-1)$ Home CM to import goods in the period-t Foreign DM. This agent rebalances his asset portfolio by exporting the CM good in the period-t Foreign CM. In period $t+1$, he will be a returning international trader and will use his assets to make purchases in the Home DM.

The Home country buyer derives utility $u(q)$ from the consumption of the DM good, where $q \in \mathbb{R}_+$ is the amount consumed. Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, increasing, and strictly concave, with $u(0) = 0$, $u'(0) = \infty$, and $-qu''(q) < u'(q)$ for all $q > 0$. The disutility of DM production for the Home and Foreign seller is $-q$, where $q \in \mathbb{R}_+$ denotes the amount produced. The technology to produce the CM good is linear in the agent's labor supply, and all agents have access to this technology in the CM. Each agent derives linear utility from the consumption of the CM good. Given these preferences and technologies, a positive utility value indicates the net consumption of the CM good, and a negative utility value means the net production of the CM good. All agents have the same discount factor $\beta \in (0, 1)$ across periods.

In addition, there exists a durable good (gold) that does not depreciate, has zero storage costs, and can be held in any divisible quantity. All agents can perfectly recognize any amount of gold presented in a bilateral match and carry any quantity of it across markets. Agents do not derive direct utility from owning gold. We later provide a version of the model in which a subset of agents will derive utility from the ownership of gold to generate a non-monetary demand for gold and show that all the model properties remain the same.

Buyers' and sellers' identities are unknown to each other, and their trading histories are private information, which precludes credit in the DM and makes a medium of exchange essential. Because gold is a durable commodity, agents can use it as a medium of exchange to trade in the Home and Foreign DM. Throughout the analysis, the value of gold in the Foreign CM will remain constant.[4] The SOE assumption implies that gold movements between the Home country and the rest of the world do not affect the international price of gold.

We formalize trade flows asymmetrically. As we have seen, an international trader can meet a Foreign country seller when traveling abroad to the Foreign DM. However, a Foreign country buyer cannot meet a Home country seller in the Home DM. This assumption is important because it allows us to describe the SOE without specifying the individual holdings of gold of agents in the rest of the world. Later in the paper, we present a model with two large economies in which trade flows are symmetric across countries, which is suitable for analyzing international trade between two large regions. In that version of the model, the value of gold will be endogenously determined in each country.[5]

Bellman equations. Let $\bar{\rho} \in \mathbb{R}_+$ denote the price of gold in terms of the CM good in the Foreign CM, and let $\rho_t \in \mathbb{R}_+$ denote the price of gold in terms of the CM good in the Home CM. Let $W_t(m)$ denote the period-t CM value function of a domestic trader with $m \in \mathbb{R}_+$ units of money (i.e., gold holdings), let $\widehat{W}_t(m)$ denote the CM value function of an international trader, and let $W_t^F(m)$ denote the CM value function of a returning international trader. Let $V_t(m)$ denote the period-t DM value function of a domestic trader with $m \in \mathbb{R}_+$ units of money, let $\widehat{V}_t(m)$ denote the DM value function of an international trader, and let $V_t^F(m)$ denote the DM value function of a returning international trader. The Home country buyer's CM value functions

[4] We could have developed the analysis in a scenario in which the price of gold appreciated or depreciated at a constant rate. Mathematically, this model could be solved by following the same steps as below.

[5] In this version of the model, Home country buyers can meet Foreign country sellers, and Foreign country buyers can meet Home country sellers at each date. Nonetheless, we continue to assume that sellers do not move across countries.

are:

$$\widehat{W}_t(m) = \max_{(x,m')\in\mathbb{R}\times\mathbb{R}_+} \left[x + \beta\widehat{V}_{t+1}(m')\right] \text{ s.t. } x + \rho_t m' = \rho_t m$$
$$W_t^F(m) = \max_{(x,m')\in\mathbb{R}\times\mathbb{R}_+} \left[x + \beta V_{t+1}^F(m')\right] \text{ s.t. } x + \bar{\rho} m' = \bar{\rho} m$$
$$W_t(m) = \max_{(x,m')\in\mathbb{R}\times\mathbb{R}_+} \left[x + \beta V_{t+1}(m')\right] \text{ s.t. } x + \rho_t m' = \rho_t m.$$

In the Home CM, the buyer learns whether he will be a domestic or international trader in the following period. After learning his status, his period-t CM value function describes his continuation value. An international trader in the period-t Foreign CM knows he will be a returning international trader at $t+1$.

The Home country buyer's DM value functions are given by:

$$\begin{aligned}
\widehat{V}_t(m) &= \delta\left[u(\widehat{q}(m)) + W_t^F\left(m - \widehat{d}(m)\right)\right] + (1-\delta) W_t^F(m) \\
V_t^F(m) &= \omega\left[u(q_t(m)) + \lambda W_t(m - d_t(m)) + (1-\lambda)\widehat{W}_t(m - d_t(m))\right] \\
&\quad + (1-\omega)\left[\lambda W_t(m) + (1-\lambda)\widehat{W}_t(m)\right] \\
V_t(m) &= \alpha\left[u(q_t(m)) + \lambda W_t(m - d_t(m)) + (1-\lambda)\widehat{W}_t(m - d_t(m))\right] \\
&\quad + (1-\alpha)\left[\lambda W_t(m) + (1-\lambda)\widehat{W}_t(m)\right].
\end{aligned}$$

The functions $\widehat{q}(m)$, $\widehat{d}(m)$, $q_t(m)$, and $d_t(m)$ denote the terms of trade in the Foreign and Home DM, respectively, as a function of the buyer's money holdings. An international trader with m units of money gets $\widehat{q}(m)$ units of the DM good from the Foreign country seller with whom he is currently matched in exchange for the money payment $\widehat{d}(m)$. A domestic trader (or a returning international trader) with m units of money gets $q_t(m)$ units of the DM good from the Home country seller with whom he is currently matched in exchange for the money payment $d_t(m)$.

Because the buyer's preferences are quasi-linear with respect to the effort level in the CM, we can write the Home country buyer's beginning-of-the-period Bellman equations as:

$$\begin{aligned}
\widehat{V}_t(m) &= \delta\left[u(\widehat{q}(m)) - \bar{\rho}\widehat{d}(m)\right] + \max_{m'\in\mathbb{R}_+}\left[-\bar{\rho}(m' - m) + \beta V_{t+1}^F(m')\right] \\
V_t^F(m) &= \omega\left[u(q_t(m)) - \rho_t d_t(m)\right] + \lambda \max_{m'\in\mathbb{R}_+}\left[-\rho_t(m' - m) + \beta V_{t+1}(m')\right] \\
&\quad + (1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\rho_t(m' - m) + \beta\widehat{V}_{t+1}(m')\right] \\
V_t(m) &= \alpha\left[u(q_t(m)) - \rho_t d_t(m)\right] + \lambda \max_{m'\in\mathbb{R}_+}\left[-\rho_t(m' - m) + \beta V_{t+1}(m')\right] + \\
&\quad (1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\rho_t(m' - m) + \beta\widehat{V}_{t+1}(m')\right].
\end{aligned}$$

These equations fully describe the Home country buyer's transitions across different states over time in the SOE.

Bargaining. Throughout the analysis, we assume that the buyer makes a take-it-or-leave-it offer to the seller in the DM. This bargaining protocol greatly simplifies the analysis of the dynamic behavior of the model. Given this assumption, we can derive:

$$\widehat{q}(m) = \begin{cases} \bar{\rho}m \text{ if } m < \frac{q^*}{\bar{\rho}} \\ q^* \text{ if } m \geq \frac{q^*}{\bar{\rho}} \end{cases}$$

$$\widehat{d}(m) = \begin{cases} m \text{ if } m < \frac{q^*}{\bar{\rho}} \\ \frac{q^*}{\bar{\rho}} \text{ if } m \geq \frac{q^*}{\bar{\rho}} \end{cases}$$

$$q_t(m) = \begin{cases} \rho_t m \text{ if } m < \frac{q^*}{\rho_t} \\ q^* \text{ if } m \geq \frac{q^*}{\rho_t} \end{cases}$$

$$d_t(m) = \begin{cases} m \text{ if } m < \frac{q^*}{\rho_t} \\ \frac{q^*}{\rho_t} \text{ if } m \geq \frac{q^*}{\rho_t} \end{cases}$$

where $q^* \in \mathbb{R}_+$ is the surplus-maximizing quantity: $u'(q^*) = 1$. If the value of the buyer's money holdings is large enough to induce the seller to produce q^*, the efficient level of trade occurs. If not, the buyer spends all he can, and the seller produces an amount smaller than q^*.

The next step to derive the equilibrium of the model is to solve the portfolio problems of all types of buyers: domestic traders, international traders, and returning international traders. Because each type will trade in different markets going forward, they will hold different quantities of money when the price level differs across countries.

Domestic traders. Let $m_{t+1} \in \mathbb{R}_+$ denote the domestic trader's money holdings at the end of the period-t Home CM. The portfolio problem for this agent can be written as:

$$\max_{m_{t+1} \in \mathbb{R}_+} \{-\rho_t m_{t+1} + \beta \{\alpha [u(q_{t+1}(m_{t+1})) - \rho_{t+1} d_{t+1}(m_{t+1})] + \rho_{t+1} m_{t+1}\}\}.$$

This optimization problem is well defined provided $\rho_t \geq \beta \rho_{t+1}$. If this condition holds as a strict inequality, the first-order condition is:

$$-\rho_t + \beta \rho_{t+1} [\alpha u'(\rho_{t+1} m_{t+1}) + 1 - \alpha] \leq 0, \text{ with equality if } m_{t+1} > 0. \quad (1)$$

A domestic trader acquires gold in the Home CM to potentially trade with a Home country seller in the following DM. When condition (1) holds with equality, it defines the domestic trader's demand for money balances in the Home CM.

International traders. Let $\widehat{m}_{t+1} \in \mathbb{R}_+$ denote the international trader's money holdings at the end of the period-t Home CM. The portfolio problem for this agent can be written as:

$$\max_{\widehat{m}_{t+1}\in\mathbb{R}_+} \left\{-\rho_t\widehat{m}_{t+1} + \beta\left\{\delta\left[u\left(\widehat{q}\left(\widehat{m}_{t+1}\right)\right) - \bar{\rho}\widehat{d}\left(\widehat{m}_{t+1}\right)\right] + \bar{\rho}\widehat{m}_{t+1}\right\}\right\}.$$

This optimization problem is well defined provided $\rho_t \geq \beta\bar{\rho}$. If this condition holds as a strict inequality, the first-order condition is:

$$-\rho_t + \beta\bar{\rho}\left[\delta u'\left(\bar{\rho}\widehat{m}_{t+1}\right) + 1 - \delta\right] \leq 0, \text{ with equality if } \widehat{m}_{t+1} > 0. \tag{2}$$

An international trader acquires gold in the Home CM to potentially trade with a Foreign country seller in the subsequent Foreign DM. Because both the international trader and the Foreign country seller trade in the Foreign CM at date $t+1$, they value gold equally going forward.

Returning international traders. Let $m^F_{t+1} \in \mathbb{R}_+$ denote the returning international trader's money holdings at the end of the period-t Foreign CM. The portfolio problem for this agent can be written as:

$$\max_{m^F_{t+1}\in\mathbb{R}_+} \left\{-\bar{\rho}m^F_{t+1} + \beta\left\{\omega\left[u\left(q_{t+1}\left(m^F_{t+1}\right)\right) - \rho_{t+1}d_{t+1}\left(m^F_{t+1}\right)\right] + \rho_{t+1}m^F_{t+1}\right\}\right\}.$$

This optimization problem is well defined provided $\bar{\rho} \geq \beta\rho_{t+1}$. If this condition holds as a strict inequality, the first-order condition is:

$$-\bar{\rho} + \beta\rho_{t+1}\left[\omega u'\left(\rho_{t+1}m^F_{t+1}\right) + 1 - \omega\right] \leq 0, \text{ with equality if } m^F_{t+1} > 0. \tag{3}$$

A returning international trader acquires gold in the period-t Foreign CM to potentially trade with a Home country seller in the period-$(t+1)$ Home DM. Both agents will subsequently trade in the period-$(t+1)$ Home CM, so they value gold equally going forward.

Market clearing. Let $Q_t \in \mathbb{R}_+$ denote the total gold stock in the Home country available in period t. Under the gold standard, Q_t is also the total money supply in the Home country. The market-clearing condition in the Home country implies:

$$Q_t = \lambda m_{t+1} + (1-\lambda)\,\widehat{m}_{t+1}. \tag{4}$$

In the period-t Home CM, two types of buyers demand gold: domestic traders (with total demand λm_{t+1}) and international traders (with total demand $(1-\lambda)\,\widehat{m}_{t+1}$). Recall that returning international traders rebalance their portfolio in the period-t Foreign CM.

We also keep track of the evolution of the total gold stock in the Home country. The law of motion for the Home gold stock is $Q_{t+1} = Q_t + (1-\lambda)\left(m^F_{t+1} - \widehat{m}_{t+1}\right)$. A returning international trader will bring gold from the Foreign country, and an international trader leaving the Home country in the following period will take gold out of the Home country. Hence, the Home gold stock grows (shrinks) from period t to $t+1$ if the difference $m_{t+1}^F - \widehat{m}_{t+1}$ is positive (negative).

Demand functions. Using the previously derived first-order conditions, we construct the demand functions for each type of buyer. Consider the international trader. Provided $\rho_t > \beta\bar{\rho}$, the first-order condition (2) implicitly defines $\widehat{m}_{t+1} = \widehat{m}\left(\rho_t\right)$. The slope of this implicit function is given by:

$$\widehat{m}'\left(\rho_t\right) = \frac{1}{\delta\beta\bar{\rho}^2 u''\left(\widehat{q}_{t+1}\right)} < 0,$$

where $\widehat{q}_{t+1} = \bar{\rho}\widehat{m}\left(\rho_t\right)$. The international trader's demand for money is strictly decreasing in the period-t value of gold in the Home CM. Because the value of gold in the Foreign country is constant over time, a higher period-t value of gold in the Home CM implies a lower real rate of return on gold, which leads international traders to reduce their desired gold holdings.

Consider now the returning international trader. Provided $\bar{\rho} > \beta\rho_{t+1}$, the first-order condition (3) implicitly defines $m_{t+1}^F = m_F\left(\rho_{t+1}\right)$. The slope of this implicit function is given by:

$$m_F'\left(\rho_{t+1}\right) = -\frac{\omega u'\left(q_{t+1}^F\right) + 1 - \omega + \omega q_{t+1}^F u''\left(q_{t+1}^F\right)}{\omega\rho_{t+1}^2 u''\left(q_{t+1}^F\right)} > 0,$$

where $q_{t+1}^F = \rho_{t+1} m_F\left(\rho_{t+1}\right)$. A returning international trader acquires gold in the period-t Foreign CM at the fixed price $\bar{\rho}$. If the value of gold in the period-$(t+1)$ Home CM is higher, so is the real return on gold holdings going forward. Because of the higher return on gold, returning international traders will increase their demand for gold in the period-t Foreign CM.

DM output. The quantities produced and consumed in the matches involving Home country buyers in the Home and Foreign DM are:

$$\begin{aligned} q_{t+1} &= \lambda^{-1}\rho_{t+1}\left[Q_t - (1-\lambda)\,\widehat{m}\left(\rho_t\right)\right] \leq q^* \\ q_{t+1}^F &= \rho_{t+1} m_F\left(\rho_{t+1}\right) \leq q^* \\ \widehat{q}_{t+1} &= \bar{\rho}\widehat{m}\left(\rho_t\right) \leq q^*. \end{aligned}$$

The quantity $q_t \in \mathbb{R}_+$ is the period-t DM consumption of a domestic trader in the Home DM, $q_t^F \in \mathbb{R}_+$ is the period-t DM consumption of a returning international trader in the Home DM, and $\widehat{q}_t \in \mathbb{R}_+$ is the period-t DM consumption of an international trader in the Foreign DM. Additionally, we have $Q_t \geq (1-\lambda)\,\widehat{m}\left(\rho_t\right)$ in any equilibrium.

Equilibrium. We can use the market-clearing condition (4) and the previously derived demand functions to obtain the equilibrium relation:

$$m_{t+1} = m\left(Q_t, \rho_t\right) \equiv \lambda^{-1}\left[Q_t - (1-\lambda)\,\widehat{m}\left(\rho_t\right)\right].$$

The function $m\left(Q_t, \rho_t\right)$ is increasing in both arguments. If $\rho_t > \beta\rho_{t+1}$, we can use the first-order condition (1) to derive the equilibrium condition:

$$\rho_t = \beta \rho_{t+1} \left[\alpha u' \left(\rho_{t+1} m \left(Q_t, \rho_t \right) \right) + 1 - \alpha \right]. \tag{5}$$

This equation implicitly defines $\rho_{t+1} = g\left(Q_t, \rho_t\right)$. The implicit function theorem then implies:

$$g_1\left(Q_t, \rho_t\right) = \frac{-\alpha \lambda^{-1} \rho_{t+1}^2 u''\left(q_{t+1}\right)}{\alpha \left[u'\left(q_{t+1}\right) + q_{t+1} u''\left(q_{t+1}\right)\right] + 1 - \alpha} > 0$$
$$g_2\left(Q_t, \rho_t\right) = \frac{1}{\beta} \frac{\alpha \beta \lambda^{-1} \left(1 - \lambda\right) \rho_{t+1}^2 u''\left(q_{t+1}\right) \widehat{m}'\left(\rho_t\right) + 1}{\alpha \left[u'\left(q_{t+1}\right) + q_{t+1} u''\left(q_{t+1}\right)\right] + 1 - \alpha} > 0,$$

where g_i denotes the partial derivative of $g\left(\cdot\right)$ with respect to argument $i = 1, 2$. As we can see, the implicit function $g\left(Q_t, \rho_t\right)$ is strictly increasing in both arguments.

Notice that we have a laissez-faire arrangement in our model. The only reason the money supply in the Home country will change over time is because of international trade between the Home country and the rest of the world without trade barriers. In other words, there is no government agency that controls the quantity of money in the domestic economy.[6]

We are ready to define an equilibrium of the model formally. An equilibrium of the SOE under the gold standard is a sequence $\{Q_t, \rho_t\}$ satisfying the Euler equation (5) and the law of motion:

$$Q_{t+1} = Q_t + \left(1 - \lambda\right) \left[m_F\left(g\left(Q_t, \rho_t\right)\right) - \widehat{m}\left(\rho_t\right)\right], \tag{6}$$

together with the boundary conditions:

$$\rho_t > \beta \rho_{t+1}$$
$$\beta \bar{\rho} \leq \rho_t \leq \beta^{-1} \bar{\rho}$$
$$0 \leq \rho_{t+1} m\left(Q_t, \rho_t\right) \leq q^*,$$

and the initial condition for the Home gold stock $Q_0 > 0$. Let $Q_{t+1} = f\left(Q_t, \rho_t\right)$ represent the difference equation defined by condition (6).

In what follows, it is helpful to think about the equilibrium of the model as a dynamic system $\left(Q_{t+1}, \rho_{t+1}\right) = \left(f\left(Q_t, \rho_t\right), g\left(Q_t, \rho_t\right)\right)$ in the $\left(Q_t, \rho_t\right)$ space, subject to the previously defined boundary conditions and the initial condition for the Home gold stock. We now characterize the properties of this dynamic system.

Steady state. Our first step is to find its steady state. Any steady state $\left(Q^s, \rho^s\right)$ must satisfy:

$$\rho^s \left[Q^s - \left(1 - \lambda\right) \widehat{m}\left(\rho^s\right)\right] = \lambda \bar{q}_\alpha \tag{7}$$

[6] In reality, central banks of small open economies under the gold standard had some control of the quantity of money in their economies due to transportation costs and regulations. We abstract from those as they would not alter the spirit of our main results beyond complicating algebra.

and $\widehat{m}(\rho^s) = m_F(\rho^s)$, where the constant $\bar{q}_\alpha \in \mathbb{R}_+$ is implicitly defined by $\beta^{-1} = \alpha u'(\bar{q}_\alpha) + 1 - \alpha$. Because $m_F(\cdot)$ is strictly increasing and $\widehat{m}(\cdot)$ is strictly decreasing (and given the Inada conditions), there is a unique value of gold $\rho^s > 0$ that perfectly balances gold inflows and outflows in the SOE:

$$\widehat{m}(\rho) = m_F(\rho) \Leftrightarrow \rho = \rho^s.$$

If $\delta = \omega$, then the only way to have the gold inflows match the gold outflows in the Home country (so that the Home gold stock remains constant over time) is to set the value of gold in the Home country equal to the international value of gold.

To derive the steady-state Home gold stock consistent with the steady-state value of gold, ρ^s, notice that condition (7) implies:

$$Q = \frac{\lambda \bar{q}_\alpha}{\rho} + (1 - \lambda)\widehat{m}(\rho) \tag{8}$$

for any value of ρ. Because the right-hand side of equation (8) is strictly decreasing in ρ and goes to infinity as ρ approaches zero, there is a unique level of the Home gold stock consistent with the steady-state value of gold, ρ^s, which we denote by Q^s. Hence, the SOE's steady state is unique.

If $\delta = \omega$, we have $\rho^s = \bar{\rho}$ and:

$$Q^s = \frac{1}{\bar{\rho}}\left[\lambda \bar{q}_\alpha + (1 - \lambda)\bar{q}_\delta\right] \equiv \bar{Q}, \tag{9}$$

where $\bar{q}_\delta \in \mathbb{R}_+$ is implicitly defined by $\beta^{-1} = \delta u'(\bar{q}_\delta) + 1 - \delta$. In this case, the Home country value of gold equals the international value of gold, and the quantity $\bar{Q}$ provides the Home gold stock consistent with that price. We summarize these findings in the following proposition.

Proposition 1 *The small open economy in the Home country has a unique steady state with $\rho_t = \rho^s$ and $Q_t = Q^s$ at all dates. If $\delta = \omega$, the steady-state value of gold is $\bar{\rho}$, and the steady-state Home gold stock, given by $\bar{Q}$, is defined in equation (9).*

The steady-state DM output is $\bar{q}_\alpha$ if the match involves a domestic trader, $\bar{q}_\delta$ if it involves an international trader and $\bar{q}_\omega$ if it involves a returning international trader, with the quantity $\bar{q}_\omega \in \mathbb{R}_+$ implicitly defined by $\beta^{-1} = \omega u'(\bar{q}_\omega) + 1 - \omega$. Because $\bar{q}_\delta, \bar{q}_\omega \leq \bar{q}_\alpha < q^*$, the unique steady state is inefficient because the quantity traded in any bilateral match is below the surplus-maximizing quantity.

If $Q_0 = Q^s$, the domestic economy remains in the steady state forever. If $Q_0 \neq Q^s$, then we need to study the behavior of the dynamic system outside the steady state. Because the Home gold stock, Q_t, is a predetermined variable, it will not adjust immediately to the level at which gold inflows and gold outflows are perfectly balanced, so there are interesting dynamics that characterize the gold standard economy. Also, the price level in this economy is given by $\frac{1}{\rho_t}$. A central question in the study of monetary economies under a commodity standard is whether the price level converges to a steady state regardless of the initial position.

We now turn to the analysis of dynamic equilibria. In what follows, we assume that $\delta = \omega$ to simplify the dynamic analysis.

Non-stationary trajectories. Consider the system $(Q_{t+1}, \rho_{t+1}) = (f(Q_t, \rho_t), g(Q_t, \rho_t))$. The locus $Q_{t+1} - Q_t = 0$ is implicitly defined by $m_F(g(Q_t, \rho_t)) = \widehat{m}(\rho_t)$ in the (Q_t, ρ_t) space. The slope of this curve is:

$$\frac{d\rho_t}{dQ_t} = \frac{m_F'(g(Q_t, \rho_t))\, g_1(Q_t, \rho_t)}{\widehat{m}'(\rho_t) - m_F'(g(Q_t, \rho_t))\, g_2(Q_t, \rho_t)} < 0.$$

Notice that $Q_{t+1} < Q_t \Leftrightarrow \widehat{m}(\rho_t) > m_F(g(Q_t, \rho_t))$, which implies that any path for Q_t is increasing to the right of the downward-sloping locus $Q_{t+1} - Q_t = 0$ in the (Q_t, ρ_t) space. Figure 2 illustrates this phaseline. The lower bound Q^m depicted in Figure 2 is implicitly defined by: $m_F(g(Q^m, \beta^{-1}\bar{\rho})) = \widehat{m}(\beta^{-1}\bar{\rho})$.

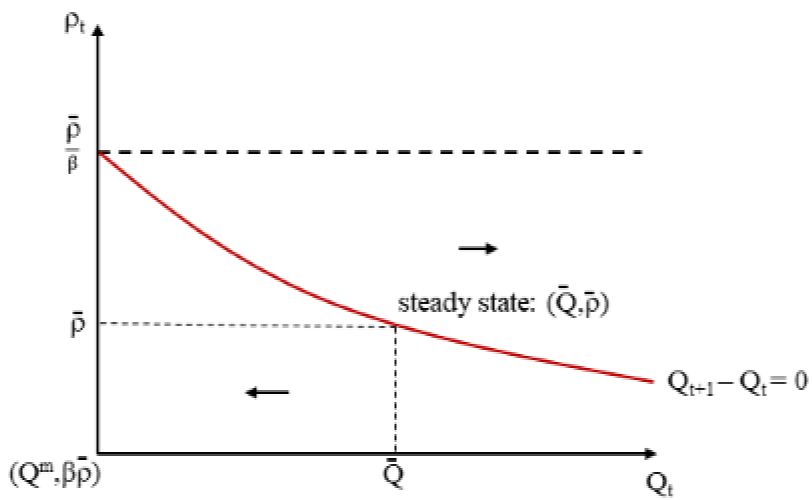

Figure 2: Locus $Q_{t+1} - Q_t = 0$

Focus now on the locus $\rho_{t+1} - \rho_t = 0$, given by $Q_t = \frac{\lambda \bar{q}_\alpha}{\rho_t} + (1-\lambda)\widehat{m}(\rho_t)$. The slope of this curve is:

$$\frac{d\rho_t}{dQ_t} = \frac{1}{-\frac{\lambda \bar{q}_\alpha}{\rho_t^2} + (1-\lambda)\widehat{m}'(\rho_t)} < 0.$$

From equation (5), we find that $\rho_{t+1} > \rho_t \Leftrightarrow q_{t+1} > \bar{q}_\alpha$. We can use $\rho_{t+1} = g(Q_t, \rho_t)$ to show that:

$$\rho_{t+1} > \rho_t \Leftrightarrow Q_t > \frac{\lambda \bar{q}_\alpha}{\rho_t} + (1-\lambda)\widehat{m}(\rho_t),$$

which implies that any path for ρ_t is increasing to the right of the downward-sloping locus $\rho_{t+1} - \rho_t = 0$ in the (Q_t, ρ_t) space. See Figure 3.

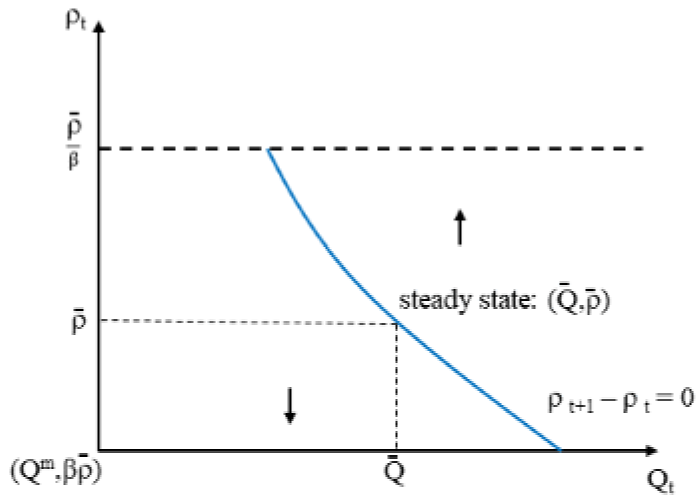

Figure 3: Locus $\rho_{t+1} - \rho_t = 0$

The dynamic equilibrium analysis so far points to the existence of a saddle path through the steady state $\left(\bar{Q}, \bar{\rho}\right)$, provided the slope of the locus $\rho_{t+1} - \rho_t = 0$ is steeper in absolute value than that of the locus $Q_{t+1} - Q_t = 0$. We now establish this property by studying the linear approximation of the dynamic system near the steady state. The Jacobian matrix at the steady state has entries:

$$
\begin{aligned}
f_1\left(\bar{Q}, \bar{\rho}\right) &= 1 + (1-\lambda) \frac{\alpha \lambda^{-1} \delta^{-1} \left[\beta^{-1} + \delta \bar{q}_\delta u''\left(\bar{q}_\delta\right)\right]}{\beta^{-1} + \alpha \bar{q}_\alpha u''\left(\bar{q}_\alpha\right)} \frac{u''\left(\bar{q}_\alpha\right)}{u''\left(\bar{q}_\delta\right)} > 1 \\
f_2\left(\bar{Q}, \bar{\rho}\right) &= (1-\lambda) \left[\frac{\left[\beta^{-1} + \delta \bar{q}_\delta u''\left(\bar{q}_\delta\right)\right]}{\delta \bar{\rho}^2 u''\left(\bar{q}_\delta\right)} g_2\left(\bar{Q}, \bar{\rho}\right) - \frac{1}{\delta \beta \bar{\rho}^2 u''\left(\bar{q}_\delta\right)}\right] > 0 \\
g_1\left(\bar{Q}, \bar{\rho}\right) &= -\frac{\alpha \lambda^{-1} \bar{\rho}^2 u''\left(\bar{q}_\alpha\right)}{\beta^{-1} + \alpha \bar{q}_\alpha u''\left(\bar{q}_\alpha\right)} > 0 \\
g_2\left(\bar{Q}, \bar{\rho}\right) &= \frac{1}{\beta} \frac{1 + \frac{(1-\lambda))\alpha \lambda^{-1} u''\left(\bar{q}_\alpha\right)}{\delta \bar{\rho}^2 u''\left(\bar{q}_\delta\right)}}{\beta^{-1} + \alpha \bar{q}_\alpha u''\left(\bar{q}_\alpha\right)} > 1.
\end{aligned}
$$

All these entries are positive, with the elements in the main diagonal strictly greater than 1. This means that the trace of the Jacobian matrix is greater than 2, so one of its eigenvalues is always outside the unit circle.

Hence, the dynamic system can be stable only if there is a saddle path through the unique steady state. The following proposition provides the conditions for the existence of a unique downward-sloping saddle path for the dynamic system.

Proposition 2 *Let $D \equiv f_1\left(\bar{Q}, \bar{\rho}\right) g_2\left(\bar{Q}, \bar{\rho}\right) - f_2\left(\bar{Q}, \bar{\rho}\right) g_1\left(\bar{Q}, \bar{\rho}\right)$ denote the determinant of the Ja-*

cobian matrix. Assume $\left[f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)\right]^2>4D$*. Then, a unique downward-sloping saddle path through the steady state exists provided:*

$$-1<\frac{f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)-\sqrt{\left[f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)\right]^2-4D}}{2}<1.$$

Given the initial condition $Q_0>0$*, one can find the initial value* ρ_0 *on the saddle path to construct the unique equilibrium trajectory, which converges to the steady state. If* $Q_0<\bar{Q}$*, then* $\{\rho_t\}$ *is strictly decreasing and* $\{Q_t\}$ *is strictly increasing along the equilibrium trajectory. If* $Q_0>\bar{Q}$*, then* $\{\rho_t\}$ *is strictly increasing and* $\{Q_t\}$ *is strictly decreasing along the equilibrium trajectory.*

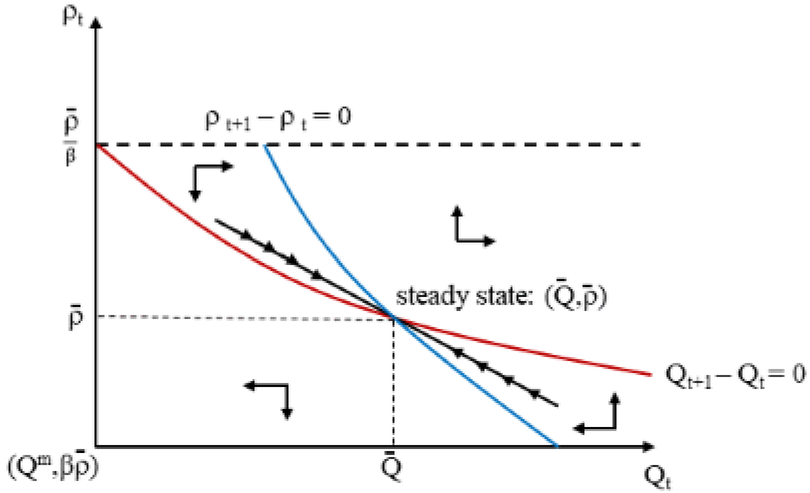

Figure 4: Dynamic system under the gold standard

Figure 4 depicts the dynamic system when the unique downward-sloping saddle path exists. Although the conditions for the existence of the saddle path may not seem intuitive, our numerical explorations suggest that this property of the dynamic system is robust; that is, as we vary the preference parameters within a wider range in the parameter space, the downward-sloping saddle path is preserved.[7]

Given this saddle-path property of the dynamic system, the following analysis highlights a fundamental feature of the gold standard: long-run price stability. Depending on the initial gold stock in the Home country, it can experience either inflation or deflation. These price

[7] Suppose that $u\left(q\right)=A\frac{q^{1-\sigma}}{1-\sigma}$, with $A>0$ and $0<\sigma<1$, which is a standard functional form in the literature. Note that $u'\left(q\right)=Aq^{-\sigma}$. In the numerical analysis, we set $A=1$ and vary σ between 0 and 1. The other parameters are: $\beta=.96$, $\alpha=.8$, $\delta=\omega=.5$, and $\bar{\rho}=1$. As we vary σ between 0 and 1, the largest eigenvalue of the Jacobian matrix always remains outside the unit circle, and the smallest eigenvalue of the Jacobian matrix always remains inside the unit circle.

movements, however, are temporary. The domestic price level always converges to its steady-state level regardless of the initial gold stock in the Home country.

Home gold stock is below the long-run level. If $Q_0 < \bar{Q}$, the saddle path implies an increasing price-level trajectory, given by $1/\rho_t$, in the Home country. However, this inflationary process is temporary because the price level converges to its steady-state value, $1/\bar{\rho}$, which means that the inflation rate necessarily declines along the equilibrium trajectory and converges to zero. The quantity of money in the Home country rises along the equilibrium path as the gold inflows exceed the gold outflows. Although gold flows into the Home country when $Q_0 < \bar{Q}$, the rate of increase in the Home gold stock (and money supply) is decreasing. The difference between gold inflows and outflows converges to zero, so the money supply in the Home country converges to a level consistent with the international value of gold.

The following proposition describes the equilibrium evolution of DM consumption for all types of traders when $Q_0 < \bar{Q}$.

Proposition 3 *If $Q_0 < \bar{Q}$, we have: (i) $q_t \to \bar{q}_\alpha$ from below; (ii) $q_t^F \to \bar{q}_\omega$ monotonically from above; and (iii) $\widehat{q}_t \to \bar{q}_\delta$ monotonically from below.*

Proof. The equilibrium evolution of DM output in each type of meeting is determined by:

$$\begin{aligned}\frac{\rho_t}{\beta\rho_{t+1}} &= \alpha u'\left(q_{t+1}\right) + 1 - \alpha \\ \frac{\rho_t}{\beta\bar{\rho}} &= \delta u'\left(\widehat{q}_{t+1}\right) + 1 - \delta \\ \frac{\bar{\rho}}{\beta\rho_{t+1}} &= \omega u'\left(q_{t+1}^F\right) + 1 - \omega.\end{aligned}$$

Because $Q_0 < \bar{Q}$, the saddle path implies a declining price sequence $\{\rho_t\}$. Because u' is strictly decreasing, it follows that $\{q_t\}$ converges to $\bar{q}_\alpha$ from below. The equations for $\widehat{q}_{t+1}$ and q_{t+1}^F imply that $\{\widehat{q}_t\}$ is strictly increasing and converges to $\bar{q}_\delta$ from below and that $\left\{q_t^F\right\}$ is strictly decreasing and converges to $\bar{q}_\omega$ from above. ■

Table 1 summarizes the evolution of the quantities traded in the DM when the economy starts at $Q_0 < \bar{Q}$. Although all variables converge to the steady state, the direction of convergence can vary. Depending on the type of buyer in a match, the quantity traded in the DM can converge to the steady state from below or from above. In some cases, convergence is monotonic.

Table 1: Declining inflation

$q_t - \bar{q}_\alpha < 0$ at all dates	$q_t \to \bar{q}_\alpha$ from below
$q_t^F - \bar{q}_\omega > 0$ at all dates	$q_t^F \to \bar{q}_\omega$ monotonically from above
$\widehat{q}_t - \bar{q}_\delta < 0$ at all dates	$\widehat{q}_t \to \bar{q}_\delta$ monotonically from below
$m_{t+1}^F - \widehat{m}_{t+1} > 0$ at all dates	$m_{t+1}^F - \widehat{m}_{t+1} \to 0$ from above

The DM output in a match containing a returning international trader monotonically converges to its steady-state level from above. A returning international trader rebalances his portfolio in the Foreign CM, where the value of gold remains constant. Because the price level $1/\rho_t$ in the Home country converges to its steady-state level from below, a returning international trader earns a real (gross) return on his gold holdings greater than 1, which implies a value for DM output greater than its steady-state level. Because the price level converges monotonically to the steady state, the period-t DM consumption of a returning international trader is greater than the period-$(t+1)$ DM consumption of a returning international trader. Although q_t^F decreases over time, it remains above its steady-state level in the transition path.

The DM consumption of the international trader, given by $\widehat{q}_t$, is increasing over time, even though it always remains below its steady-state level along the non-stationary equilibrium trajectory. The declining inflation rate in the Home country implies rising consumption amounts for the international trader as it gradually reduces his opportunity cost of holding money balances for transaction purposes. The DM consumption of the domestic trader remains below its steady-state level along the transition path because of the inflationary process associated with the Home economy's dynamic adjustment.

Home gold stock is above the long-run level. If $Q_0 > \bar{Q}$, the saddle path implies a temporary deflation in the Home country. The rate of deflation declines along the equilibrium trajectory and converges to zero. The Home country loses gold to the rest of the world in the transition path to the steady state. The money supply falls over time and converges to the level consistent with the international value of gold. The following proposition describes the equilibrium evolution of DM consumption across all types of matches.

Proposition 4 *If $Q_0 > \bar{Q}$, we have: (i) $q_t \to \bar{q}_\alpha$ from above; (ii) $q_t^F \to \bar{q}_\omega$ monotonically from below; and (iii) $\widehat{q}_t \to \bar{q}_\delta$ monotonically from above.*

Table 2 summarizes the evolution of the quantities produced and traded in the Home and Foreign DM when the economy starts at $Q_0 > \bar{Q}$. It also shows the evolution of net gold outflows from the Home country resulting from the individual money holdings by different traders.

Table 2: Declining deflation

$q_t - \bar{q}_\alpha > 0$ at all dates	$q_t \to \bar{q}_\alpha$ from above
$q_t^F - \bar{q}_\omega < 0$ at all dates	$q_t^F \to \bar{q}_\omega$ monotonically from below
$\widehat{q}_t - \bar{q}_\delta > 0$ at all dates	$\widehat{q}_t \to \bar{q}_\delta$ monotonically from above
$m_{t+1}^F - \widehat{m}_{t+1} < 0$ at all dates	$m_{t+1}^F - \widehat{m}_{t+1} \to 0$ from below

Along the non-stationary equilibrium path, the real (gross) return on money for a domestic trader is always greater than 1, so the quantity of DM goods traded in a bilateral match involving a domestic trader remains above its steady-state level in the dynamic equilibrium. Because the saddle path implies that the value of gold in the Home country starts below the international value of gold, the international trader also obtains a quantity of DM goods from a Foreign seller that is always greater than the steady-state level, given by $\bar{q}_\delta$. Over time, q_t and $\widehat{q}_t$ approach their steady-state levels from above as the real (gross) return on money holdings approaches 1 for both traders. The DM consumption of a returning international trader remains below the steady-state level in the transition path when $Q_0 > \bar{Q}$.

The previous analysis also implies that any non-stationary equilibrium is necessarily inefficient because DM output is below the surplus-maximizing quantity for at least a subset of agents in all bilateral matches.

Long-run price stability. A remarkable feature of our model of the gold standard is that the price level always converges to the steady state. Inflation or deflation can occur along the equilibrium path depending on the initial domestic gold stock, but these price movements are temporary. The quantity of money in the Home country is a predetermined variable, so it cannot adjust immediately to the level at which gold inflows and outflows are perfectly balanced. Thus, depending on the initial Home gold stock, it is necessary to have either temporary inflation or temporary deflation along the transition path to the steady state. This feature of the model is in line with empirical studies of gold standard economies. For instance, Bordo (1981), Friedman and Schwartz (1963), and Schwartz (1986) document the long-run price stability associated with the gold standard in the data.

Monetary non-neutrality. Another distinct feature of our model is the non-neutrality of money. When the Home gold stock starts below the steady-state level, the Home country accumulates gold along the transition path, so the money supply rises over time. As we have seen, the DM output in the matches involving domestic traders is increasing along the transition path, and the price level increases at a decreasing rate along the way to the steady state. Money is non-neutral because the only way new money can enter the Home country is if exporters (i.e., returning international traders) bring in more gold to the Home country than importers (i.e., international traders) take out of the country.

Conversely, when the Home gold stock starts above the steady-state level, the Home country loses gold to the rest of the world in the transition path to the steady state, and the money supply declines over time. The DM output in the matches involving domestic traders falls, and the price level declines in the Home country.

These findings present intriguing possibilities for further experiments. For instance, what are the repercussions of a decrease in the global price of gold, potentially resulting from the discovery of substantial gold deposits in the Americas between the 16th and 18th centuries? What is the impact of changes in productivity in the DM and CM markets at home and abroad?

Among all the possible experiments we could run, we will focus on an exercise inspired by Kiyotaki et al. (1993) to gauge the effects of changes in the degree of economic integration between the SOE and the rest of the world under the gold standard.

International economic integration. In the spirit of Kiyotaki et al. (1993), we can interpret a higher value for the matching probability in the Foreign country, δ, as a higher degree of economic integration in the model. A higher δ implies a higher steady-state Home gold stock. That is, the Home country needs more gold to perfectly balance gold inflows and outflows when the international traders are more likely to find Foreign sellers when trading in the Foreign DM.

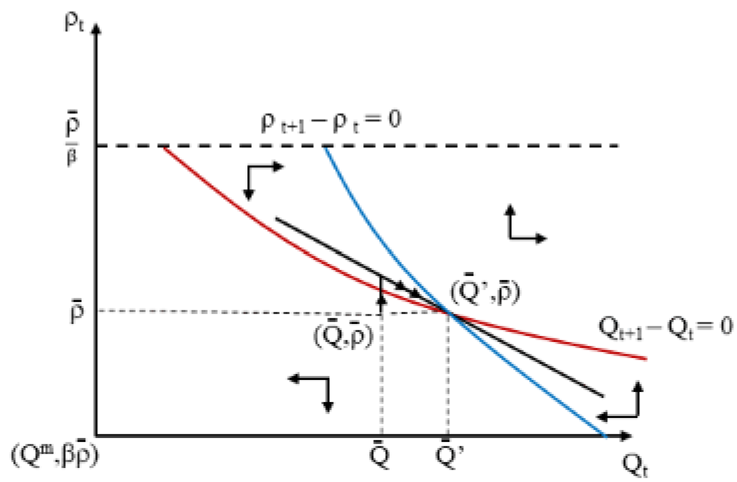

Figure 5: Effects of an unanticipated and permanent trade shock

More precisely, we construct the following experiment. Suppose the Home economy starts at the steady state described in Proposition 1 with $\rho^s = \bar{\rho}$ and $\delta = \omega$. Then, there is a one-time unanticipated and permanent increase in δ and ω by the same small amount. Figure 5 illustrates the transition path to the new steady state.

The value of gold in the Home country will jump from its initial value $\bar{\rho}$ to the downward-sloping saddle path. The Home country will experience a temporary inflationary process along the transition path to the new steady state, resulting in a net gold inflow to the domestic economy. This dynamic equilibrium will favor returning international traders, who will experience a higher DM consumption along the non-stationary equilibrium trajectory than that of the stationary equilibrium. The temporary inflation caused by the unanticipated and permanent trade shock will harm domestic and international traders in the transition path to the new steady state. A higher value for δ increases the extensive margin for international traders. Thus, the international trader will be better off in the long run because of the increased extensive margin and the unchanged steady-state consumption amount. However, the aggregate output of all matches involving international traders in the Foreign DM might decline temporarily because of the fall in the intensive margin, even though the number of matches is larger.

This experiment shows that an (unanticipated and permanent) increase in the degree of international economic integration can lead to a decline in the aggregate value of imports (i.e., the sum of real money holdings exchanged in all matches involving international traders) in the short run as the Home country accumulates gold to eventually balance gold inflows and outflows at the new steady state. On the other hand, the value of aggregate exports (i.e., the sum of real money holdings exchanged in all matches involving returning international traders) rises along the transition path, which is consistent with the required gold accumulation in the Home country to balance international gold movements with the rest of the gold standard countries.

Inelastic money supply. Since the only available medium of exchange in our economy is a commodity money, the money supply is inelastic under a laissez-faire regime. As we have seen, an unanticipated and permanent trade shock will result in temporary inflation in the Home country. In the transition path, domestic and international traders will be worse off because their DM consumption will fall below their long-run level.

Many economists have pointed out this feature of the gold standard as a fundamental flaw of commodity money systems. In particular, many critics of the gold standard have argued that creating a central bank and issuing fiat money without external constraints can result in a superior monetary system because it can potentially avoid undesirable price movements that necessarily follow from external shocks under the gold standard. Our analysis confirms that this criticism of the gold standard is indeed valid, provided the domestic authorities do not intervene in the adjustment process. Although the gold standard produces long-run price stability, it cannot avoid short-run movements in the price level induced by external disturbances. Whether a fiat money system under the auspices of a benevolent central bank can ensure both short- and long-run price stability is beyond the scope of this paper.

4 Policy Options

Because the stationary equilibrium is inefficient in the laissez-faire economy, can policy implement the efficient allocation without abandoning the gold standard?

The policy options available for policymakers under the international monetary system based on gold are more limited when compared with those of a fiat money system. The Home country government cannot change the quantity of money in the Home country without violating the basic premise of the gold standard. But governments can subsidize gold holdings by providing a flow return on gold without abandoning the gold standard.

Let $\tau_F > 0$ be the policy-induced flow return (in terms of the CM good) per unit of gold in the Foreign CM, and let $\tau_H > 0$ be the policy-induced flow return (in terms of the CM good) per unit of gold in the Home CM. Gold holders receive their flow return from the government upon arriving in the CM. The governments impose a lump-sum tax in the CM to balance their budget.

Although most dynamic properties of the laissez-faire equilibrium will be preserved under policy intervention, policy-induced flow returns will change the equilibrium of the model non-trivially. Thus, it is worth providing some details for constructing equilibrium under policy intervention.

Bargaining. In the presence of policy action, the terms of trade in a bilateral match are:

$$\widehat{q}\left(m;\tau_F\right) = \begin{cases} \left(\bar{\rho}+\tau_F\right) m \text{ if } m < \frac{q^*}{\bar{\rho}+\tau_F} \\ q^* \text{ if } m \geq \frac{q^*}{\bar{\rho}+\tau_F} \end{cases}$$

$$\widehat{d}\left(m;\tau_F\right) = \begin{cases} m \text{ if } m < \frac{q^*}{\bar{\rho}+\tau_F} \\ \frac{q^*}{\bar{\rho}+\tau_F} \text{ if } m \geq \frac{q^*}{\bar{\rho}+\tau_F} \end{cases}$$

$$q_t\left(m;\tau_H\right) = \begin{cases} \left(\rho_t+\tau_H\right) m \text{ if } m < \frac{q^*}{\rho_t+\tau_H} \\ q^* \text{ if } m \geq \frac{q^*}{\rho_t+\tau_H} \end{cases}$$

$$d_t\left(m;\tau_H\right) = \begin{cases} m \text{ if } m < \frac{q^*}{\rho_t+\tau_H} \\ \frac{q^*}{\rho_t+\tau_H} \text{ if } m \geq \frac{q^*}{\rho_t+\tau_H}. \end{cases}$$

Notice that $\bar{\rho}+\tau_F$ provides the effective value of money in the Foreign CM and that $\rho_t+\tau_H$ provides the effective value of money in the Home CM. Given the same value of gold in the subsequent CM, the threshold for the buyer's money holdings that allows him to purchase the efficient quantity is lower in the presence of government intervention.

Next, we solve the portfolio problem of all types of buyers.

Domestic traders. The portfolio problem for a domestic trader is now:

$$\max_{m_{t+1}\in\mathbb{R}_+} \left\{ -\rho_t m_{t+1} + \beta \left\{ \begin{array}{c} \alpha\left[u\left(q_{t+1}\left(m_{t+1};\tau_H\right)\right) - \left(\rho_{t+1}+\tau_H\right) d_{t+1}\left(m_{t+1};\tau_H\right)\right] \\ +\left(\rho_{t+1}+\tau_H\right) m_{t+1} \end{array} \right\} \right\}.$$

This optimization problem is well defined provided $\rho_t \geq \beta\left(\rho_{t+1}+\tau_H\right)$. If this condition holds as a strict inequality, we can write the first-order condition as:

$$-\rho_t + \beta\left(\rho_{t+1}+\tau_H\right)\left[\alpha u'\left(\left(\rho_{t+1}+\tau_H\right) m_{t+1}\right) + 1 - \alpha\right] \leq 0, \text{ with equality if } m_{t+1} > 0. \tag{10}$$

The only change from the previous section is that the real return on money for a domestic trader is now given by $\left(\rho_{t+1}+\tau_H\right)/\rho_t$, which implies that domestic policy influences the quantity of money a domestic trader will hold going forward.

International traders. The portfolio problem for an international trader is:

$$\max_{\widehat{m}_{t+1}\in\mathbb{R}_+} \left\{ -\rho_t \widehat{m}_{t+1} + \beta \left\{ \delta \left[u\left(\widehat{q}\left(\widehat{m}_{t+1};\tau_F\right)\right) - \left(\bar{\rho}+\tau_F\right) \widehat{d}\left(\widehat{m}_{t+1};\tau_F\right) \right] + \left(\bar{\rho}+\tau_F\right) \widehat{m}_{t+1} \right\} \right\}.$$

This problem is well defined provided $\rho_t \geq \beta(\bar{\rho} + \tau_F)$. If this condition holds as a strict inequality, we can write the first-order condition as:

$$-\rho_t + \beta(\bar{\rho} + \tau_F)\left[\delta u'\left((\bar{\rho} + \tau_F)\widehat{m}_{t+1}\right) + 1 - \delta\right] \leq 0, \text{ with equality if } \widehat{m}_{t+1} > 0. \tag{11}$$

The real return on money for an international trader is now given by $(\bar{\rho} + \tau_F)/\rho_t$, which implies that foreign policy influences the quantity of money an international trader will take out of the Home country for buying DM goods from a Foreign seller.

Returning international traders. The portfolio problem for a returning international trader is:

$$\max_{m^F_{t+1} \in \mathbb{R}_+} \left\{ -\bar{\rho} m^F_{t+1} + \beta \left\{ \begin{array}{c} \omega\left[u\left(q_{t+1}\left(m^F_{t+1}; \tau_H\right)\right) - (\rho_{t+1} + \tau_H)\, d_{t+1}\left(m^F_{t+1}; \tau_H\right)\right] \\ + (\rho_{t+1} + \tau_H)\, m^F_{t+1} \end{array} \right\} \right\}.$$

This problem is well defined provided $\bar{\rho} \geq \beta(\rho_{t+1} + \tau_H)$. If this condition holds as a strict inequality, the first-order condition becomes:

$$-\bar{\rho} + \beta(\rho_{t+1} + \tau_H)\left[\omega u'\left((\rho_{t+1} + \tau_H)\, m^F_{t+1}\right) + 1 - \omega\right] \leq 0, \text{ with equality if } m^F_{t+1} > 0. \tag{12}$$

The real return on money for a returning international trader is now given by $(\rho_{t+1} + \tau_H)/\bar{\rho}$. Not only does domestic policy affect the money holdings of domestic traders but also that of returning international traders, influencing gold inflows into the Home country.

Demand functions. We can now derive the demand functions for the international trader and the returning international trader by following the same steps as in Section 3. If $\rho_t > \beta(\bar{\rho} + \tau_F)$, the first-order condition (11) implicitly defines $\widehat{m}_{t+1} = \widehat{m}(\rho_t; \tau_F)$, which is strictly decreasing in ρ_t as before. If $\bar{\rho} > \beta(\rho_{t+1} + \tau_H)$, the first-order condition (12) implicitly defines $m^F_{t+1} = m_F(\rho_{t+1}; \tau_H)$, which is strictly increasing in ρ_{t+1} as before.

Equilibrium. The market-clearing condition for the Home country can be used to define the equilibrium equation: $m_{t+1} = m(Q_t, \rho_t; \tau_F) \equiv \lambda^{-1}\left[Q_t - (1-\lambda)\widehat{m}(\rho_t; \tau_F)\right]$. Following the same steps as in Section 3, we use equation (10) to derive:

$$\rho_t = \beta(\rho_{t+1} + \tau_H)\left[\alpha u'\left((\rho_{t+1} + \tau_H)\, m(Q_t, \rho_t; \tau_F)\right) + 1 - \alpha\right], \tag{13}$$

which implicitly defines $\rho_{t+1} = \widehat{g}(Q_t, \rho_t; \tau_H, \tau_F)$. As before, the implicit function theorem then implies $\widehat{g}_1 > 0$ and $\widehat{g}_2 > 0$.

We can define an equilibrium of the SOE under the gold standard as a sequence $\{Q_t, \rho_t\}$ satisfying the Euler equation (13) and the law of motion:

$$Q_{t+1} = Q_t + (1-\lambda)\left[m_F\left(\widehat{g}(Q_t, \rho_t; \tau_H, \tau_F)\right) - \widehat{m}(\rho_t; \tau_F)\right],$$

together with the boundary conditions:

$$\rho_t > \beta\left(\rho_{t+1} + \tau_H\right)$$
$$\beta\left(\bar{\rho} + \tau_F\right) \leq \rho_t \leq \beta^{-1}\bar{\rho} - \tau_H$$
$$0 \leq \lambda^{-1}\rho_{t+1}m\left(Q_t, \rho_t; \tau_F\right) \leq q^*$$

and the initial Home gold stock $Q_0 > 0$. In this equilibrium, the governments of the Home and Foreign countries set the policy variables (τ_H, τ_F) and the agents take them as given.

From the equilibrium definition under policy intervention, we can see that an important consequence of policy action is the reduced range for the value of gold in the Home country. The lower bound for the value of gold is now given by $\beta\left(\bar{\rho} + \tau_F\right) > \beta\bar{\rho}$, and the upper bound by $\beta^{-1}\bar{\rho} - \tau_H < \beta^{-1}\bar{\rho}$. A necessary condition for the existence of equilibrium is:

$$\beta\left(\bar{\rho} + \tau_F\right) < \frac{\bar{\rho}}{\beta} - \tau_H \Leftrightarrow \beta\tau_F + \tau_H < \frac{\bar{\rho}}{\beta} - \beta\bar{\rho}. \tag{14}$$

This inequality implies that the policy-induced flow returns (τ_H, τ_F) cannot be too large to ensure the existence of equilibrium under government intervention.

To simplify the analysis, assume that $\delta = \omega$. Then, it is straightforward to show that there is a unique steady state, denoted by $\left(\rho^s\left(\tau_H, \tau_F\right), Q^s\left(\tau_H, \tau_F\right)\right)$, provided the policy-induced flow returns (τ_H, τ_F) satisfy condition (14). Moreover, we have $\rho^s\left(\tau_H, \tau_F\right) = \bar{\rho}$ if and only if $\tau_H = \tau_F$. In other words, the Home price level is equal to the Foreign price level if and only if domestic and foreign policies are perfectly synchronized.

Proposition 5 *For any policy choices (τ_H, τ_F) satisfying condition (14), there exists a unique steady state denoted by $\left(\rho^s\left(\tau_H, \tau_F\right), Q^s\left(\tau_H, \tau_F\right)\right)$. We have $\rho^s\left(\tau_H, \tau_F\right) = \bar{\rho}$ if and only if $\tau_H = \tau_F$. There exists a unique downward-sloping saddle path through the steady state provided the policy choices (τ_H, τ_F) are sufficiently small and close to zero.*

The dynamic properties of the model are preserved under policy intervention provided the flow returns (τ_H, τ_F) are sufficiently close to zero. The existence of a unique steady state requires only condition (14), which is milder. Given these results, our next step is to investigate whether policy intervention can lead to the efficient allocation in the Home country.

Optimal policy. We now consider the optimal choice of domestic policy in the Home country, given the foreign policy set in the rest of the world. Specifically, the Home government chooses the domestic flow return τ_H, taking the foreign flow return τ_F as given. In what follows, we restrict attention to steady states, so our analysis focuses on long-run policy efficiency.

Let $q\left(\tau_H, \tau_F\right) \in \mathbb{R}_+$ denote the steady-state DM consumption of a domestic trader in the presence of policy intervention. The quantity $q(\tau_H, \tau_F)$ is given by:

$$\frac{\rho^s(\tau_H, \tau_F)}{\beta\left[\rho^s(\tau_H, \tau_F) + \tau_H\right]} = \alpha u'\left(q(\tau_H, \tau_F)\right) + 1 - \alpha.$$

Let $\widehat{q}(\tau_H, \tau_F) \in \mathbb{R}_+$ denote the steady-state DM consumption of an international trader, which is given by:

$$\frac{\rho^s(\tau_H, \tau_F)}{\beta(\bar{\rho} + \tau_F)} = \delta u'\left(\widehat{q}(\tau_H, \tau_F)\right) + 1 - \delta.$$

Let $q_F(\tau_H, \tau_F) \in \mathbb{R}_+$ denote the steady-state DM consumption of a returning international trader, which is given by:

$$\frac{\bar{\rho}}{\beta\left[\rho^s(\tau_H, \tau_F) + \tau_H\right]} = \delta u'\left(q_F(\tau_H, \tau_F)\right) + 1 - \delta.$$

Notice that we kept the symmetry assumption $\delta = \omega$. Suppose the Home-country flow return differs from that of the Foreign country. In that case, the steady-state consumption of international and returning international traders will not be the same, even though the probability of being matched with a seller is the same.

Let $\tau_F = \tau > 0$ denote the fixed flow return in the Foreign country, where $\tau < \beta^{-1}(1-\beta)\bar{\rho}$. Suppose that $\tau_H = \tau$ holds initially. It is easy to show that there is $0 < \varepsilon < \tau$ such that $\partial m_F / \partial \tau_H > 0$ for any $\tau_H \in (\tau - \varepsilon, \tau + \varepsilon)$. Then, for any τ_H in a small neighborhood of τ, we have:

$$\left\{ \begin{array}{c} \rho^s(\tau_H, \tau_F) > \bar{\rho} \\ \rho^s(\tau_H, \tau_F) < \bar{\rho} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \tau_H < \tau \\ \tau_H > \tau \end{array} \right\}.$$

This result is intuitive. If the policy-induced flow return is lower in the Home country relative to the rest of the world, the only way for the Home country to induce gold inflows that match gold outflows one-for-one is to have a value of gold in the Home country higher than that of the Foreign country. The opposite occurs when $\tau_H > \tau$. Figure 6 draws the net gold inflow in the Home country as a function of the stationary value of gold in the Home country. The blue line represents the difference $m_F - \widehat{m}$ when both flow returns are set at τ, and the red line plots that difference when the domestic flow return is lower than the foreign flow return.

The consumption amounts across traders will not be the same in the steady state when the policy choices of the Home and Foreign countries are not synchronized. That is, implementing the efficient allocation will not be possible unless the policies are synchronized. A corollary from this proposition is that the Home country can attain efficiency through policy intervention only if foreign policy is set at a level consistent with the surplus-maximizing quantity being traded in all matches.

Given that synchronicity is a necessary condition for efficiency, let $\tau > 0$ be the common policy-induced flow return when government policy is synchronized across countries, and let $q(\tau) \in \mathbb{R}_+$ denote the common DM consumption amount across matches when policy is synchro-

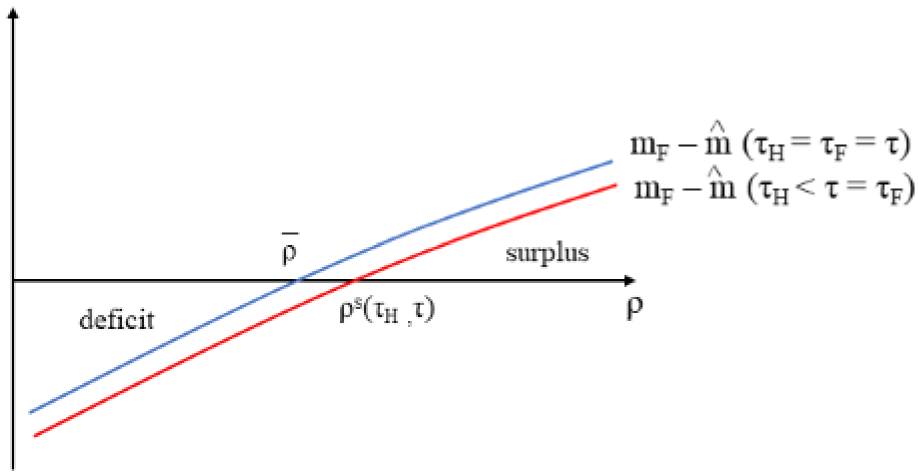

Figure 6: Net gold inflow under policy intervention

nized at the level τ. The quantity $q(\tau)$ is implicitly defined by:

$$\frac{\bar{\rho}}{\beta(\bar{\rho}+\tau)} = \delta u'(q(\tau)) + 1 - \delta. \tag{15}$$

The left-hand side of equation (15) is strictly decreasing in τ in the range $[0, \beta^{-1}(1-\beta)\bar{\rho})$, equals β^{-1} at $\tau = 0$, and approaches 1 as $\tau \to \beta^{-1}(1-\beta)\bar{\rho}$.

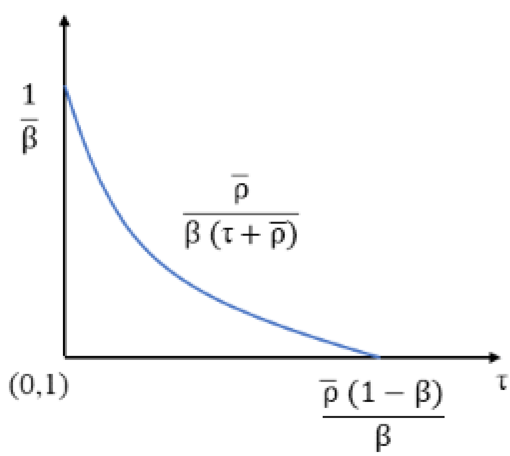

Figure 7: Optimal policy

Figure 7 plots the left-hand side of equation (15) as a function of τ. These properties of the model imply that, by taking the limit $\tau \to \beta^{-1}(1-\beta)\bar{\rho}$, we get $q(\tau) \to q^*$. That is, the steady-state DM consumption approaches the efficient level as the synchronized flow returns approach the upper bound, given by $\beta^{-1}(1-\beta)\bar{\rho}$. The following proposition summarizes these findings.

Proposition 6 *A necessary condition for efficiency in the Home country is that $\tau_H = \tau_F = \tau$ for some $\tau > 0$. Moreover, $q(\tau_H, \tau_F) = \widehat{q}(\tau_H, \tau_F) = q_F(\tau_H, \tau_F) = q(\tau)$ when $\tau_H = \tau_F = \tau$, where $q(\tau)$ is defined by equation (15). Finally, $q(\tau) \to q^*$ as $\tau \to \beta^{-1}(1-\beta)\bar{\rho}$ from below.*

The previous proposition implies that the Home country cannot attain the efficient allocation on its own. Foreign policy in the rest of the world must be set in tandem with domestic policy to attain the previously defined upper bound for the policy-induced flow returns. In other words, if international cooperation through synchronized policy decisions is unattainable, the gold standard monetary system is inefficient in the long run.[8]

Thus, the Home government can consider issuing fiat money to alter the total money supply in the Home country and implement the efficient allocation. In this hybrid system, commodity money and fiat money would coexist as media of exchange. Whether a hybrid system can achieve efficiency will depend, among other things, on whether fiat money issued in the Home country can circulate internationally and, if so, at what value. Let us now analyze this situation.

5 Fiat Money

We now consider a version of the model in which the Home country government issues fiat money and controls its supply by imposing lump-sum taxes/transfers in the Home and Foreign CMs to balance the budget each period. In this monetary arrangement, the agents can hold two distinct assets for payment in DM matches: gold and fiat money. Let $\phi_t^H \in \mathbb{R}_+$ denote the value of fiat money in the period-t Home CM, let $\phi_t^F \in \mathbb{R}_+$ denote the value of fiat money in the period-t Foreign CM, let $\rho_t^H \in \mathbb{R}_+$ denote the value of gold in the period-t Home CM, and let $\rho_t^F \in \mathbb{R}_+$ denote the value of gold in the period-t Foreign CM. The SOE assumption for the Home country, combined with the assumption that the Foreign country represents the rest of the world, implies $\rho_t^F = \bar{\rho}$ for all $t \geq 0$. The asset flows between the Home country and the rest of the world do not alter the real rate of return on other internationally traded assets, including gold.[9]

Let $q_t(m,n)$ denote the quantity of the DM good produced in a bilateral match in the Home country when the buyer has $m \in \mathbb{R}_+$ units of gold and $n \in \mathbb{R}_+$ units of fiat money, let $d_t(m,n)$

[8] This policy scheme may not be incentive-feasible in certain environments in which money holdings are privately observable. Indeed, implementing a flow return requires the government to observe an individual's money holdings. Also, it is necessary to devise a scheme to prevent the same unit of gold from being presented for the payment of a real return more than once, which may be difficult given the anonymity of agents.

[9] The international price of gold can be solely supported by the non-monetary uses of gold. We could further assume that the demand for gold from non-monetary uses is stable, which would imply a constant international value of gold.

denote the quantity of gold the buyer transfers to the seller, and let $h_t(m,n)$ denote the quantity of fiat money the buyer transfers to the seller. The quantities $\widehat{q}_t(m,n)$, $\widehat{d}_t(m,n)$, and $\widehat{h}_t(m,n)$ are their equivalent in the Foreign country. The terms of trade in the Home DM are now:

$$q_t(m,n) = \rho_t^H d_t(m,n) + \phi_t^H h_t(m,n) = \begin{cases} \rho_t^H m + \phi_t^H n \text{ if } \rho_t^H m + \phi_t^H n < q^* \\ q^* \text{ if } \rho_t^H m + \phi_t^H n \geq q^*. \end{cases}$$

The terms of trade in the Foreign country are given by:

$$\widehat{q}_t(m,n) = \bar{\rho}\widehat{d}_t(m,n) + \phi_t^F \widehat{h}_t(m,n) = \begin{cases} \bar{\rho} m + \phi_t^F n \text{ if } \bar{\rho} m + \phi_t^F n < q^* \\ q^* \text{ if } \bar{\rho} m + \phi_t^F n \geq q^*. \end{cases}$$

Although the composition of the asset portfolio transferred from the buyer to the seller is indeterminate, the real value of the total asset transferred is uniquely determined.

Following the same steps as in the model with a single asset, we can write the Bellman equations for the Home country buyer as:

$$\begin{aligned}
\widehat{V}_t(m,n) &= \delta\left[u(\widehat{q}_t(m,n)) - \bar{\rho}\widehat{d}_t(m,n) - \phi_t^F \widehat{h}_t(m,n)\right] + \phi_t^F T_t^F \\
&\quad + \bar{\rho} m + \phi_t^F n + \max_{(m',n')\in\mathbb{R}_+^2} \left[-\bar{\rho} m' - \phi_t^F n' + \beta V_{t+1}^F(m',n)\right] \\
V_t^F(m,n) &= \omega\left[u(q_t(m,n)) - \rho_t^H d_t(m,n) - \phi_t^H h_t(m,n)\right] + \rho_t^H m + \phi_t^H n \\
&\quad + \phi_t^H T_t^H + \lambda \max_{(m',n')\in\mathbb{R}_+^2} \left[-\rho_t^H m' - \phi_t^H n' + \beta V_{t+1}(m',n')\right] \\
&\quad + (1-\lambda) \max_{(m',n')\in\mathbb{R}_+^2} \left[-\rho_t^H m' - \phi_t^H n' + \beta \widehat{V}_{t+1}(m',n')\right] \\
V_t(m,n) &= \alpha\left[u(q_t(m,n)) - \rho_t^H d_t(m,n) - \phi_t^H h_t(m,n)\right] + \rho_t^H m + \phi_t^H n \\
&\quad + \phi_t^H T_t^H + \lambda \max_{(m',n')\in\mathbb{R}_+^2} \left[-\rho_t^H m' - \phi_t^H n' + \beta V_{t+1}(m',n')\right] \\
&\quad + (1-\lambda) \max_{(m',n')\in\mathbb{R}_+^2} \left[-\rho_t^H m' - \phi_t^H n' + \beta \widehat{V}_{t+1}(m',n')\right],
\end{aligned}$$

where $T_t^F \in \mathbb{R}$ and $T_t^H \in \mathbb{R}$ denote lump-sum transfers from the Home country government in the Foreign and Home CM, respectively. These equations describe the Home country buyer's transitions across different states over time in the economy with multiple assets. We now describe the optimal portfolio problem for each type of buyer in this economy by considering the case in which Home country fiat money is positively valued in both Home and Foreign CM.

Domestic traders. The portfolio problem for the domestic trader when both gold and fiat money are available in the Home CM is given by:

$$\max_{(m_{t+1},n_{t+1})\in\mathbb{R}_+^2}\left\{-\rho_t^H m_{t+1}-\phi_t^H n_{t+1}+\beta\left\{\begin{array}{c}\alpha\left[\begin{array}{c}u\left(q_{t+1}\left(m_{t+1},n_{t+1}\right)\right)\\-\rho_{t+1}^H d_{t+1}\left(m_{t+1},n_{t+1}\right)\\-\phi_{t+1}^H h_{t+1}\left(m_{t+1},n_{t+1}\right)\end{array}\right]\\+\rho_{t+1}^H m_{t+1}+\phi_{t+1}^H n_{t+1}\end{array}\right\}\right\}.$$

This optimization problem is well defined provided $\rho_t^H \geq \beta\rho_{t+1}^H$ and $\phi_t^H \geq \beta\phi_{t+1}^H$. If both conditions hold as strict inequalities, we can write the first-order conditions as:

$$-\rho_t^H+\beta\rho_{t+1}^H\left[\alpha u'\left(\rho_{t+1}^H m_{t+1}+\phi_{t+1}^H n_{t+1}\right)+1-\alpha\right]\leq 0,\text{ with equality if } m_{t+1}>0 \tag{16}$$

$$-\phi_t^H+\beta\phi_{t+1}^H\left[\alpha u'\left(\rho_{t+1}^H m_{t+1}+\phi_{t+1}^H n_{t+1}\right)+1-\alpha\right]\leq 0,\text{ with equality if } n_{t+1}>0. \tag{17}$$

If the domestic trader holds both assets in portfolio between dates t and $t+1$, then the rate of return equality $\frac{\phi_{t+1}^H}{\phi_t^H}=\frac{\rho_{t+1}^H}{\rho_t^H}$ in the domestic asset market must hold in equilibrium. The domestic trader is indifferent between the two assets going forward, and the asset portfolio composition is indeterminate in equilibrium.

Returning international traders. The portfolio problem for the returning international trader is given by:

$$\max_{\left(m_{t+1}^F,n_{t+1}^F\right)\in\mathbb{R}_+^2}\left\{-\bar{\rho}m_{t+1}^F-\phi_t^F n_{t+1}^F+\beta\left\{\begin{array}{c}\omega\left[\begin{array}{c}u\left(q_{t+1}\left(m_{t+1}^F,n_{t+1}^F\right)\right)\\-\rho_{t+1}^H d_{t+1}\left(m_{t+1}^F,n_{t+1}^F\right)\\-\phi_{t+1}^H h_{t+1}\left(m_{t+1}^F,n_{t+1}^F\right)\end{array}\right]\\+\rho_{t+1}^H m_{t+1}^F+\phi_{t+1}^H n_{t+1}^F\end{array}\right\}\right\}.$$

This optimization problem is well defined provided $\bar{\rho}\geq\beta\rho_{t+1}^H$ and $\phi_t^F\geq\beta\phi_{t+1}^H$. If both conditions hold as strict inequalities, we can write the first-order conditions as:

$$-\bar{\rho}+\beta\rho_{t+1}^H\left[\omega u'\left(\rho_{t+1}^H m_{t+1}^F+\phi_{t+1}^H n_{t+1}^F\right)+1-\omega\right]\leq 0,\text{ with equality if } m_{t+1}^F>0 \tag{18}$$

$$-\phi_t^F+\beta\phi_{t+1}^H\left[\omega u'\left(\rho_{t+1}^H m_{t+1}^F+\phi_{t+1}^H n_{t+1}^F\right)+1-\omega\right]\leq 0,\text{ with equality if } n_{t+1}^F>0. \tag{19}$$

If the returning international trader holds both assets in portfolio from period t to $t+1$, then the rate of return equality $\frac{\phi_{t+1}^H}{\phi_t^F}=\frac{\rho_{t+1}^H}{\bar{\rho}}$ must hold in equilibrium.

International traders. The portfolio problem for the international trader is given by:

$$\max_{(\widehat{m}_{t+1},\widehat{n}_{t+1})\in\mathbb{R}_+^2}\left\{-\rho_t^H\widehat{m}_{t+1}-\phi_t^H\widehat{n}_{t+1}+\beta\left\{\begin{array}{c}\delta\left[\begin{array}{c}u\left(\widehat{q}_{t+1}\left(\widehat{m}_{t+1},\widehat{n}_{t+1}\right)\right)\\-\bar{\rho}\widehat{d}_{t+1}\left(\widehat{m}_{t+1},\widehat{n}_{t+1}\right)\\-\phi_{t+1}^F\widehat{h}_{t+1}\left(\widehat{m}_{t+1},\widehat{n}_{t+1}\right)\end{array}\right]\\+\bar{\rho}\widehat{m}_{t+1}+\phi_{t+1}^F\widehat{n}_{t+1}\end{array}\right\}\right\}.$$

This optimization problem is well defined provided $\rho_t^H \geq \beta\bar{\rho}$ and $\phi_t^H \geq \beta\phi_{t+1}^F$. If both conditions hold as strict inequalities, we can write the first-order conditions as:

$$-\rho_t^H + \beta\bar{\rho}\left[\delta u'\left(\bar{\rho}\widehat{m}_{t+1} + \phi_{t+1}^F\widehat{n}_{t+1}\right) + 1 - \delta\right] \leq 0, \text{ with equality if } \widehat{m}_{t+1} > 0 \tag{20}$$

$$-\phi_t^H + \beta\phi_{t+1}^F\left[\delta u'\left(\bar{\rho}\widehat{m}_{t+1} + \phi_{t+1}^F\widehat{n}_{t+1}\right) + 1 - \delta\right] \leq 0, \text{ with equality if } \widehat{n}_{t+1} > 0. \tag{21}$$

If the international trader holds both assets in his portfolio from period t to $t+1$, then the rate of return equality $\frac{\phi_{t+1}^F}{\phi_t^H} = \frac{\bar{\rho}}{\rho_t^H}$ must hold in equilibrium.

Market clearing. The market-clearing conditions in the Home country asset markets are:

$$Q_t = \lambda m_{t+1} + (1-\lambda)\,\widehat{m}_{t+1} \tag{22}$$

$$N_t^H = \lambda n_{t+1} + (1-\lambda)\,\widehat{n}_{t+1}, \tag{23}$$

where $N_t^H \in \mathbb{R}_+$ denotes the total quantity of fiat money in the Home country in period t. The market-clearing condition for fiat money in the Foreign country is:

$$N_t^F = (1-\lambda)\,n_{t+1}^F, \tag{24}$$

where $N_t^F \in \mathbb{R}_+$ denotes the total quantity of fiat money in the Foreign country in period t.

The total fiat money supply is denoted by $N_t \in \mathbb{R}_+$. Then, we have:

$$N_t = N_t^H + N_t^F \tag{25}$$

at all dates. We assume the Home government follows a policy rule of the form: $N_{t+1} = \mu N_t$, where $\mu \geq \beta$ represents the money growth rate. The Home country's central bank intervenes in the Home and Foreign CM to implement its policy rule.

The laws of motion for the Home gold stock and the Home fiat money supply are:

$$Q_{t+1} = Q_t + (1-\lambda)\left(m_{t+1}^F - \widehat{m}_{t+1}\right) \tag{26}$$

$$N_{t+1}^H = N_t^H + (1-\lambda)\left(n_{t+1}^F - \widehat{n}_{t+1}\right) + T_{t+1}^H, \tag{27}$$

respectively, where $T_t^H \in \mathbb{R}$ denotes the period-t lump-sum transfer from the Home government in the Home CM. The law of motion for the fiat money supply in the Foreign country is:

$$N_{t+1}^F = N_t^F + (1-\lambda)\left(\widehat{n}_{t+1} - n_{t+1}^F\right) + T_{t+1}^F, \tag{28}$$

where $T_t^F \in \mathbb{R}$ denotes the period-t lump-sum transfer from the Home government in the Foreign CM. Under the pure gold standard, equation (26) describes the evolution of the money supply in the Home country, which consists solely of gold and depends only on the asset flows associated with international commerce. In a dual monetary system, the total money supply is *elastic* because the Home government can issue or retire fiat money in the Home and Foreign CM to alter the aggregate monetary stock.

Equilibrium. Given the money growth rule in the Home country, we can define a monetary equilibrium of the SOE with multiple assets as a sequence

$$\left\{\rho_t^H, \phi_t^H, \phi_t^F, Q_t, N_t^H, N_t^F, n_{t+1}, \widehat{n}_{t+1}, n_{t+1}^F, m_{t+1}, \widehat{m}_{t+1}, m_{t+1}^F\right\}$$

satisfying the first-order conditions (16)-(21), the market-clearing conditions (22)-(25), and the laws of motion (26)-(28), together with the boundary conditions, $\rho_t^H \geq \beta \rho_{t+1}^H$, $\beta\bar{\rho} \leq \rho_t^H \leq \beta^{-1}\bar{\rho}$, $\phi_t^H \geq \beta\phi_{t+1}^H$, $\phi_t^H \geq \beta\phi_{t+1}^F$, $\phi_t^F \geq \beta\phi_{t+1}^H$, given the initial conditions $Q_0 > 0$ and $N_0 > 0$.

We now characterize a class of equilibria in this monetary economy that is relevant to our analysis of the gold standard. Specifically, we consider the endogenous demonetization of gold in the Home country due to domestic monetary policy.

Demonetization of gold. We show the existence of a subset of stationary equilibria characterized by the fact that fiat money completely displaces gold as a medium of exchange for the residents of the Home country. In particular, the demonetization of gold occurs endogenously if the Home government chooses the money growth rate to be in the range $\beta < \mu < 1$. In this equilibrium, fiat money issued by the Home country circulates both domestically and internationally. We can think of the circulation of domestically issued fiat money in the Foreign CM as an offshore money market for the domestic currency. Importers and exporters from the Home country trade the Home currency offshore to conduct their international transactions, which can result in a positive value for that currency in the Foreign CM.

Assume that $\delta = \omega$. Suppose the Home government sets the money growth rate such that $\beta < \mu < 1$, which requires a lump-sum tax in the Home and Foreign CM to balance the budget. Let $\bar{q}_\alpha(\mu) \in \mathbb{R}_+$ denote the quantity satisfying:

$$\frac{\mu}{\beta} = \alpha u'\left(\bar{q}_\alpha(\mu)\right) + 1 - \alpha,$$

and let $\bar{q}_\delta(\mu) \in \mathbb{R}_+$ denote the quantity satisfying:

$$\frac{\mu}{\beta} = \delta u'\left(\bar{q}_\delta(\mu)\right) + 1 - \delta.$$

Notice that $\bar{q}_\alpha(\mu) > \bar{q}_\alpha(1) = \bar{q}_\alpha$ and that $\bar{q}_\delta(\mu) > \bar{q}_\delta(1) = \bar{q}_\delta$ when $\beta < \mu < 1$. Suppose that:

$$\frac{\phi_{t+1}^H}{\phi_t^H} = \frac{\phi_{t+1}^F}{\phi_t^F} = \frac{1}{\mu}$$

holds at all dates, which implies $\phi_t^H = \mu^{-t}\phi_0^H$ and $\phi_t^F = \mu^{-t}\phi_0^F$ at any date t. Set $\phi_0^H = \phi_0^F =$

$\phi_0 > 0$. Also, suppose that $\rho_t^H = \bar{\rho}$ for all t. Then, it is straightforward to use the first-order conditions (16)-(21) to show that $m_{t+1} = \widehat{m}_{t+1} = m_{t+1}^F = 0$, $n_{t+1} = \phi_0^{-1}\mu^{(t+1)}\bar{q}_\alpha(\mu)$, and $\widehat{n}_{t+1} = n_{t+1}^F = \phi_0^{-1}\mu^{(t+1)}\bar{q}_\delta(\mu)$ for any $\phi_0 > 0$.

We now show that there is an initial distribution of endowments that is consistent with the individual choices in the proposed equilibrium. For any $Q_0 > 0$ initially available in the Home country, one can allocate this amount evenly to the international traders after making their portfolio decision in the Home CM at $t = 0$. It is clear, then, that they will use their initial gold endowment to increase their DM consumption in the period-1 Foreign DM. If their endowment allows them to get q^* from the Foreign seller with whom they are paired, the international traders can sell all remaining gold holdings in the period-1 Foreign CM at the price $\bar{\rho}$. The value of gold in the Home CM will be $\bar{\rho}$ at all dates. At this price, no Home country buyer will choose to hold gold in his portfolio. We state this result as follows.

Proposition 7 *Suppose that $\delta = \omega$. If the Home government sets the money growth rate such that $\beta < \mu < 1$, there is an initial distribution of endowments and a value $\phi_0 > 0$ such that the prices and quantities $\rho_t^H = \bar{\rho}$, $\phi_t^H = \mu^{-t}\phi_0$, $\phi_t^F = \mu^{-t}\phi_0$, $Q_{t+1} = 0$, $N_t^H = \phi_0^{-1}\mu^{(t+1)}\left[\lambda\bar{q}_\alpha(\mu) + (1-\lambda)\bar{q}_\delta(\mu)\right]$, $N_t^F = (1-\lambda)\phi_0^{-1}\mu^{(t+1)}\bar{q}_\delta(\mu)$, $n_{t+1} = \phi_0^{-1}\mu^{(t+1)}\bar{q}_\alpha(\mu)$, $\widehat{n}_{t+1} = n_{t+1}^F = \phi_0^{-1}\mu^{(t+1)}\bar{q}_\delta(\mu)$, and $m_{t+1} = \widehat{m}_{t+1} = m_{t+1}^F = 0$ for all t are an equilibrium of the model. In this equilibrium, $q_t = \bar{q}_\alpha(\mu) > \bar{q}_\alpha$ and $\widehat{q}_t = q_t^F = \bar{q}_\delta(\mu) > \bar{q}_\delta$ at all dates.*

The efficient allocation is obtained when the Home government sets $\mu = \beta$, given that $\bar{q}_\alpha(\mu) \to q^*$ and $\bar{q}_\delta(\mu) \to q^*$ as $\mu \to \beta$. Does this result render the gold standard inessential? The gold standard can implement the efficient allocation only if Home and Foreign policies are synchronized at the right level. In comparison, the Friedman rule (attained by setting $\mu = \beta$ in this environment) can implement the efficient allocation in the SOE under a fiat currency regime. However, we cannot guarantee unique implementation. In other words, we cannot guarantee that the mapping between the policy instrument (i.e., the money growth rate) and the set of allocations is unique.

6 Non-Monetary Demand for Gold

In this section, we present a version of the model where gold has non-monetary uses. We aim to show that all equilibrium properties derived in previous sections are preserved in this extended model.

Starting from the benchmark model developed above, we add a third type of agent: investors (one can think about these agents as goldsmiths as well). In the Home country, there is a measure $[0, \gamma]$ of investors, with $0 \leq \gamma \leq 1$. The investors do not participate in the decentralized market (they only trade in the CM) and derive direct utility $v(k)$ from the ownership of k units of gold between two consecutive dates (for example, as jewelry or from its use in an industrial process from which it can be recovered at the end of it). Assume that $v : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, increasing, and strictly concave. Suppose that $v'(0) = \infty$ and $v'(\infty) = 0$ (i.e., the standard Inada conditions).

We describe next the investor's portfolio problem in the Home CM and characterize the equilibrium of the extended model.

Investors. The investor always stays in the Home country and trades only in the Home CM on each date. The role of investors in the model is to provide a non-monetary demand for gold, a crucial feature of a commodity money system. We can write the investor's portfolio problem as:

$$\max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[v\left(k_{t+1}\right) + \rho_t \left(k_t - k_{t+1}\right) \right].$$

Suppose that $\rho_t > \beta\rho_{t+1}$ holds at all dates. Given that $v'(0) = \infty$ and $v'(\infty) = 0$, the first-order condition for the investor's problem is:

$$\rho_t - \beta\rho_{t+1} = v'\left(k_{t+1}\right). \tag{29}$$

Additionally, we have the transversality condition $\lim_{t\to\infty} \beta^t k_{t+1} = 0$.

Market clearing. The market-clearing condition in the Home country is now given by:

$$Q_t = \gamma k_{t+1} + \lambda m_{t+1} + (1-\lambda)\,\widehat{m}_{t+1}. \tag{30}$$

In the period-t Home CM, three different types of agents demand gold: investors (with total demand γk_{t+1}), domestic traders (with total demand λm_{t+1}), and international traders (with total demand $(1-\lambda)\,\widehat{m}_{t+1}$). The law of motion for the Home gold stock is the same as that of the benchmark model.

Demand functions. The demand functions $\widehat{m}\left(\rho_t\right)$ and $m_F\left(\rho_{t+1}\right)$ are the same as in the benchmark model. Provided $\rho_t > \beta\rho_{t+1}$, the first-order condition (29) can be used to derive the demand function $k_{t+1} = \psi\left(\rho_t - \beta\rho_{t+1}\right)$, where $\psi \equiv \left(v'\right)^{-1}$ denotes the inverse of v'. Because $\psi(\cdot)$ is strictly decreasing, the investor's demand for gold is a decreasing function of the difference $\rho_t - \beta\rho_{t+1}$.

DM output. The quantities produced in the matches involving Home country buyers in the Home and Foreign DM are given by:

$$\begin{gathered}
q_{t+1} = \lambda^{-1}\rho_{t+1}\left[Q_t - \gamma\psi\left(\rho_t - \beta\rho_{t+1}\right) - (1-\lambda)\,\hat{m}\left(\rho_t\right)\right] \le q^* \\
q^F_{t+1} = \rho_{t+1} m_F\left(\rho_{t+1}\right) \le q^* \\
\hat{q}_{t+1} = \bar{\rho}\widehat{m}\left(\rho_t\right) \le q^*.
\end{gathered}$$

The quantity $q_t \in \mathbb{R}_+$ is the period-t DM consumption of a domestic trader in the Home DM, $q_t^F \in \mathbb{R}_+$ is the period-t DM consumption of a returning international trader in the Home DM, and $\hat{q}_t \in \mathbb{R}_+$ is the period-t DM consumption of an international trader in the Foreign DM. The non-negativity of the DM consumption of the domestic trader implies that $Q_t \geq \gamma\psi\left(\rho_t - \beta\rho_{t+1}\right) + \left(1-\lambda\right)\widehat{m}\left(\rho_t\right)$ holds in equilibrium.

Equilibrium. We use the market-clearing condition (30) and the previously derived demand functions to obtain the equilibrium relation:

$$m_{t+1} = m\left(Q_t, \rho_t, \rho_{t+1}\right) \equiv \lambda^{-1}\left[Q_t - \gamma\psi\left(\rho_t - \beta\rho_{t+1}\right) - \left(1-\lambda\right)\widehat{m}\left(\rho_t\right)\right].$$

The function $m\left(Q_t, \rho_t, \rho_{t+1}\right)$ is increasing in the first and second arguments and decreasing in the third. If $\rho_t > \beta\rho_{t+1}$, we can use the domestic trader's first-order condition to derive the Euler equation:

$$\rho_t = \beta\rho_{t+1}\left[\alpha u'\left(\rho_{t+1} m\left(Q_t, \rho_t, \rho_{t+1}\right)\right) + 1 - \alpha\right]. \tag{31}$$

This equation implicitly defines $\rho_{t+1} = g\left(Q_t, \rho_t\right)$. The implicit function theorem implies:

$$g_1 = \frac{-\alpha\lambda^{-1}\rho_{t+1}^2 u''\left(q_{t+1}\right)}{\alpha\left[u'\left(q_{t+1}\right) + q_{t+1}u''\left(q_{t+1}\right)\right] + 1 - \alpha + \alpha\beta\gamma\lambda^{-1}\rho_{t+1}^2 u''\left(q_{t+1}\right)\psi'\left(\rho_t - \beta\rho_{t+1}\right)} > 0$$

$$g_2 = \frac{1}{\beta}\frac{\alpha\beta\gamma\lambda^{-1}\rho_{t+1}^2 u''\left(q_{t+1}\right)\psi'\left(\rho_t - \beta\rho_{t+1}\right) + \alpha\beta\lambda^{-1}\left(1-\lambda\right)\rho_{t+1}^2 u''\left(q_{t+1}\right)\widehat{m}'\left(\rho_t\right) + 1}{\alpha\left[u'\left(q_{t+1}\right) + q_{t+1}u''\left(q_{t+1}\right)\right] + 1 - \alpha + \alpha\beta\gamma\lambda^{-1}\rho_{t+1}^2 u''\left(q_{t+1}\right)\psi'\left(\rho_t - \beta\rho_{t+1}\right)} > 0,$$

where g_i denotes the partial derivative of $g\left(\cdot\right)$with respect to argument $i = 1, 2$. As we can see, the implicit function $g\left(Q_t, \rho_t\right)$ is strictly increasing in both arguments.

We can now formally define an equilibrium of the extended model. An equilibrium of the SOE under the gold standard is a sequence $\{Q_t, \rho_t\}$ satisfying the Euler equation (31) and the law of motion:

$$Q_{t+1} = Q_t + \left(1-\lambda\right)\left[m_F\left(g\left(Q_t, \rho_t\right)\right) - \widehat{m}\left(\rho_t\right)\right],$$

together with the boundary conditions:

$$\rho_t > \beta\rho_{t+1}$$
$$\beta\bar{\rho} \leq \rho_t \leq \beta^{-1}\bar{\rho}$$
$$0 \leq \rho_{t+1} m\left(Q_t, \rho_t, \rho_{t+1}\right) \leq q^*,$$

and the initial condition $Q_0 > 0$. Additionally, the equilibrium sequence must satisfy the transversality condition: $\lim_{t\to\infty} \beta^t \psi\left(\rho_t - \beta\rho_{t+1}\right) = 0$.

Steady state. Any steady state $\left(Q, \rho\right)$ must satisfy:

$$\rho\left[Q - \gamma\psi\left(\left(1-\beta\right)\rho\right) - \left(1-\lambda\right)\widehat{m}\left(\rho\right)\right] = \lambda\bar{q}_\alpha \tag{32}$$

and $\widehat{m}(\rho) = m_F(\rho)$.

Because $m_F(\cdot)$ is strictly increasing and $\widehat{m}(\cdot)$ is strictly decreasing, there is a unique value of gold $\rho^s > 0$ that balances gold inflows and outflows in the SOE. If $\delta = \omega$, then the only way to have the gold inflows match the gold outflows (so that the Home gold stock remains constant over time) is to set the value of gold in the Home country equal to the international value of gold.

To derive the steady-state Home gold stock consistent with the steady-state value of gold ρ^s, note that condition (32) implies:

$$Q = \frac{\lambda \bar{q}_\alpha}{\rho} + \gamma\psi\left((1-\beta)\rho\right) + (1-\lambda)\widehat{m}(\rho), \tag{33}$$

for any value of ρ. Because the right-hand side of equation (33) is strictly decreasing in ρ and goes to infinity as ρ approaches zero, there is a unique value of the Home gold stock consistent with the steady-state price ρ^s, represented by Q^s. Hence, the SOE's steady state is unique. If $\delta = \omega$, we have $\rho^s = \bar{\rho}$ and:

$$Q^s = \frac{1}{\bar{\rho}}\left[\lambda \bar{q}_\alpha + (1-\lambda)\bar{q}_\delta\right] + \gamma\psi\left((1-\beta)\bar{\rho}\right) \equiv \bar{Q}. \tag{34}$$

In this case, the Home country value of gold equals the international value of gold, and the quantity $\bar{Q}$ provides the Home gold stock consistent with that price. We summarize these findings in the following proposition.

Proposition 8 *The domestic economy in the Home country has a unique steady state with $\rho_t = \rho^s$ and $Q_t = Q^s$ at all dates. If $\delta = \omega$, the steady-state value of gold is $\bar{\rho}$, and the steady-state Home gold stock, given by $\bar{Q}$, is defined in equation (34).*

The only change from the benchmark model is that the steady-state gold stock in the Home country is now larger because of the domestic demand for gold from investors. As in the benchmark model, we simplify the dynamic analysis by assuming that $\delta = \omega$.

Non-stationary trajectories. We now turn to the formal study of the dynamic system: $(Q_{t+1}, \rho_{t+1}) = (f(Q_t, \rho_t), g(Q_t, \rho_t))$. Consider the locus $Q_{t+1} - Q_t = 0$, which is implicitly defined by $m_F(g(Q_t, \rho_t)) = \widehat{m}(\rho_t)$ in the (Q_t, ρ_t) space. The slope of this curve is given by:

$$\frac{d\rho_t}{dQ_t} = \frac{m_F'(g(Q_t, \rho_t))\, g_1(Q_t, \rho_t)}{\widehat{m}'(\rho_t) - m_F'(g(Q_t, \rho_t))\, g_2(Q_t, \rho_t)} < 0.$$

Notice that $Q_{t+1} < Q_t \Leftrightarrow \widehat{m}(\rho_t) > m_F(g(Q_t, \rho_t))$, which implies that any path for Q_t is increasing to the right of the downward-sloping locus $Q_{t+1} - Q_t = 0$ in the (Q_t, ρ_t) space.

Consider now the locus $\rho_{t+1} - \rho_t = 0$ given by:

$$Q_t = \frac{\lambda \bar{q}_\alpha}{\rho_t} + \gamma\psi\left((1-\beta)\rho_t\right) + (1-\lambda)\widehat{m}(\rho_t).$$

The slope of this curve is given by:

$$\frac{d\rho_t}{dQ_t}=\frac{1}{-\frac{\lambda\bar{q}_\alpha}{\rho_t^2}+\gamma\left(1-\beta\right)\psi'\left(\left(1-\beta\right)\rho_t\right)+\left(1-\lambda\right)\widehat{m}'\left(\rho_t\right)}<0.$$

From (31), we find that $\rho_{t+1}>\rho_t\Leftrightarrow q_{t+1}>\bar{q}_\alpha$. We can use $\rho_{t+1}=g\left(Q_t,\rho_t\right)$ to show that:

$$\rho_{t+1}>\rho_t\Leftrightarrow Q_t>\frac{\lambda\bar{q}_\alpha}{\rho_t}+\gamma\psi\left(\left(1-\beta\right)\rho_t\right)+\left(1-\lambda\right)\widehat{m}\left(\rho_t\right),$$

which implies that any path for ρ_t is increasing to the right of the downward-sloping locus $\rho_{t+1}-\rho_t=0$ in the $\left(Q_t,\rho_t\right)$ space.

As in the benchmark model, the dynamic equilibrium analysis points to the existence of a downward-sloping saddle path through the steady state $\left(\bar{Q},\bar{\rho}\right)$. We can formally establish this property by studying the linear approximation of the dynamic system near the steady state. Our next step, then, is to construct the Jacobian matrix at the steady state, which has entries:

$$\begin{aligned}
f_1\left(\bar{Q},\bar{\rho}\right)&=1+\left(1-\lambda\right)\frac{\alpha\lambda^{-1}\delta^{-1}\left[\beta^{-1}+\delta\bar{q}_\delta u''\left(\bar{q}_\delta\right)\right]}{\beta^{-1}+\alpha\bar{q}_\alpha u''\left(\bar{q}_\alpha\right)+\alpha\beta\gamma\bar{\rho}^2u''\left(\bar{q}_\alpha\right)\psi'\left(\left(1-\beta\right)\bar{\rho}\right)}\frac{u''\left(\bar{q}_\alpha\right)}{u''\left(\bar{q}_\delta\right)}>1\\
f_2\left(\bar{Q},\bar{\rho}\right)&=\left(1-\lambda\right)\left[\frac{\left[\beta^{-1}+\delta\bar{q}_\delta u''\left(\bar{q}_\delta\right)\right]}{\delta\bar{\rho}^2u''\left(\bar{q}_\delta\right)}g_2\left(\bar{Q},\bar{\rho}\right)-\frac{1}{\delta\beta\bar{\rho}^2u''\left(\bar{q}_\delta\right)}\right]>0\\
g_1\left(\bar{Q},\bar{\rho}\right)&=\frac{-\alpha\lambda^{-1}\bar{\rho}^2u''\left(\bar{q}_\alpha\right)}{\beta^{-1}+\alpha\bar{q}_\alpha u''\left(\bar{q}_\alpha\right)+\alpha\beta\gamma\lambda^{-1}\bar{\rho}^2u''\left(\bar{q}_\alpha\right)\psi'\left(\left(1-\beta\right)\bar{\rho}\right)}>0\\
g_2\left(\bar{Q},\bar{\rho}\right)&=\frac{1}{\beta}\frac{\alpha\beta\lambda^{-1}u''\left(\bar{q}_\alpha\right)\left[\gamma\psi'\left(\left(1-\beta\right)\bar{\rho}\right)+\frac{1-\lambda}{\delta\beta\bar{\rho}^2u''\left(\bar{q}_\delta\right)}\right]+1}{\beta^{-1}+\alpha\bar{q}_\alpha u''\left(\bar{q}_\alpha\right)+\alpha\beta\gamma\lambda^{-1}\bar{\rho}^2u''\left(\bar{q}_\alpha\right)\psi'\left(\left(1-\beta\right)\bar{\rho}\right)}>1.
\end{aligned}$$

All entries in the Jacobian matrix are positive, with all elements in the main diagonal strictly greater than one. This means that the trace of the Jacobian matrix is greater than two, so one of its eigenvalues is always outside the unit circle. Thus, the only hope for obtaining a stable dynamic system is through the existence of a saddle path through the steady state. We next establish a version of Proposition 2 for the extended model to ensure a unique downward-sloping saddle path for the dynamic system.

Proposition 9 *Let $D\equiv f_1\left(\bar{Q},\bar{\rho}\right)g_2\left(\bar{Q},\bar{\rho}\right)-f_2\left(\bar{Q},\bar{\rho}\right)g_1\left(\bar{Q},\bar{\rho}\right)$ denote the determinant of the Jacobian matrix. Assume $\left[f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)\right]^2>4D$. Then, a unique downward-sloping saddle path through the steady state exists provided:*

$$-1<\frac{f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)-\sqrt{\left[f_1\left(\bar{Q},\bar{\rho}\right)+g_2\left(\bar{Q},\bar{\rho}\right)\right]^2-4D}}{2}<1.$$

Given the initial condition $Q_0>0$, one can find the initial value ρ_0 on the saddle path to construct the unique equilibrium trajectory, which converges to the steady state. If $Q_0<\bar{Q}$, then $\{\rho_t\}$ is *strictly decreasing and $\{Q_t\}$ is strictly increasing along the equilibrium trajectory. If $Q_0 > \bar{Q}$, then $\{\rho_t\}$ is strictly increasing and $\{Q_t\}$ is strictly decreasing along the equilibrium trajectory.*

The previous result shows that the saddle-path property of the dynamic system is also obtained when we add to the model the non-monetary demand for gold. All the dynamic properties of the extended model are the same as those in the benchmark model. Although the dynamic properties remain essentially the same, it is important to mention that the welfare analysis will change slightly because we need to include the welfare of investors in the extended model. The implications for buyers and sellers will not change substantially because their welfare will still depend on the price dynamics in the Home country.

7 Two Large Economies

We now develop a two-country model of the world economy in which gold circulates domestically and internationally. Although it will be more difficult to characterize the dynamic equilibrium of the world economy, this setup will allow us to gain further insights into the evolution of prices, money, and trade flows across two large regions.

Assume that the world economy contains two large countries: Home and Foreign. Each country consists of a measure $[0, 1]$ of buyers, a measure $[0, 1]$ of sellers, and a measure $[0, \gamma]$ of investors (or goldsmiths), where $\gamma \leq 1$. The investors have the same preferences as those described in Section 6, and all other preferences and technologies are the same as those previously described in the SOE.

The international movement of agents is assumed to be symmetric across countries. At the beginning of each period, a random measure $1 - \lambda$ of all buyers currently in the Home (Foreign) country CM will travel abroad to the Foreign (Home) DM in the next period. Specifically, each buyer in the Home (Foreign) country finds out whether he will move to the Foreign (Home) country at date $t + 1$ in the period-t CM. Figure 8 summarizes the movement of buyers across borders. Sellers and investors never move across countries.

A buyer arriving from the Foreign (Home) country will find a Home (Foreign) country seller with probability $\delta \in (0, 1)$ in the following DM. A buyer who stays in the Home (Foreign) country from period t to $t + 1$ will find a Home (Foreign) country seller with probability $\alpha \in (0, 1)$ in the following DM. As before, we assume that $\delta \leq \alpha$ (i.e., international trade matches are harder to find than domestic ones).

Bellman equations. We now turn to constructing a world equilibrium for the model with two large economies. Let $\hat{V}_t^h\left(\hat{m}^h\right)$ denote the period-t Foreign DM value function for a buyer arriving from the period-$(t-1)$ Home CM with $\hat{m}^h \in \mathbb{R}_+$ units of money, and let $V_t^h\left(m^h\right)$ denote the period-t Home DM value function for a buyer arriving from the period-$(t-1)$ Home CM with $m^h \in \mathbb{R}_+$ units of money. Let $\hat{V}_t^f\left(\hat{m}^f\right)$ denote the period-t Home DM value function for a buyer

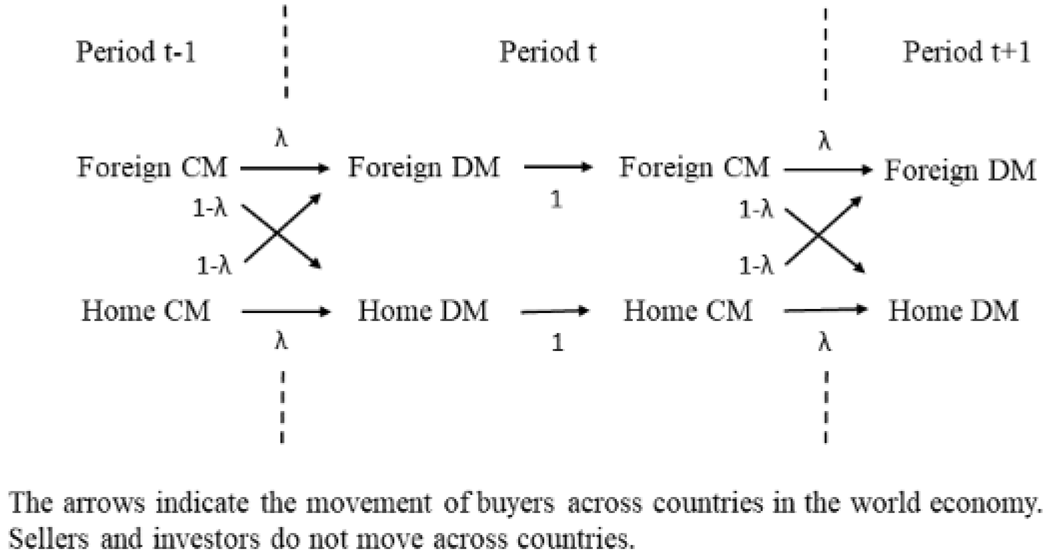

Figure 8: Movement of buyers in the model with two large economies

arriving from the period-$(t-1)$ Foreign CM with $\hat{m}^f$ units of money, and let $V_t^f\left(m^f\right)$ denote the period-t Foreign DM value function for a buyer arriving from the period-$(t-1)$ Foreign CM with m^f units of money. Let $\phi_t^h \in \mathbb{R}_+$ denote the value of gold in the period-t Home CM, and let $\phi_t^f \in \mathbb{R}_+$ denote the value of gold in the period-t Foreign CM. The Bellman equations for all buyers in the world economy are:

$$
\begin{aligned}
\hat{V}_t^h\left(\hat{m}^h\right) &= \delta\left[u\left(q_t^f\left(\hat{m}^h\right)\right)-\phi_t^f d_t^f\left(\hat{m}^h\right)\right]+\lambda \max_{m'\in\mathbb{R}_+}\left[-\phi_t^f\left(m'-\hat{m}^h\right)+\beta V_{t+1}^f\left(m'\right)\right] \\
&\quad +(1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\phi_t^f\left(m'-\hat{m}^h\right)+\beta \hat{V}_{t+1}^f\left(m'\right)\right] \\
V_t^f\left(m^f\right) &= \alpha\left[u\left(q_t^f\left(m^f\right)\right)-\phi_t^f d_t^f\left(m^f\right)\right]+\lambda \max_{m'\in\mathbb{R}_+}\left[-\phi_t^f\left(m'-m^f\right)+\beta V_{t+1}^f\left(m'\right)\right] \\
&\quad +(1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\phi_t^f\left(m'-m^f\right)+\beta \hat{V}_{t+1}^f\left(m'\right)\right] \\
\hat{V}_t^f\left(\hat{m}^f\right) &= \delta\left[u\left(q_t^h\left(\hat{m}^f\right)\right)-\phi_t^h d_t^h\left(\hat{m}^f\right)\right]+\lambda \max_{m'\in\mathbb{R}_+}\left[-\phi_t^h\left(m'-\hat{m}^f\right)+\beta V_{t+1}^h\left(m'\right)\right] \\
&\quad +(1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\phi_t^h\left(m'-\hat{m}^f\right)+\beta \hat{V}_{t+1}^h\left(m'\right)\right] \\
V_t^h\left(m^h\right) &= \alpha\left[u\left(q_t^h\left(m^h\right)\right)-\phi_t^h d_t^h\left(m^h\right)\right]+\lambda \max_{m'\in\mathbb{R}_+}\left[-\phi_t^h\left(m'-m^h\right)+\beta V_{t+1}^h\left(m'\right)\right] \\
&\quad +(1-\lambda)\max_{m'\in\mathbb{R}_+}\left[-\phi_t^h\left(m'-m^h\right)+\beta \hat{V}_{t+1}^h\left(m'\right)\right].
\end{aligned}
$$

These equations fully describe the buyer's transitions across different states in the world economy over time. From a buyer's perspective, there are four possible individual states. A buyer in the period-t Home CM can either move to the Home DM in period $t+1$ (an event that occurs with probability λ) or move to the Foreign DM in period $t+1$ (an event that occurs with probability $1-\lambda$). A buyer in the period-t Foreign CM can either move to the Foreign DM in period $t+1$ (an event that occurs with probability λ) or move to the Home DM in period $t+1$ (an event that occurs with probability $1-\lambda$).

Bargaining. We continue to assume that the buyer makes a take-it-or-leave-it offer to the seller in each bilateral match. The terms of trade in the Bellman equations are given by:

$$q_t^i(m) = \begin{cases} \phi_t^i m \text{ if } m < \frac{q^*}{\phi_t^i} \\ q^* \text{ if } m \geq \frac{q^*}{\phi_t^i} \end{cases}$$

$$d_t^i(m) = \begin{cases} m \text{ if } m < \frac{q^*}{\phi_t^i} \\ \frac{q^*}{\phi_t^i} \text{ if } m \geq \frac{q^*}{\phi_t^i} \end{cases}$$

for each country $i \in \{h, f\}$. As before, we let $q^* \in \mathbb{R}_+$ denote the surplus-maximizing quantity in a bilateral match.

Optimal portfolio choices. In this monetary model with two large economies, the necessary conditions for the existence of a world equilibrium are: $\phi_t^f \geq \beta\phi_{t+1}^f$, $\phi_t^f \geq \beta\phi_{t+1}^h$, $\phi_t^h \geq \beta\phi_{t+1}^h$, and $\phi_t^h \geq \beta\phi_{t+1}^f$. Suppose that all these conditions hold as strict inequalities at all dates. Then, by solving the portfolio problem for each type of buyer in the world economy, we obtain the following optimality conditions:

$$\phi_t^f = \beta\phi_{t+1}^f\left[\alpha u'\left(\phi_{t+1}^f m_{t+1}^f\right) + 1 - \alpha\right] \tag{35}$$

$$\phi_t^f = \beta\phi_{t+1}^h\left[\delta u'\left(\phi_{t+1}^h \hat{m}_{t+1}^f\right) + 1 - \delta\right] \tag{36}$$

$$\phi_t^h = \beta\phi_{t+1}^h\left[\alpha u'\left(\phi_{t+1}^h m_{t+1}^h\right) + 1 - \alpha\right] \tag{37}$$

$$\phi_t^h = \beta\phi_{t+1}^f\left[\delta u'\left(\phi_{t+1}^f \hat{m}_{t+1}^h\right) + 1 - \delta\right]. \tag{38}$$

In these equations, $m_{t+1}^h \in \mathbb{R}_+$ denotes the money holdings of a buyer in the period-t Home CM heading to the Home DM in period $t+1$, $m_{t+1}^f \in \mathbb{R}_+$ denotes the money holdings of a buyer in the period-t Foreign CM heading to the Foreign DM in period $t+1$, $\hat{m}_{t+1}^h \in \mathbb{R}_+$ denotes the money holdings of a buyer in the period-t Home CM heading to the Foreign DM in period $t+1$, and $\hat{m}_{t+1}^f \in \mathbb{R}_+$ denotes the money holdings of a buyer in the period-t Foreign CM heading to the Home DM in period $t+1$. In all cases, the buyer spends all his money holdings in a bilateral trade match.

Market clearing. The period-t market-clearing condition in the Home CM is given by:

$$Q_t^h = \lambda m_{t+1}^h + (1-\lambda)\,\hat{m}_{t+1}^h + \gamma k_{t+1}^h,$$

where $Q_t^h \in \mathbb{R}_+$ denotes the period-t total Home gold stock and $k_{t+1}^h \in \mathbb{R}_+$ denotes the quantity of gold Home investors hold from period t to $t+1$. The period-t market-clearing condition in the Foreign CM is:

$$Q_t^f = \lambda m_{t+1}^f + (1-\lambda)\,\hat{m}_{t+1}^f + \gamma k_{t+1}^f,$$

where $Q_t^f \in \mathbb{R}_+$ denotes the period-t total Foreign gold stock and $k_{t+1}^f \in \mathbb{R}_+$ denotes the quantity of gold Foreign investors hold from period t to $t+1$. At all dates, the world gold stock is given by $2Q \in \mathbb{R}_+$. Hence, we have $Q_t^h + Q_t^f = 2Q$ for all $t \geq 0$.

Laws of motion for gold. The laws of motion for the Home and Foreign gold stocks are:

$$\begin{aligned} Q_{t+1}^h &= Q_t^h + (1-\lambda)\left(\hat{m}_{t+1}^f - \hat{m}_{t+1}^h\right) \\ Q_{t+1}^f &= Q_t^f + (1-\lambda)\left(\hat{m}_{t+1}^h - \hat{m}_{t+1}^f\right), \end{aligned}$$

respectively. The gold stock in each country changes over time according to the international trade flow associated with that country, which is determined by the price level sequences in both countries in a perfect-foresight equilibrium.

Demand functions. It is straightforward to show that equation (36) implicitly defines $\hat{m}_{t+1}^f = \hat{m}^f\left(\phi_t^f, \phi_{t+1}^h\right)$ for any pair $\left(\phi_t^f, \phi_{t+1}^h\right)$ satisfying $\phi_t^f > \beta\phi_{t+1}^h > 0$. Moreover, it can be easily shown that $\hat{m}_1^f < 0$ and $\hat{m}_2^f > 0$, where $\hat{m}_i^f$ denotes the derivative of $\hat{m}^f(\cdot)$ with respect to argument $i \in \{1,2\}$. Similarly, we can use equation (38) to implicitly define $\hat{m}_{t+1}^h = \hat{m}^h\left(\phi_t^h, \phi_{t+1}^f\right)$ for any pair $\left(\phi_t^h, \phi_{t+1}^f\right)$ satisfying $\phi_t^h > \beta\phi_{t+1}^f > 0$. In this case, we have $\hat{m}_1^h < 0$ and $\hat{m}_2^h > 0$, where $\hat{m}_i^h$ denotes the derivative of $\hat{m}^h(\cdot)$ with respect to argument $i \in \{1,2\}$. Given these implicitly defined functions, we can use equations (35) and (37), together with the market-clearing conditions, to define the following system of equations:

$$\begin{aligned} \phi_t^f &= \beta\phi_{t+1}^f\left[\alpha u'\left(\lambda^{-1}\phi_{t+1}^f\left[2Q - Q_t^h - (1-\lambda)\,\hat{m}^f\left(\phi_t^f, \phi_{t+1}^h\right) - \gamma\psi\left(\phi_t^f - \beta\phi_{t+1}^f\right)\right]\right) + 1 - \alpha\right] \\ \phi_t^h &= \beta\phi_{t+1}^h\left[\alpha u'\left(\lambda^{-1}\phi_{t+1}^h\left[Q_t^h - (1-\lambda)\,\hat{m}^h\left(\phi_t^h, \phi_{t+1}^f\right) - \gamma\psi\left(\phi_t^h - \beta\phi_{t+1}^h\right)\right]\right) + 1 - \alpha\right], \end{aligned}$$

where the demand function $k_{t+1}^i = \psi\left(\phi_t^i - \beta\phi_{t+1}^i\right)$, with $i \in \{h, f\}$, is derived from the investor's optimization problem described in Section 6.

We can solve the previous system of equations to obtain a solution of the form: $\mathbf{\Phi}_{t+1} = \mathbf{G}\left(\mathbf{\Phi}_t, Q_t^h\right)$, where $\mathbf{\Phi}_t = \left(\phi_t^h, \phi_t^f\right) \in \mathbb{R}_+^2$ denotes the vector of prices and $\mathbf{G}: \mathbb{R}_+^3 \to \mathbb{R}_+^2$ denotes the implicitly defined mapping in this solution.

Equilibrium. The next step toward world equilibrium is to define the function $F: \mathbb{R}_+^3 \to \mathbb{R}_+$ by:

$$F\left(\mathbf{\Phi}_t, Q_t^h\right) \equiv Q_t^h + (1-\lambda)\left[\hat{m}^f\left(\phi_t^f, G^h\left(\mathbf{\Phi}_t, Q_t^h\right)\right) - \hat{m}^h\left(\phi_t^h, G^f\left(\mathbf{\Phi}_t, Q_t^h\right)\right)\right],$$

where $\mathbf{G} = \left(G^h, G^f\right)$. Then, we can formally define an equilibrium for the world economy under the gold standard as a three-dimensional dynamic system in the $\left(\mathbf{\Phi}_t, Q_t^h\right)$ space satisfying the dynamic equations:

$$\begin{aligned} Q_{t+1}^h &= F\left(\mathbf{\Phi}_t, Q_t^h\right) \\ \mathbf{\Phi}_{t+1} &= \mathbf{G}\left(\mathbf{\Phi}_t, Q_t^h\right) \end{aligned}$$

together with the boundary conditions $\phi_t^f \geq \beta\phi_{t+1}^f$, $\phi_t^f \geq \beta\phi_{t+1}^h$, $\phi_t^h \geq \beta\phi_{t+1}^h$, and $\phi_t^h \geq \beta\phi_{t+1}^f$ at all dates, given the initial gold stock $Q_0^h > 0$ in the Home country. This dynamic system contains one predetermined variable, Q_t^h, and two forward-looking (or jump) variables, ϕ_t^h and ϕ_t^f.

Symmetric steady state. One can easily show that a unique symmetric steady state exists in which the world gold stock is evenly distributed between the two countries. This symmetric steady state is characterized by the fact that $Q_t^h = Q$, $\phi_t^h = \phi_t^f$, and $\hat{m}_{t+1}^h = \hat{m}_{t+1}^f$ hold at all dates. At each date, gold inflows and outflows are perfectly matched within each country, given the constant equilibrium price level in each economy. In any bilateral match involving a buyer arriving from the Foreign (Home) country and a Home (Foreign) country seller, the quantity traded is $\bar{q}_\delta$ at any date. In any bilateral match involving a buyer from the Home (Foreign) country and a Home (Foreign) country seller, the quantity traded is $\bar{q}_\alpha$ at any date.

Non-stationary trajectories. If the initial gold stock in the Home country, Q_0^h, is different from its symmetric steady-state value, Q, there can be a dynamic adjustment to the steady state. Suppose a dynamic path to the steady state exists. In that case, it is clear that the country that starts with a per capita gold stock greater than Q will eventually lose gold along the transition path. In contrast, the other country will accumulate gold until the per capita gold stock in the world economy is equalized across countries. However, this transition path may not be monotonic for the equilibrium prices and quantities, and there can be multiple equilibrium trajectories.

One possible approach to characterizing non-stationary equilibrium trajectories for the world economy under the gold standard is to linearize the dynamic system around the symmetric steady state. A sufficient condition for the linearized system to have a unique solution for a transition path to the steady state is that two eigenvalues of the Jacobian matrix are outside the unit circle and that one eigenvalue is inside the unit circle. However, we cannot guarantee that a monotonic path of convergence for the price level and the quantity of money exists in this case (something we could accomplish in the SOE). We could further study the Jacobian matrix's properties by imposing restrictions on preferences and technologies that deliver monotonicity in the transition path. We leave this task for future work.

8 Conclusion

We have constructed a small open economy model in the New Monetarist tradition to study inflation and output dynamics under the gold standard. Our analysis has identified three important properties of the gold standard: (i) long-run price stability; (ii) the non-neutrality of money; and (iii) the necessity of short-run price fluctuations to accommodate trade shocks as part of the adjustment mechanism for prices and the quantity of money. We have also demonstrated that achieving efficiency under the gold standard relies on international policy cooperation, which may pose political challenges. We have also considered the properties of fiat money in this environment and have shown that efficiency can be attained by an optimal monetary policy that implies the demonetization of gold. Finally, we have developed a version of the model with two large countries.

The model is highly tractable and formally describes the price-specie adjustment mechanism. This mechanism, initially developed by David Hume, has remained the cornerstone of subsequent theoretical analyses of the gold standard and other international monetary arrangements. We believe that the framework developed in this paper can also be used for studying other issues in international economics, such as the properties of different exchange rate regimes under fiat currencies and the relationship between international capital flows and monetary policy implementation in a small open economy or with two large economies.

References

Altermatt, L., Iwasaki, K., and Wright, R. (2021). Asset pricing in monetary economies. *Journal of International Money and Finance*, 115:102352.

Bordo, M., Eichengreen, B., Klingebiel, D., Martinez-Peria, M. S., and Rose, A. K. (2001). Is the crisis problem growing more severe? *Economic Policy*, 16(32):53–82.

Bordo, M. D. (1981). The classical gold standard: some lessons for today. *Federal Reserve Bank of St. Louis Review*, 63(May):2–17.

Bordo, M. D. and Kydland, F. E. (1995). The gold standard as a rule: An essay in exploration. *Explorations in Economic History*, 32(4):423–464.

Choi, M. and Rocheteau, G. (2021). Money mining and price dynamics. *American Economic Journal: Macroeconomics*, 13(4):246–294.

Choi, M. and Rocheteau, G. (2022). Money mining and price dynamics: The case of divisible currencies. *Journal of Economic Dynamics and Control*, 142:1–20.

Devereux, M. and Shi, S. (2013). Vehicle currency. *International Economic Review*, 54(1):97–133.

Eichengreen, B. (2019). *Globalizing Capital: A History of the International Monetary System.* Princeton University Press, 3rd edition.

Farhi, E. and Maggiori, M. (2018). A model of the international monetary system. *Quarterly Journal of Economics*, 133(1):295–355.

Ford, A. (1960). Notes on the working of the gold standard before 1914. *Oxford Economic Papers*, 11(1):52–76.

Friedman, M. and Schwartz, A. (1963). *A Monetary History of the United States, 1867-1960.* Princeton University Press.

Gallarotti, G. M. (1995). *The Anatomy of an International Monetary Regime: The Classical Gold Standard, 1880-1914.* Oxford University Press.

Geromichalos, A. and Herrenbrueck, L. (2016). Monetary policy, asset prices, and liquidity in over-the-counter markets. *Journal of Money, Credit and Banking*, 48(1):35–79.

Geromichalos, A., Lee, J., Lee, S., and Oikawa, K. (2016). Over-the-counter trade and the value of assets as collateral. *Economic Theory*, 62:443–475.

Geromichalos, A., Licari, J. M., and Suárez-Lledó, J. (2007). Monetary policy and asset prices. *Review of Economic Dynamics*, 10(4):761–779.

Geromichalos, A. and Simonovska, I. (2014). Asset liquidity and international portfolio choice. *Journal of Economic Theory*, 151(3):342–380.

Gomis-Porqueras, P., Kam, T., and Waller, C. (2017). Nominal exchange rate determinacy under the threat of counterfeiting. *American Economic Journal: Macroeconomics*, 9(2):256–273.

Hume, D. ([1752], 1977). Of the balance of trade. In *Essays Moral, Political, Literary (LF ed.)*. Liberty Fund.

Kiyotaki, N., Matsuyama, K., and Matsui, A. (1993). Toward a theory of international currency. *Review of Economic Studies*, 60(2):283–307.

Kiyotaki, N. and Wright, R. (1989). On money as a medium of exchange. *Journal of Political Economy*, 97(4):927–954.

Lagos, R. (2010). Asset prices and liquidity in an exchange economy. *Journal of Monetary Economics*, 57(8):913–930.

Lagos, R., Rocheteau, G., and Wright, R. (2017). Liquidity: A new monetarist perspective. *Journal of Economic Literature*, 55(3):1–70.

Lagos, R. and Wright, R. (2003). Dynamics, cycles, and sunspot equilibria in 'genuinely dynamic, fundamentally disaggregative' models of money. *Journal of Economic Theory*, 109(2):156–171.

Lagos, R. and Wright, R. (2005). A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3):463–484.

Palma, N. (2022). The real effects of monetary expansions: Evidence from a large scale historical experiment. *Review of Economic Studies*, 89(3):1593–1627.

Redish, A. (1990). The evolution of the gold standard in England. *Journal of Economic History*, 50(4):789–805.

Rocheteau, G. and Wright, R. (2005). Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202.

Sargent, T. (2019). Commodity and token monies. *Economic Journal*, 129(619):1457–1476.

Sargent, T. J. and Wallace, M. (1983). A model of commodity money. *Journal of Monetary Economics*, 12(1):163–187.

Schwartz, A. J. (1986). Alternative monetary regimes: The gold standard. *In Colin Campbell and William Dougan, eds. Alternative Monetary Regimes*, pages 44–72.

Shi, S. (1995). Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67(2):467–498.

Trejos, A. and Wright, R. (1995). Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141.

Trejos, A. and Wright, R. (2001). International currency. *B.E. Journal of Macroeconomics*, 1(0).

Velde, F. R. and Weber, W. E. (2000). A model of bimetallism. *Journal of Political Economy*, 108(6):1210–1234.

Velde, F. R., Weber, W. E., and Wright, R. (1999). A model of commodity money, with applications to Gresham's law and the debasement puzzle. *Review of Economic Dynamics*, 2(1):291–323.

Wallace, N. (2001). Whither monetary economics? *International Economic Review*, 42(4):847–869.

Williamson, S. and Wright, R. (2010). New monetarist economics: Methods. *Federal Reserve Bank of St. Louis Review*, 92(4):265–302.

Zhang, C. (2014). An information-based theory of international currency. *Journal of International Economics*, 93(2):286–301.

Zhu, T. (2022). On non-neutrality of the exchange-rate regime. *Unpublished manuscript*.